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Abstract. Mortar methods provide a flexible nonconforming domain decomposition method for the numerical approximation of non linear multibody contact problems. Here, we use dual Lagrange multipliers and work with the saddle point formulation. Under some regularity assumption on the solution, an optimal convergence rate of $\mathcal{O}(h^{0.5+\nu})$ can be established if the weak solution $\mathbf{u}$ is in $H^{3/2+\nu}(\Omega_1) \times H^{3/2+\nu}(\Omega_2)$, $0 < \nu \leq 0.5$. This optimal a priori bound is new and holds in contrast to other results in 2D and 3D. Compared with a standard linear saddle point formulation, two additional terms which provide a measure for the nonconformity of the approach have to be taken into account.

Abstract. Nous intéressons à une méthode d'éléments finis non conformes appliquée à un problème de contact sans frottement entre deux solides déformables. On se propose d'approcher les déplacements sur des maillages différents et de les coupler par la méthode des éléments avec joints. Nous donnons des nouveaux résultats optimaux pour l'erreur de discrétisations en dimension deux et trois.

La simulation numérique des problèmes de contact entre plusieurs solides déformables est assez difficile. Le but de cette note est de démontrer des résultats optimaux sur la majoration d'erreur a priori de la méthode des joints appliquée au problème de contact sans frottement. La méthode des joints a été indroduite en [6] et a été également appliquée aux inégalités variationelles, cf. [4, 5, 8]. On considère le problème de contact sans frottement entre deux solides déformables. Les deux solides $\Omega_i \subset \mathbb{R}^d$, $1 \leq i \leq 2$, $d = 2, 3$ sont initialement en contact sur le segment Γ_C. Le bord de $\Omega = \Omega_1 \cup \Omega_2$ est composé de trois parties disjointes Γ_D, Γ_N et Γ_C. Sur Γ_D, on impose des conditions aux limites de type Dirichlet homogène et des forces surfaciques p_i, $1 \leq i \leq d$, sont appliquées sur Γ_N. Les deux solides ne peuvent pas se pénétrer. La condition de contact entre Ω_1 et Ω_2 est decrite par (1.2) et (1.3). La condition complémentaire indique que les deux solides se touchent ou que $\sigma_n(\mathbf{u}_1)$ est égale à zéro. Le vecteur normal unitaire sur $\partial\Omega_i$ est noté par $\mathbf{n}_i$. Comme il n'y a pas de frottement, la composante tangentielle $\sigma_T(\mathbf{u}_1)$ est nulle sur tout Γ_C. Le problème consiste à trouver un champ de déplacements $\mathbf{u}$ vérifiant (1.1)–(1.3). On est alors amené à chercher la solution faible du problème de point selle (1.4), où $\mathbf{X}$ est un espace Hilbertien et $\mathbf{M}^+$ est un cône convexe. Ce problème est équivalent à une inégalité variationnelle: Trouver $\mathbf{u} \in \mathcal{K}$ telle que

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}), \quad \mathbf{v} \in \mathcal{K} \ ,$$

où le convexe $\mathcal{K}$ est donné par $\mathcal{K} := \{\mathbf{u} \in \mathbf{X}, [\mathbf{u} \cdot \mathbf{n}_1] \leq g\}$. On dénote le saut de champ de déplacement en direction de la normale par $[\mathbf{u} \cdot \mathbf{n}_1]$. Les définitions de $\mathbf{X}$ et de la forme bilinéaire $a(\cdot, \cdot)$ sont données dans la section 1. Pour discrétiser le problème, chaque solide est maillé avec une triangulation régulière et $\mathbf{u}$ est approchée par des éléments finis P1. Des fonctions de base biorthogonales sont utilisées pour définir l'espace des multiplicateurs de Lagrange, [9].

THÉORÈME 0.1. *Soit $(\mathbf{u}, \lambda)$ la solution du problème variationnel (1.4) et $(\mathbf{u}_h, \lambda_h)$ la solution discrète. Supposons que la solution $\mathbf{u}$ appartienne à $\mathbf{H}^{3/2+\nu}(\Omega_1) \times \mathbf{H}^{3/2+\nu}(\Omega_2)$, $0 < \nu \leq \frac{1}{2}$. Supposons également qu'en 2D, le nombre des points pour lesquels on passe de $[\mathbf{u} \cdot \mathbf{n}_1] < 0$ à $[\mathbf{u} \cdot \mathbf{n}_1] = 0$ soit fini et qu'en 3D le nombre des courbes pour lesquelles on passe de $[\mathbf{u} \cdot \mathbf{n}_1] < 0$ à $[\mathbf{u} \cdot \mathbf{n}_1] = 0$ soit fini et qu'elles soient assez régulières. On trouve alors une borne optimale pour l'erreur de la discrétisation*

$$\|\mathbf{e}_h\|^2 := \|\mathbf{u} - \mathbf{u}_h\|_1^2 + \|\lambda - \lambda_h\|_{-\frac{1}{2}}^2 \leq C h^{1+2\nu} \|\mathbf{u}\|_{3/2+\nu}^2 \ . \tag{0.1}$$

1. Introduction and variational formulation. The numerical approximation of non linear multibody contact problems is a challenging task. Modern discretization techniques are very often based on flexible nonconforming approaches. Here, we provide new optimal a priori estimates for a nonconforming domain decomposition technique based on mortar methods. We focus on dual Lagrange multipliers which have been introduced for the linear case in [9]. Dual Lagrange multiplier spaces are based on a biorthogonal basis resulting in a diagonal mass matrix. But all our results can be easily generalized for standard Lagrange multipliers, see, e.g., [6]. The main advantage of dual Lagrange multipliers is that fast and efficient monotone multigrid methods for multibody contact problems can be applied as iterative solvers, [11]. This is not possible for the case of standard Lagrange multipliers.

For simplicity, we restrict ourselves to the case of two deformable linear elastic bodies in contact. The two bodies in their reference configuration are identified with the domains $\Omega_k \subset \mathbb{R}^d$, $k = 1, 2$, $d = 2, 3$, and we decompose the solution $\mathbf{u}$ in $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, and write $(\mathbf{u}_k)_n := \mathbf{u}_k \cdot \mathbf{n}_k$, $k = 1, 2$, where $\mathbf{n}_k$ is the outer unit normal on $\partial\Omega_k$. The non-mortar side is associated with subdomain Ω_1. We start with the decomposition of the boundary of Ω into three disjoint parts, Γ_D is the Dirichlet part, Γ_N denotes the Neumann part and Γ_C stands for the contact boundary. On both subdomains, the possible contact boundary Γ_C is associated with a suitable parametrization. The actual contact zone between the two bodies is a priori unknown and is assumed to be a subset of Γ_C. We denote tensor and vector quantities by bold symbols, e.g., τ and $\mathbf{v}$, and its components by τ_{ij} and v_i, $1 \leq i, j \leq d$. The partial derivative with respect to x_j is abbreviated with the index $_{,j}$. Furthermore, we enforce the summation convention on all repeated indices ranging from 1 to d.

The non linear contact problem can be written as a boundary value problem. Here, we consider the case without friction. In addition to the equilibrium conditions in Ω_1 and Ω_2 and the boundary conditions on $\partial\Omega$

$$\begin{array}{rcll} -\sigma_{ij}(\mathbf{u})_{,j} &=& f_i, & \text{in } \Omega_1 \cup \Omega_2 \ , \\ \mathbf{u} &=& 0, & \text{on } \Gamma_D \ , \\ \sigma_{ij}(\mathbf{u}) \cdot n_j &=& p_i, & \text{on } \Gamma_N \ , \end{array} \tag{1.1}$$

we have the following conditions on the possible contact boundary Γ_C

$$\sigma_T(\mathbf{u}_1) = \sigma_T(\mathbf{u}_2) = 0 \ , \qquad \sigma_n(\mathbf{u}_1) = \sigma_n(\mathbf{u}_2) \leq 0 \ , \tag{1.2}$$

and the linearized non penetration condition on Γ_C

$$g \geq ((\mathbf{u}_1)_n + (\mathbf{u}_2)_n) =: [\mathbf{u} \cdot \mathbf{n}_1] \ , \qquad 0 = ((\mathbf{u}_1)_n + (\mathbf{u}_2)_n - g)\, \sigma_n(\mathbf{u}_1) \ , \tag{1.3}$$

where the function $g \colon \Gamma_C \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ is the distance between the two bodies in normal direction taken with respect to the reference configuration; see, e.g., [7]. Here, we assume for simplicity that $g = 0$ and that Γ_C is a straight line in 2D and a straight segment in 3D. Based on the equation of equilibrium (1.1), the strain-displacement relation and the constitutive law a weak formulation can be obtained. In the case of a linear elastic material, the stress tensor $\sigma := E_{ijlm}\, u_{l,m}$ depends linearly on the infinitesimal strain tensor $\epsilon(\mathbf{u}) := 1/2(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$, where Hooke's tensor $\mathbf{E} := (E_{ijlm})_{ijlm=1}^d$, $E_{ijlm} \in L^\infty(\Omega)$, is assumed to be sufficiently smooth, symmetric and uniformly positive definite.

The weak solution of the non linear contact problem (1.1)–(1.3) can be obtained by a minimization problem on a convex set giving rise to a variational inequality or equivalently to a saddle point formulation. The unconstrained product space is denoted by $\mathbf{X} := \mathbf{H}^1_*(\Omega_1) \times \mathbf{H}^1_*(\Omega_2)$, where $\mathbf{H}^1_*(\Omega_i) := \{\mathbf{v} \in (H^1(\Omega_i))^d \ ; \ \mathbf{v}_{|\Gamma_D} = 0\}$. To obtain the saddle point problem, we introduce a Lagrange multiplier space $\mathbf{M} := M^d$ being the dual space of the trace space W^d of $\mathbf{H}^1_*(\Omega_1)$ restricted to Γ_C. The Lagrange multiplier λ is equal to $-\sigma_n(\mathbf{u}_1)$. Here, we assume that Γ_C is compact embedded in $\partial\Omega_1 \setminus \Gamma_D$. We note that the approach can be generalized to the case that $\Gamma_C = \partial\Omega_1 \setminus \Gamma_D$. In that case, we have to work with $H^{1/2}_{00}(\Gamma_C)$ instead of $H^{1/2}(\Gamma_C)$, and in the discrete setting we have to use a Lagrange multiplier space having the same modification as in the case of mortar finite elements in the neighborhood of crosspoints in 2D and of the wirebasket in 3D. Then the weak form of the non linear contact problem can be rewritten as: Find $(\mathbf{u}, \lambda) \in (\mathbf{X}, \mathbf{M}^+)$

$$\begin{array}{lcll} a(\mathbf{u}, \mathbf{v}) + b(\lambda, \mathbf{v}) &=& f(\mathbf{v}) \ , & \mathbf{v} \in \mathbf{X}, \\ b(\mu - \lambda, \mathbf{u}) &\leq& 0, & \mu \in \mathbf{M}^+, \end{array} \tag{1.4}$$

where $\mathbf{M}^+ := \{\mu \in \mathbf{M} \,|\, < \mu \cdot \mathbf{n}_1, w >_{\Gamma_C} \geq 0, w \in W^+\}$, $W^+ := \{w \in W \,|\, w \geq 0\}$. The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a(\mathbf{v}, \mathbf{w}) := \sum_{k=1}^{2} \int_{\Omega_k} E_{ijlm} w_{i,j} v_{l,m}\, dx \ , \qquad b(\mu, \mathbf{v}) := < (\mathbf{v}_1)_n + (\mathbf{v}_2)_n, \mu \cdot \mathbf{n}_1 >_{\Gamma_C} \ ,$$

and $< \cdot,\cdot >_{\Gamma_C}$ denotes the duality pairing between M and W. Moreover, it is easy to see that $b(\lambda, \mathbf{u}) = 0$ and $\lambda_T := \lambda - \lambda^n \mathbf{n}_1 = 0$, where $\lambda^n := \lambda \cdot \mathbf{n}_1$ is the normal component of λ. We denote by $\gamma_a \subset \Gamma_C$ the actual contact set, i.e., $[\mathbf{u} \cdot \mathbf{n}_1] = 0$ on γ_a and by $\gamma_c := \Gamma_C \setminus \gamma_a$ the complement. If the displacement $\mathbf{u}$ is continuous, the set γ_a is a closed subset of Γ_C.

2. Optimal a priori estimates. To discretize the non linear contact problem, we use on each subdomain Ω_i a shape regular and quasi uniform triangulation and conforming finite elements $\mathbf{X}_{h_i} := (X_{h_i})^d$ of lowest order, i.e., piecewise linear finite elements on simplicial triangulations and piecewise bilinear and bicubic elements on rectangular and hexahedral triangulations, respectively. On both subdomains independent triangulations can be used resulting generally in non-matching triangulations at Γ_C. We assume that there are two constants $0 < c_1 \leq c_2 < \infty$ such that $c_1 h_1 \leq h_2 \leq c_2 h_1$, $h := \max(h_1, h_2)$. The discrete Lagrange multiplier space $\mathbf{M}_h := (M_h)^d$ being defined on the non-mortar side of the possible contact boundary Γ_C is associated with the $(d-1)$-dimensional mesh inherited from the triangulation on Ω_1. Its basis functions are locally supported and biorthogonal to the nodal basis functions of the discrete trace space W_h of X_{h_1}, see, e.g., [9], with respect to the L^2-scalar product on Γ_C. We define the discrete convex cone $\mathbf{M}_h^+$ by $\mathbf{M}_h^+ := \{\mu \in \mathbf{M}_h \,|\, < \mu \cdot \mathbf{n}_1, w >_{\Gamma_C} \geq 0, w \in W_h^+\}$, $W_h^+ := W^+ \cap W_h$. Replacing $(\mathbf{X}, \mathbf{M}^+)$ in (1.4) by the discrete space $(\mathbf{X}_h, \mathbf{M}_h^+)$, where $\mathbf{X}_h := \mathbf{X}_{h_1} \times \mathbf{X}_{h_2}$, we obtain the discrete solution $(\mathbf{u}_h, \lambda_h)$. The broken Sobolev norms on $\mathbf{H}^s(\Omega_1) \times \mathbf{H}^s(\Omega_2)$ are denoted by $\|\cdot\|_s$. If we consider a norm on Γ_C, we add the additional lower index Γ_c

THEOREM 2.1. *Under the regularity assumption that* $\mathbf{u} \in \mathbf{H}^{3/2+\nu}(\Omega_1) \times \mathbf{H}^{3/2+\nu}(\Omega_2)$, $0 < \nu \leq \frac{1}{2}$, *and the number of points in 2D and the number of curves in 3D of* $\gamma_a \cap \bar{\gamma}_c$ *is finite and that* $\bar{\gamma}_c$ *in 3D satisfies a cone condition, we obtain the following optimal a priori estimate*

$$\|\mathbf{e}_h\|^2 := \|\mathbf{u} - \mathbf{u}_h\|_1^2 + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C}^2 \leq C h^{1+2\nu} \|\mathbf{u}\|_{3/2+\nu}^2, \quad h \leq h_0 \ . \tag{2.1}$$

Proof. To obtain a priori estimates for a saddle point problem, the approximation properties of the discrete spaces and a uniform discrete inf-sup condition are essential. These results are well known for mortar finite elements. In contrast to the linear case, two additional terms which provide a measure for the nonconformity of the approach have to be taken into account

$$\|\mathbf{e}_h\|^2 \leq C \left(\inf_{\mathbf{v}_h \in \mathbf{X}_h} \|\mathbf{u} - \mathbf{v}_h\|_1^2 + \inf_{\mu_h \in \mathbf{M}_h} \|\lambda - \mu_h\|_{-\frac{1}{2}}^2 + \max(0, b(\lambda, \mathbf{u}_h)) + \max(0, b(\lambda_h, \mathbf{u})) \right) \ ,$$

see, e.g., [2, 8]. The upper bound for the first two terms is standard. We only have to consider the third term $\max(0, b(\lambda, \mathbf{u}_h))$ and the fourth term $\max(0, b(\lambda_h, \mathbf{u})$ in more detail. In a first step, we provide some well known results for the mortar projection and its dual operator in the case of dual Lagrange multipliers. The mortar projection $\Pi_h : L^2(\Gamma_c) \to W_h$ and its dual $\Pi_h^* : L^2(\Gamma_c) \to M_h$ are defined by

$$\int_{\Gamma_C} (\Pi_h w)\, \mu_h \, ds = \int_{\Gamma_C} w\, \mu_h \, ds, \quad \mu_h \in M_h, \quad \int_{\Gamma_C} (\Pi_h^* \mu)\, w_h \, ds = \int_{\Gamma_C} \mu w_h \, ds, \quad w_h \in W_h.$$

For both operators, the following approximation properties hold

$$\begin{aligned} \|w - \Pi_h w\|_{0,\Gamma_C} &\leq c h^s |w|_{s,\Gamma_C}, \quad w \in H^s(\Gamma_C), 0 \leq s \leq 2, \\ \|\mu - \Pi_h^* \mu\|_{0,\Gamma_C} &\leq c h^s |\mu|_{s,\Gamma_C}, \quad \mu \in H^s(\Gamma_C), 0 \leq s \leq 1, \end{aligned}$$

see [10].

Using the definition of the bilinear form $b(\cdot,\cdot)$, $\mathbf{M}_h^+$ and W_h^+, the second line of the saddle point problem (1.4) yields $\int_{\Gamma_C} \psi_j [\mathbf{u}_h \mathbf{n}_1]\, d\sigma \leq 0$ for all nodal basis functions ψ_j of M_h. By means of the definition of Π_h, we find

$$\int_{\Gamma_C} \psi_j [\mathbf{u}_h \mathbf{n}_1]\, d\sigma = \int_{\Gamma_C} \psi_j \Pi_h [\mathbf{u}_h \mathbf{n}_1]\, d\sigma \leq 0.$$

Using the fact that $\Pi_h[\mathbf{u}_h\mathbf{n}_1] \in W_h$, we can write $\Pi_h[\mathbf{u}_h\mathbf{n}_1] = \sum_i \alpha_i\phi_i$, where ϕ_i are the standard nodal basis functions of the trace space W_h. Now, the biorthogonality between the basis functions gives that $\alpha_i \le 0$ and thus $\Pi_h[\mathbf{u}_h\mathbf{n}_1] \le 0$. Note that this result does not require that the Lagrange multiplier is continuous nor that it is greater equal zero. Introducing $\lambda_n := \lambda \cdot \mathbf{n}_1$ and observing that $\lambda_n \in H^\nu(\Gamma_C)$, $\lambda_n \ge 0$, we find

$$\begin{aligned} b(\lambda, \mathbf{u}_h) &= \int_{\Gamma_C} \lambda_n\big([\mathbf{u}_h\,\mathbf{n}_1] - \Pi_h[\mathbf{u}_h\,\mathbf{n}_1] + \Pi_h[\mathbf{u}_h\,\mathbf{n}_1]\big)\, d\sigma \le \int_{\Gamma_C} \lambda_n\big([\mathbf{u}_h\,\mathbf{n}_1] - \Pi_h[\mathbf{u}_h\,\mathbf{n}_1]\big)\, d\sigma \\ &= \int_{\Gamma_C} (\lambda_n - \Pi_h^*\lambda_n)\big([\mathbf{u}_h\,\mathbf{n}_1] - \Pi_h[\mathbf{u}_h\,\mathbf{n}_1]\big)\, d\sigma \\ &\le Ch^\nu|\lambda_n|_{\nu,\Gamma_C}\|[\mathbf{u}_h\,\mathbf{n}_1] - \Pi_h[\mathbf{u}_h\mathbf{n}_1]\|_{0,\Gamma_C} \le Ch^\nu|\mathbf{u}|_{\frac{3}{2}+\nu}\|[\mathbf{u}_h\,\mathbf{n}_1] - \Pi_h[\mathbf{u}_h\mathbf{n}_1]\|_{0,\Gamma_C} \end{aligned}$$

In a next step, we consider the term $\|[\mathbf{u}_h\,\mathbf{n}_1] - \Pi_h[\mathbf{u}_h\mathbf{n}_1]\|_{0,\Gamma_C}$ in more detail. Applying the triangle inequality and the approximation property of the mortar projection gives

$$\begin{aligned} \|[\mathbf{u}_h\,\mathbf{n}_1] - \Pi_h[\mathbf{u}_h\mathbf{n}_1]\|_{0,\Gamma_C} &\le \|[(\mathbf{u}_h - \mathbf{u})\,\mathbf{n}_1] - \Pi_h[(\mathbf{u}_h - \mathbf{u})\,\mathbf{n}_1]\|_{0,\Gamma_C} + \|[\mathbf{u}\,\mathbf{n}_1] - \Pi_h[\mathbf{u}\mathbf{n}_1]\|_{0,\Gamma_C} \\ &\le C(h^{\frac{1}{2}}|[(\mathbf{u}_h - \mathbf{u})\,\mathbf{n}_1]|_{\frac{1}{2},\Gamma_C} + h^{1+\nu}|[\mathbf{u}\,\mathbf{n}_1]|_{1+\nu,\Gamma_C}). \end{aligned}$$

Combining these results and applying Young's inequality, we find for $\epsilon > 0$

$$b(\lambda, \mathbf{u}_h) \le C(h^{1+2\nu}|\mathbf{u}|^2_{\frac{3}{2}+\nu} + h^{\frac{1}{2}+\nu}|\mathbf{u}|_{\frac{3}{2}+\nu}|\mathbf{u}_h - \mathbf{u}|_1) \le \frac{C}{\epsilon}h^{1+2\nu}|\mathbf{u}|^2_{\frac{3}{2}+\nu} + \epsilon|\mathbf{u}_h - \mathbf{u}|_1^2. \tag{2.2}$$

Using the best approximation properties in the upper bound for $\|\mathbf{e}_h\|$ and (2.2) with a fixed $\epsilon > 0$ small enough, we obtain

$$\|\mathbf{e}_h\|^2 \le C\left(h^{1+2\nu}|\mathbf{u}|^2_{\frac{3}{2}+\nu} + \max(0, b(\lambda_h, \mathbf{u}))\right) \;.$$

To obtain an optimal bound for the fourth term $\max(0, b(\lambda_h, \mathbf{u}))$, we introduce a discrete truncation operator $\tilde{I}_h : \mathbf{H}^{3/2+\nu}(\Omega_1) \times \mathbf{H}^{3/2+\nu}(\Omega_2) \longrightarrow W_h$ which is defined in terms of its values at the nodes p. We set $\tilde{I}_h\mathbf{v}(p) := I_h[\mathbf{v}\cdot\mathbf{n}_1](p)$ if supp $\phi_p \subset \gamma_a$ or if supp $\phi_p \subset \bar{\gamma}_c$, and $\tilde{I}_h\mathbf{v}(p) := 0$ for all other nodes. Here, ϕ_p stands for the nodal hat function associated with the node p, and I_h denotes a suitable locally defined quasi interpolation operator such that

$$I_h[\mathbf{v}\cdot\mathbf{n}_1](p) \le 0 \text{ if } [\mathbf{v}\cdot\mathbf{n}_1] \le 0 \quad\text{and}\quad \|[\mathbf{v}\cdot\mathbf{n}_1] - I_h[\mathbf{v}\cdot\mathbf{n}_1]\|_{\frac{1}{2};\Gamma_C} \le Ch^{\frac{1}{2}+\nu}\|\mathbf{u}\|_{3/2+\nu} \;. \tag{2.3}$$

Moreover the value of $I_h[\mathbf{v}\cdot\mathbf{n}_1](p)$ depends only on the values of $[\mathbf{v}\cdot\mathbf{n}_1]$ restricted to the support of ϕ_p. It is easy to see that the Lagrange interpolation operator satisfies the conditions on I_h. Then, we find that $\int_{\Gamma_C} \lambda^n\tilde{I}_h\mathbf{u}\; d\sigma = 0$ and by definition $-\tilde{I}_h\mathbf{u} \in W_h^+$. Due to this observation, we obtain the following bound

$$\begin{aligned} b(\lambda_h, \mathbf{u}) &= \int_{\Gamma_C} \lambda_h^n[\mathbf{u}\cdot\mathbf{n}_1]\; d\sigma \le \int_{\Gamma_C} \lambda_h^n([\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u})\; d\sigma = \int_{\Gamma_C} (\lambda_h^n - \lambda^n)([\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u})\; d\sigma \\ &\le \|\lambda - \lambda_h\|_{-\frac{1}{2};\Gamma_C}\|[\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u})\|_{\frac{1}{2};\Gamma_C} \\ &\le \|\lambda - \lambda_h\|_{-\frac{1}{2};\Gamma_C}(\|[\mathbf{u}\cdot\mathbf{n}_1] - I_h[\mathbf{u}\cdot\mathbf{n}_1]\|_{\frac{1}{2};\Gamma_C} + \|I_h[\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u}\|_{\frac{1}{2};\Gamma_C}) \\ &\le C\|\lambda - \lambda_h\|_{-\frac{1}{2};\Gamma_C}(h^{0.5+\nu}\|\mathbf{u}\|_{3/2+\nu} + \|I_h[\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u}\|_{\frac{1}{2};\Gamma_C}) \;. \end{aligned}$$

Now, we have to consider $\|I_h[\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u}\|_{\frac{1}{2};\Gamma_C}$ in more detail. To do so, we introduce the scalar function $s(\cdot)$ by $s := [\mathbf{u}\cdot\mathbf{n}_1]$. Using an inverse inequality, it is sufficient to bound the L^2-norm

$$\|I_h[\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u}\|^2_{\frac{1}{2};\Gamma_C} \le \frac{C}{h}\|I_h[\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u}\|^2_{0;\Gamma_C} \le Ch^{d-2}\sum_{p\in\mathcal{N}_h^c}(I_h s(p))^2 \;, \tag{2.4}$$

where the set $\mathcal{N}_h^c$ denotes all nodes p such that $\tilde{I}_h\mathbf{u}(p) \neq I_h[\mathbf{u}\cdot\mathbf{n}_1](p)$. Due to the regularity assumptions on γ_c, γ_a, we find that the number of points in $\mathcal{N}_h^c$ is finite in 2D and of order h^{-1} in 3D. In a first step, we consider the situation in 2D. Here, we use the standard Lagrange interpolation operator on W_h for I_h. It is well known that it satisfies the conditions (2.3). We set $\mathcal{N}^c := \bar{\gamma}_c \cap \gamma_a$ and recall that the number of points in $\mathcal{N}^c$ is finite by assumption. Let $d_{\min} > 0$ be the minimal distance between two different points in $\mathcal{N}^c$. Moreover, we assume that $h \leq d_{\min}/2$. This assumption can be replaced by a weaker local one. Due to the regularity assumptions, $s(\cdot)$ is continuous, and we can write $I_h s(p) = s(p)$, $p \in \mathcal{N}_h^c$, as

$$s(p) = \int_{p_c}^{p} s'(\tau)\; d\tau = \frac{1}{\|p_s - p_c\|} \int_{p_s}^{p_c} \left(\int_{p_c}^{p} s'(\tau) d\tau \right) d\sigma \ ,$$

where for a given $p \in \mathcal{N}_h^c$ the two points p_s and p_c are defined as follows: The point $p_c \in \mathcal{N}^c$ is given by $p_c \in \text{supp}\ \phi_p$, and p_s is a discrete node such that $\|p - p_s\| \leq 2h$, $p_s \notin \text{supp}\ \phi_p$ and $p_c \in J_p := \{p + t(p_s - p), 0 \leq t \leq 1\}$. For each $p \in \mathcal{N}_h^c$ such p_s and p_c exist and are uniquely defined. The right picture in Figure 2.1 illustrates the notation. The left picture shows $\tilde{I}_h$ for a given $[\mathbf{u}\cdot\mathbf{n}_1]$, the nodes for which $\tilde{I}_h\mathbf{u} \neq I_h[\mathbf{u}\cdot\mathbf{n}_1]$ are marked by empty circles.

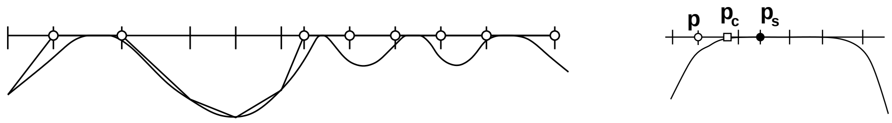

FIG. 2.1. *Nodes in $\mathcal{N}_h^c$ (left) and local notation (right)*

We note that $p_c \in \mathcal{N}^c$ does not have to be a discrete node. By construction, $\|p - p_c\| \leq h$, $\|p_c - p_s\| \leq 2h \leq d_{\min}$ and thus $I_p := \{p_c + t(p_s - p_c), 0 \leq t \leq 1\} \subset \gamma_a$. Moreover, the function $s(\cdot)$ restricted to I_p is zero. In terms of the quasi uniformity of the triangulation, we find $\|p_s - p_c\| \geq ch$ and thus

$$\begin{aligned}
(s(p))^2 &= \left(\frac{1}{\|p_s - p_c\|} \int_{p_s}^{p_c} \left(\int_{p_c}^{p} \frac{s'(\tau) - s'(\sigma)}{\|\tau - \sigma\|^{0.5+\nu}} \|\tau - \sigma\|^{0.5+\nu} d\tau \right) d\sigma \right)^2 \\
&\leq \frac{C}{h^2} \int_{p_s}^{p_c} \Big(\int_{p_c}^{p} \frac{(s'(\tau) - s'(\sigma))^2}{\|\tau - \sigma\|^{1+2\nu}} d\tau \Big) d\sigma \int_{p_s}^{p_c} \Big(\int_{p_c}^{p} \|\tau - \sigma\|^{1+2\nu} d\tau \Big) d\sigma \\
&\leq \frac{C}{h^2} h^{1+2\nu} h^2 \|s'\|_{\nu;J_p}^2 \leq C h^{1+2\nu} \|s'\|_{\nu;J_p}^2 \ .
\end{aligned}$$

Now, we can sum over all $p \in \mathcal{N}_h^c$ and obtain in 2D

$$\begin{aligned}
\|I_h[\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u}\|_{\frac{1}{2};\Gamma_C}^2 &\leq C h^{1+2\nu} \sum_{p \in \mathcal{N}_h^c} \|s'\|_{\nu;J_p}^2 \leq C h^{1+2\nu} \|[\mathbf{u}\cdot\mathbf{n}_1]\|_{1+\nu;\Gamma_c}^2 \leq C h^{1+2\nu} \|[\mathbf{u}]\|_{1+\nu;\Gamma_c}^2 \\
&\leq C h^{1+2\nu} \|\mathbf{u}\|_{3/2+\nu}^2 \ .
\end{aligned}$$

In 3D, the proof is more technically, and we do not work out all details but provide the basic steps. Here, we use a quasi interpolation operator for I_h which is defined locally by its values at the nodes on Γ_c

$$I_h s(p) := \frac{1}{2c_0 h} \int_{S_p} s(\sigma)\; d\sigma_z \ ,$$

where p is the midpoint of the one dimensional interval S_p of length $2c_0h$ and direction z. The direction z is arbitrary but should be fixed for each node p. The constant $c_0 > 0$ depends on the shape regularity of the mesh, but not on h. It is taken such that $B_p(c_0h) \subset$ supp ϕ_p for all nodes p, where $B_p(c_0h)$ is the circle with radius c_0h and center p. We note that I_h restricted to W_h is not the identity, but it reproduces a linear function. This operator is similar to the ones of Clément and Scott–Zhang, and thus it is easy to verify (2.3). Under the assumption that h is small enough, there exists for each $p \in \mathcal{N}_h^c$ a shape regular quadrilateral R_p such that S_p is one edge, see the left picture of Figure 2.2. Moreover $R_p \cap \gamma_a$ contains a shape regular quadrilateral having the opposite edge of S_p as one edge and having an area $\geq c_R h^2$, where all regularity constants depend only on the regularity of the triangulation and of $\bar{\gamma}_c$ but not on the node p. This quadrilateral can be now mapped by F_p to a reference square $\hat{R}$ with length h such that $\hat{s}(\hat{\sigma}) := s(F_p^{-1}(\hat{\sigma}))$ restricted to the lower half $\hat{R}_2$ of the square is zero, and such that the Jacobian and its inverse is bounded. The upper half of the square is denoted by $\hat{R}_1$, see Figure 2.2.

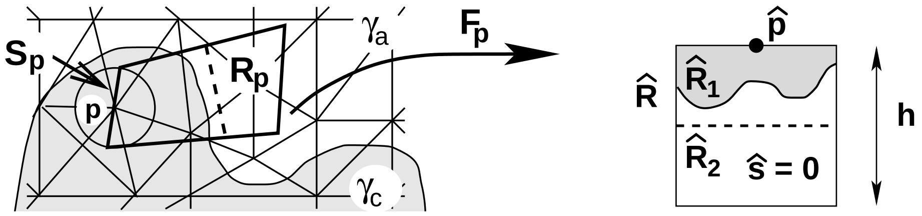

FIG. 2.2. *Edge S_p and quadrilateral R_p (left) and reference square (right)*

Now, we can proceed as in 2D and find an upper bound for $I_h s(p)$

$$
\begin{aligned}
(I_h s(p))^2 &= \frac{1}{h^2}\left(\int_0^h \hat{s}(\sigma_1, h)\; d\sigma_1\right)^2 = \frac{1}{h^2}\left(\int_{\hat{R}_1} \hat{s}_{\sigma_2}(\sigma)\; d\sigma\right)^2 \\
&= \frac{4}{h^6}\left(\int_{\hat{R}_2}\left(\int_{\hat{R}_1} \frac{\hat{s}_{\sigma_2}(\sigma) - \hat{s}_{\sigma_2}(\sigma')}{\|\sigma - \sigma'\|^{1+\nu}}\|\sigma - \sigma'\|^{1+\nu}\; d\sigma\right) d\sigma'\right)^2 \\
&\leq \frac{4}{h^6}\int_{\hat{R}_2}\int_{\hat{R}_1} \frac{(\hat{s}_{\sigma_2}(\sigma) - \hat{s}_{\sigma_2}(\sigma'))^2}{\|\sigma - \sigma'\|^{2+2\nu}} d\sigma d\sigma' \int_{\hat{R}_2}\int_{\hat{R}_1} \|\sigma - \sigma'\|^{2+2\nu} d\sigma d\sigma' \\
&\leq \frac{C}{h^6} h^{2+2\nu} h^4 \int_{\hat{R}}\int_{\hat{R}} \frac{(\hat{s}_{\sigma_2}(\sigma) - \hat{s}_{\sigma_2}(\sigma'))^2}{\|\sigma - \sigma'\|^{2+2\nu}} d\sigma d\sigma' \leq C h^{2\nu} \|\hat{s}_{\sigma_2}\|^2_{\nu;\hat{R}} \leq C h^{2\nu} \|s\|^2_{1+\nu;R_p} \quad .
\end{aligned}
$$

A coloring argument and (2.4) yield $\|I_h[\mathbf{u}\cdot\mathbf{n}_1] - \tilde{I}_h\mathbf{u}\|_{\frac{1}{2};\Gamma_C} \leq C h^{0.5+\nu} \|[\mathbf{u}\cdot\mathbf{n}_1\|_{1+\nu;\Gamma_C}$. □

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