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Artificial Multilevel Boundary Element Preconditioners

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Abstract

A hierarchical multilevel preconditioner is constructed for an efficient solution of a first kind boundary integral equation with the single layer potential operator discretized by a boundary element method. This technique is based on a hierarchical clustering of all boundary elements as used in fast boundary element methods. This hierarchy is applied to define a sequence of nested boundary element spaces of piecewise constant basis functions as used in the definition of the preconditioning multilevel operator.

1 Boundary Element Methods

As a model problem we consider the first kind boundary integral equation

$$(Vw)(x) := \frac{1}{4\pi} \int_\Gamma \frac{1}{|x-y|} w(y) ds_y = f(x) \quad \text{for } x \in \Gamma, \tag{1}$$

where $\Gamma = \partial\Omega$ is the boundary of a closed Lipschitz domain $\Omega \subset I\!R^3$. Let $Z_h = \text{span}\{\psi_k\}_{k=1}^N$ be a family of boundary element spaces of piecewise constant basis functions ψ_k which are defined with respect to a uniform boundary discretization into N plane triangular boundary elements τ_k with mesh size h. The Galerkin discretization of the boundary integral equation (1) is equivalent to the linear system of equations, $V_h \underline{w} = \underline{f}$, where the stiffness matrix V_h and the last vector $\underline{f}$ are given by

$$V_h[\ell, k] = \frac{1}{4\pi} \int_{\tau_\ell} \int_{\tau_k} \frac{1}{|x-y|} ds_y ds_x, \quad f_\ell = \int_{\tau_\ell} f(x) ds_x, \quad k, \ell = 1, \ldots, N.$$

The matrix V_h is symmetric and positive definite, but its spectral condition number behaves like $\mathcal{O}(h^{-1})$. Therefore there is a need to use some efficient preconditioner when applying the conjugate gradient scheme to solve $V_h \underline{w} = \underline{f}$. One possible approach is based on the discretization of the hypersingular boundary integral operator

$$(Du)(x) := -\frac{1}{4\pi} \frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} \frac{1}{|x-y|} w(y) ds_y \quad \text{for } x \in \Gamma$$

with respect to a boundary element space $W_h = \text{span}\{\varphi_i\}_{i=1}^M$, see [9]. To obtain a spectrally equivalent preconditioning matrix for V_h the boundary element spaces Z_h and W_h have to satisfy a certain stability condition as in mixed finite element methods [6]. Theoretical and numerical results for two–dimensional model problems are given in [9]. For three–dimensional problems, however, the construction of the boundary element space W_h is not a obvious task when Z_h is the space of piecewise constant basis functions. Other preconditioning strategies are multigrid [8] or multilevel methods [3] which require a nested hierarchy of the boundary element spaces Z_h and the underlying boundary discretizations. Here we will describe a preconditioning algorithm for the case when such a mesh hierarchy is not given a priori. Such a situation appears usually in engineering and industrial applications where the number of boundary element is already large just to describe the geometry.

2 Multilevel Preconditioners

Let $Z_h = Z_J$ be a boundary element space of piecewise constant basis functions which is nested in a sequence

$$Z_0 \subset Z_1 \subset \cdots \subset Z_J = Z_h \subset \cdots \tag{2}$$

of boundary element spaces Z_j with mesh size $h_j = \frac{1}{2} h_{j-1}$. By $Q_j : L_2(\Gamma) \to Z_j$ we denote the L_2 Galerkin projection onto Z_j defined by

$$\langle Q_j w, \tau \rangle_{L_2(\Gamma)} \; = \; \langle w, \tau \rangle_{L_2(\Gamma)} \quad \text{for all } \tau \in Z_j, \; j = 0, \ldots, J. \tag{3}$$

For $s \in I\!R$ and $w \in Z_J$ we define the multilevel operator [2]

$$A^s w \; := \; \sum_{j=0}^{J} h_j^{-2s} \, (Q_j - Q_{j-1}) w \tag{4}$$

where $Q_{-1} := 0$. From [7, Theorem 2] it is known that there hold the spectral equivalence inequalities

$$c_1 ||w||^2_{H^{-1/2}(\Gamma)} \; \leq \; \langle A^{-1/2} w, w \rangle_{L_2(\Gamma)} \; \leq \; c_2 J^2 ||w||^2_{H^{-1/2}(\Gamma)} \tag{5}$$

for all $w \in Z_J \subset H^{-1/2}(\Gamma)$. The resulting preconditioning matrix can be written as

$$C_V^{-1} \; = \; M_h^{-1} A_h^{-1/2} M_h^{-1} \tag{6}$$

with

$$A_h^{-1/2}[\ell, k] \; = \; \langle A^{-1/2} \psi_k, \psi_\ell \rangle_{L_2(\Gamma)}, \quad M_h[\ell, k] \; = \; \langle \psi_k, \psi_\ell \rangle_{L_2(\Gamma)}, \quad k, \ell = 1, \ldots, N.$$

Note that the mass matrix M_h is diagonal and therefore easily invertible. Moreover, the application of the multilevel operator $A^{-1/2}$ can be described explicitely when using piecewise constant test and trial functions. In this case the L_2 projections $Q_j w$ can be computed directly by inverting diagonal matrices.

The crucial issue in a multilevel preconditioning approach is the avaibility of the hierarchy (2) of boundary element spaces Z_j. Starting from a coarse mesh with an associated space Z_0 one can define all boundary element spaces Z_j and therefore $Z_h = Z_L$ by a recursive refinement procedure. Now we consider the case where already the boundary element space Z_h is given. From this space Z_h and from the underlying mesh we have to construct an artificial mesh hierarchy to define the multilevel operator (4). Here we will use ideas from fast boundary element methods to define an appropriate clustering hierarchy of all boundary elements. In particular, the fast multipole method [4], the panel clustering method [5] as well as the Algebraic Clustering Approximation method [1] are based on such a hierarchy which may be constructed by a bisection algorithm.

Let $\{\tau_k\}_{k=1}^N$ be the set of all boundary elements associated with the boundary element space $Z_h = Z_J$ and let $I_1^0 = \{1, \ldots, M\}$ be the Index set of $Z_h = Z_J$. With respect to the boundary element τ_k we denote by x_k the midpoint of τ_k. By averaging we then obtain the center $\hat{x}$ of the point set $\{x_k\}_{k=1}^M$ by

$$\hat{x} \; = \; \frac{1}{M} \sum_{k=1}^{M} x_k.$$

Then we have to find the mean vector $w \in I\!R^3$ satisfying

$$\sum_{k=1}^{M} |w^\top (x_k - \hat{x})| \; = \; \max_{v \in I\!R^3, ||v||_2 = 1} \sum_{k=1}^{M} |v^\top (x_k - \hat{x})|$$

which is the eigenvector associated to the maximal eigenvalue of the covariance matrix given by

$$K := \sum_{k=1}^{M} (x_k - \hat{x})(x_k - \hat{x})^\top .$$

The initial index set I_1^0 can now be decomposed due to

$$I_1^0 = I_1^1 \cup I_2^1, \quad I_2^1 := I \backslash I_1^1, \quad I_1^1 := \left\{ k \in I : w^\top (x_k - \hat{x}) > 0 \right\}. \tag{7}$$

A recursive application of this bisection algorithm then defines a hierarchic decomposition of the index set I_1^0 and therefore of the boundary element space $Z_h = Z_J$, see Figure 1. This algorithm is repeated until the index sets I_j^p contain not more than n_0 boundary elements. Hence the boundary elements τ_k of the fine grid boundary element space Z_h are clustered first within the index sets I_j^p and later due to the hierarchy defined by the bisection algorithm. In the same way one can define the element areas of all levels just by summing up. We assume that for the coarse grid mesh size $h_0 < 1$. This may require to use coarse grid spaces I_j^q for some $q > 0$ instead of I_1^0. While $Z_h = Z_L$ is the boundary element space given a priori, all coarser boundary element spaces Z_j, $j < J$ are given due to the hierarchic construction described. Note that due to the choice of piecewise constant basis functions almost no restrictions in the coarsening appear, in particular there are no restrictions to avoid hanging nodes.

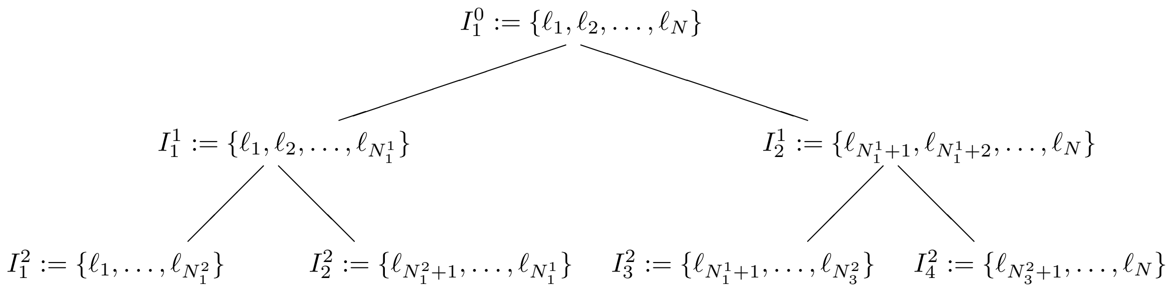

Figure 1: Hierarchy of the index set I_1^0.

3 Numerical Results

As numerical example we consider the solution of the first kind boundary integral equation

$$\frac{1}{4\pi} \int_\Gamma \frac{1}{|x-y|} w(y) ds_y = 1 \quad \text{for } x \in \Gamma,$$

where $\Gamma = \partial\Omega$ is the boundary of a L shaped domain as depicted in Figure 2.

The coarse grid consists of 28 uniform triangles and we consider up to five uniform refinements yielding 24576 boundary elements on the finest level. By construction we obtain a hierarchically nested sequence of boundary element spaces and therefore we can apply also the standard geometric multilevel preconditioner for comparison. For the latter we also consider a simple diagonal scaling. In Table 1 we give the estimated extremal eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$ and the resulting spectral condition number κ of all preconditioned systems. We see that the simple diagonal scaling gives us a spectral condition number κ which behaves like $\mathcal{O}(h^{-1})$ while both multilevel approaches result in a polylogarithmic behaviour of κ. Since in the geometric multilevel approach the number of

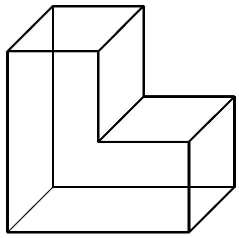

Figure 2: L shaped domain Ω.

	Diagonal Scaling			Geometric Multilevel			Artificial Multilevel		
N	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ
96	1.96 –2	7.37 –1	37.57	7.81 –2	2.95	37.73	5.85 –2	8.52 –1	14.55
384	9.52 –3	7.37 –1	77.47	5.58 –2	2.95	52.91	4.10 –2	8.52 –1	20.77
1536	4.69 –3	7.37 –1	157.34	4.25 –2	2.95	69.38	3.29 –2	8.49 –1	25.84
6144	2.33 –3	7.37 –1	315.92	3.35 –2	2.95	88.14	2.66 –2	8.52 –1	32.01
24576	1.17 –3	7.37 –1	632.44	2.69 –2	2.95	109.86	2.26 –2	8.52 –1	37.65

Table 1: Estimated spectral condition numbers for the preconditioned systems.

levels is fixed due to the construction from a given coarse grid level, the artificial approach gives a deeper hierarchy and therefore even better results.

The first numerical results for an academic test example confirm the efficiency of the proposed preconditioning approach. At present we work on the application of this approach when solving engineering and industrial problems in complicated structures where no geometric multilevel methods can be applied. Moreover, we can easily combine our preconditioning strategy with the use of fast boundary element methods such as the fast multipole method [4]. In particular, we can use the hierarchy of all boundary element clusters in both the fast multipole method and in the application of the preconditioner. Therefore, the additional amount of work to construct the preconditioner is negliable. The proposed preconditioning strategy can be used in a similar way for an efficient solution of the hypersingular boundary integral equation as well as for the symmetric boundary integral approach when dealing with mixed boundary value problems. Moreover, it can also be applied for more complicated problems such as in linear elasticity. Related results and a more detailed mathematical theory will be reported in forecomming papers.

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