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Abstract

The transformation method has been proposed as a practical tool for the simulation and the analysis of systems with uncertain parameters using fuzzy arithmetic. A major advantage of this method, among others, is the fact that in the analysis part of the method, the degree of influence for each uncertain parameter can be computed with a rather simple formula. However, a thorough examination of how accurate this measure of the degree of influence really is has not yet been performed. In the paper, the influence measure provided by the transformation method is compared to a rather classical one derived from differential calculus. Furthermore, an additional measure is introduced which can serve as an indicator for the effectiveness and the reliability of the influence measure calculated by the transformation method.

Keywords: fuzzy numbers, fuzzy arithmetic, sensitivity analysis

1 Introduction

To achieve reliable results for the numerical solution of real-world problems, exact values for the parameters of the model equations should be available. In practice, however, those exact values can often not be provided, and the models usually exhibit a rather high degree of uncertainty. Normally, these uncertainties arise from incomplete or imprecise information or from problems of identification, and are finally reflected in uncertain model parameters, sometimes also in uncertain initial or boundary conditions. Consequently, the results that are obtained for solutions which only use one specific crisp value as the most likely value for an uncertain parameter cannot be considered to be representative for the whole spectrum of possible results.

As a very practical approach to solve this limitation, the use of fuzzy arithmetic based on the transformation method [3] has been introduced. Here, the uncertain parameters of the model are numerically implemented as fuzzy numbers [2, 5], and the arithmetical operations for the evaluation of the model are carried out by using the transformation method as an extended or generalized arithmetic for fuzzy numbers. Basically, this method can be considered as an advanced and extended version of the so-called vertex method [1], showing, however, a couple of merits [3]. By this technique, the complete information about the uncertainties in the model can be included and one can demonstrate how these uncertainties are processed through the calculation procedure. The transformation method avoids the possibly serious drawbacks of conventional fuzzy arithmetic [4, 3], and, as an additional advantage, it can also be used to determine the degrees of influence for each fuzzy parameter, i.e. the proportion to which the uncertainty of each model parameter contributes to the overall uncertainty of the model output.

2 Transformation method

Recalling the main topics of [3], the implementation of fuzzy arithmetic using the transformation method is outlined in the following.

2.1 Simulation of fuzzy-parameterized models

In general, a fuzzy-parameterized model consists of three key components:

1. A set of n independent fuzzy-valued input parameters p_i with the membership functions $\mu_{p_i}(x_i)$, $i = 1, 2, \ldots, n$.
2. The model itself, which can be interpreted as a set of N functions f_r, $r = 1, 2, \ldots, N$, that perform some operations on the fuzzy input variables.
3. A set of N fuzzy-valued output parameters q_r with the membership functions $\mu_{q_r}(z_r)$, $r = 1, 2, \ldots, N$, that are obtained as the result of the functions f_r.

For the simulation of a fuzzy-parameterized model using the transformation method in its *general form* [3], each fuzzy parameter p_i, $i = 1, 2, \ldots, n$ is first being decomposed into a set P_i of $(m+1)$ intervals $X_i^{(j)}$, $j = 0, 1, \ldots, m$, of the form

$$P_i = \left\{ X_i^{(0)}, X_i^{(1)}, \ldots, X_i^{(m)} \right\} \tag{1}$$

with

$$X_i^{(j)} = \left[a_i^{(j)}, b_i^{(j)} \right], \quad a_i^{(j)} \leq b_i^{(j)}, \tag{2}$$
$$i = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, m.$$

For the purpose of decomposition, the μ-axis is subdivided into m segments, equally spaced by $\Delta\mu = 1/m$ (Fig. 1). The $(m+1)$ levels of membership μ_j are then given by

$$\mu_j = \frac{j}{m}, \quad j = 0, 1, \ldots, m. \tag{3}$$

The intervals of each level of membership μ_j, $j = 0, 1, \ldots, m$, can now be transformed into arrays $\widehat{X}_i^{(j)}$ of the following form:

$$\widehat{X}_i^{(j)} = \big(\overbrace{\gamma_{1,i}^{(j)}, \gamma_{2,i}^{(j)} \ldots, \gamma_{(m+1-j),i}^{(j)}, \ldots, \gamma_{1,i}^{(j)}, \gamma_{2,i}^{(j)} \ldots, \gamma_{(m+1-j),i}^{(j)}}^{(m+1-j)^{i-1} \quad \text{(m+1-j)-tuples}} \big) \tag{4}$$

with

$$\gamma_{l,i}^{(j)} = \underbrace{\left(c_{l,i}^{(j)}, \ldots, c_{l,i}^{(j)} \right)}_{(m+1-j)^{n-i} \text{ elements}} \tag{5}$$

and

$$c_{l,i}^{(j)} = \begin{cases} a_i^{(j)} & \text{for} \quad l = 1 \quad \text{and} \quad j = 0, 1, \ldots, m, \\ \frac{1}{2}\left(c_{l-1,i}^{(j+1)} + c_{l,i}^{(j+1)} \right) & \text{for} \quad l = 2, 3, \ldots, m-j \quad \text{and} \quad j = 0, 1, \ldots, m-2, \\ b_i^{(j)} & \text{for} \quad l = m-j+1 \quad \text{and} \quad j = 0, 1, \ldots, m, \end{cases} \tag{6}$$

Note that $a_i^{(j)}$ and $b_i^{(j)}$ are the lower and upper bounds of the interval $X_i^{(j)}$ at the membership level μ_j for the ith uncertain model parameter.

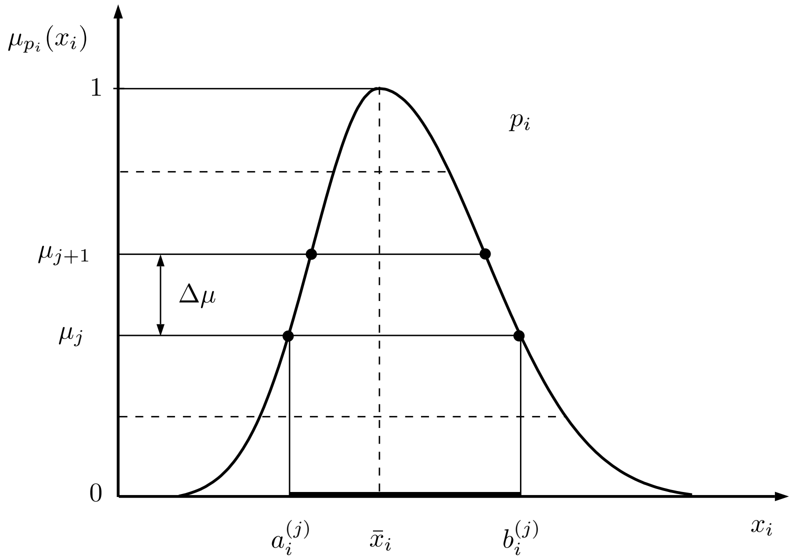

Figure 1: Implementation of the ith uncertain parameter as a fuzzy number p_i decomposed into intervals.

Assuming that the fuzzy-parameterized model is given by the arithmetical expression f with the functional form

$$q = f\left(p_1, p_2, \ldots, p_n\right) , \tag{7}$$

its estimation is then carried out by evaluating the expression separately at each of the positions of the arrays using the conventional arithmetic for crisp numbers. Thus, if the output q of the system can be expressed in its decomposed and transformed forms by the arrays $\widehat{Z}^{(j)}$, $j = 0, 1, \ldots, m$, the kth element ${}^k\hat{z}^{(j)}$ of the array $\widehat{Z}^{(j)}$ is given by

$${}^k\hat{z}^{(j)} = f\left({}^k\hat{x}_1^{(j)}, {}^k\hat{x}_2^{(j)}, \ldots, {}^k\hat{x}_n^{(j)}\right) , \tag{8}$$

where ${}^k\hat{x}_i^{(j)}$ denotes the kth element of the array $\widehat{X}_i^{(j)}$. Finally, the fuzzy-valued result q of the problem can be achieved in its decomposed form

$$Z^{(j)} = \left[a^{(j)}, b^{(j)}\right] , \quad j = 0, 1, \ldots, m , \tag{9}$$

by retransforming the arrays $\widehat{Z}^{(j)}$ according to the recursive formulae

$$\left.\begin{aligned} a^{(j)} &= \min_k \left(a^{(j+1)}, {}^k\hat{z}^{(j)}\right) \\ b^{(j)} &= \max_k \left(b^{(j+1)}, {}^k\hat{z}^{(j)}\right) \end{aligned}\right. , \quad j = 0, 1, \ldots, m-1 , \tag{10}$$

and

$$a^{(m)} = \min_k \left({}^k\hat{z}^{(m)}\right) = \max_k \left({}^k\hat{z}^{(m)}\right) = b^{(m)} . \tag{11}$$

2.2 Analysis of fuzzy-parameterized models

Obviously, the degree of fuzziness of the output parameter q, or in general, of the N output parameters q_r, $r = 1, 2, \ldots, N$, will depend on the degree of fuzziness of the input parameters

p_i, $i = 1, 2, \ldots, n$. However, without additional calculations, we cannot determine to which extend the degree of uncertainty of each input parameter contributes to the overall degree of uncertainty of the model output.

To illustrate this, let us consider, as a simple example, a model with two fuzzy input variables p_1 and p_2 with their membership functions $\mu_{p_1}(x_1)$ and $\mu_{p_2}(x_2)$ according to Fig. 2, and one output variable q. As the model function, we assume

$$f(p_1, p_2) = p_2 \cos(\pi p_1) \; . \tag{12}$$

The fuzzy-valued result q is shown in Fig. 2.

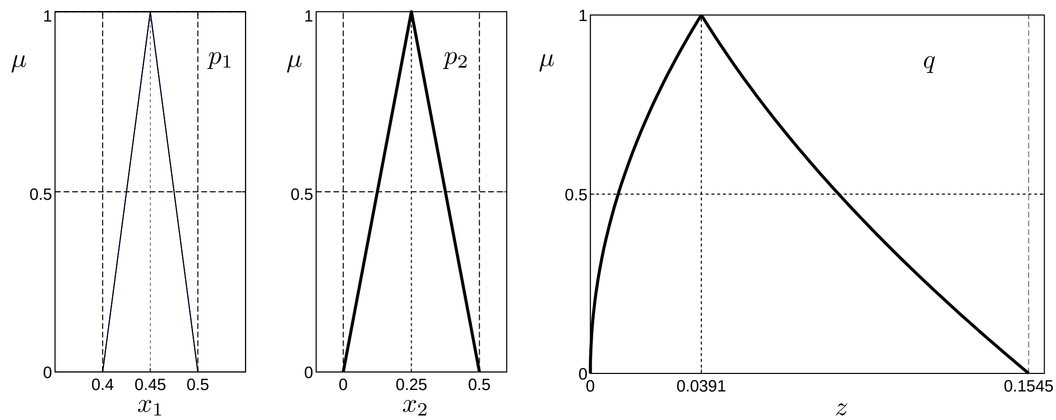

Figure 2: Membership functions of the fuzzy input parameters p_1 and p_2 and of the fuzzy output parameter $q = f(p_1, p_2)$.

As for the question to which degree each of the two input parameters contributes to the overall uncertainty of the result, one might guess that p_2 has a larger effect on the uncertainty of the result, since the worst-case variation from the peak value of the membership function is $\pm 100\%$, which is considerably larger than the variation of $\pm 11\%$ of parameter p_1. However, the examination below shows that this is not the case.

A qualitative answer to this question can be obtained by evaluating the function f again, but this time, we assume one of the parameters to be crisp, i.e. we evaluate the function f by using the crisp peak values $\bar{x}_1$ and $\bar{x}_2$ of the membership functions $\mu_{p_1}(x_1)$ and $\mu_{p_2}(x_2)$ instead of p_1 and p_2, respectively. We obtain two different results

$$q_1 = f(p_1, \bar{x}_2) \quad \text{and} \quad q_2 = f(\bar{x}_1, p_2) \; , \tag{13}$$

each depending on only one fuzzy variable. The results are shown in Fig. 3. Comparing the two plots, it can be concluded that despite the fact that parameter p_1, in worst case, varies just 11% compared to its peak value, it contributes to about the same extend to the overall uncertainty as parameter p_2. Therefore, the degree of influence of parameter p_1 appears to be about nine times larger than the degree of influence of parameter p_2 for the examined fuzzy numbers p_1 and p_2.

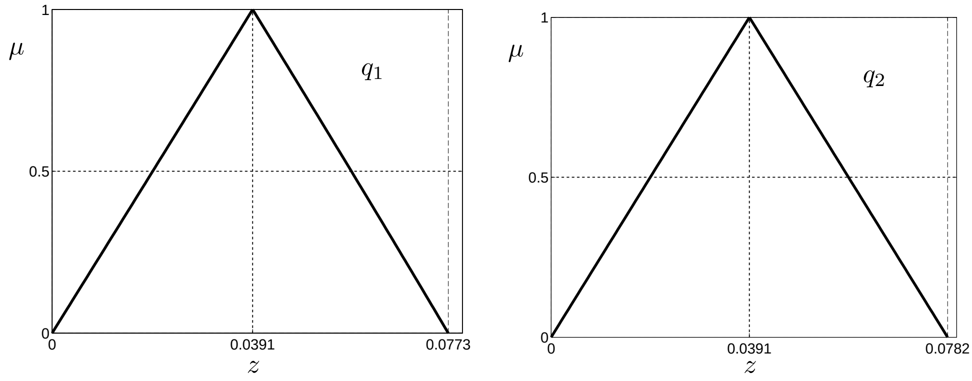

Figure 3: Membership functions of the fuzzy output parameters $q_1 = f(p_1, \bar{x}_2)$ and $q_2 = f(\bar{x}_1, p_2)$.

To quantitatively determine the degree of influence of each input parameter on the overall uncertainty of the fuzzy-valued model output, the transformation method provides a procedure which is presented as the analysis part of the method in [3] and will be described in Section 4. In order to rate the reliability and the efficiency of this measure, the results of the influence measure based on the transformation method will be compared to an influence measure that is derived from differential calculus and can quite easily be determined as long as the model function f is available in analytical form.

3 Measuring the degree of influence using the total differential

The underlying idea is to use the total differential $\mathrm{d}f$ of the model function $f(p_1, p_2, \ldots, p_n)$ to compute the degrees of influence for each fuzzy input parameter p_i, $i = 1, 2, \ldots, n$.

3.1 Description of the procedure

Let $\bar{x}_i$ be the variable where the membership function $\mu_{p_i}(x_i)$ of each fuzzy parameter p_i, $i = 1, 2, \ldots, n$, has its peak. According to differential calculus, the total differential at the point $\bar{P} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is then defined as

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{P})\,\mathrm{d}x_i \tag{14}$$

where $\mathrm{d}f$ can be interpreted as an approximation of the overall change rate of the function value $f(\bar{P})$ when the input parameters x_i are changed by $\mathrm{d}x_i$ around $\bar{x}_i$. Each input change $\mathrm{d}x_i$ separately contributes by $(\mathrm{d}f)_i$ to the overall change rate $\mathrm{d}f$. If the change rate $\mathrm{d}x_i$ of the ith input parameters is now assumed to be a constant percentage c of its corresponding peak value $\bar{x}_i$, we can substitute $\mathrm{d}x_i$ by $c\,\bar{x}_i$ to obtain

$$\mathrm{d}f = \sum_{i=1}^{n} (\mathrm{d}f)_i = c \sum_{i=1}^{n} \bar{x}_i \, \frac{\partial f}{\partial x_i}(\bar{P}) \; . \tag{15}$$

As a relative measure of influence, we can now define the normalized change rates ρ_i^* of each parameter as

$$\rho_i^* = \frac{|(\mathrm{d}f)_i|}{\sum\limits_{q=1}^{n} |(\mathrm{d}f)_q|} = \frac{\bar{x}_i \left|\frac{\partial f}{\partial x_i}(\bar{P})\right|}{\sum\limits_{q=1}^{n} \left|\bar{x}_q \frac{\partial f}{\partial x_q}(\bar{P})\right|} \tag{16}$$

satisfying the consitency condition

$$\sum_{i=1}^{n} \rho_i^* = 1 \ . \tag{17}$$

Since the fuzziness of the model output $q = f(p_1, p_2)$ can be seen as the overall range of variation $(\mathrm{d}f)$ that is applied to its peak value, the normalized change rates ρ_i^* in (16) can be interpreted as a measure for the degree of influence of the fuzziness of the parameter p_i on the uncertainty of the model output q.

To illustrate the procedure, let us consider again the example from Section 2.2. The partial derivatives of the function $f(x_1, x_2) = x_2 \cos(\pi x_1)$ at $\bar{P} = (\bar{x}_1, \bar{x}_2)$ are

$$\frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2) = -\pi\, \bar{x}_2 \sin(\pi \bar{x}_1) \quad \text{and} \quad \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2) = \cos(\pi \bar{x}_1). \tag{18}$$

Thus, we obtain for the degree of influence

$$\rho_1^* = \frac{\left|\bar{x}_1 \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\right|}{\left|\bar{x}_1 \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)\right| + \left|\bar{x}_2 \frac{\partial f}{\partial x_2}(\bar{x}_1, \bar{x}_2)\right|} \tag{19}$$

Substituting $\bar{x}_1 = 0.45$ and $\bar{x}_2 = 0.25$ gives $\rho_1^* = 89.9\%$. Similarly, ρ_2^* can be obtained to 10.1%.

In the two-dimensional case, the degree of influence can nicely be illustrated using a contour plot. As one can see in Fig. 4, the variation of x_1 around $\bar{x}_1 = 0.45$ has a higher impact on the variation of the result than the variation of x_2 around $\bar{x}_2 = 0.25$.

3.2 Properties and limitations of the procedure

The degrees of influence obtained with the procedure above are independent of the fuzziness of the input parameters. This guarantees that no matter how different the uncertainty ranges of the input parameters are, the proper degrees of influence can be achieved.

However, the degrees of influence are dependent on the actual peak values of the input parameters. Therefore, when using the same model function, completely different degrees of influence may be obtained if the sets of input parameters differ significantly. Whether this is the case depends on the model function. Especially, when considering one of the fuzzy variables as crisp, e.g. after its degree of influence proves negligible, one has to keep in mind that for another set of fuzzy input parameters, the variable may no longer be negligible at all.

For the example from Section 2.2, the degrees of influence are dependent on the actual peak value $\bar{x}_1$, but independent of $\bar{x}_2$, since we can eliminate $\bar{x}_2$ from the calculation of the degree of influence ρ_1^*, as can easily be seen when substituting (18) into (19). Fig. 5 shows the degree of influence of ρ_1^* in dependence of the actual peak value $\bar{x}_1$.

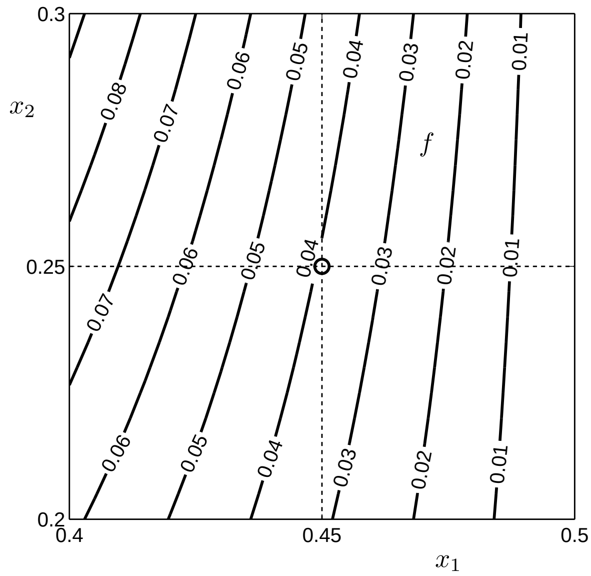

Figure 4: Contour line plot of the function $f(x_1, x_2) = x_2 \cos(\pi x_1)$ around $\bar{x}_1 = 0.45$ and $\bar{x}_2 = 0.25$.

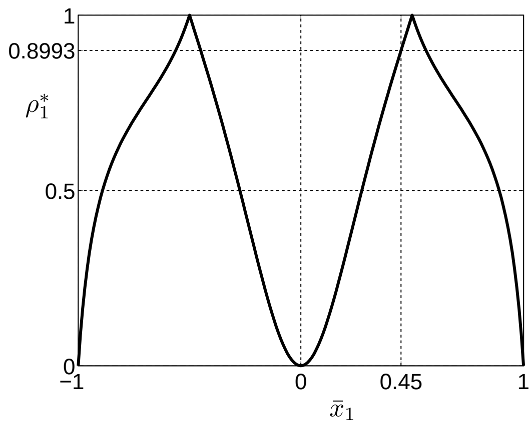

Figure 5: Degree of influence of ρ_1^* in dependence of the peak value $\bar{x}_1 \in [-1, 1]$.

To achieve the desired independence of the degrees of influence from the dimensions of the fuzzy parameters p_i, some scaling is performed in (19) using the peak values of the membership functions. Therefore, an input parameter that has its peak value at zero will automatically receive a degree of influence of zero. This can also be seen from Fig. 5.

4 Measuring the degree of influence using the transformation method

As a second output of the transformation method, in addition to the simulation of fuzzy-parameterized models, the degrees of influence of each fuzzy-valued model parameter on the overall fuzziness of the model output can be determined. The so-obtained measures of influence are basically an approximation of the ones computed on the basis of the total differential, as discussed in Section 3, which will be shown below.

4.1 Description of the procedure

In the following, the procedure of computing the degrees of influence of the fuzzy-valued model parameters is formulated for the general form of the transformation method. Nevertheless, the conclusions drawn from this case are in equal manner valid for the transformation method in its reduced form [3].

In a first step, the coefficients $\eta_i^{(j)}$, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, (m-1)$, are to be determined according to the following formulae:

$$\eta_i^{(j)} = \left[(m+1-j)^{n-1}(b_i^{(j)} - a_i^{(j)})\right]^{-1} \times \sum_{k=1}^{(m+1-j)^{n-i}} \sum_{l=1}^{(m+1-j)^{i-1}} \left({}^{k_2(k,l)}\hat{z}^{(j)} - {}^{k_1(k,l)}\hat{z}^{(j)} \right) \tag{20}$$

with

$$\begin{aligned} k_1(k,l) &= k + (m+1-j)(l-1)(m+1-j)^{n-i} \\ &= k + (l-1)(m+1-j)^{n-i+1} \\ k_2(k,l) &= k + ((m+1-j)l - 1)\,(m+1-j)^{n-i} \,. \end{aligned} \tag{21}$$

The values $a_i^{(j)}$ and $b_i^{(j)}$ denote the lower and upper bound of the interval $X_i^{(j)}$, and ${}^k\hat{z}^{(j)}$ is the kth elements of the array $\widehat{Z}^{(j)}$ as calculated in (8) of the simulation part of the transformation method. The coefficients $\eta_i^{(j)}$ can be interpreted as gain factors which express the effect of the uncertainty of the ith parameter p_i on the uncertainty of the model output q at the membership level μ_j.

To achieve a form that is independent of the dimensions of the uncertain parameters p_i, the standardized mean gain factors κ_i can be determined as an overall measure of influence according to

$$\kappa_i = \frac{\sum_{j=1}^{m-1} \mu_j \left| \eta_i^{(j)} \, (a_i^{(j)} + b_i^{(j)}) \right|}{2 \sum_{j=1}^{m-1} \mu_j} = \frac{1}{m-1} \sum_{j=1}^{m-1} \mu_j \left| \eta_i^{(j)} \, (a_i^{(j)} + b_i^{(j)}) \right| \,. \tag{22}$$

In a final step, as an relative measure of influence, the normalized values ρ_i can be determined for $i = 1, 2, \ldots, n$ according to

$$\rho_i = \frac{\kappa_i}{\sum_{q=1}^{n} \kappa_q} = \frac{\sum_{j=1}^{m-1} \mu_j \left| \eta_i^{(j)} \, (a_i^{(j)} + b_i^{(j)}) \right|}{\sum_{q=1}^{n} \sum_{j=1}^{m-1} \mu_j \left| \eta_q^{(j)} \, (a_q^{(j)} + b_q^{(j)}) \right|} \tag{23}$$

satisfying the consistency condition

$$\sum_{i=1}^{n} \rho_i = 1 \; . \tag{24}$$

The standardized mean gain factors κ_i and, as a relative measure, the degrees of influence ρ_i, quantify the effect of the ith varying parameter p_i on the overall variation of the problem output q, assuming every parameter to be varied relatively to the same extent.

4.2 Application and analysis of the procedure

To illustrate the procedure, we will apply the calculation of the influence measures using the general transformation method to the example from Section 2.2. We will show that actually the well-known scheme of central differences is used to compute the gain factors $\eta_i^{(j)}$, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, (m-1)$, which can in turn be interpreted as an approximation to the partial derivatives of the fuzzy-parameterized model function. We will conclude that in general, for symmetric membership functions, the influence measures of the general transformation method closely correspond to the ones computed with the total differential as presented in Section 3.

Fig. 6 shows the decomposition of the two fuzzy parameters p_1 and p_2 into sets of $m + 1 = 5$ intervals each. We will use this decomposed form in the following.

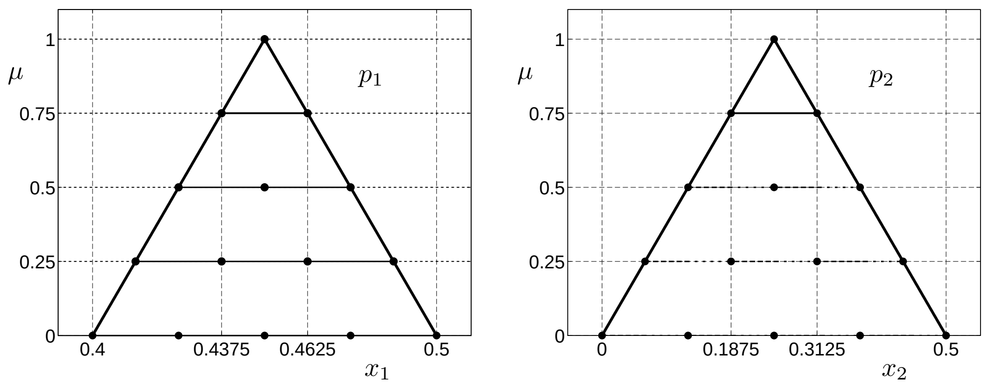

Figure 6: Decomposition of the fuzzy parameters p_1 and p_2 into sets of $m + 1 = 5$ intervals using the general transformation method.

In a first step, the coefficients $\eta_i^{(j)}$ (or gain factors) must be computed, for each parameter p_i, $i = 1, 2$, and each membership level μ_j, $j = 1, 2, \ldots, 4$. Having, for example, a closer look at the membership level $\mu_3 = 0.75$, we obtain for the gain factor of parameter p_1 using (20)

$$\eta_1^{(3)} = \frac{1}{2\,(b_1^{(3)} - a_1^{(3)})} \sum_{k=1}^{2} \sum_{l=1}^{1} \left({}^{k_2(k,l)}\hat{z}^{(3)} \; - \; {}^{k_1(k,l)}\hat{z}^{(3)} \right)$$

$$= \frac{1}{2}\,(-0.969 - 0.582) = -0.7755 \qquad (25)$$

with

$$a_1^{(3)} = 0.4375\,, \qquad b_1^{(3)} = 0.4625\,,$$
$$\left({}^{k_2(1,1)}\hat{z}^{(3)} \; - \; {}^{k_1(1,1)}\hat{z}^{(3)} \right) = f(0.4625, 0.1875) - f(0.4375, 0.1875) \qquad (26)$$
$$\left({}^{k_2(2,1)}\hat{z}^{(3)} \; - \; {}^{k_1(2,1)}\hat{z}^{(3)} \right) = f(0.4625, 0.3125) - f(0.4375, 0.3125)\;.$$

A graphical interpretation of the calculation of $\eta_1^{(3)}$ as an approximation of $\partial f/\partial x_1\,(\bar{x}_1, \bar{x}_2)$ is shown in Fig. 7. The outer dashed lines are secants to the model function at $f(x_1, 0.1875)$ and $f(x_1, 0.3125)$ with the variable g denoting the gradients. The inner dashed line is the tangent in $f(\bar{P})$ in x_1-direction, the gradient of which corresponds to the value of the derivative $\partial f/\partial x_1\,(\bar{x}_1, \bar{x}_2) = g_{\text{exact}}$.

If the membership function $\mu_{p_1}(x_1)$ of the parameter p_1 is symmetric, we see that the gain factor $\eta_1^{(3)}$ is a good approximation of the partial derivative in x_1 of the model function $f(x_1, x_2)$ at $\bar{P} = (\bar{x}_1, \bar{x}_2) = (0.45, 0.25)$:

$$\frac{\partial f}{\partial x_1}(0.45, 0.25) \approx -0.7755\;. \qquad (27)$$

To illustrate how the approximation is performed, we rewrite (20), assuming the membership functions to be symmetric. It shows that actually, an approximation by central differences is performed where the mean value of the computed central differential quotients then gives the final approximation of the partial derivative at $\bar{P}$.

For symmetric fuzzy parameters, where $\bar{x}$ denotes the center of the intervals after decomposition, we have

$$a_i^{(j)} = \bar{x}_i - \Delta x_i^{(j)} \quad \text{and} \quad b_i^{(j)} = \bar{x}_i + \Delta x_i^{(j)}\;. \qquad (28)$$

If we replace the array elements $\hat{z}^{(j)}$ by its corresponding function evaluations, we can write (20) as

$$\eta_i^{(j)} = \left[(m+1-j)^{n-1} \right]^{-1}$$
$$\times \sum_{k=1}^{(m+1-j)^{n-1}} \frac{1}{2\Delta x_i^{(j)}} \left[f \begin{pmatrix} {}^k\hat{x}_1^{(j)} \\ \vdots \\ {}^k\hat{x}_{i-1}^{(j)} \\ \bar{x}_i + \Delta x_i^{(j)} \\ {}^k\hat{x}_{i+1}^{(j)} \\ \vdots \\ {}^k\hat{x}_n^{(j)} \end{pmatrix} - f \begin{pmatrix} {}^k\hat{x}_1^{(j)} \\ \vdots \\ {}^k\hat{x}_{i-1}^{(j)} \\ \bar{x}_i - \Delta x_i^{(j)} \\ {}^k\hat{x}_{i+1}^{(j)} \\ \vdots \\ {}^k\hat{x}_n^{(j)} \end{pmatrix} \right]. \qquad (29)$$

There are $(m+1-j)$ different values ${}^k\hat{x}_i^{(j)}$ for each variable x_i, $i = 1, 2, \ldots, n$, for each discretized membership level μ_j, $j = 0, 1, \ldots, m$. Altogether, we have $(m+1-j)^{n-1}$ possible combinations at each membership level for the above formula. For each of these combinations, the central difference quotient is computed, which becomes obvious when we recall the formula for the central difference quotient of an analytic function in one variable:

$$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + \mathcal{O}(\Delta x)^2 \,. \tag{30}$$

Since the accuracy of the central difference quotient increases with the decrease of Δx_i, we can conclude that for functions that are sufficiently smooth within a close range of $\bar{P} = (\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n)$, the most accurate approximation of the $\partial f/\partial x_i(\bar{P})$ should be obtained for the membership level μ_{m-1}, i.e. for the one that corresponds to the α-cut directly below the top level.

Once the gain factors $\eta_i^{(j)}$ have been determined, the normalized degrees of influence ρ_i, $i = 1, 2, \ldots, n$, can be computed by use of (23). That formula basically performs two tasks:

1. Computation of a weighted average for the approximations of the partial derivatives. The weighting is performed according to the degree of membership μ_j, which gives the α-cuts with a higher level of membership a higher weight. Note that $\eta_i^{(0)}$ is excluded from the formula since it does not contribute to the degree of influence due to its chosen weight of zero. Although this weighting is somehow arbitrary, it can be motivated, for the approximations of the partial derivatives usually become less accurate with decreasing level of membership, as explained above. However, since the membership level μ_{m-1} is most likely to give the best approximation, one might argue that just the gain factor $\eta_i^{(m-1)}$ could be used to compute the influence measures.
2. Normalization similar to the procedure explained in Section 2.1 to achieve independency of the dimension and the fuzziness of the input parameters. The close resemblance of (23) and (16) is obvious.

To conclude the analysis of measuring the degrees of influence with the transformation method, let us calculate the remaining degrees of influence for the example above. In addition to $\eta_1^{(3)}$, we obtain

$$\eta_1^{(2)} = -0.7749, \quad \eta_1^{(1)} = -0.7739 \quad \text{and} \quad \eta_1^{(0)} = -0.7725 \,. \tag{31}$$

As expected, the accuracy of the approximation of the exact gradient $\partial f/\partial x_1(\bar{P})$, which amounts to $\partial f/\partial x_1(0.45, 0.25) = -0.776$, decreases with decreasing membership level μ_j. After computing the coefficients $\eta_2^{(j)}$, $j = 0, 1, 2, \ldots, 4$, for parameter the p_2 in the same way, we can use (23) to obtain the normalized degrees of influence $\rho_1 = 89.9\%$ and $\rho_2 = 10.1\%$.

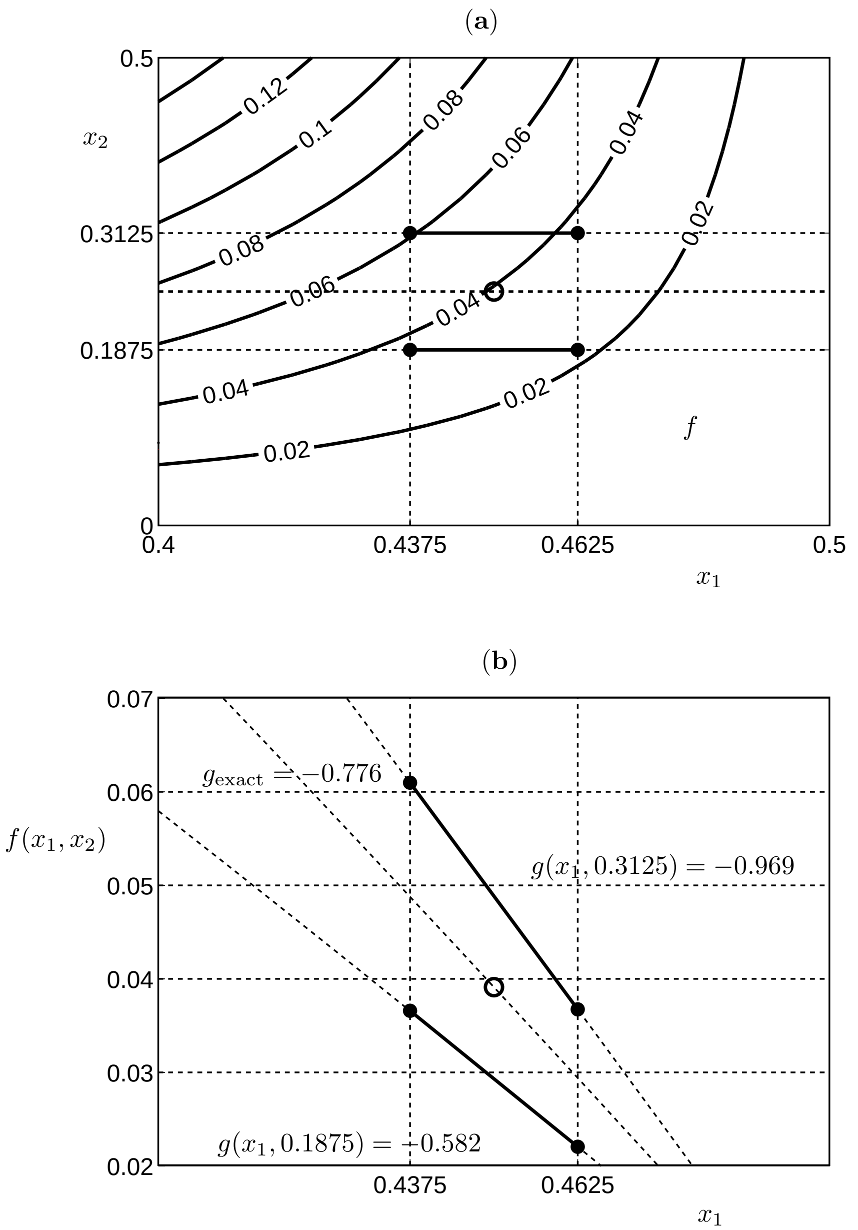

Figure 7: (**a**): contour plot of $f(x_1, x_2)$, and (**b**): graphical interpretation of the calculation of $\eta_1^{(3)}$ as an approximation of $\partial f/\partial x_1\,(\bar{x}_1, \bar{x}_2)$.

4.3 Interpretation and verification of the results

So far, we have demonstrated that measuring the degrees of influence with the general transformation method resembles a strategy using the total differential of the model function, and we observe the following:

- In case of symmetric membership functions, we can expect a result for the degrees of influence which is very close to the one obtained using the total differential. However, the larger the fuzziness of the model parameters becomes and the "less linear" the model function is, the greater will be the difference of the results achieved with the two methods. For a high fuzziness of the model parameters, independency of the degrees of influence from the fuzziness of the model parameters can no longer be guaranteed for the transformation method. On the other hand, in those cases it is questionable anyway if the total differential around the peak values can still be considered as a reasonable basis for the degree of influence.

- In case of non-symmetric membership functions, an analogy to the total differential cannot be drawn that easily. We have rather to be aware of the fact that the centers of the intervals $X_i^{(j)}$, $j = 0, 1, \ldots, (m-1)$, of the decomposed fuzzy number p_i do no longer correspond to the peak value of the membership function. Therefore, we cannot explicitly replace the interval bounds $a_i^{(j)}$ and $b_i^{(j)}$, as we did in (28) and (29), to obtain a formulation that is based on the central difference quotient and approximates the partial derivatives. However, if the derivative of the model function with respect to the non-symmetric parameter does not vary significantly within the considered interval, i.e. the curvature is small, the results are still very similar to the ones obtained using the total differential since the variation of the gain factors will still be very close the actual partial derivatives of the fuzzy-parameterized model function.

The observations above imply that it would be useful to be able to measure the quality of the obtained influence measures. For simple analytical model functions, we could of course compare the approximate result directly to the exact result using the total differential. However, for more complex functions, this might not be an option.

We suggest to first determine the scatter of the gain factors $\eta_i^{(j)}$, $i = 1, 2, \ldots, n$, for the different membership levels μ_j, $j = 0, 1, \ldots, (m-1)$, using their unbiased standard deviation. Then we introduce a relative standard deviation s_i for each uncertain parameter p_i that can help making an informed decision whether the results for the degrees of influence are useful or not. It can easily be computed according to

$$s_i = \frac{1}{|\bar{\eta}_i|} \sqrt{\frac{\sum\limits_{j=0}^{m-1} \left(\eta_i^{(j)} - \bar{\eta}_i \right)^2}{m-1}} \,, \quad i = 1, 2, \ldots, n \tag{32}$$

with

$$\bar{\eta}_i = \frac{1}{m} \sum_{j=0}^{m-1} \eta_i^{(j)} \,. \tag{33}$$

Instead of the arithmetic mean in (33), one may also use the weighted gain factors as proposed in [3], which should render similar results.

If a "high" relative standard deviation s_i is obtained – from experience, we suggest to consider variations greater than 10% as high –, one can conclude that

- either the fuzziness of the input parameter is too large with respect to the function characteristics in order to receive a good approximation of the total differential, which is the basis of the influence calculation,
- or, if in addition to a high standard deviation s_i, we have a mean gain factor η_i close to zero, it is very likely that the model function exhibits local extrema with respect to p_i close to the peak value of p_i.

For both cases, an implementation of the measure of the degree of influence should consider these special occurrences. While the former case can help to expose unreliable results for the degrees of influence measurements, the latter can indicate that a thorough discretization of the inner part of the intervals is necessary to obtain correct results.

5 Numerical examples

In this section, we consider a couple of test models with fuzzy-parameterized model functions of the form $q = f(p_1, p_2)$ to compare the approach of using the total differential with the one of approximatively measuring the degrees of influence using the transformation method. We define the function input parameters as symmetric triangular fuzzy numbers p_1 and p_2 according to the LR-Notation $p_i = \langle \bar{x}_i, \alpha_i, \alpha_i \rangle_{LR}$, where the membership functions $\mu_{p_i}(x_i)$, $i = 1, 2$, are given by

$$\mu_{p_i}(x_i) = \begin{cases} L\left(\frac{\bar{x}_i - x_i}{\alpha_i}\right) & \text{for} \quad x_i < \bar{x}_i \\ R\left(\frac{x_i - \bar{x}_i}{\alpha_i}\right) & \text{for} \quad x_i \geq \bar{x}_i \end{cases} \tag{34}$$

with the reference functions

$$L(u) = \max(0, 1 - u) \qquad \text{and} \qquad R(u) = \max(0, 1 - u) \; . \tag{35}$$

The stretching parameter

$$\alpha_i = |r_i \, \bar{x}_i| \tag{36}$$

represents the absolute value of the upper and lower worst-case deviation from the peak value, respectively, and r_i quantifies the worst-case deviation relative to the peak value $\bar{x}_i$. As an example, the triangular fuzzy number $\langle 4, 2, 2 \rangle_{LR}$ has its peak value at $\bar{x} = 4$ with a worst-case range of uncertainty $r = \pm 50\%$ of the peak value.

Since the effectiveness of the influence measure based on the transformation method usually depends on the fuzziness of the model parameters, we provide results for varying ranges of uncertainty, if necessary. This influence measure also depends on the refinement of decomposition of the fuzzy numbers, i.e. on the number of membership levels. However, this effect proves to be relatively small, so it is sufficient to use a refinement of just $m+1 = 5$ membership levels in the examples presented below.

In the following, six examples are itemized together with a discussion of the numerical results, as illustrated in Figs 8 to 13. Each of the examples shows three plots:

1. The degree of influence of p_1 on the overall fuzziness of the output $q = f(p_1, p_2)$ for p_1 and p_2 for varying peak values $\bar{x}_1$ and $\bar{x}_2$, $\bar{x}_1, \bar{x}_2 \in [0.01, 1]$, of the fuzzy parameters p_1 and p_2, but fixed ranges of uncertainty r_1 and r_2. This plot allows an analysis of how the degree of influence changes with changing peak values of the fuzzy-valued arguments of the function f.
2. The relative standard deviation s_1 for the parameter p_1, calculated according to (33) and (32), which serves as a measure for the effectiveness of the approximation compared to the exact gradient $\partial f/\partial x_1(\bar{x}_1, \bar{x}_2)$, and thus also as a measure for the reliability of the computed degree of influence.
3. The relative error
$$e_1 = \left| \frac{\bar{\eta}_1 - \frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)}{\frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)} \right| \tag{37}$$
of the average approximation $\bar{\eta}_1$ of the partial derivative with respect to x_1 at $\bar{P}$, to be compared with the error estimator s_1.

Example 1: $q = f(p_1, p_2) = p_1 + p_2$ (Fig. 8).
This function $f(x_1, x_2)$ is linear in x_1 and x_2, so both partial derivatives $\partial f/\partial x_1$ and $\partial f/\partial x_2$ have a constant value of unity. The approximations of the derivatives are exact within floating point accuracy, since all evaluations of the central difference quotient give the exact result.

Example 2: $q = f(p_1, p_2) = p_1\, p_2$ (Fig. 9).
As one can see, the degree of influence ρ_1 of the parameter p_1 is of constant value $\rho_1 = 0.5$, which can be explained by looking at the total differential of the function $f(x_1, x_2)$ at $\bar{P} = (\bar{x}_1, \bar{x}_2)$:
$$\mathrm{d}f = \bar{x}_2 \ \mathrm{d}x_1 + \bar{x}_1 \ \mathrm{d}x_2 \ . \tag{38}$$
With change rates $\mathrm{d}x_1$ and $\mathrm{d}x_2$ being some constant percentage c of $\bar{x}_1$ and $\bar{x}_2$, we obtain
$$\mathrm{d}f = \underbrace{\bar{x}_2\, c\, \bar{x}_1}_{(\mathrm{d}f)_1} + \underbrace{\bar{x}_1\, c\, \bar{x}_2}_{(\mathrm{d}f)_2} , \tag{39}$$
which indicates that each of the parameters p_1 and p_2 contributes by the same amount $(\mathrm{d}f)_1 = (\mathrm{d}f)_2 = c\, \bar{x}_1\, \bar{x}_2$ to the overall fuzziness $\mathrm{d}f$ of the output q. The approximations of the derivatives $\partial f/\partial x_1(\bar{P})$ are still exact within floating point accuracy, since for $n = 2$ parameters, we have $k = (m + 1 - j)$ evaluations of the central difference quotient at each membership level $\mu_j, j = 0, \ldots, (m - 1)$:
$$\begin{aligned}\frac{\partial f}{\partial x_1}(\bar{x}_1,\ {}^k\hat{x}_2^{(j)})^{(j)} &= \frac{f(\bar{x}_1 + \Delta x_1^{(j)},\ {}^k\hat{x}_2^{(j)}) - f(\bar{x}_1 - \Delta x_1^{(j)},\ {}^k\hat{x}_2^{(j)})}{2\Delta x_1^{(j)}} + \mathcal{O}(\Delta x_1^{(j)})^2 \\ &= \frac{(\bar{x}_1 + \Delta x_1^{(j)})\ {}^k\hat{x}_2^{(j)} - (\bar{x}_1 - \Delta x_1^{(j)})\ {}^k\hat{x}_2^{(j)}}{2\Delta x_1^{(j)}} = \ {}^k\hat{x}_2^{(j)} .\end{aligned}$$
Since we have an equidistant distribution of the values ${}^k\hat{x}_2^{(j)}$ for each α-cut and $\bar{x}_2$ is the peak value of the symmetric membership function of p_2, we have
$$\frac{\partial f}{\partial x_1}(\bar{x}_1, \bar{x}_2)^{(j)} = (m + 1 - j)^{-1} \sum_{k=1}^{m+1-j} {}^k\hat{x}_2^{(j)} = \bar{x}_2,$$

which is equal to the exact partial derivative at $\bar{P}$ and independent of the number of membership levels.

Example 3: $q = f(p_1, p_2) = (p_1)^3\, p_2$ (Fig. 10).
Like in Example 2, we have degrees of influence that are constant with respect to the fuzzy parameters p_1 or p_2. The total differential $\mathrm{d}f$ of the function $f(x_1, x_2)$ at $\bar{P} = (\bar{x}_1, \bar{x}_2)$ at is obtained to

$$\mathrm{d}f = 3\,\bar{x}_1^2\,\bar{x}_2\,\mathrm{d}x_1 + \bar{x}_1^3\,\mathrm{d}x_2\ . \tag{40}$$

Assuming again the change rates $\mathrm{d}x_1$ and $\mathrm{d}x_2$ to be some constant percentage c of $\bar{x}_1$ and $\bar{x}_2$, we get

$$\mathrm{d}f = \underbrace{3\,c\,\bar{x}_1^3\,\bar{x}_2}_{(\mathrm{d}f)_1} + \underbrace{c\,\bar{x}_1^3\,\bar{x}_2}_{(\mathrm{d}f)_2}\ , \tag{41}$$

which, based on (16), would suggest that p_1 contributes by $\rho_1^* = (\mathrm{d}f)_1/\mathrm{d}f = 75\%$ and p_2 by $\rho_2^* = (\mathrm{d}f)_2/\mathrm{d}f = 25\%$ to the overall fuzziness $\mathrm{d}f$ of the result q. Unfortunately, the degrees of influence computed with the transformation method do only approximately correspond to these values. Here, the approximative character of the calculation procedure affects the effectiveness of the result to some degree. As we can see from the plots of the relative standard deviation s_1, the result does not depend on the peak values of the fuzzy numbers, but it depends on the range of uncertainty of the fuzzy parameters. This is due to the fact that the central differential quotient produces a larger error for the approximation of the partial derivative with increasing Δx. So we can expect more reliable results for smaller ranges of uncertainty of the fuzzy parameters.

Example 4: $q = f(p_1, p_2) = p_2\,\cos(\pi p_1)$ (Fig. 11).
Example 5: $q = f(p_1, p_2) = p_1 + p_2/p_1$ (Fig. 12).
When looking at the plots of s_1 and e_1, we notice that for functions that have local extrema within the examined interval, we usually obtain a larger value for the relative standard deviation s_1. This is due to the fact that when the partial derivative of the function gets close to zero, variations of the central difference quotient become more significant. Although the absolute differences might be small – all are close to zero here –, the relative difference can still be large.

Example 6: $q = f(p_1, p_2) = \sqrt{(p_1 - 0.1)^4 + (p_2 - 0.1)^4}$ (Fig. 13).
This function is another example that demonstrates that for non-linear functions, the approximations of the partial derivatives computed by the transformation method become less reliable with increasing fuzziness of the input parameters p_1 and p_2. We can see a significant difference in the degrees of influence when considering uncertainty ranges of $\pm 200\%$ on the one side, and $\pm 5\%$ on the other. In contrast to the results for an uncertainty of $\pm 5\%$, the ones $\pm 200\%$ can no longer be considered as a reliable approximation of the partial derivatives.

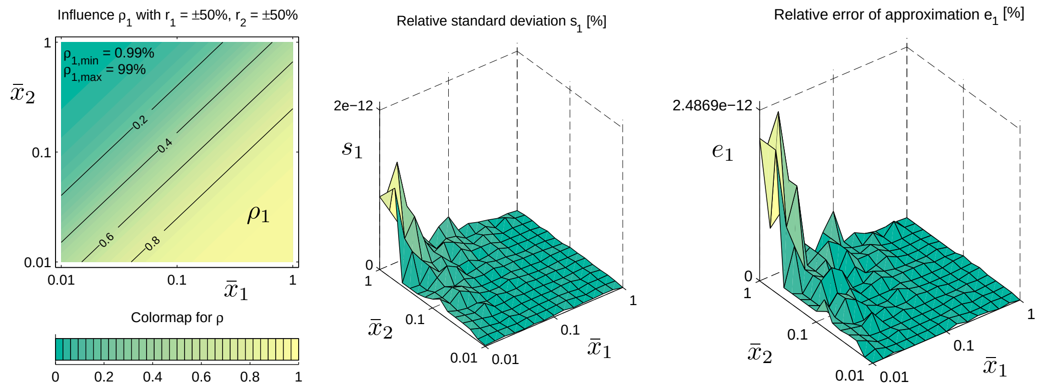

Figure 8: Example 1 – $q = f(p_1, p_2) = p_1 + p_2$.

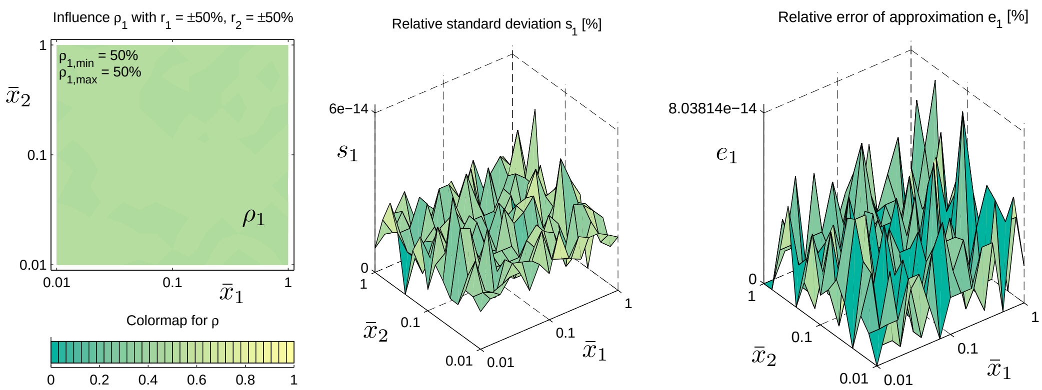

Figure 9: Example 2 – $q = f(p_1, p_2) = p_1\, p_2$.

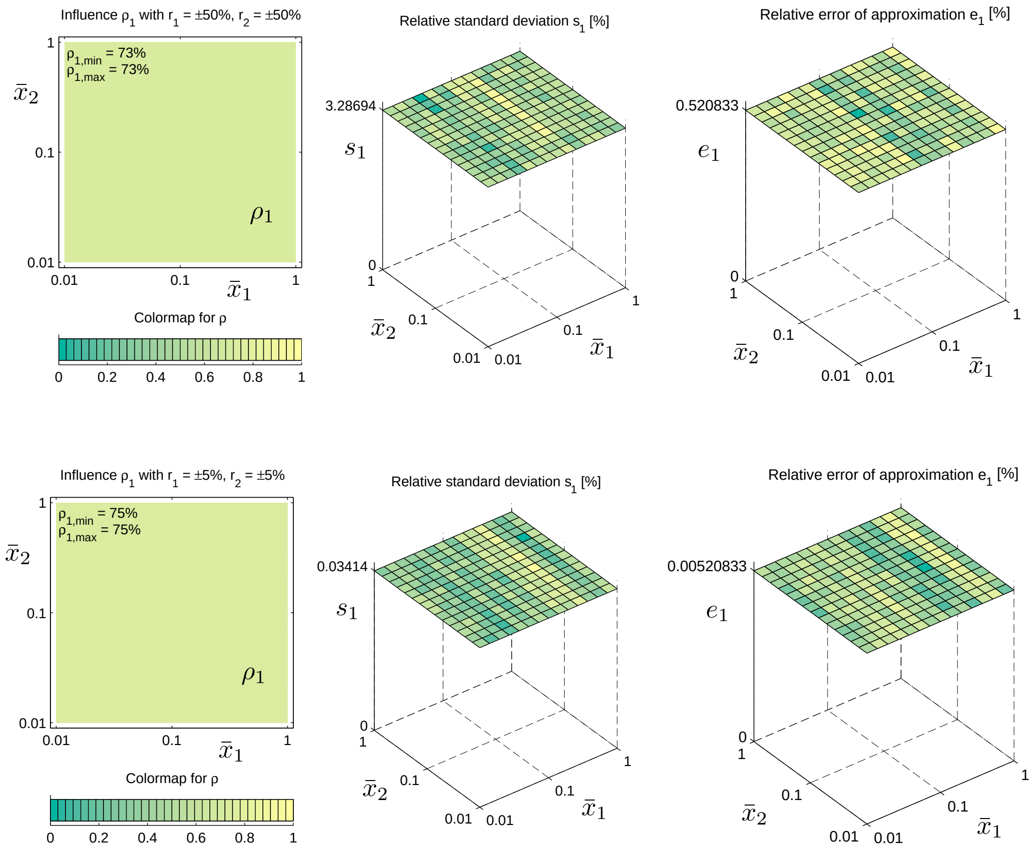

Figure 10: Example 3 – $q = f(p_1, p_2) = (p_1)^3\, p_2$.

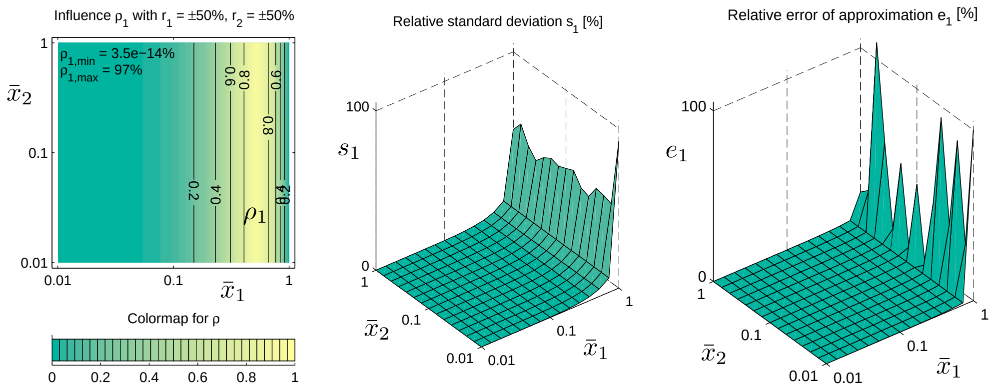

Figure 11: Example 4 – $q = f(p_1, p_2) = p_2\, \cos(\pi p_1)$.

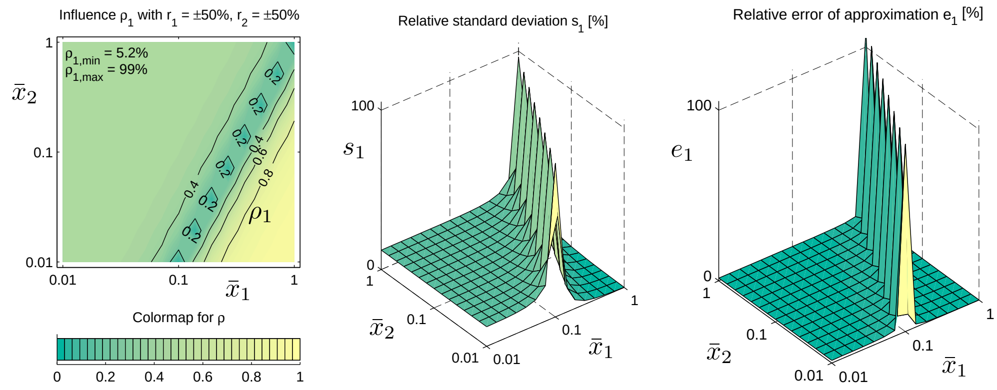

Figure 12: Example 5 – $q = f(p_1, p_2) = p_1 + p_2/p_1$.

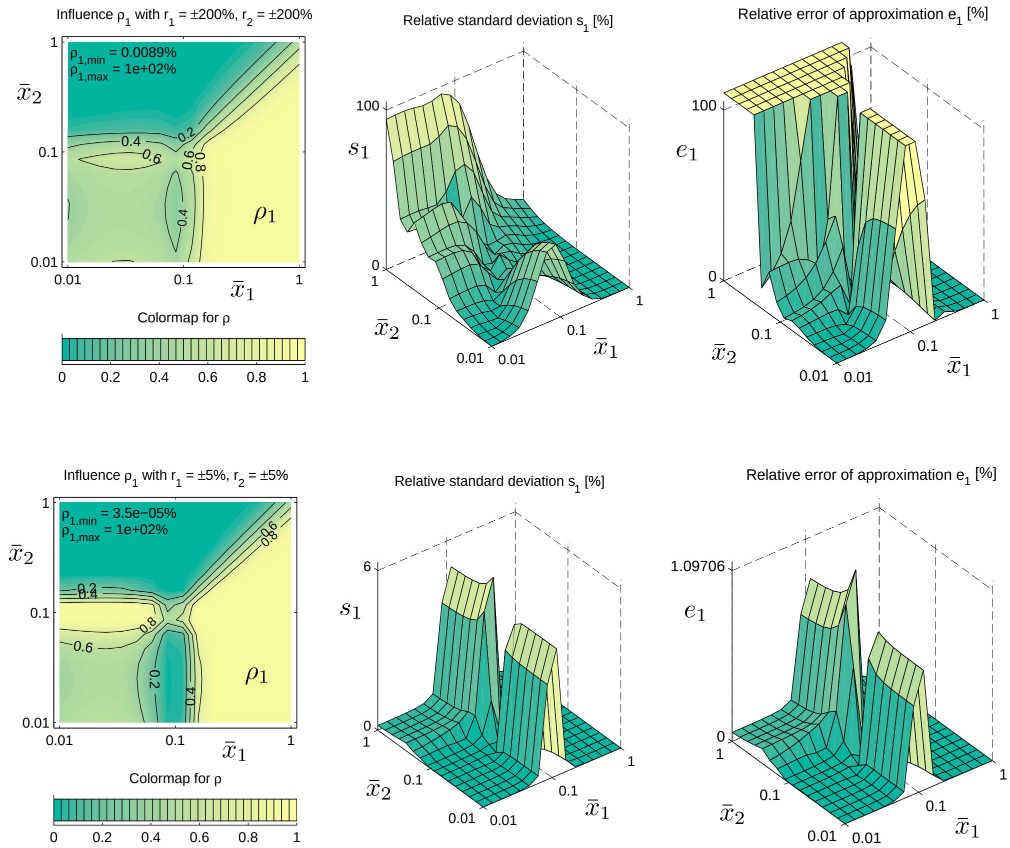

Figure 13: Example 6 – $q = f(p_1, p_2) = \sqrt{(p_1 - 0.1)^4 + (p_2 - 0.1)^4}$.

6 Conclusions

It has been shown that the influence measure proposed as a part of the transformation method in fuzzy arithmetic is actually based on an approximation of the total differential of the model function at the peak values of the fuzzy-valued model parameters. In most cases, this procedure provides a very reliable tool for the determination of the degree of influence of each uncertain model parameter on the overall fuzziness of the model output. However, the so-obtained degrees of influence get less reliable with increasing fuzziness of the model parameters. For this reason, we propose to compute the relative standard deviation of the gain factors, as introduced in the paper, as an indicator for the effectiveness and the reliability of the results.

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