Universität Stuttgart

A Note on the Ellipticity of the Single Layer Potential in two–dimensional Linear Elastostatics

Olaf Steinbach

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Preprint 2003/010

Universität Stuttgart

A Note on the Ellipticity of the Single Layer Potential in two–dimensional Linear Elastostatics

Olaf Steinbach

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Preprint 2003/010

Institut für Angewandte Analysis und Numerische Simulation (IANS)
Fakultät Mathematik und Physik
Fachbereich Mathematik
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: ians-preprints@mathematik.uni-stuttgart.de
WWW: http://preprints.ians.uni-stuttgart.de

ISSN **1611-4176**

A Note on the Ellipticity of the Single Layer Potential in two–dimensional Linear Elastostatics

O. Steinbach

Institut für Angewandte Analysis und Numerische Simulation
Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
steinbach@mathematik.uni-stuttgart.de

Abstract

We prove that the single layer potential operator of planar linear elastostatics is elliptic in $[H^{-1/2}(\Gamma)]^2$, if the domain Ω is scaled appropriately.

Key words: Planar linear elasticity, single layer potential, ellipticity.

1 Introduction

The solution of the Dirichlet boundary value problem in planar linear elastostatics,

$$\begin{aligned} -\frac{E}{2(1+\nu)}\Delta \underline{u}(x) - \frac{E}{2(1+\nu)(1-2\nu)}\operatorname{grad}\operatorname{div}\underline{u}(x) &= \underline{0} \qquad \text{for } x \in \Omega \subset I\!R^2, \\ \underline{u}(x) &= \underline{g}(x) \quad \text{for } x \in \Gamma = \partial\Omega, \end{aligned}$$

can be described by using the indirect single–layer potential ansatz

$$u_i(\widetilde{x}) := (\widetilde{V}\underline{w})_i(\widetilde{x}) = \int_\Gamma \sum_{j=1}^2 U^*_{ij}(\widetilde{x},y)w_j(y)ds_y \quad \text{for } \widetilde{x} \in \Omega, i = 1,2 \tag{1.1}$$

with the Kelvin tensor as fundamental solution,

$$U^*_{ij}(x,y) = \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}\left[(4\nu-3)\log|x-y|\delta_{ij} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right] \tag{1.2}$$

for $i,j = 1,2$. It is assumed that $\Omega \subset I\!R^2$ is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, $E > 0$ is the Young modulus and $\nu \in (0, \frac{1}{2})$ is the Poisson ratio. To find the unknown density function $\underline{w} \in [H^{-1/2}(\Gamma)]^2$ we consider the limiting process $\Omega \ni \widetilde{x} \to x \in \Gamma$ which gives the boundary integral equation

$$(V\underline{w})_i(x) := \int_\Gamma \sum_{j=1}^2 U^*_{ij}(x,y)w_j(y)ds_y = g_i(x) \quad \text{for } x \in \Gamma, i = 1,2. \tag{1.3}$$

It is well known [1] that $V : [H^{-1/2}(\Gamma)]^2 \to [H^{1/2}(\Gamma)]^2$ is bounded. Let

$$[H^{-1/2}_*(\Gamma)]^2 := \left\{\underline{w} \in [H^{-1/2}(\Gamma)]^2 : \langle w_i, 1\rangle_{L_2(\Gamma)} = 0 \quad \text{for } i = 1,2\right\}. \tag{1.4}$$

As in the three–dimensional case, see [3, Lemma 4.5], one can prove the $[H^{-1/2}_*(\Gamma)]^2$–ellipticity of the single layer potential V,

$$\langle V\underline{\widetilde{w}}, \underline{\widetilde{w}}\rangle_{L_2(\Gamma)} \geq \widetilde{c}^V_1 \,||\underline{\widetilde{w}}||^2_{[H^{-1/2}(\Gamma)]^2} \quad \text{for all } \underline{\widetilde{w}} \in [H^{-1/2}_*(\Gamma)]^2. \tag{1.5}$$

The proof of the ellipticity estimate (1.5) is based on related ellipticity estimates of the associated domain bilinear forms defined with respect to the interior domain Ω and the exterior domain $\Omega^c := I\!R^2 \backslash \overline{\Omega}$, respectively. For the latter the far field estimate

$$|(V\underline{w})_i(x)| \;=\; \mathcal{O}(\frac{1}{|x|}) \quad \text{as } |x| \to \infty, i = 1,2 \tag{1.6}$$

is to be assumed. While in the three–dimensional case (1.6) is satisfied for all $\underline{w} \in [H^{-1/2}(\Gamma)]^3$, in the two–dimensional case (1.6) is valid for $\underline{\widetilde{w}} \in [H_*^{-1/2}(\Gamma)]^2$ only. But to ensure the unique solvability of the first kind boundary integral equation (1.3) by the Lax–Milgram theorem we need to have the ellipticity estimate (1.5) for all $\underline{w} \in [H^{-1/2}(\Gamma)]^2$.

In the case of the two–dimensional Laplace operator the corresponding single layer potential is $H^{-1/2}(\Gamma)$–elliptic, if the domain Ω satisfies the scaling condition $\operatorname{diam}\Omega < 1$, see [4]. In the case of the system of planar linear elastostatics it seems to be an open problem, how to scale the computational domain Ω to ensure the $[H^{-1/2}(\Gamma)]^2$–ellipticity of the single layer potential V. While the system of linear elastostatics is strongly related to the Bilaplacian, one may extract such results from [2].

In this note we apply the ideas of the proof for the single layer potential of the Laplace operator to show the ellipticity of the single layer potential V if a suitable scaling of the computational domain Ω is applied. This approach is based on the definition of appropriate natural density functions and related Lagrange parameters which correspond to the capacity in the case of the Laplace operator [5]. Then the $[H^{-1/2}(\Gamma)]^2$–ellipticity of the single layer potential V follows as in the case of the Laplace operator.

2 Scaling of the computational domain

For some positive parameter $\alpha \in I\!R_+$ we define the scaled boundary

$$\hat{\Gamma} \ni \hat{x} \;:=\; \frac{1}{\alpha}\, x \quad \text{for } x \in \Gamma. \tag{2.1}$$

and consider the boundary integral equation (1.3) in the new coordinates,

$$(V_\alpha \underline{\hat{w}})_i(\hat{x}) \;:=\; (V\underline{w})_i(\alpha\hat{x}) \;=\; g_i(\alpha\hat{x}) \;=:\; \hat{g}_i(\hat{x}) \quad \text{for } \hat{x} \in \hat{\Gamma}, i = 1,2,$$

with the scaled single layer potential

$$(V_\alpha \underline{\hat{w}})_i(\hat{x}) \;=\; \int\limits_{\hat{\Gamma}} \sum_{j=1}^{2} U_{ij}^\alpha(\hat{x},\hat{y})\hat{w}_j(\hat{y}) ds_{\hat{y}} \quad \text{for } \hat{x} \in \hat{\Gamma}, \tag{2.2}$$

and with the modified Kelvin tensor

$$U_{ij}^\alpha(\hat{x},\hat{y}) \;=\; \frac{1}{4\pi}\,\frac{1}{E}\,\frac{1+\nu}{1-\nu}\left[(4\nu-3)\log\alpha|\hat{x}-\hat{y}|\delta_{ij} + \frac{(\hat{x}_i-\hat{y}_i)(\hat{x}_j-\hat{y}_j)}{|\hat{x}-\hat{y}|^2}\right] \tag{2.3}$$

for $i,j = 1,2$. Obviously, for $\underline{\hat{w}} \in [H_*^{-1/2}(\hat{\Gamma})]^2$ there holds

$$\langle V_\alpha \underline{\hat{w}}, \underline{\hat{w}}\rangle_{L_2(\hat{\Gamma})} \;=\; \langle V\underline{\hat{w}}, \underline{\hat{w}}\rangle_{L_2(\hat{\Gamma})}$$

and from (1.5) the $[H_*^{-1/2}(\hat{\Gamma})]^2$–ellipticity of V_α follows.

3 Natural density functions

For simplicity in the presentation we now skip the index $\hat{}$ and consider the single layer potential V_α with respect to the boundary Γ.

To show the $[H^{-1/2}(\Gamma)]^2$–ellipticity of the single layer potential V_α we proceed as in the case of the Laplace operator to define some natural density functions. We start to consider the saddle point problem to find $(\underline{w}^1, \underline{\lambda}^1) \in [H^{-1/2}(\Gamma)]^2 \times I\!R^2$ such that

$$\begin{aligned}
\langle V_\alpha \underline{w}^1, \underline{\tau}\rangle_{L_2(\Gamma)} - \lambda_1^1 \langle 1, \tau_1\rangle_{L_2(\Gamma)} - \lambda_2^1 \langle 1, \tau_2\rangle_{L_2(\Gamma)} &= 0 \\
\langle w_1^1, 1\rangle_{L_2(\Gamma)} &= 1 \\
\langle w_2^1, 1\rangle_{L_2(\Gamma)} &= 0
\end{aligned} \tag{3.1}$$

is satisfied for all $\underline{\tau} \in [H^{-1/2}(\Gamma)]^2$. By putting $w_1^1 := \widetilde{w}_1^1 + 1/|\Gamma|, w_2^1 := \widetilde{w}_2^1$, $\underline{\widetilde{w}}^1 \in [H_*^{-1/2}(\Gamma)]^2$ is the unique solution of the variational problem

$$\langle V_\alpha \underline{\widetilde{w}}^1, \underline{\tau}\rangle_{L_2(\Gamma)} = -\frac{1}{|\Gamma|} \langle V_\alpha (1,0)^\top, \underline{\tau}\rangle_{L_2(\Gamma)} \tag{3.2}$$

for all $\underline{\tau} \in [H_*^{-1/2}(\Gamma)]^2$. Afterwards we can compute the first Lagrange parameter

$$\lambda_1^1 = \langle V_\alpha \underline{w}^1, \underline{w}^1\rangle_{L_2(\Gamma)}.$$

In the same way we obtain $(\underline{w}^2, \underline{\lambda}^2) \in [H^{-1/2}(\Gamma)]^2 \times I\!R^2$ by solving the saddle point problem

$$\begin{aligned}
\langle V_\alpha \underline{w}^2, \underline{\tau}\rangle_{L_2(\Gamma)} - \lambda_1^2 \langle 1, \tau_1\rangle_{L_2(\Gamma)} - \lambda_2^2 \langle 1, \tau_2\rangle_{L_2(\Gamma)} &= 0 \\
\langle w_1^2, 1\rangle_{L_2(\Gamma)} &= 0 \\
\langle w_2^2, 1\rangle_{L_2(\Gamma)} &= 1
\end{aligned} \tag{3.3}$$

for all $\underline{\tau} \in [H^{-1/2}(\Gamma)]^2$ and therefore

$$\lambda_2^2 = \langle V_\alpha \underline{w}^2, \underline{w}^2\rangle_{L_2(\Gamma)}.$$

Moreover,

$$\lambda_2^1 = \lambda_1^2 = \langle V_\alpha \underline{w}^1, \underline{w}^2\rangle_{L_2(\Gamma)}.$$

Lemma 3.1 *The Lagrange parameters λ_i^i $(i = 1, 2)$ can be written as*

$$\lambda_i^i = \langle V_1 \underline{w}^i, \underline{w}^i\rangle_{L_2(\Gamma)} + \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}(4\nu - 3)\log\alpha,$$

while the Lagrange parameters $\lambda_1^2 = \lambda_2^1$ are independent of α,

$$\lambda_1^2 = \lambda_2^1 = \langle V_1 \underline{w}^1, \underline{w}^2\rangle_{L_2(\Gamma)}.$$

Proof. A direct calculation gives for $i = 1$

$$\begin{aligned}
\lambda_1^1 &= \langle V_\alpha \underline{w}^1, \underline{w}^1\rangle_{L_2(\Gamma)} \\
&= \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}\int_\Gamma\int_\Gamma \sum_{i=1}^2 (4\nu-3)\log\alpha|x-y| w_i^1(y) w_i^1(x) ds_x ds_y \\
&\quad + \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}\int_\Gamma\int_\Gamma \sum_{i,j=1}^2 \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2} w_i^1(y) w_j^1(x) ds_x ds_y \\
&= \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}\int_\Gamma\int_\Gamma \sum_{i=1}^2 (4\nu-3)\log|x-y| w_i^1(y) w_i^1(x) ds_x ds_y \\
&\quad + \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}\int_\Gamma\int_\Gamma \sum_{i,j=1}^2 \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2} w_i^1(y) w_j^1(x) ds_x ds_y \\
&\quad + \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}(4\nu-3)\log\alpha \sum_{i=1}^2 \left[\int_\Gamma w_i^1(x) ds_x\right]^2 \\
&= \langle V_1 \underline{w}^1, \underline{w}^1\rangle_{L_2(\Gamma)} + \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}(4\nu-3)\log\alpha
\end{aligned}$$

due to $\langle w_1^1, 1\rangle_{L_2(\Gamma)} = 1$ and $\langle w_2^1, 1\rangle_{L_2(\Gamma)} = 0$, see the saddle point problem (3.1). For λ_2^2, the assertion follows in the same way. For $\lambda_1^2 = \lambda_2^1$ we obtain

$$\begin{aligned}
\lambda_1^2 &= \langle V_\alpha \underline{w}^1, \underline{w}^2\rangle_{L_2(\Gamma)} \\
&= \langle V_1 \underline{w}^1, \underline{w}^2\rangle_{L_2(\Gamma)} + \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}(4\nu-3)\log\alpha \sum_{i=1}^{2}\int_\Gamma w_i^1(x)ds_x \int_\Gamma w_i^2(y)ds_y \\
&= \langle V_1 \underline{w}^1, \underline{w}^2\rangle_{L_2(\Gamma)},
\end{aligned}$$

since $\langle w_1^2, 1\rangle_{L_2(\Gamma)} = \langle w_2^1, 1\rangle_{L_2(\Gamma)} = 0$. ■

Corollary 3.1 *The scaling parameter $\alpha \in I\!R_+$ can be chosen such that*

$$\min\{\lambda_1^1, \lambda_2^2\} > 2\,|\lambda_2^1| \tag{3.4}$$

is satisfied.

4 Ellipticity estimate

For an arbitrary given $\underline{w} \in [H^{-1/2}(\Gamma)]^2$ we consider the splitting

$$\underline{w} = \underline{\widetilde{w}} + \alpha_1 \underline{w}^1 + \alpha_2 \underline{w}^2, \quad \alpha_i = \langle w_i, 1\rangle_{L_2(\Gamma)} \quad (i=1,2) \tag{4.1}$$

yielding $\underline{\widetilde{w}} \in [H_*^{-1/2}(\Gamma)]^2$. Now we give the main result of this note.

Theorem 4.1 *Let the scaling parameter $\alpha \in I\!R_+$ be chosen such that (3.4) is satisfied. Then the single layer potential V_α is $[H^{-1/2}(\Gamma)]^2$–elliptic,*

$$\langle V_\alpha \underline{w}, \underline{w}\rangle_{L_2(\Gamma)} \geq c_1^V\, ||\underline{w}||^2_{[H^{-1/2}(\Gamma)]^2} \quad \textit{for all } \underline{w} \in [H^{-1/2}(\Gamma)]^2. \tag{4.2}$$

Proof. For $\underline{w} \in [H^{-1/2}(\Gamma)]^2$ we consider the splitting (4.1). By applying the triangle and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
||\underline{w}||^2_{[H^{-1/2}(\Gamma)]^2} &= ||\underline{\widetilde{w}} + \alpha_1\underline{w}^1 + \alpha_2\underline{w}^2||^2_{[H^{-1/2}(\Gamma)]^2} \\
&\leq \Big[||\underline{\widetilde{w}}||_{[H^{-1/2}(\Gamma)]^2} + |\alpha_1|\,||\underline{w}^1||^2_{[H^{-1/2}(\Gamma)]^2} + |\alpha_2|\,||\underline{w}^2||^2_{[H^{-1/2}(\Gamma)]^2}\Big]^2 \\
&\leq 3\,\Big[||\underline{\widetilde{w}}||^2_{[H^{-1/2}(\Gamma)]^2} + \alpha_1^2\,||\underline{w}^1||^2_{[H^{-1/2}(\Gamma)]^2} + \alpha_2^2\,||\underline{w}^2||^2_{[H^{-1/2}(\Gamma)]^2}\Big] \\
&\leq 3\,\max\Big\{1, ||\underline{w}^1||^2_{[H^{-1/2}(\Gamma)]^2}, ||\underline{w}^2||^2_{[H-1/2(\Gamma)]^2}\Big\}\Big[||\underline{\widetilde{w}}||^2_{[H^{-1/2}(\Gamma)]^2} + \alpha_1^2 + \alpha_2^2\Big].
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\langle V_\alpha \underline{w}, \underline{w}\rangle_{L_2(\Gamma)} &= \langle V_\alpha[\underline{\widetilde{w}} + \alpha_1\underline{w}^1 + \alpha_2\underline{w}^2], \underline{\widetilde{w}} + \alpha_1\underline{w}^1 + \alpha_2\underline{w}^2\rangle_{L_2(\Gamma)} \\
&= \langle V_\alpha\underline{\widetilde{w}}, \underline{\widetilde{w}}\rangle_{L_2(\Gamma)} + \alpha_1^2\,\langle V_\alpha\underline{w}^1, \underline{w}^1\rangle_{L_2(\Gamma)} + \alpha_2^2\,\langle V_\alpha\underline{w}^2, \underline{w}^2\rangle_{L_2(\Gamma)} \\
&\quad +2\alpha_1\langle V_\alpha\underline{w}^1, \underline{\widetilde{w}}\rangle_{L_2(\Gamma)} + 2\alpha_2\langle V_\alpha\underline{w}^2, \underline{\widetilde{w}}\rangle_{L_2(\Gamma)} + 2\alpha_1\alpha_2\langle V_\alpha\underline{w}^1, \underline{w}^2\rangle_{L_2(\Gamma)} \\
&= \langle V_\alpha\underline{\widetilde{w}}, \underline{\widetilde{w}}\rangle_{L_2(\Gamma)} + \alpha_1^2\,\lambda_1^1 + \alpha_2^2\,\lambda_2^2 + 2\alpha_1\alpha_2\,\lambda_1^2
\end{aligned}$$

due to the definition of $\underline{w}^1$ and $\underline{w}^2$, respectively. Using the $[H_*^{-1/2}(\Gamma)]^2$–ellipticity of V_α and (3.4) we further conclude

$$\langle V_\alpha \underline{w}, \underline{w}\rangle_{L_2(\Gamma)} \geq \widetilde{c}_1^V\,||\underline{\widetilde{w}}||^2_{[H^{-1/2}(\Gamma)]^2} + \alpha_1^2\,\lambda_1^1 + \alpha_2^2\,\lambda_2^2 - 2|\alpha_1|\,|\alpha_2|\,|\lambda_1^2|$$

$$\geq \widetilde{c}_1^V \, ||\underline{\widetilde{w}}||^2_{[H^{-1/2}(\Gamma)]^2} + \min\{\lambda_1^1, \lambda_2^2\} \, \left[\alpha_1^2 + \alpha_2^2 - |\alpha_1| \, |\alpha_2|\right]$$

$$\geq \widetilde{c}_1^V \, ||\underline{\widetilde{w}}||^2_{[H^{-1/2}(\Gamma)]^2} + \frac{1}{2} \min\{\lambda_1^1, \lambda_2^2\} \, \left[\alpha_1^2 + \alpha_2^2\right]$$

$$\geq \min\left\{\widetilde{c}_1^V, \frac{1}{2}\lambda_1^1, \frac{1}{2}\lambda_2^2\right\} \left[||\underline{\widetilde{w}}||^2_{[H^{-1/2}(\Gamma)]^2} + \alpha_1^2 + \alpha_2^2\right].$$

This completes the proof. ■

Remark 4.1 *When an arbitrary domain Ω with Lipschitz boundary $\Gamma = \partial\Omega$ is given, one can solve the saddle point problems (3.1) and (3.3) by a numerical scheme to obtain approximate values for the Lagrange parameters λ_i^j, $i, j = 1, 2$. Using Lemma 3.1 then one can find an appropriate scaling parameter $\alpha \in \mathbb{R}_+$ such that (3.4) is satisfied.*

Remark 4.2 *In [6] it was shown that the ellipticity constant c_1^V tends to zero as $\nu \to \frac{1}{2}$. However, in [6] a splitting approach is described which gives a robust boundary element method also for nearly incompressible materials.*

References

[1] M. Costabel, Boundary integral operators on Lipschitz domains: Elementary results. SIAM J. Math. Anal. 19 (1988) 613–626.

[2] M. Costabel, M. Dauge, Invertibility of the biharmonic single layer potential operator, Integral Equations Operator Theory 24 (1996) 46–67.

[3] M. Costabel, E. P. Stephan, Coupling of finite and boundary element methods for an elastoplastic interface problem. SIAM J. Numer. Anal. 27 (1990) 1212–1226.

[4] G. C. Hsiao, W. L. Wendland, A finite element method for some integral equations of the first kind. J. Math. Anal. Appl. 58 (1977) 449–481.

[5] W. McLean, Strong Elliptic Systems and Boundary Integral Equations. Cambridge University Press, 2000.

[6] O. Steinbach, A robust boundary element method for nearly incompressible linear elasticity. Numer. Math., published electronically, 2002.

Erschienene Preprints ab Nummer 2003/001

Komplette Liste: `http://preprints.ians.uni-stuttgart.de`

2003/001 *Lamichhane, B. P., Wohlmuth, B. I.*: Mortar Finite Elements for Interface Problems.

2003/002 *Dryja, M., Gantner, A., Widlund, O. B., Wohlmuth, B. I.*: Multilevel Additive Schwarz Preconditioner For Nonconforming Mortar Finite Element Methods.

2003/003 *Klimke, A., Hanss, M.*: On the Reliability of the Influence Measure in the Transformation Method of Fuzzy Arithmetic.

2003/004 *Klimke, A.*: RANDEXPR: A Random Symbolic Expression Generator.

2003/005 *Klimke, A.*: How to Access Matlab from Java.

2003/006 *Merkle, T.*: Phase separation in solid mixtures under elastic loadings with application to solder materials.

2003/007 *Lamichhane, B. P., Wohlmuth, B. I.*: Second Order Lagrange Multiplier Spaces for Mortar Finite Elements in 3D.

2003/008 *Fritz, A., Hüeber, S., Wohlmuth, B. I.*: A comparison of mortar and Nitsche techniques for linear elasticity.

2003/009 *Klimke, A.*: An Efficient Implementation of the Transformation Method of Fuzzy Arithmetic

2003/010 *Steinbach, O.*: A Note on the Ellipticity of the Single Layer Potential in two-dimensional Linear Elastostatics