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AN ENERGY METHOD FOR THE STRONGLY NONLINEAR CAHN-LARCHÉ EQUATION SYSTEM

1. Introduction

In this paper, we study a mathematical model of diffusive phase separation taking mechanical interactions into account. The model contains a fourth order parabolic equation for the mass concentration c and a second order elliptic equation for the displacement $\boldsymbol{u}$. We allow, that the mobility tensor $\boldsymbol{M}$ and the surface stress tensor $\boldsymbol{\Gamma}$ depend on the mass concentration c. The existence proof of weak solutions of the following type of a Cahn-Larché equation system represents a new result. Our model has the form

$$\dot{c} - \operatorname{div}_{\boldsymbol{x}}(\boldsymbol{M}(c)\nabla_{\boldsymbol{x}}\mu) = 0,$$

$$\mu = -\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{\Gamma}(c)\nabla_{\boldsymbol{x}}c) + \frac{1}{2}\boldsymbol{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c \cdot \nabla_{\boldsymbol{x}}c + \psi_{,c}(c) + W_{,c}(c, \boldsymbol{\varepsilon}(\boldsymbol{u})),$$

$$-\operatorname{div}_{\boldsymbol{x}}(W_{,\boldsymbol{\varepsilon}}(c, \boldsymbol{\varepsilon}(\boldsymbol{u}))) = \boldsymbol{0}.$$

Diffusive phase separation without elastic misfit is studied in [4, 5] and is classically described by Cahn-Hilliard's equation. From [4] follows, that the free energy is a Ginsburg-Landau functional defined in terms of the mass concentration c. The classical Cahn-Hilliard theory describes interfaces as transition layers of finite but small thickness. Therefore the surface energy term is quadratic in the gradient of the mass concentration c, including the surface stress tensor $\boldsymbol{\Gamma}$ as constitutive modulus. The homogeneous free energy density ψ takes the form of a non-convex function with respect to the mass concentration c in the case of phase separation.

The consideration of elastic misfit affecting the diffusive phase separation is done in [19], while in [18] a thermodynamical consistent derivation is presented. In this case the free energy is a Ginsburg-Landau functional in terms of the mass concentration c and additionally of the displacement $\boldsymbol{u}$. Phenomenological studies of the influence of mechanicals effects on the phase separation are outlined in [14, 15].

In general, these models consider only equations with constant material parameters. In [21] the above field equations with concentration depending material parameters are derived in a thermodynamical consistent way. Furthermore, from [9] we get constitutive equations for the concentration depending material parameters.

There are a lot of papers on Cahn-Hilliard and Cahn-Larché equations, which are related to this one. In [12] there is proved an existence result of weak solutions of Cahn-Hilliard's equation without elasticity, constant material parameters and smooth homogeneous free energy densities. The more practical case of a logarithmic homogeneous free energy is considered in [11], which holds also for systems of Cahn-Hilliard equations. There is an existence result for the Cahn-Hilliard equation with concentration depending and degenerating mobilities in [10].

Furthermore, we refer to existence results of Cahn-Larché's equation system, which takes elastic effects into account. An existence result for this system with constant material parameters is formulated in [6]. In [16] this result is generalised to case of concentration depending elasticity, but considers constant surface stress tensor as well as a constant mobility tensor. The result from [16] holds also for systems as well as for logarithmic homogeneous free energies. A viscous Cahn-Larché equation system with concentration depending material parameters is observed in [3]. But this model differs from that one discussed in this paper, because the field equations in [3] do not satisfy a gradient flow structure.

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This paper is organised as follows:
In section 2 we derive the generalised Cahn-Larché equation system, whereas we focus the attention on the gradient flow structure. This property presents the key to the following analysis.
Moreover, in section 3 we formulate the function spaces, which are used to calculate weak solutions. We extend the assumptions formulated in [16], such that the nonlinearities in the surface stress tensor $\boldsymbol{\Gamma}(c)$ and in the mobility $\boldsymbol{M}(c)$ can be attended. Finally, we formulate different types of weak solutions of the Cahn-Larché equation system.
Section 4 is the main section of this paper, it contains the existence proof of weak solutions. Our existence proof consists of an extension of the ideas of [16] in order to deal with the nonlinearities in the mobility tensor $\boldsymbol{M}(c)$ and in the surface stress tensor $\boldsymbol{\Gamma}(c)$. We apply a discretisation of the Cahn-Larché's equation system with respect to time and calculate a discrete weak solution by using an energy minimisation method and taking the gradient flow structure into account. Finally we determine an uniform a-priori energy estimate in order to apply the classical compactness methods from [20] to get a weak solution.

2. Statement of the problem

We consider a binary alloy, for example a Sn-Pb or a Sn-Ag system, which is modelled as a mixture of two different components. The alloy occupies a spatial domain $\Omega \subset \mathbb{R}^d$ with an outer normal vector $\boldsymbol{n}$, whereas the space dimensions $d = 2; 3$ are observed. We describe the phase separation process inside the binary system in a time interval $(0, T)$ and we introduce the time space cylinder $\Omega_T := (0, T) \times \Omega$, furthermore we denote its cylindrical shell by $\Gamma_T := (0, T) \times \partial\Omega$.
The microstructure of the binary alloy is characterised by the mass concentration of the first component, denoted by c, whereas the mass concentration of the second component can be calculated by $1 - c$. The diffusion equation follows from the continuity equation by observing a diffusive flux $\boldsymbol{J}$ and a transformation into Lagrangian coordinates, see [21]. Additionally, we consider a no flux boundary condition and an initial condition c_0. These considerations lead to

$$\dot{c} + \operatorname{div}_{\boldsymbol{x}}(\boldsymbol{J}) = 0 \qquad \text{in } \Omega_T, \tag{2.1}$$

$$\boldsymbol{J} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_T, \tag{2.2}$$

$$c(0, \boldsymbol{x}) = c_0(\boldsymbol{x}) \qquad \text{in } \Omega. \tag{2.3}$$

We remark, that we use for the time derivative the notation $\dot{c} := \frac{d}{dt}c$. The diffusion equation (2.1) combined with the no flux boundary condition (2.2) guarantees mass conservation in Ω, which is proven in [21],

$$\int_\Omega c(t, \boldsymbol{x}) \, d\boldsymbol{x} = \int_\Omega c_0(\boldsymbol{x}) \, d\boldsymbol{x} =: m > 0. \tag{2.4}$$

In order to take mechanical effects into account, for example elastic misfit due to thermal mismatch, we have to consider the balance of linear momentum of the mixture. We denote the displacement of the mixture by $\boldsymbol{u}$ and the linearised second Piola-Kirchoff stress tensor of the alloy by $\boldsymbol{\sigma}$. A dimensional analysis in [21] shows, that a quasi steady equilibrium holds for the binary system.
Due to the fact, that we consider mixed boundary conditions, we decompose the boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ into a Dirichlet boundary Γ_D, where the system is hard clamped, and a Neumann boundary Γ_N, where a mechanical loading $\boldsymbol{g}$ is observed. Finally, we introduce $\Gamma_T^D := (0, T) \times \Gamma_D$, $\Gamma_T^N := (0, T) \times \Gamma_N$ and formulate the quasi steady equilibrium

$$-\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{\sigma}) = \boldsymbol{0} \qquad \text{in } \Omega_T, \tag{2.5}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_T^D, \tag{2.6}$$

$$\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{g} \qquad \text{on } \Gamma_T^N. \tag{2.7}$$

In the following, we derive equations for the quantities $\boldsymbol{J}$ and $\boldsymbol{\sigma}$ from the free energy of the binary alloy. The formulation of the free energy considering the phase separation process under mechanical loadings in a binary system is done in [19]. Generalising the results from [19], we get the following free energy density $e(c, \nabla_{\boldsymbol{x}} c, \boldsymbol{\varepsilon}(\boldsymbol{u}))$ and the energy functional $E(c, \boldsymbol{u})$,

$$e(c, \nabla_{\boldsymbol{x}} c, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \frac{1}{2}\boldsymbol{\Gamma}(c)\nabla_{\boldsymbol{x}} c \cdot \nabla_{\boldsymbol{x}} c + \psi(c) + W(c, \boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{2.8}$$

$$E(c, \boldsymbol{u}) = \int_\Omega e(c, \nabla_{\boldsymbol{x}} c, \boldsymbol{\varepsilon}(\boldsymbol{u}))\, d\boldsymbol{x} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}\, d\boldsymbol{a}_{\boldsymbol{x}}. \tag{2.9}$$

The first term of the free energy density contains the surface energy density of the phases and is constitutively described by a second order surface stress tensor $\boldsymbol{\Gamma}(c)$. The surface energy of the phases describes Cahn-Hilliard diffusion and is derived in [4]. A detailed formula for the surface stress tensor $\boldsymbol{\Gamma}(c)$ used for instance in Sn-Pb systems is given in [9]. In this paper we assume, that the boundary $\partial\Omega$ does not affect Cahn-Hilliard diffusion, which provides a homogeneous Neumann boundary condition for the mass concentration c.

The second term of the free energy density takes the homogeneous free energy into account, which describes classical Fickean diffusion and its density is denoted by $\psi(c)$. For practical considerations we have to observe a homogeneous free energy, with a logarithmic entropy term and an interaction energy given by a non positive definite second order tensor $\boldsymbol{A}$. We point out, that even for non positive definite tensors $\boldsymbol{A}$ phase separation occurs. From [14] we get the homogeneous free energy density including the gas constant R and the absolute temperature θ as a parameter

$$\psi(c) = R\theta(c\ln(c) + (1-c)\ln(1-c)) + \frac{1}{2}\boldsymbol{A}\begin{pmatrix} c \\ 1-c \end{pmatrix} \cdot \begin{pmatrix} c \\ 1-c \end{pmatrix}.$$

Nevertheless, in the context of this paper we consider for mathematical reasons only homogeneous free energy densities of the form

$$\psi(c) := \psi_1(c) + \psi_2(c),$$

where $\psi_1(c)$ is a convex function and $\psi_2(c)$ is a polynomial.

From a practical point of view, it is suitable to restrict our considerations on small strains. Due to that fact we work with a linearised strain of the mixture, usually given by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\mathrm{D}_{\boldsymbol{x}}\boldsymbol{u} + (\mathrm{D}_{\boldsymbol{x}}\boldsymbol{u})^T).$$

In that case the strain energy is given by a Hook's law with a fourth order elasticity tensor $\boldsymbol{C}(c)$ and additionally takes eigenstrains $\overline{\boldsymbol{\varepsilon}}(c)$ into account. Constitutive relations for the elasticity tensor as well as for the eigenstrains concerning thermal mismatch are formulated in [9], used for Sn-Pb systems. The strain energy density therefore is calculated by

$$W(c, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \frac{1}{2}\boldsymbol{C}(c)(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \overline{\boldsymbol{\varepsilon}}(c)) : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \overline{\boldsymbol{\varepsilon}}(c)).$$

Adapting classical thermodynamical relations, we introduce the chemical potential μ, which in [24] is defined as the variational derivative of the free energy E with respect to the mass concentration c. We calculate from the energy density (2.8) the chemical potential

$$\begin{aligned}\mu &= -\operatorname{div}_{\boldsymbol{x}}(\mathrm{D}_{\nabla_{\boldsymbol{x}} c}\, e(c, \nabla_{\boldsymbol{x}} c, \boldsymbol{\varepsilon}(\boldsymbol{u}))) + \mathrm{D}_c\, e(c, \nabla_{\boldsymbol{x}} c, \boldsymbol{\varepsilon}(\boldsymbol{u})) \\ &= -\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{\Gamma}(c)\nabla_{\boldsymbol{x}} c) + \frac{1}{2}\boldsymbol{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}} c \cdot \nabla_{\boldsymbol{x}} c + \psi_{,c}(c) \\ &\qquad + \frac{1}{2}\boldsymbol{C}_{,c}(c)(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \overline{\boldsymbol{\varepsilon}}(c)) : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \overline{\boldsymbol{\varepsilon}}(c)) - \boldsymbol{C}(c)(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \overline{\boldsymbol{\varepsilon}}(c)) : \overline{\boldsymbol{\varepsilon}}_{,c}(c).\end{aligned} \tag{2.10}$$

We remark, that in this paper we use for the partial derivative the notation $\boldsymbol{\Gamma}_{,c}(c) := \frac{\partial}{\partial c}\boldsymbol{\Gamma}(c)$. Further, from classical thermodynamics we conclude, that the stress tensor of the alloy $\boldsymbol{\sigma}$ enters the model as the derivative of the free energy density e with respect to the strain $\boldsymbol{\varepsilon}$ and we get

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\varepsilon}}\, e(c, \nabla_{\boldsymbol{x}} c, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \boldsymbol{C}(c)(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \overline{\boldsymbol{\varepsilon}}(c)). \tag{2.11}$$

Thermodynamical restrictions evaluated in [18, 21] show, that $-\nabla_{\boldsymbol{x}}\mu$ is the driving force for the diffusive flux $\boldsymbol{J}$, where the relation between both quantities is given by a second order constitutive modulus named mobility $\boldsymbol{M}(c)$. A constitutive formula for the mobility $\boldsymbol{M}(c)$ is outlined in [9],

$$\boldsymbol{J} = -\boldsymbol{M}(c)\nabla_{\boldsymbol{x}}\mu. \tag{2.12}$$

The constitutive relation (2.12) in view of (2.10) and (2.11) show the flow gradient structure of the diffusion process. Finally, we know from [18, 21], that the dissipation of the process is a functional of the mass concentration c and the chemical potential μ, calculated by

$$D(c,\mu) = -\int_\Omega \boldsymbol{M}(c)\nabla_{\boldsymbol{x}}\mu \cdot \nabla_{\boldsymbol{x}}\mu \, d\boldsymbol{x} \le 0. \tag{2.13}$$

From the dissipation inequality (2.13) follows directly, that the mobility $\boldsymbol{M}(c)$ must be a positive definite second order tensor.
Inserting the constitutive relation (2.12) into the diffusion equation (2.14) and taking the chemical potential (2.10), as well as the quasi steady equilibrium (2.5) and the stress relation (2.11) into account, we get the generalised Cahn-Larché equation system with concentration depending material equations

$$\dot{c} - \operatorname{div}_{\boldsymbol{x}}(\boldsymbol{M}(c)\nabla_{\boldsymbol{x}}\mu) = 0 \quad \text{in } \Omega_T, \tag{2.14}$$

$$\mu = -\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{\Gamma}(c)\nabla_{\boldsymbol{x}}c) + \frac{1}{2}\boldsymbol{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c \cdot \nabla_{\boldsymbol{x}}c + \psi_{,c}(c) + W_{,c}(c,\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega_T, \tag{2.15}$$

$$-\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{C}(c)(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \overline{\boldsymbol{\varepsilon}}(c))) = \boldsymbol{0} \quad \text{in } \Omega_T, \tag{2.16}$$

$$\boldsymbol{\Gamma}(c)\nabla_{\boldsymbol{x}}c \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_T, \tag{2.17}$$

$$\boldsymbol{M}(c)\nabla_{\boldsymbol{x}}c \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_T, \tag{2.18}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_T^D, \tag{2.19}$$

$$\boldsymbol{C}(c)(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \overline{\boldsymbol{\varepsilon}}(c))\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma_T^N, \tag{2.20}$$

$$c(0,\boldsymbol{x}) = c_0(\boldsymbol{x}) \quad \text{in } \Omega. \tag{2.21}$$

We remark, that in this model the unknowns are the mass concentration c and the displacement $\boldsymbol{u}$. Furthermore, we handle the chemical potential μ also as an unknown in this paper.

3. Weak formulation

In this context, we restrict our considerations as mentioned above on space dimensions $d = 2; 3$. In order to develop an existence theory for weak solutions of the generalised Cahn-Larché equation system (2.14) - (2.21) we assume further restriction on the domain:
Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with a Lipschitz boundary. The decomposition of the boundary $\partial\Omega$ into a Dirichlet part Γ_D and a Neumann part Γ_N has to satisfy the following properties:

$$\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N; \quad \Gamma_D \cap \Gamma_N = \emptyset; \quad \operatorname{meas}(\Gamma_D) > 0. \tag{3.1}$$

3.1. Function spaces.

In general, let X be a Banach space equipped with the norm $\|\cdot\|_X$, then we denote its dual space with X'. The conjugate pairing of an element $f \in X'$ with an element $v \in X$ is described by $\langle f, v\rangle$. X' is also a Banach space endowed with an operator norm

$$\|f\|_{X'} := \sup_{v\in X} \frac{|\langle f, v\rangle|}{\|v\|_X}.$$

Additionally, we use the notation $\boldsymbol{X}$ for a Banach space of vector valued functions.
Let $(v_n)_{n\in\mathbb{N}} \subset X$ be a sequence, which converges weakly to $v \in X$, in a reflexive Banach space, then we use the notation

$$v_n \rightharpoonup v.$$

Furthermore, if the sequence $(v_n)_{n\in\mathbb{N}} \subset X'$ is a weak-* convergent sequence in a non reflexive Banach space, then we use the notation

$$v_n \overset{*}{\rightharpoonup} v.$$

Throughout this paper we write $L_p(\Omega)$ with $1 \le p \le \infty$ for the usual Lebesgue spaces as defined in [13]. These spaces are Banach spaces equipped with the classical norm denoted by $\|\cdot\|_{L_p(\Omega)}$. Moreover, the space $L_2(\Omega)$ is a separable Hilbert space with the usual inner product $\langle\cdot,\cdot\rangle_{L_2(\Omega)}$.

For functions in $L_p(\Omega)$ Hölder's inequality holds as proved in [13] and especially yields a Cauchy-Schwartz inequality on the Hilbert space $L_2(\Omega)$.
Let $l \in \mathbb{N}$, the we make use of the Sobolev spaces denoted by $H^l(\Omega)$ as defined in [25]. These spaces are separable Hilbert spaces endowed with the usual norm $\|\cdot\|_{H^l(\Omega)}$ or semi-norm $|\cdot|_{H^l(\Omega)}$. Furthermore, let $s = l + \sigma$ where $l \in \mathbb{N}$, $0 < \sigma < 1$, then $H^s(\Omega)$ denotes the Sobolev-Slobodeckij spaces as defined in [25], which are also separable Hilbert spaces equipped with the usual norm $\|\cdot\|_{H^s(\Omega)}$ or semi-norm $|\cdot|_{H^s(\Omega)}$.
In order to show the existence of weak solutions of the generalised Cahn-Larché equation system, we have to take the mass conservation (2.4) into account. This condition is included into the function space used for the mass concentration c and therefore we define

$$X_m(\Omega) := \left\{ v \in H^1(\Omega) \ : \ \int_\Omega v \, d\boldsymbol{x} = m, \quad m \geq 0 \text{ const} \right\}.$$

The space $X_m(\Omega)$ is equipped with the usual Sobolev norm $\|\cdot\|_{H^1(\Omega)}$ or semi-norm $|\cdot|_{H^1(\Omega)}$. We remark, that if $m = 0$, $X_0(\Omega)$ is a linear subspace of $H^1(\Omega)$, and if $m > 0$, $X_m(\Omega)$ is an affine subspace of $H^1(\Omega)$.
As a special case of [25] theorem 7.7 with $s = 1$, we have Poincaré's inequality in $X_m(\Omega)$.

Lemma 3.1 (Poincaré's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain, then there exists a constant $c_p > 0$, such that for all $v \in X_m(\Omega)$ with $m \geq 0$ holds*

$$\|v\|^2_{H^1(\Omega)} \leq c_p(|v|^2_{H^1(\Omega)} + m^2). \tag{3.2}$$

We further need trace spaces corresponding to $H^l(\Omega)$ on a boundary part $\Gamma \subset \partial\Omega$, Γ an open subset of $\partial\Omega$. Trace spaces are defined in the sense of Sobolev-Slobodeckij spaces on compact manifolds, see [17] chapter 1.5.2. Here we need $C^{k,1}$-smoothness of $\partial\Omega$ for the definition of $H^s(\Gamma)$, where s and k are related by $k \in \mathbb{N}$, $|s| \leq k + 1$.
As a special case of [17] theorem 1.5.2.1 we have the following trace theorem with $s = 1$ and $k = 0$:

Theorem 3.2 (Trace theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with a Lipschitz boundary, $\Gamma \subset \partial\Omega$ be an open subset. Then the mapping*

$$\gamma|_\Gamma : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\Gamma),$$
$$\gamma|_\Gamma(v) = v|_\Gamma.$$

exists and is understood in the sense of traces. Furthermore, this operator is continuous and there exists a constant $c_\gamma > 0$, such that the following estimate holds

$$\|\gamma|_\Gamma(v)\|_{H^{\frac{1}{2}}(\Gamma)} = \|v\|_{H^{\frac{1}{2}}(\Gamma)} \leq c_\gamma \|v\|_{H^1(\Omega)}. \tag{3.3}$$

In order to consider the generalised Cahn-Larché equation with mixed boundary conditions, we need the trace space $H^{\frac{1}{2}}(\Gamma_D)$ to describe Dirichlet data. Additionally, we introduce on the Neumann boundary Γ_N the trace space

$$\widetilde{H}^{\frac{1}{2}}(\Gamma_N) := \left\{ v : \ v = \tilde{v}|_{\Gamma_N}, \text{ where } \tilde{v} \in H^{\frac{1}{2}}(\partial\Omega) \text{ with } \operatorname{supp}(\tilde{v}) \subset \overline{\Gamma}_N \right\},$$

which is endowed with the norm $\|v\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_N)} := \|\tilde{v}\|_{H^{\frac{1}{2}}(\partial\Omega)}$.
The Neumann boundary condition acting on the boundary part Γ_N is understood in a weak sense. Therefore we define the dual spaces of the trace spaces by

$$H^{-\frac{1}{2}}(\Gamma_N) := \left(\widetilde{H}^{\frac{1}{2}}(\Gamma_N)\right)', \qquad \widetilde{H}^{-\frac{1}{2}}(\Gamma_D) := \left(H^{\frac{1}{2}}(\Gamma_D)\right)',$$

which are equipped with the usual operator norms.
The function space used for the displacement $\boldsymbol{u}$ must take the Dirichlet boundary condition (2.19) into account. In order to consider the hard clamped part of the system we define the space

$$\boldsymbol{X}(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \ \gamma|_{\Gamma_D}(\boldsymbol{v}) = \boldsymbol{0} \right\},$$

which is endowed with the usual Sobolev norm $\|\cdot\|_{H^1(\Omega)}$ or semi-norm $|\cdot|_{H^1(\Omega)}$.

Lemma 3.3 (Korn's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain satisfying condition* (3.1)*, then there exists a constant $c_k > 0$, such that for all $\boldsymbol{v} \in \boldsymbol{X}(\Omega)$ holds*

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} \leq c_k \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, d\boldsymbol{x}. \tag{3.4}$$

Proof. See for instance [22] page 79. □

As a special case of [1] theorem 6.2 we have the following compact embedding result:

Theorem 3.4 (Rellich-Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with Lipschitz boundary, then the following compact embedding holds*

$$H^l(\Omega) \Subset L_q(\Omega); \quad q \leq \frac{d}{\frac{d}{2} - l}.$$

Finally, we introduce Lebesgue spaces with Banach space valued functions, which are necessary to consider time depending weak problems. Let X be a Banach space, then we get from [8] chapter 18

$$L_p(0,T;X) := \left\{ v : \ (0,T) \longrightarrow X : \ \text{weak measurable}, \|v\|_{L_p(0,T;X)} < \infty \right\},$$

which is in view of [8] also a Banach space equipped with the norm

$$\|v\|^p_{L_p(0,T;X)} := \int_0^T \|v\|^p_X dt, \qquad 1 \leq p < \infty,$$
$$\|v\|_{L_\infty(0,T;X)} := \operatorname*{ess\,sup}_{t \in (0,T)} \|v\|_X, \qquad p = \infty.$$

The existence proof of weak solutions of the generalised Cahn-Larché equation system needs essentially the theorem of Arselà-Ascoli, see [23].

Theorem 3.5 (Arzelà-Ascoli)**.** *Let X be a Banach space, then a set $F \subset C([0,T];X)$ is relatively compact if and only if:*

(1) $F(t) := \{f(t) \ : \ f \in F\}$ is relatively compact in X for all $0 < t < T$,
(2) F is uniformly equicontinuous: $\forall \varepsilon > 0$, $\exists \eta > 0$ such that

$$\|f(t_2) - f(t_1)\|_X \leq \varepsilon, \quad \forall f \in F, \quad \forall 0 \leq t_1 \leq t_2 \leq T \quad \textit{such that} \quad |t_2 - t_1| \leq \eta.$$

3.2. **Assumptions.** In this section we formulate all sufficient conditions in order to show the existence of weak solutions of the generalised Cahn-Larché equation system. On the one hand these assumptions are of physical nature, on the other hand they are formulated for technical reasons. We assume the following constraints:

(A-1) **Surface stress tensor**: Let $\boldsymbol{\Gamma} \in C^1(\mathbb{R}, \mathbb{R}^{d \times d}_{\text{sym}})$ be a second order tensor valued function, such that there exist constants $C_\Gamma, c_\Gamma, C_{\Gamma'}, c_{\Gamma'} > 0$ with the property, that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ and for all $c \in \mathbb{R}$ hold

$$|\boldsymbol{\Gamma}(c)\boldsymbol{a} \cdot \boldsymbol{b}| \leq C_\Gamma |\boldsymbol{a}||\boldsymbol{b}|, \qquad |\boldsymbol{\Gamma}_{,c}(c)\boldsymbol{a} \cdot \boldsymbol{b}| \leq C_{\Gamma'} |\boldsymbol{a}||\boldsymbol{b}|,$$
$$\boldsymbol{\Gamma}(c)\boldsymbol{a} \cdot \boldsymbol{a} \geq c_\Gamma |\boldsymbol{a}|^2, \qquad \boldsymbol{\Gamma}_{,c}(c)\boldsymbol{a} \cdot \boldsymbol{a} \geq c_{\Gamma'} |\boldsymbol{a}|^2.$$

(A-2) **Homogeneous free energy**: Let $\psi(c) := \psi_1(c) + \psi_2(c)$ be a function with $\psi_1, \psi_2 \in C^1(\mathbb{R}, \mathbb{R})$, whereas ψ_1 is a convex function. Additionally, we assume, that for all $c \in X_m(\Omega)$ follows $\psi(c) \in L_1(\Omega)$. Furthermore, there exists a constant $c_\psi \geq 0$, such that the homogeneous free energy is bounded below by

$$\psi(c) \geq -c_\psi.$$

Additionally, we assume, that for all $\delta > 0$ exist a constant $C_\delta > 0$ and an estimate

$$|\psi_{1,c}(c)| \leq \delta \psi_1(c) + C_\delta.$$

Moreover, there exists a constant $C_{\psi'} > 0$, such that a polynomial estimate holds

$$|\psi_{2,c}(c)| \leq C_{\psi'}(|c|^q + 1),$$

where in the case $d = 2$ we have $1 \leq q < \infty$ and in the case $d = 3$, we have $1 \leq q < 6$.

(A-3) **Mobility**: Let $\boldsymbol{M} \in C^0(\mathbb{R}, \mathbb{R}^{d\times d}_{\text{sym}})$ be a second order tensor valued function, such that there exist constants $C_M, c_M > 0$, with the property, that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ and for all $c \in \mathbb{R}$ hold

$$|\boldsymbol{M}(c)\boldsymbol{a}\cdot\boldsymbol{b}| \le C_M|\boldsymbol{a}||\boldsymbol{b}|, \qquad \boldsymbol{M}(c)\boldsymbol{a}\cdot\boldsymbol{a} \ge c_M|\boldsymbol{a}|^2.$$

This assumption corresponds to the dissipation inequality (2.13).

(A-4) **Initial condition**: The initial mass distribution has to satisfy

$$c_0 \in X_m(\Omega), \quad \text{with} \quad m = \int_\Omega c_0(\boldsymbol{x})\, d\boldsymbol{x}.$$

(A-5) **Elasticity tensor**: This function is assumed to be a fourth order tensor valued function $\boldsymbol{C}(c) := \{C_{ijkl}(c)\}^d_{i,j,k,l=1}$, such that $C_{i,j,k,l} \in C^1(\mathbb{R}, \mathbb{R})$ and the property $C_{i,j,k,l} = C_{j,i,k,l} = C_{i,j,l,k}$ must be valid. Additionally, there exist constants $C_C, c_C, C_{C'} > 0$, such that for all $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d\times d}\setminus\{\boldsymbol{0}\}$ and for all $c \in \mathbb{R}$ hold

$$|\boldsymbol{C}(c)\boldsymbol{A} : \boldsymbol{B}| \le C_C|\boldsymbol{A}||\boldsymbol{B}|, \qquad |\boldsymbol{C}_{,c}(c)\boldsymbol{A} : \boldsymbol{B}| \le C_{C'}|\boldsymbol{A}||\boldsymbol{B}|,$$
$$\boldsymbol{C}(c)\boldsymbol{A} : \boldsymbol{A} \ge c_C|\boldsymbol{A}|^2.$$

(A-6) **Eigenstrain**: Let $\overline{\boldsymbol{\varepsilon}} \in C^1(\mathbb{R}, \mathbb{R}^{d\times d}_{\text{sym}})$ be a second order valued function, such that there exist constants $C_{\overline{\boldsymbol{\varepsilon}}}, C_{\overline{\boldsymbol{\varepsilon}}'} > 0$, that for all $c \in \mathbb{R}$ hold

$$|\overline{\boldsymbol{\varepsilon}}(c)| \le C_{\overline{\boldsymbol{\varepsilon}}}(|c| + 1), \qquad |\overline{\boldsymbol{\varepsilon}}_{,c}(c)| \le C_{\overline{\boldsymbol{\varepsilon}}'}.$$

(A-7) **Mechanical loading**: The mechanical loading is understood in a distributional sense and it has to satisfy

$$\boldsymbol{g} \in L_\infty(0, T; \boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N)).$$

We remark, that in view of assumption (A-1) the field equation (2.14) and (2.15) form a fourth order parabolic equation with respect to the mass concentration c, while assumption (A-5) forms a second order elliptic equation for the displacement $\boldsymbol{u}$ by equation (2.16).

3.3. **Different weak formulations.** In this paper we distinguish between semi-weak and weak formulation of the generalised Cahn-Larché equation system (2.14) - (2.21). The semi-weak formulation considers only a weak formulation of the field equations (2.14) - (2.16) with respect to space.

Definition 3.6 (Semi-weak solution)**.** At each time $t \in (0, T)$ a triple $c(t, \cdot) \in X_m(\Omega)$ with $\dot{c}(t, \cdot) \in X_0'(\Omega)$, $\mu(t, \cdot) \in H^1(\Omega)$ and $\boldsymbol{u}(t, \cdot) \in \boldsymbol{X}(\Omega)$ is called semi-weak solution of the field equations (2.14) - (2.16), if it holds

$$\int_\Omega \dot{c}\zeta\, d\boldsymbol{x} + \int_\Omega \boldsymbol{M}(c)\nabla_{\boldsymbol{x}}\mu\cdot\nabla_{\boldsymbol{x}}\zeta\, d\boldsymbol{x} = 0 \qquad \forall\zeta \in X_0(\Omega), \tag{3.5}$$

$$\int_\Omega \mu\xi\, d\boldsymbol{x} = \int_\Omega \boldsymbol{\Gamma}(c)\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}\xi + \frac{1}{2}\boldsymbol{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}c\xi + \psi_{,c}(c)\xi + W_{,c}(c, \boldsymbol{\varepsilon}(\boldsymbol{u}))\xi\, d\boldsymbol{x} \qquad \forall\xi \in H^1(\Omega)\cap L_\infty(\Omega), \tag{3.6}$$

$$\int_\Omega W_{,\boldsymbol{\varepsilon}}(c, \boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x} = \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{\eta}\, d\boldsymbol{a}_{\boldsymbol{x}} \qquad \forall\boldsymbol{\eta} \in \boldsymbol{X}(\Omega), \tag{3.7}$$

$$c(0, \boldsymbol{x}) = c_0(\boldsymbol{x}) \qquad \text{almost everywhere.}$$

Formally, we calculate the semi-weak formulation by multiplying the field equations (2.14) - (2.16) with the corresponding test function, integrating over the domain Ω, using Gauß's theorem and taking the corresponding boundary conditions (2.17) - (2.20) into account.

In this formulation the time derivative of the mass concentration $\dot{c}$ is understood in a distributional sense, where we have to use the space $X_0'(\Omega)$ in order to guarantee mass conservation. We further remark, that all integrals in definition 3.6 exist, due to assumption (A-1) - (A-3) and (A-5) - (A-7), and the fact, that we use almost everywhere bounded test functions for the chemical potential.

For technical reasons, we consider the weak formulation of the quasi steady momentum balance (3.7) with an arbitrary, but fixed mass concentration $c \in X_m(\Omega)$.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain satisfying condition* (3.1), *let the assumption (A-5) - (A-7) be satisfied and let $c \in X_m(\Omega)$ arbitrary, but fixed. Then for the semi-weak momentum balance* (3.7) *the following estimates hold*

$$\text{(1)} \quad \left| \int_\Omega \boldsymbol{C}(c)\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x} \right| \le C_C \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{\eta}\|_{\boldsymbol{H}^1(\Omega)} \qquad \forall \boldsymbol{u}, \boldsymbol{\eta} \in \boldsymbol{X}(\Omega),$$

$$\text{(2)} \quad \int_\Omega \boldsymbol{C}(c)\boldsymbol{\varepsilon}(\boldsymbol{\eta}) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x} \ge \frac{c_C}{c_k} \|\boldsymbol{\eta}\|^2_{\boldsymbol{H}^1(\Omega)} \qquad \forall \boldsymbol{\eta} \in \boldsymbol{X}(\Omega),$$

$$\text{(3)} \quad \left| \int_\Omega \boldsymbol{C}(c)\overline{\boldsymbol{\varepsilon}}(c) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{\eta}\, d\boldsymbol{a_x} \right|$$
$$\le C \left(\|c\|_{L_2(\Omega)} + \|\boldsymbol{g}\|_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))} + \operatorname{meas}^{\frac{1}{2}}(\Omega) \right) \|\boldsymbol{\eta}\|_{H^1(\Omega)} \qquad \forall \boldsymbol{\eta} \in \boldsymbol{X}(\Omega),$$

with a constant $C = C(C_C, C_{\overline{\boldsymbol{\varepsilon}}}, c_\gamma)$.

Proof. (1) follows directly from assumption (A-5) and from Hölder's inequality. *(2)* is a direct consequence from assumption (A-5) and from Korn's inequality (3.4). *(3)* is calculated by using assumptions (A-5) - (A-7), the trace theorem with estimate (3.3) and Hölder's inequality. We get

$$\left| \int_\Omega \boldsymbol{C}(c)\overline{\boldsymbol{\varepsilon}}(c) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{\eta}\, d\boldsymbol{a_x} \right|$$
$$\le C_C C_{\overline{\boldsymbol{\varepsilon}}} \int_\Omega (|c|+1)|\boldsymbol{\varepsilon}(\boldsymbol{\eta})|\, d\boldsymbol{x} + c_\gamma \|\boldsymbol{g}\|_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))} \|\boldsymbol{\eta}\|_{\boldsymbol{H}^1(\Omega)}$$
$$\le C \left(\|c\|_{L_2(\Omega)} + \|\boldsymbol{g}\|_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))} + \operatorname{meas}^{\frac{1}{2}}(\Omega) \right) \|\boldsymbol{\eta}\|_{\boldsymbol{H}^1(\Omega)},$$

with a constant $C = C(C_C, C_{\overline{\boldsymbol{\varepsilon}}}, c_\gamma)$. □

For an arbitrary, but fixed mass concentration $c \in X_m(\Omega)$, there exists due to lemma 3.7 and Lax-Milgram's theorem, see [25] theorem 17.9, a unique displacement $\boldsymbol{u} \in \boldsymbol{X}(\Omega)$ satisfying the semi-weak formulation of the quasi steady equilibrium (3.7). This displacement $\boldsymbol{u}$ satisfies by Lax-Milgram's theorem the following a-priori estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C \left(\|c\|_{L_2(\Omega)} + \|\boldsymbol{g}\|_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))} + \operatorname{meas}^{\frac{1}{2}}(\Omega) \right), \tag{3.8}$$

with a constant $C = C(C_C, C_{\overline{\boldsymbol{\varepsilon}}}, c_C, c_k, c_\gamma)$. As a special case of this result, there exists a unique initial displacement $\boldsymbol{u}_0 \in \boldsymbol{X}(\Omega)$ corresponding to the initial mass concentration $c_0 \in X_m(\Omega)$.
Finally, we give the complete weak formulation of the field equations (2.14) - (2.16), which is a weak formulation with respect to time and space.

Definition 3.8 (Weak solution)**.** A triple $c \in L_2(0,T;X_m(\Omega))$ with $\dot{c} \in L_2(0,T;X_0'(\Omega))$, $\mu \in L_2(0,T;H^1(\Omega))$ and $\boldsymbol{u} \in L_2(0,T;\boldsymbol{X}(\Omega))$ is called weak solution of the field equations (2.14) - (2.16), if it holds

$$-\int_{\Omega_T} \dot{\zeta}(c - c_0)\, d\boldsymbol{x}dt + \int_{\Omega_T} \boldsymbol{M}(c)\nabla_{\boldsymbol{x}}\mu \cdot \nabla_{\boldsymbol{x}}\zeta\, d\boldsymbol{x}dt = 0$$
$$\forall \zeta \in L_2(0,T;X_0(\Omega)) \text{ with } \dot{\zeta} \in L_2(\Omega_T) \text{ and } \zeta(T,\boldsymbol{x}) = 0 \text{ a.e.}, \tag{3.9}$$

$$\int_{\Omega_T} \mu\xi\, d\boldsymbol{x}dt = \int_{\Omega_T} \boldsymbol{\Gamma}(c)\nabla_{\boldsymbol{x}}c \cdot \nabla_{\boldsymbol{x}}\xi + \frac{1}{2}\boldsymbol{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c \cdot \nabla_{\boldsymbol{x}}c\xi + \psi_{,c}(c)\xi$$
$$+ W_{,c}(c,\boldsymbol{\varepsilon}(\boldsymbol{u}))\xi\, d\boldsymbol{x}dt \qquad \forall \xi \in L_2(0,T;H^1(\Omega)) \cap L_\infty(\Omega_T), \tag{3.10}$$

$$\int_{\Omega_T} W_{,\boldsymbol{\varepsilon}}(c,\boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x}dt = \int_{\Gamma_T^N} \boldsymbol{g} \cdot \boldsymbol{\eta}\, d\boldsymbol{a_x}dt \qquad \forall \boldsymbol{\eta} \in L_2(0,T;\boldsymbol{X}(\Omega)). \tag{3.11}$$

Formally, we get this weak formulation by multiplying the field equations (2.14) - (2.16) with the corresponding test functions, integrating over the domain Ω and time $(0,T)$, using Gauß's theorem and taking the boundary conditions (2.17) - (2.20) as well as the initial condition (2.21) into account. Similar as above, all integrals in definition 3.8 exist due to assumption (A-1) - (A-7) and the fact, that we use bounded test functions for the chemical potential.

4. Existence of weak solutions

In this section we prove the existence of weak solutions of the strong nonlinear Cahn-Larché equation system in the sense of definition 3.8.

Theorem 4.1 (Existence of weak solutions)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain satisfying condition* (3.1) *and let the assumption (A-1) - (A-7) be satisfied. Then there exists at least one triple $c \in L_\infty(0,T;X_m(\Omega))$ with $\dot{c} \in L_2(0,T;X_0'(\Omega))$, $\mu \in L_2(0,T;H^1(\Omega))$ and $\boldsymbol{u} \in L_\infty(0,T;\boldsymbol{X}(\Omega))$ satisfying the field equations* (2.14) - (2.16) *in the sense of definition 3.8.*

At this point we give a small overview about the following existence proof.
In subsection 4.1 we derive a time discretisation of the semi-weak Cahn-Larché equation system (3.5) - (3.7) similar to [16]. Moreover, we use the time discretisation in order to linearise the mobility tensor. Finally, we are able to eliminate the diffusion equation, such that we end up with a time discrete equation system with two equations for the discrete mass concentration c^n and the discrete displacement $\boldsymbol{u}^n$ at each discrete time. A corresponding discrete energy functional is also formulated.
In subsection 4.2 we proof the existence of a time discrete solution c^n, μ^n and $\boldsymbol{u}^n$ of the Cahn-Larché equation system. This is done by minimising the discrete energy. We show for the minimiser, that the weak Euler-Lagrange differential equations corresponds to the discrete semi-weak formulation of the Cahn-Larché equation system. This is possible, because the time discrete semi-weak formulation of the Cahn-Larché equation system and the discrete energy possess a flow gradient structure.
In subsection 4.3 we derive uniform a-priori estimates for the discrete solutions of the Cahn-Larché equation system. Here we use the Lyapunov property of the energy functional essentially.
In subsection 4.4 we show, that the piecewise constant expansion c_N, μ_N and $\boldsymbol{u}_N$ of the time discrete solutions of the Cahn-Larché equation system converge to a weak solution in the sense of definition 3.8.

4.1. Time discretisation.

In the first step of this existence proof, we observe a uniform decomposition of the time interval $(0,T)$ into N parts and define

$$\Delta t := \frac{T}{N}.$$

We introduce the time discrete solution of the generalised Cahn-Larché equation system at time $t = n\Delta t$ for all $1 \le n \le N$ by

$$c^n(\boldsymbol{x}) := c(n\Delta t, \boldsymbol{x}); \quad \mu^n(\boldsymbol{x}) := \mu(n\Delta t, \boldsymbol{x}); \quad \boldsymbol{u}^n(\boldsymbol{x}) := \boldsymbol{u}(n\Delta t, \boldsymbol{x}). \tag{4.1}$$

In the following, the functions c^n, μ^n and $\boldsymbol{u}^n$ are considered as unknowns. Moreover, we formulate a piecewise constant expansion of these functions with respect to time and define for all $1 \le n \le N$ at time $t \in ((n-1)\Delta t, n\Delta t]$

$$c_N(t,\boldsymbol{x}) := c^n(\boldsymbol{x}); \quad \mu_N(t,\boldsymbol{x}) := \mu^n(\boldsymbol{x}); \quad \boldsymbol{u}_N(t,\boldsymbol{x}) := \boldsymbol{u}^n(\boldsymbol{x}). \tag{4.2}$$

The time derivative of the mass concentration in the diffusion equation (2.14) will be approximated by an implicit Eulerian scheme. We get for all $1 \le n \le N$

$$\dot{c}_N(n\Delta t, \boldsymbol{x}) \approx \frac{c^n(\boldsymbol{x}) - c^{n-1}(\boldsymbol{x})}{\Delta t}. \tag{4.3}$$

Finally, we apply the time discretisation (4.2) to linearise the mobility tensor in order to get a linear diffusion equation. Therefore we define the linearised mobility tensor

$$\boldsymbol{M}_{N-1}(t,\boldsymbol{x}) := \boldsymbol{M}(c_{N-1}(t,\boldsymbol{x})). \tag{4.4}$$

Inserting the time discretisation (4.1) and (4.3) as well as the linearised mobility tensor (4.4) into the semi-weak problem given by definition 3.6, we end up with the definition of a discrete semi-weak solution. The time t is observed as a parameter in the definition of a discrete semi-weak solution.

Definition 4.2. At each discrete time $t = n\Delta t$, $1 \leq n \leq N$, a triple $c^n \in X_m(\Omega)$, $\mu^n \in H^1(\Omega)$ and $\boldsymbol{u}^n \in \boldsymbol{X}(\Omega)$ is called discrete semi-weak solution of the field equations (2.14) - (2.16), if it holds

$$\int_\Omega \left(\frac{c^n - c^{n-1}}{\Delta t}\right) \zeta \, d\boldsymbol{x} + \int_\Omega \boldsymbol{M}_{N-1} \nabla_{\boldsymbol{x}} \mu^n \cdot \nabla_{\boldsymbol{x}} \zeta \, d\boldsymbol{x} = 0 \qquad \forall \zeta \in X_0(\Omega), \quad (4.5)$$

$$\int_\Omega \mu^n \xi \, d\boldsymbol{x} = \int_\Omega \boldsymbol{\Gamma}(c^n) \nabla_{\boldsymbol{x}} c^n \cdot \nabla_{\boldsymbol{x}} \xi + \frac{1}{2} \boldsymbol{\Gamma}_{,c}(c^n) \nabla_{\boldsymbol{x}} c^n \cdot \nabla_{\boldsymbol{x}} c^n \xi + \psi_{,c}(c^n) \xi + W_{,c}(c^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n)) \xi \, d\boldsymbol{x} \qquad \forall \xi \in H^1(\Omega) \cap L_\infty(\Omega), \quad (4.6)$$

$$\int_\Omega W_{,\boldsymbol{\varepsilon}}(c^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n)) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) \, d\boldsymbol{x} = \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{\eta} \, d\boldsymbol{a}_{\boldsymbol{x}} \qquad \forall \boldsymbol{\eta} \in \boldsymbol{X}(\Omega). \quad (4.7)$$

In order to handle the strong nonlinearities of this type of Cahn-Larché equation system, we use the gradient flow structure of the problem given by (2.10) and (2.11). We calculated a discrete semi-weak solution by an energy minimisation method.
In view of this idea the discrete semi-weak diffusion equation (4.5) does not correspond to the gradient flow structure. We eliminate equation (4.5) by using a Riesz operator corresponding to the linearised mobility tensor (4.4), where the time t is also considered as a parameter.

Definition 4.3. Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with a Lipschitz boundary, then the operator $\mathcal{M}_N$ is defined by

$$\mathcal{M}_N : \; X_0(\Omega) \longrightarrow X_0'(\Omega),$$

$$\langle \mathcal{M}_N \mu, \zeta \rangle := \int_\Omega \boldsymbol{M}_{N-1} \nabla_{\boldsymbol{x}} \mu \cdot \nabla_{\boldsymbol{x}} \zeta \, d\boldsymbol{x} \qquad \forall \zeta \in X_0(\Omega).$$

The operator $\mathcal{M}_N$ corresponds to a Laplace operator with a material tensor $\boldsymbol{M}_{N-1}$ and homogeneous Neumann boundary conditions.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let assumption (A-3) be satisfied, then for the operator $\mathcal{M}_N$ the following estimates hold*

(1) $|\langle \mathcal{M}_N \mu, \zeta \rangle| \leq C_M \|\mu\|_{H^1(\Omega)} \|\zeta\|_{H^1(\Omega)} \qquad \forall \mu, \zeta \in X_0(\Omega),$

(2) $\langle \mathcal{M}_N \zeta, \zeta \rangle \geq \dfrac{c_M}{c_p} \|\zeta\|^2_{H^1(\Omega)} \qquad \forall \zeta \in X_0(\Omega),$

(3) $\mathcal{M}_N$ is a self-adjoint operator.

Proof. *(1)* follows directly from (A-3) and Hölder's inequality. *(2)* is a consequence of (A-3) and Poincaré's inequality (3.2). *(3)* holds due to the symmetry of $\boldsymbol{M}$ assumed in (A-3). □

In view of Lax-Milgram's theorem the inverse operator of $\mathcal{M}_N$ exists and it is denoted by $\mathcal{M}_N^{-1}$. By using the operator $\mathcal{M}_N^{-1}$ we introduce an inner product as well as a norm on the space $X_0'(\Omega)$.

Definition 4.5. Let $f, g \in X_0'(\Omega)$, then the inner product with respect to $\mathcal{M}_N^{-1}$ is defined by

$$\langle f, g \rangle_{\mathcal{M}_N} := \int_\Omega \boldsymbol{M}_{N-1} \nabla_{\boldsymbol{x}} \mathcal{M}_N^{-1} f \cdot \nabla_{\boldsymbol{x}} \mathcal{M}_N^{-1} g \, d\boldsymbol{x},$$

the corresponding norm is given by

$$\|f\|^2_{\mathcal{M}_N} := \langle f, f \rangle_{\mathcal{M}_N}.$$

Applying Riesz's representation theorem, see [2] theorem 4.1, we identify a function $v \in X_0(\Omega)$ with a functional in $X_0'(\Omega)$ via,

$$\mathcal{R} : \; X_0(\Omega) \longrightarrow X_0'(\Omega),$$

$$\langle \mathcal{R} v, \zeta \rangle = \int_\Omega v \zeta \, d\boldsymbol{x}, \qquad \forall \zeta \in X_0(\Omega).$$

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain and let assumption (A-3) be satisfied. Then for all $v \in X_0(\Omega)$ and for all $\delta > 0$ holds*

$$\|v\|^2_{L_2(\Omega)} \le \frac{C_M}{\delta}\|v\|^2_{\mathcal{M}_N} + \delta|v|^2_{H^1(\Omega)}.$$

Proof. We consider the following weak problem

$$\int_\Omega \boldsymbol{M}_{N-1}\nabla_{\boldsymbol{x}} w \cdot \nabla_{\boldsymbol{x}}\zeta \, d\boldsymbol{x} = \int_\Omega v\zeta \, d\boldsymbol{x} \qquad \forall \zeta \in X_0(\Omega). \tag{4.8}$$

Due to Riesz's representation theorem and the existence of the inverse operator $\mathcal{M}_N^{-1}$, the solution of the weak problem (4.8) is given by

$$w = \mathcal{M}_N^{-1} v.$$

Using weak problem (4.8) in combination with the symmetry and the positive definiteness of $\boldsymbol{M}_{N-1}$, Hölder's inequality, definition 4.5 and Young's inequality, we get

$$\begin{aligned}
\|v\|^2_{L_2(\Omega)} &= \int_\Omega \boldsymbol{M}_{N-1}\nabla_{\boldsymbol{x}} w \cdot \nabla_{\boldsymbol{x}} v \, d\boldsymbol{x} = \int_\Omega \boldsymbol{M}_{N-1}\nabla_{\boldsymbol{x}}\mathcal{M}_N^{-1}v \cdot \nabla_{\boldsymbol{x}} v \, d\boldsymbol{x} \\
&= \int_\Omega \boldsymbol{M}^{\frac{1}{2}}_{N-1}\nabla_{\boldsymbol{x}}\mathcal{M}_N^{-1}v \cdot \boldsymbol{M}^{\frac{1}{2}}_{N-1}\nabla_{\boldsymbol{x}} v \, d\boldsymbol{x} \\
&\le \left(\int_\Omega \left(\boldsymbol{M}^{\frac{1}{2}}_{N-1}\nabla_{\boldsymbol{x}}\mathcal{M}_N^{-1}v\right)^2 d\boldsymbol{x}\right)^{\frac{1}{2}} \left(\int_\Omega \left(\boldsymbol{M}^{\frac{1}{2}}_{N-1}\nabla_{\boldsymbol{x}} v\right)^2 d\boldsymbol{x}\right)^{\frac{1}{2}} \\
&\le C_M^{\frac{1}{2}}\|v\|_{\mathcal{M}_N}|v|_{H^1(\Omega)} \le \frac{C_M}{\delta}\|v\|^2_{\mathcal{M}_N} + \delta|v|^2_{H^1(\Omega)}.
\end{aligned}$$

□

Finally, we are able to eliminate the discrete semi-weak diffusion equation (4.5), in order to reduce the equation system (4.5) - (4.7) onto a system with only two equations for the unknowns c^n and $\boldsymbol{u}^n$. Therefore, we apply the operator $\mathcal{M}_N^{-1}$ and Riesz representation theorem to the discrete semi-weak diffusion equation (4.5) and deduce

$$\mu^n = -\mathcal{M}_N^{-1}\left(\frac{c^n - c^{n-1}}{\Delta t}\right) \in X_0(\Omega). \tag{4.9}$$

We remark, that for any Lagrangian multiplier $\lambda^n \in \ker(\mathcal{M}_N) = \mathbb{R}$ holds

$$\mu^n = -\mathcal{M}_N^{-1}\left(\frac{c^n - c^{n-1}}{\Delta t}\right) + \lambda^n \in H^1(\Omega) \tag{4.10}$$

is also a solution of the discrete semi-weak diffusion equation (4.5).
By this technique, the discrete semi-weak diffusion equation (4.5) is eliminated and we insert (4.9) into the discrete semi-weak equation of the chemical potential (4.6). Let $\xi \in X_0(\Omega)$ be a test function, then we use the fact, that $\mathcal{M}_N^{-1}$ is a self-adjoint operator, which follows from lemma 4.4 and from [2] theorem 10.5. We take definition 4.5 into account and calculate

$$\begin{aligned}
\int_\Omega \mu^n\xi \, d\boldsymbol{x} &= -\int_\Omega \mathcal{M}_N^{-1}\left(\frac{c^n - c^{n-1}}{\Delta t}\right)\xi \, d\boldsymbol{x} = -\int_\Omega \mathcal{M}_N^{-1}\mathcal{M}_N\mathcal{M}_N^{-1}\left(\frac{c^n - c^{n-1}}{\Delta t}\right)\xi \, d\boldsymbol{x} \\
&= -\int_\Omega \mathcal{M}_N\mathcal{M}_N^{-1}\left(\frac{c^n - c^{n-1}}{\Delta t}\right)\mathcal{M}_N^{-1}\xi \, d\boldsymbol{x} = -\left\langle \frac{c^n - c^{n-1}}{\Delta t}, \xi\right\rangle_{\mathcal{M}_N}.
\end{aligned} \tag{4.11}$$

In view of calculation (4.11) we restrict the discrete semi-weak problem given by definition 4.2 to a weak problem with only two equations for the mass concentration c^n and the displacement $\boldsymbol{u}^n$.

Problem 4.7. At each discrete time $t = n\Delta t$, $1 \le n \le N$, we have to find $c^n \in X_m(\Omega)$ and $\boldsymbol{u}^n \in \boldsymbol{X}(\Omega)$, such that holds

$$\int_\Omega \boldsymbol{\Gamma}(c^n)\nabla_{\boldsymbol{x}} c^n \cdot \nabla_{\boldsymbol{x}}\xi + \frac{1}{2}\boldsymbol{\Gamma}_{,c}(c^n)\nabla_{\boldsymbol{x}} c^n \cdot \nabla_{\boldsymbol{x}} c^n\xi + \psi_{,c}(c^n)\xi + W_{,c}(c^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n))\xi \, d\boldsymbol{x}$$

$$= -\left\langle \frac{c^n - c^{n-1}}{\Delta t}, \xi \right\rangle_{\mathcal{M}_N} \qquad \forall \xi \in X_0(\Omega) \cap L_\infty(\Omega),$$

$$\int_\Omega W_{,\boldsymbol{\varepsilon}}(c^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n)) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) \, d\boldsymbol{x} = \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{\eta} \, d\boldsymbol{a_x} \qquad \boldsymbol{\eta} \in \boldsymbol{X}(\Omega).$$

The corresponding energy to problem 4.7 is called discrete energy and is given by

$$E_N^n(d, \boldsymbol{v}) := E(d, \boldsymbol{v}) + \frac{1}{2\Delta t}\|d - c^{n-1}\|^2_{\mathcal{M}_N} \qquad \forall d \in X_m(\Omega), \quad \forall \boldsymbol{v} \in \boldsymbol{X}(\Omega). \tag{4.12}$$

We directly see, that problem 4.7 and the discrete energy (4.12) satisfy a gradient flow structure with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}_N}$.

4.2. **Energy minimisation.** In the second step of this existence proof we calculate a solution of problem 4.7 by minimising the discrete energy (4.12).

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain satisfying condition* (3.1)*, let assumption (A-1) - (A-7) be satisfied. Furthermore, we assume for the time discretisation a stability condition*

$$\Delta t \leq \frac{c_\Gamma}{8 c_C^2 C_{\overline{\boldsymbol{\varepsilon}}}^2 C_M}.$$

Then there exists at least one tuple $c^n \in X_m(\Omega)$ and $\boldsymbol{u}^n \in \boldsymbol{X}(\Omega)$ such that

$$E_N^n(c^n, \boldsymbol{u}^n) = \min_{\substack{d \in X_m(\Omega), \\ \boldsymbol{v} \in \boldsymbol{X}(\Omega)}} E_N^n(d, \boldsymbol{v}).$$

We remark, that the tuple consisting of the mass concentration c^n and the displacement $\boldsymbol{u}^n$ is called the minimiser of the discrete energy E_N^n.

Proof. **Step 1**: In order to prove lemma 4.8 we first show, that E_N^n is a coercive functional on the space $X_m(\Omega) \times \boldsymbol{X}(\Omega)$, see [7] chapter 3.0.
We consider the surface energy and take (A-1) into account in order to calculate

$$\int_\Omega \frac{1}{2} \boldsymbol{\Gamma}(d) \nabla_{\boldsymbol{x}} d \cdot \nabla_{\boldsymbol{x}} d \, d\boldsymbol{x} \geq \frac{c_\Gamma}{2} |d|^2_{H^1(\Omega)}.$$

The homogeneous free energy is bounded below as outlined in assumption (A-2) and therefore we get

$$\int_\Omega \psi(d) \, d\boldsymbol{x} \geq -c_\psi \operatorname{meas}(\Omega).$$

In order to consider the strain energy, we have to observe a technical estimate. For all $d \in X_m(\Omega)$ holds $d - c^{n-1} \in X_0(\Omega)$, therefore we can apply lemma 4.6 and yield

$$\begin{aligned}
\|d\|^2_{L_2(\Omega)} &= \|d - c^{n-1} + c^{n-1}\|^2_{L_2(\Omega)} \leq 2\|d - c^{n-1}\|^2_{L_2(\Omega)} + 2\|c^{n-1}\|^2_{L_2(\Omega)} \\
&\leq \frac{4C_M}{\delta_1}\|d - c^{n-1}\|^2_{\mathcal{M}_N} + \delta_1 |d - c^{n-1}|^2_{H^1(\Omega)} + 2\|c^{n-1}\|^2_{L_2(\Omega)} \\
&\leq \frac{4C_M}{\delta_1}\|d - c^{n-1}\|^2_{\mathcal{M}_N} + 2\delta_1 |d|^2_{H^1(\Omega)} + C(\delta_1, c^{n-1}).
\end{aligned} \tag{4.13}$$

The strain energy is estimated below by using assumption (A-5) and (A-6), Young's inequality, Hölder's inequality, Korn's inequality (3.4) and estimate (4.13). We therefore get

$$\begin{aligned}
\int_\Omega W(d, \boldsymbol{\varepsilon}(\boldsymbol{v})) \, d\boldsymbol{x} &\geq \frac{c_C}{2} \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - 2\overline{\boldsymbol{\varepsilon}}(d) : \boldsymbol{\varepsilon}(\boldsymbol{v}) + \overline{\boldsymbol{\varepsilon}}(d) : \overline{\boldsymbol{\varepsilon}}(d) \, d\boldsymbol{x} \\
&\geq \frac{c_C}{2} \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - \overline{\boldsymbol{\varepsilon}}(d) : \overline{\boldsymbol{\varepsilon}}(d) \, d\boldsymbol{x} \\
&\geq \frac{c_C}{4c_k}\|\boldsymbol{v}\|^2_{H^1(\Omega)} - \frac{c_C C_{\overline{\boldsymbol{\varepsilon}}}}{2}\|d\|^2_{L_2(\Omega)} - \frac{c_C C_{\overline{\boldsymbol{\varepsilon}}}}{2}\operatorname{meas}(\Omega) \\
&\geq \frac{c_C}{4c_k}\|\boldsymbol{v}\|^2_{H^1(\Omega)} - \frac{2c_C C_{\overline{\boldsymbol{\varepsilon}}} C_M}{\delta_1}\|d - c^{n-1}\|^2_{\mathcal{M}_N} - \delta_1 c_C C_{\overline{\boldsymbol{\varepsilon}}} |d|^2_{H^1(\Omega)} - C(\delta_1, c^{n-1}).
\end{aligned}$$

The energy of the boundary loading is estimated by using assumption (A-7), the trace theorem with estimate (3.3) and Young's inequality. We estimate

$$\begin{aligned}\int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v}\ d\boldsymbol{a_x} &\le c_\gamma \|\boldsymbol{g}\|_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\\ &\le \frac{c_\gamma^2}{\delta_2}\|\boldsymbol{g}\|^2_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))} + \delta_2\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)}\end{aligned}$$

Finally, we choose $\delta_1 = \frac{c_\Gamma}{4c_C C_{\overline{\varepsilon}}}$ and $\delta_2 = \frac{c_C}{8c_k}$, summarise the above estimates and take the stability condition as well as Poincaré's inequality (3.2) into account. Then we get

$$\begin{aligned}E_N^n(d,\boldsymbol{v}) &\ge \frac{c_\Gamma}{4}|d|^2_{H^1(\Omega)} + \frac{c_C}{8c_k}\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} + \left(\frac{1}{2\Delta t} - \frac{4c_C^2C_{\overline{\varepsilon}}^2C_M}{c_\Gamma}\right)\|d-c^{n-1}\|^2_{\mathcal{M}_N} - C\\ &\ge \frac{c_\Gamma}{4c_p}\|d\|^2_{H^1(\Omega)} + \frac{c_C}{8c_k}\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} - C,\end{aligned}$$

with a constant $C = C(C_{\overline{\varepsilon}}, c_\gamma, c_c, c_\psi, m, c_k, c^{n-1}, \boldsymbol{g}, \mathrm{meas}(\Omega))$.

Step 2: In the next step we show, that the discrete energy is sequentially weakly lower semi-continuous, see [7] chapter 3.1.1. Let $(d_k)_{k\in\mathbb{N}} \subset X_m(\Omega)$ and $(\boldsymbol{v}_k)_{k\in\mathbb{N}} \subset \boldsymbol{X}(\Omega)$ be weak-convergent sequences, this reads

$$d_k \rightharpoonup d \quad \text{in } X_m(\Omega), \qquad \boldsymbol{v}_k \rightharpoonup \boldsymbol{v} \quad \text{in } \boldsymbol{X}(\Omega), \qquad k\to\infty.$$

We remark, that weak-convergent sequences are always bounded, see [2] remark 6.3.4, which is an often used property. Due to Rellich-Kondrachov's embedding theorem, we know, that $X_m(\Omega) \Subset L_2(\Omega)$ and therefore $(d_k)_{k\in\mathbb{N}}$ converges strongly in $L_2(\Omega)$. Furthermore, we can extract by Weyl's corollary, see [13] chapter 6 corollary 4.8, an almost everywhere convergent subsequence, denoted without loss of generality also by $(d_k)_{k\in\mathbb{N}}$. This reads

$$d_k \to d \quad \text{in } L_2(\Omega), \qquad d_k(\boldsymbol{x}) \to d(\boldsymbol{x}) \quad \text{almost everywhere}, \qquad k\to\infty.$$

We first consider the surface energy and take the convexity in $\nabla_{\boldsymbol{x}} d$ into account. Moreover, we use the boundedness of $(d_k)_{k\in\mathbb{N}}$, the almost everywhere convergence of $(d_k)_{k\in\mathbb{N}}$, assumption (A-1) with Lebesgue's convergence theorem, [13] chapter 4 theorem 5.2, and the weak convergence of $(d_k)_{k\in\mathbb{N}}$ in order to calculate

$$\begin{aligned}&\lim_{k\to\infty}\int_\Omega \frac{1}{2}\boldsymbol{\Gamma}(d_k)\nabla_{\boldsymbol{x}} d_k\cdot\nabla_{\boldsymbol{x}} d_k - \frac{1}{2}\boldsymbol{\Gamma}(d)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d\ d\boldsymbol{x}\\ &\quad= \lim_{k\to\infty}\int_\Omega \frac{1}{2}(\boldsymbol{\Gamma}(d_k)\nabla_{\boldsymbol{x}} d_k\cdot\nabla_{\boldsymbol{x}} d_k - \boldsymbol{\Gamma}(d_k)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d) + \frac{1}{2}(\boldsymbol{\Gamma}(d_k)-\boldsymbol{\Gamma}(d))\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d\ d\boldsymbol{x}\\ &\quad\ge \lim_{k\to\infty}\int_\Omega \boldsymbol{\Gamma}(d_k)\nabla_{\boldsymbol{x}} d\cdot(\nabla_{\boldsymbol{x}} d_k - \nabla_{\boldsymbol{x}} d) + \frac{1}{2}(\boldsymbol{\Gamma}(d_k)-\boldsymbol{\Gamma}(d))\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d\ d\boldsymbol{x}\\ &\quad= \lim_{k\to\infty}\int_\Omega (\boldsymbol{\Gamma}(d_k)-\boldsymbol{\Gamma}(d))\nabla_{\boldsymbol{x}} d\cdot(\nabla_{\boldsymbol{x}} d_k - \nabla_{\boldsymbol{x}} d) + \boldsymbol{\Gamma}(d)\nabla_{\boldsymbol{x}} d\cdot(\nabla_{\boldsymbol{x}} d_k - \nabla_{\boldsymbol{x}} d)\\ &\qquad\qquad + \frac{1}{2}(\boldsymbol{\Gamma}(d_k)-\boldsymbol{\Gamma}(d))\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d\ d\boldsymbol{x} = 0.\end{aligned}$$

The first term of the homogeneous free energy ψ_1 is a continuous and convex function with respect to d, see assumption (A-2), and therefore the convergence follows from [7] chapter 2. The second part ψ_2 is a continuous function with respect to d and polynomial bounded by assumption (A-2), hence the convergence follows from the generalised Lebesgue convergence theorem, see [2] theorem 1.21. Therefore we get

$$\lim_{k\to\infty}\int_\Omega \psi(d_k) - \psi(d)\ d\boldsymbol{x} \ge 0.$$

The strain energy is observed by using the convexity of W with respect to the strain $\boldsymbol{\varepsilon}(\boldsymbol{v})$. Furthermore, we take the almost everywhere convergence of $(d_k)_{k\in\mathbb{N}}$ assumption (A-5) and (A-6) with Lebesgue's convergence theorem into account and additionally we use the weak convergence of $(\boldsymbol{v}_k)_{k\in\mathbb{N}}$ to calculate

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega W(d_k,\boldsymbol{\varepsilon}(\boldsymbol{v}_k)) - W(d,\boldsymbol{\varepsilon}(\boldsymbol{v}))\,d\boldsymbol{x}\\
&\quad= \lim_{k\to\infty}\int_\Omega W(d_k,\boldsymbol{\varepsilon}(\boldsymbol{v}_k)) - W(d_k,\boldsymbol{\varepsilon}(\boldsymbol{v})) + W(d_k,\boldsymbol{\varepsilon}(\boldsymbol{v})) - W(d,\boldsymbol{\varepsilon}(\boldsymbol{v}))\,d\boldsymbol{x}\\
&\quad\geq \lim_{k\to\infty}\int_\Omega W_{,\boldsymbol{\varepsilon}}(d_k,\boldsymbol{\varepsilon}(\boldsymbol{v})) : (\boldsymbol{\varepsilon}(\boldsymbol{v}_k)-\boldsymbol{\varepsilon}(\boldsymbol{v})) + W(d_k,\boldsymbol{\varepsilon}(\boldsymbol{v})) - W(d,\boldsymbol{\varepsilon}(\boldsymbol{v}))\,d\boldsymbol{x}\\
&\quad= \lim_{k\to\infty}\int_\Omega (W_{,\boldsymbol{\varepsilon}}(d_k,\boldsymbol{\varepsilon}(\boldsymbol{v})) - W_{,\boldsymbol{\varepsilon}}(d,\boldsymbol{\varepsilon}(\boldsymbol{v}))) : (\boldsymbol{\varepsilon}(\boldsymbol{v}_k)-\boldsymbol{\varepsilon}(\boldsymbol{v}))\\
&\qquad\qquad + W_{,\boldsymbol{\varepsilon}}(d,\boldsymbol{\varepsilon}(\boldsymbol{v})) : (\boldsymbol{\varepsilon}(\boldsymbol{v}_k)-\boldsymbol{\varepsilon}(\boldsymbol{v})) + W(d_k,\boldsymbol{\varepsilon}(\boldsymbol{v})) - W(d,\boldsymbol{\varepsilon}(\boldsymbol{v}))\,d\boldsymbol{x} = 0.
\end{aligned}$$

The convergence of the energy of the boundary loading is achieved due to the fact, that it provides a linear functional with respect to $\boldsymbol{v}$ and we have the weak convergence of $(\boldsymbol{v}_k)_{k\in\mathbb{N}}$. This leads to

$$\lim_{k\to\infty}\int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v}_k - \boldsymbol{g}\cdot\boldsymbol{v}\,d\boldsymbol{a}_{\boldsymbol{x}} = 0.$$

For the time discretisation part of the discrete energy, we use the convexity of the norm $\|\cdot\|_{\mathcal{M}_N}$ as well as the weak convergence of $(d_k)_{k\in\mathbb{N}}$ and derive

$$\lim_{k\to\infty}\frac{1}{2\Delta t}\|d_k - c^{n-1}\|^2_{\mathcal{M}_N} - \frac{1}{2\Delta t}\|d-c^{n-1}\|^2_{\mathcal{M}_N} \geq \lim_{k\to\infty}\frac{1}{\Delta t}\big\langle d-c^{n-1}, d_k - d\big\rangle_{\mathcal{M}_N} = 0.$$

Step 1 and step 2 present all conditions, which are necessary to apply [7] theorem 1.1 respectively [26], from which the existence of a minimiser $c^n\in X_m(\Omega)$ and $\boldsymbol{u}^n\in\boldsymbol{X}(\Omega)$ follows. □

We show now, that c^n and $\boldsymbol{u}^n$ solve the discrete problem 4.7.

Lemma 4.9. *Let $\Omega\subset\mathbb{R}^d$ be an open and bounded Lipschitz domain, satisfying condition* (3.1) *and let assumption (A-1) - (A-7) be satisfied. For an arbitrary test functions $\xi\in H^1(\Omega)\cap L_\infty(\Omega)$ and $\boldsymbol{\eta}\in\boldsymbol{X}(\Omega)$ the Gâteaux derivative of E^n_N exists and it holds*

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{E^n_N(d+\delta\xi,\boldsymbol{v}+\delta\boldsymbol{\eta}) - E^n_N(d,\boldsymbol{v})}{\delta}\\
&\quad= \int_\Omega \boldsymbol{\Gamma}(d)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}}\xi + \frac{1}{2}\boldsymbol{\Gamma}_{,c}(d)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d\xi + \psi_{,c}(d)\xi + W_{,c}(d,\boldsymbol{\varepsilon}(\boldsymbol{v}))\xi\\
&\qquad\qquad + W_{,\boldsymbol{\varepsilon}}(d,\boldsymbol{\varepsilon}(\boldsymbol{v})) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\,d\boldsymbol{x} - \int_{\Gamma_N}\boldsymbol{g}\cdot\boldsymbol{\eta}\,d\boldsymbol{a}_{\boldsymbol{x}} + \frac{1}{\Delta t}\big\langle d-c^{n-1},\xi\big\rangle_{\mathcal{M}_N}. \qquad (4.14)
\end{aligned}$$

Proof. We start this proof by considering the surface energy, using the mean value theorem, assumption (A-1) and Lebesgue's convergence theorem, to calculate

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_\Omega \frac{1}{2}\boldsymbol{\Gamma}(d+\delta\xi)\nabla_{\boldsymbol{x}}(d+\delta\xi)\cdot\nabla_{\boldsymbol{x}}(d+\delta\xi) - \frac{1}{2}\boldsymbol{\Gamma}(d)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d\,d\boldsymbol{x}\\
&\quad= \lim_{\delta\to 0}\frac{1}{\delta}\int_\Omega \frac{1}{2}\boldsymbol{\Gamma}(d+\delta\xi)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d + \delta\boldsymbol{\Gamma}(d+\delta\xi)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}}\xi + \frac{1}{2}\delta^2\boldsymbol{\Gamma}(d+\delta\xi)\nabla_{\boldsymbol{x}}\xi\cdot\nabla_{\boldsymbol{x}}\xi\\
&\qquad\qquad - \frac{1}{2}\boldsymbol{\Gamma}(d)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d\,d\boldsymbol{x}\\
&\quad= \lim_{\delta\to 0}\int_\Omega \frac{1}{2}\boldsymbol{\Gamma}_{,c}(d+\nu(\delta)\xi)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d\xi + \boldsymbol{\Gamma}(d+\delta\xi)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}}\xi + \frac{1}{2}\delta\boldsymbol{\Gamma}(d+\delta\xi)\nabla_{\boldsymbol{x}}\xi\cdot\nabla_{\boldsymbol{x}}\xi\,d\boldsymbol{x}\\
&\quad= \int_\Omega \frac{1}{2}\boldsymbol{\Gamma}_{,c}(d)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}} d\xi + \boldsymbol{\Gamma}(d)\nabla_{\boldsymbol{x}} d\cdot\nabla_{\boldsymbol{x}}\xi.
\end{aligned}$$

Furthermore, we observe ψ_1, the first term of the homogeneous free energy, which is in view of assumption (A-2) a convex function. Without loss of generality we work with a test function

$\xi \in H^1(\Omega) \cap L_\infty(\Omega)$ with $\|\xi\|_{L_\infty(\Omega)} \le 1$ and get

$$\begin{aligned} \psi_1(d+\delta\xi) &\le \psi_1(d) + \delta\psi_{1,c}(d+\delta\xi)\xi \le \psi_1(d) + \delta|\psi_{1,c}(d+\delta\xi)|\|\xi\|_{L_\infty(\Omega)} \\ &\le \psi_1(d) + \delta(\psi_1(d+\delta\xi) + C) \\ (1-\delta)\psi_1(d+\delta\xi) &\le \psi_1(d) + \delta C. \end{aligned} \tag{4.15}$$

Since we are interested in the case $\delta \to 0$, we determine for all $\delta \le \frac{1}{2}$,

$$\psi_1(d+\delta\xi) \le \psi_1(d) + C. \tag{4.16}$$

We show that for all $\delta \le \frac{1}{2}$ the difference quotient of ψ_1 is uniformly bounded by using estimate (4.15) and (4.16)

$$\left|\frac{\psi_1(d+\delta\xi) - \psi_1(d)}{\delta}\right| \le \left|\frac{\psi_1(d) + \delta(\psi_1(d+\delta\xi) + C) - \psi_1(d)}{\delta}\right| \le \psi_1(d) + 2C.$$

From assumption (A-2) we get $\psi_1 \in C^1(\mathbb{R},\mathbb{R})$, in combination with the uniform boundedness of the difference quotient and Lebesgue's convergence theorem we calculate

$$\lim_{\delta\to 0} \int_\Omega \frac{\psi_1(d+\delta\xi) - \psi_1(d)}{\delta}\, d\boldsymbol{x} = \int_\Omega \psi_{1,c}(d)\xi\, d\boldsymbol{x}.$$

Considering the non-convex term of the homogeneous free energy we know from assumption (A-2) $\psi_1 \in C^1(\mathbb{R},\mathbb{R})$ and we have a polynomial boundedness. Due to that fact, we get by using the generalised Lebesgue's convergence theorem

$$\lim_{\delta\to 0} \int_\Omega \frac{\psi_2(d+\delta\xi) - \psi_2(d)}{\delta}\, d\boldsymbol{x} = \int_\Omega \psi_{2,c}(d)\xi\, d\boldsymbol{x}.$$

Moreover, the strain energy is observed by using again the mean value theorem in combination with assumption (A-5) and (A-6) to calculate

$$\begin{aligned} &\lim_{\delta\to 0} \frac{1}{\delta} \int_\Omega W(d+\delta\xi, \boldsymbol{\varepsilon}(\boldsymbol{v}+\delta\boldsymbol{\eta})) - W(d, \boldsymbol{\varepsilon}(\boldsymbol{v}))\, d\boldsymbol{x} \\ &\quad = \lim_{\delta\to 0} \frac{1}{\delta} \int_\Omega W(d+\delta\xi, \boldsymbol{\varepsilon}(\boldsymbol{v}+\delta\boldsymbol{\eta})) - W(d, \boldsymbol{\varepsilon}(\boldsymbol{v}+\delta\boldsymbol{\eta})) \\ &\qquad\qquad + W(d, \boldsymbol{\varepsilon}(\boldsymbol{v}+\delta\boldsymbol{\eta})) - W(d, \boldsymbol{\varepsilon}(\boldsymbol{v}))\, d\boldsymbol{x} \\ &\quad = \lim_{\delta\to 0} \int_\Omega W_{,c}(d+\nu(\delta)\xi, \boldsymbol{\varepsilon}(\boldsymbol{v}+\delta\boldsymbol{\eta}))\xi + W_{,\boldsymbol{\varepsilon}}(d, \boldsymbol{\varepsilon}(\boldsymbol{v}+\nu(\delta)\boldsymbol{\eta})) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x} \\ &\quad = \int_\Omega W_{,c}(d, \boldsymbol{\varepsilon}(\boldsymbol{v}))\xi + W_{,\boldsymbol{\varepsilon}}(d, \boldsymbol{\varepsilon}(\boldsymbol{v})) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x}. \end{aligned}$$

The time discretisation term of the discrete energy directly yields

$$\begin{aligned} &\lim_{\delta\to 0} \frac{1}{2\delta\Delta t} \left(\|d+\delta\xi - c^{n-1}\|^2_{\mathcal{M}_N} - \|d - c^{n-1}\|^2_{\mathcal{M}_N}\right) \\ &\quad = \lim_{\delta\to 0} \left(\frac{1}{\Delta t}\langle d - c^{n-1}, \xi\rangle_{\mathcal{M}_N} + \delta\langle \xi, \xi\rangle_{\mathcal{M}_N}\right) = \frac{1}{\Delta t}\langle d - c^{n-1}, \xi\rangle_{\mathcal{M}_N}. \end{aligned}$$

The energy of the boundary loading is a linear functional with respect to the displacement $\boldsymbol{v}$ and therefore holds

$$\lim_{\delta\to 0} \int_{\Gamma_N} \boldsymbol{g}\cdot(\boldsymbol{v}+\delta\boldsymbol{\eta}) - \boldsymbol{g}\cdot\boldsymbol{v}\, d\boldsymbol{a}_{\boldsymbol{x}} = \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{\eta}\, d\boldsymbol{a}_{\boldsymbol{x}}.$$

The summary of all calculations shows the statement of this lemma. □

We remark, that due to [7] theorem 1.3 for the minimiser $c^n \in X_m(\Omega)$ and $\boldsymbol{u}^n \in \boldsymbol{X}(\Omega)$ the Gâteaux derivative is equal to zero, which means for test functions $\xi \in X_0(\Omega) \cap L_\infty(\Omega)$ and $\boldsymbol{\eta} \in \boldsymbol{X}(\Omega)$

$$\int_\Omega \boldsymbol{\Gamma}(c^n)\nabla_{\boldsymbol{x}} c^n \cdot \nabla_{\boldsymbol{x}} \xi + \frac{1}{2}\boldsymbol{\Gamma}_{,c}(c^n)\nabla_{\boldsymbol{x}} c^n \cdot \nabla_{\boldsymbol{x}} c^n \xi + \psi_{,c}(c^n)\xi + W_{,c}(c^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n))\xi$$
$$+ W_{,\boldsymbol{\varepsilon}}(c^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n)) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{\eta}\, d\boldsymbol{a_x} + \left\langle \frac{c^n - c^{n-1}}{2\Delta t}, \xi \right\rangle_{\mathcal{M}_N} = 0.$$

Because that the test functions $\xi \in X_0(\Omega) \cap L_\infty(\Omega)$ and $\boldsymbol{\eta} \in \boldsymbol{X}(\Omega)$ are independent, the minimiser $c^n \in X_m(\Omega)$ and $\boldsymbol{u}^n \in \boldsymbol{X}(\Omega)$ provides a solution of the discrete weak problem 4.7.
Furthermore, the minimiser $c^n \in X_m(\Omega)$ and $\boldsymbol{u}^n \in \boldsymbol{X}(\Omega)$ with a chemical potential $\mu^n \in H^1(\Omega)$ given by (4.10) forms a discrete weak solution in the sense of definition 4.2. This holds, because the Gâteaux derivative exists for all test functions $\xi \in H^1(\Omega) \cap L_\infty(\Omega)$ and $\boldsymbol{\eta} \in \boldsymbol{X}(\Omega)$ and we can choose the constant Lagrange multiplier

$$\begin{aligned} \lambda^n &= \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \mu^n \, d\boldsymbol{x} \\ &= \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \frac{1}{2}\boldsymbol{\Gamma}_{,c}(c^n)\nabla_{\boldsymbol{x}} c^n \cdot \nabla_{\boldsymbol{x}} c^n + \psi_{,c}(c^n) + W_{,c}(c^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n))\, d\boldsymbol{x}. \end{aligned} \tag{4.17}$$

Now, we have calculated the piecewise constant interpolants of the mass concentration $(c_N)_{N\in\mathbb{N}} \subset L_2(0,T;X_m(\Omega))$, of the chemical potential $(\mu_N)_{N\in\mathbb{N}} \subset L_2(0,T;H^1(\Omega))$ and of the displacement $(\boldsymbol{u}_N)_{N\in\mathbb{N}} \subset L_2(0,T;\boldsymbol{X}(\Omega))$ for sufficient large N. Finally, we have to prove, that these sequences converge to a weak solution of the field equations (2.14) - (2.16) in the sense of definition 3.8.

4.3. **A-priori estimates.** In the third step of this existence proof, we derive uniform a-priori estimates for the discrete weak solutions $(c_N)_{N\in\mathbb{N}}$, $(\mu_N)_{N\in\mathbb{N}}$ and $(\boldsymbol{u}_N)_{N\in\mathbb{N}}$. In order to calculate an a-priori estimate, we use the Lyapunov property of the system.

Lemma 4.10 (Lyapunov property). *At each discrete time $t = n\Delta t$ holds for the a discrete semi-weak solution c_N, μ_N and $\boldsymbol{u}_N$ the following energy estimate*

$$E(c^n, \boldsymbol{u}^n) + \frac{1}{2}\int_0^{n\Delta t}\int_\Omega \boldsymbol{M}_{N-1}\nabla_{\boldsymbol{x}}\mu_N \cdot \nabla_{\boldsymbol{x}}\mu_N \, d\boldsymbol{x}dt \le E(c^0, \boldsymbol{u}^0),$$

respectively

$$E(c^n, \boldsymbol{u}^n) \le E(c^0, \boldsymbol{u}^0).$$

Proof. We consider the discrete energy functional (4.12) at time $t = n\Delta t$, $1 \le n \le N$, with a corresponding minimiser $c^n \in X_m(\Omega)$ and $\boldsymbol{u}^n \in \boldsymbol{X}(\Omega)$ and use as comparison functions $c^{n-1} \in X_m(\Omega)$ and $\boldsymbol{u}^{n-1} \in \boldsymbol{X}(\Omega)$. Then we get

$$\begin{aligned} E_N^n(c^n, \boldsymbol{u}^n) &= E(c^n, \boldsymbol{u}^n) + \frac{1}{2\Delta t}\|c^n - c^{n-1}\|^2_{\mathcal{M}_N} \\ &\le E_N^n(c^{n-1}, \boldsymbol{u}^{n-1}) = E(c^{n-1}, \boldsymbol{u}^{n-1}). \end{aligned}$$

Taking definition 4.5 and equation (4.10) into account, then we calculate

$$\begin{aligned} \frac{1}{2\Delta t}\|c^n - c^{n-1}\|^2_{\mathcal{M}_N} &= \frac{1}{2\Delta t}\int_\Omega \boldsymbol{M}_{N-1}\nabla_{\boldsymbol{x}}\mathcal{M}_N^{-1}(c^n - c^{n-1}) \cdot \nabla_{\boldsymbol{x}}\mathcal{M}_N^{-1}(c^n - c^{n-1})\, d\boldsymbol{x} \\ &= \frac{\Delta t}{2}\int_\Omega \boldsymbol{M}_{N-1}\nabla_{\boldsymbol{x}}\mu^n \cdot \nabla_{\boldsymbol{x}}\mu^n \, d\boldsymbol{x}. \end{aligned}$$

From both calculation we directly deduce

$$E(c^n, \boldsymbol{u}^n) + \frac{\Delta t}{2}\int_\Omega \boldsymbol{M}_{N-1}\nabla_{\boldsymbol{x}}\mu^n \cdot \nabla_{\boldsymbol{x}}\mu^n \, d\boldsymbol{x} \le E(c^{n-1}, \boldsymbol{u}^{n-1}).$$

We apply this estimate recursively and take assumption (A-3) into account, then we get the statement of this lemma. ☐

Additionally to the piecewise constant interpolant c_N, we introduce a piecewise linear interpolant of the mass concentration $\bar{c}_N$. For all $1 \le n \le N$ we define at time $t \in ((n-1)\Delta t, n\Delta t]$

$$\bar{c}_N(t, \boldsymbol{x}) := c^n(\boldsymbol{x}) \left(\frac{t - (n-1)\Delta t}{\Delta t} \right) - c^{n-1}(\boldsymbol{x}) \left(\frac{t - n\Delta t}{\Delta t} \right), \tag{4.18}$$

$$\dot{\bar{c}}_N(t, \boldsymbol{x}) = \frac{c^n(\boldsymbol{x}) - c^{n-1}(\boldsymbol{x})}{\Delta t}. \tag{4.19}$$

It is obvious to see in view of (4.19), that $\bar{c}_N$ is a solution of the discrete semi-weak diffusion equation (4.5). For $t \in ((n-1)\Delta t, n\Delta t]$, $1 \le n \le N$ we get

$$\int_\Omega \dot{\bar{c}}_N \zeta \, d\boldsymbol{x} + \int_\Omega \boldsymbol{M}_{N-1} \nabla_{\boldsymbol{x}} \mu_N \cdot \nabla_{\boldsymbol{x}} \zeta \, d\boldsymbol{x} = 0 \qquad \forall \zeta \in X_0(\Omega). \tag{4.20}$$

In order to show the existence of a weak solution of the generalised Cahn-Larché equation system, we prove uniform a-priori estimates for the interpolants c_N, $\bar{c}_N$, μ_N and $\boldsymbol{u}_N$.

Lemma 4.11 (A-priori estimates)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain satisfying condition* (3.1) *and let the assumptions (A-1) - (A-7) be satisfied. Then for the discrete weak solution c_N, μ_N and $\boldsymbol{u}_N$ exists constants C_c, $C_{\bar{c}}$, $C_{\bar{c}'}$, C_μ, $C_{\boldsymbol{u}}$ and $C_\psi > 0$, depending on the material parameters, the boundary loading $\boldsymbol{g}$ and the initial condition c_0, such that the following uniform a-priori estimates hold*

(1) $\|c_N\|_{L_\infty(0,T;H^1(\Omega))} \le C_c$
(2) $\|\bar{c}_N\|_{L_\infty(0,T;H^1(\Omega))} \le C_{\bar{c}}$
(3) $\|\dot{\bar{c}}_N\|_{L_2(0,T;X_0'(\Omega))} \le C_{\bar{c}'}$
(4) $\|\mu_N\|_{L_2(0,T;H^1(\Omega))} \le C_\mu$
(5) $\|\boldsymbol{u}_N\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))} \le C_{\boldsymbol{u}}$
(6) $\sup\limits_{t \in [0,T]} \int_\Omega \psi(c_N) \, d\boldsymbol{x} \le C_\psi$

Proof. **Step 1**: At first we consider the discrete balance of linear momentum (4.7) and take equation (3.8) into account, from which we get

$$\|\boldsymbol{u}^n\|^2_{\boldsymbol{H}^1(\Omega)} \le C \left(\|c^n\|^2_{H^1(\Omega)} + \|\boldsymbol{g}\|^2_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))} + \text{meas}(\Omega) \right).$$

Step 2: Furthermore, we consider the energy at time $t = n\Delta t$ for a minimiser $c^n \in X_m(\Omega)$ and $\boldsymbol{u}^n \in \boldsymbol{X}(\Omega)$. We apply assumption (A-1), (A-2) and (A-7) with Poincaré's inequality (3.2) and Young's inequality. Thus we get

$$\begin{aligned} E(c^n, \boldsymbol{u}^n) &= \int_\Omega \frac{1}{2} \boldsymbol{\Gamma}(c^n) \nabla_{\boldsymbol{x}} c^n \cdot \nabla_{\boldsymbol{x}} c^n + \psi(c^n) + W(c^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n)) \, d\boldsymbol{x} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}^n \, d\boldsymbol{a}_{\boldsymbol{x}} \\ &\ge \frac{c_\Gamma}{2c_p} \|c^n\|^2_{H^1(\Omega)} - \frac{c_\Gamma}{2} m^2 - c_\psi \, \text{meas}(\Omega) - \frac{c_\gamma^2}{\delta} \|\boldsymbol{g}\|^2_{L_\infty(0,T;H^{-\frac{1}{2}}(\Gamma_N))} - \delta \|\boldsymbol{u}^n\|^2_{\boldsymbol{H}^1(\Omega)}. \end{aligned}$$

We use step 1 and choose $\delta = \frac{c_\Gamma}{4c_p C}$, from which we deduce

$$E(c^n, \boldsymbol{u}^n) \ge \frac{c_\Gamma}{4c_p} \|c^n\|^2_{H^1(\Omega)} - \tilde{C},$$

with a constant $\tilde{C} = \tilde{C}(C, c_\psi, c_\Gamma, c_\gamma, m, \text{meas}(\Omega), \boldsymbol{g})$.
Step 3: We observe the energy at time $t = 0$ by using assumption (A-1) - (A-7) and the estimate (3.8) for $\boldsymbol{u}_0$, which is the initial displacement corresponding to c_0. We calculate

$$\begin{aligned} E(c_0, \boldsymbol{u}_0) &= \int_\Omega \frac{1}{2} \boldsymbol{\Gamma}(c_0) \nabla_{\boldsymbol{x}} c_0 \cdot \nabla_{\boldsymbol{x}} c_0 + \psi(c_0) + W(c_0, \boldsymbol{\varepsilon}(\boldsymbol{u}_0)) \, d\boldsymbol{x} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}_0 \, d\boldsymbol{a}_{\boldsymbol{x}} \\ &\le \frac{C_\Gamma}{2} |c_0|^2_{H^1(\Omega)} + \|\psi(c_0)\|_{L_1(\Omega)} + \frac{C_C}{2} (|\boldsymbol{u}_0|^2_{\boldsymbol{H}^1(\Omega)} + 2C_{\overline{\boldsymbol{\varepsilon}}} (\|c_0\|^2_{L_2(\Omega)} + \text{meas}(\Omega))) \\ &\quad + c_\gamma \|\boldsymbol{g}\|_{L_\infty(0,T;H^{-\frac{1}{2}}(\Gamma_N))} \|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} \\ &\le \bar{C} (\|c_0\|^2_{H^1(\Omega)} + \|\psi(c_0)\|_{L_1(\Omega)} + \|\boldsymbol{g}\|^2_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))} + \text{meas}(\Omega)), \end{aligned}$$

with a constant $\bar{C} = \bar{C}(C, C_\Gamma, C_C, C_{\overline{\boldsymbol{\varepsilon}}}, c_\gamma)$. From step 2, step 3 and lemma 4.10 we directly follow statement *(1)* and *(2)*. Using step 1 and statement *(1)*, statement *(5)* is proved.

Step 4: From lemma 4.10 we directly deduce an estimate for the chemical potential. We consider at time $t = N\Delta t$ lemma 4.10 and get

$$\int_0^T \int_\Omega \boldsymbol{M}_{N-1} \nabla_{\boldsymbol{x}} \mu_N \cdot \nabla_{\boldsymbol{x}} \mu_N \, d\boldsymbol{x} dt \leq E(c_0, \boldsymbol{u}_0) - E(c^N, \boldsymbol{u}^N) \leq C < \infty.$$

The constant C in this estimate is independent of c^N and $\boldsymbol{u}^N$ due to statement *(1)* and *(5)*. Furthermore, we calculate a lower estimate for the chemical potential by using assumption (A-3), Poincaré's inequality (3.2) and the Lagrangian multiplier (4.17)

$$\begin{aligned}\int_0^T \int_\Omega \boldsymbol{M}_{N-1} \nabla_{\boldsymbol{x}} \mu_N \cdot \nabla_{\boldsymbol{x}} \mu_N \, d\boldsymbol{x} dt &\geq c_M \int_0^T \int_\Omega |\nabla_{\boldsymbol{x}} \mu|^2 \, d\boldsymbol{x} dt \\ &\geq \frac{c_M}{c_p} \|\mu\|^2_{L_2(0,T;H^1(\Omega))} - c_M \int_0^T \lambda^n dt \\ &\geq \frac{c_M}{c_p} \|\mu\|^2_{L_2(0,T;H^1(\Omega))} - C.\end{aligned}$$

Due to equation (4.17) in combination with statement *(1)* and *(5)*, there exists a uniform bound C for $c_M \int_0^T \lambda^n dt$. From step 4 we yield statement *(4)*.

Step 5: Moreover, we consider the discrete semi-weak diffusion equation (4.20) in order to prove statement *(3)*. Let $\zeta \in X_0(\Omega)$ an arbitrary test function, then we calculate in view of assumption (A-3)

$$\left| \int_\Omega \dot{\bar{c}}_N \zeta \, d\boldsymbol{x} \right| = \left| \int_\Omega \boldsymbol{M}_{N-1} \nabla_{\boldsymbol{x}} \mu_N \cdot \nabla_{\boldsymbol{x}} \zeta \, d\boldsymbol{x} \right| \leq C_M \|\mu_N\|_{H^1(\Omega)} \|\zeta\|_{H^1(\Omega)}.$$

Using the definition of an operator norm and statement *(4)*, then we yield

$$\|\dot{\bar{c}}_N\|_{L_2(0,T;X_0'(\Omega))} \leq C_M \|\mu_N\|_{L_2(0,T;H^1(\Omega))} \leq C_M C_\mu = C_{c'} < \infty,$$

which proves statement *(3)*.

Step 6: Finally, we prove statement *(6)* by considering again the energy at time $t = n\Delta t$ and taking assumption (A-1), (A-5) and (A-7) into account. The trace theorem and estimate (3.3) imply

$$\begin{aligned}E(c^n, \boldsymbol{u}^n) &= \int_\Omega \frac{1}{2} \boldsymbol{\Gamma}(c^n) \nabla_{\boldsymbol{x}} c^n \cdot \nabla_{\boldsymbol{x}} c^n + \psi(c^n) + W(c^n, \boldsymbol{\varepsilon}(\boldsymbol{u}^n)) \, d\boldsymbol{x} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}^n \, d\boldsymbol{a}_{\boldsymbol{x}} \\ &\geq \int_\Omega \psi(c^n) \, d\boldsymbol{x} - c_\gamma \|\boldsymbol{g}\|_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))} \|\boldsymbol{u}^n\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))}.\end{aligned}$$

Applying lemma 4.10 and statement *(5)*, then we directly get

$$\int_\Omega \psi(c^n) \, d\boldsymbol{x} \leq E(c_0, \boldsymbol{u}_0) + c_\gamma \|\boldsymbol{g}\|_{L_\infty(0,T;\boldsymbol{H}^{-\frac{1}{2}}(\Gamma_N))} \|\boldsymbol{u}^n\|_{L_\infty(0,T;\boldsymbol{H}^1(\Omega))} \leq C_\psi < \infty.$$

This estimate proves statement *(6)*. □

4.4. **Existence proof.** In this section we extract convergent subsequences of the sequences $(c_N)_{N\in\mathbb{N}}$, $(\bar{c}_N)_{N\in\mathbb{N}}$, $(\mu_N)_{N\in\mathbb{N}}$ and $(\boldsymbol{u}_N)_{n\in\mathbb{N}}$ and show, that their limits provide a weak solution of the generalised Cahn-Larché equation system in the sense of definition 3.8.

Lemma 4.12. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain satisfying condition* (3.1) *and let assumption (A-1) - (A-7) be satisfied. Then there exist at least one $c \in L_\infty(0,T;H^1(\Omega))$ with $\dot{c} \in L_2(0,T;X_0'(\Omega))$ and subsequences denoted without loss of generality with the same index*

$(c_N)_N$, $(\bar{c}_N)_N \subset L_\infty(0,T;H^1(\Omega))$, *such that*

$$\bar{c}_N \longrightarrow c \qquad \text{in} \quad C([0,T];L_2(\Omega)), \tag{4.21}$$

$$c_N \longrightarrow c \qquad \text{in} \quad L_\infty(0,T;L_2(\Omega)), \tag{4.22}$$

$$c_N(t,\boldsymbol{x}) \longrightarrow c(t,\boldsymbol{x}) \qquad \text{almost everywhere}, \tag{4.23}$$

$$c_N \overset{*}{\rightharpoonup} c \qquad \text{in} \quad L_\infty(0,T;H^1(\Omega)), \tag{4.24}$$

$$\dot{\bar{c}}_N \rightharpoonup \dot{c} \qquad \text{in} \quad L_2(0,T;X_0'(\Omega)). \tag{4.25}$$

Proof. We consider $t_1, t_2 \in (0,T)$ with $t_1 < t_2$ and choose the test function in (4.20) as $\zeta(\boldsymbol{x}) = \bar{c}_N(t_2,\boldsymbol{x}) - \bar{c}_N(t_1,\boldsymbol{x})$. After integration over (t_1,t_2), we apply integration by parts with respect to time, use assumption (A-3) and lemma 4.11 in order to estimate

$$\begin{aligned}
\|\bar{c}_N(t_2,\cdot) - \bar{c}_N(t_1,\cdot)\|^2_{L_2(\Omega)} &\leq \left| \int_{t_1}^{t_2} \int_\Omega \boldsymbol{M}_{N-1} \nabla_{\boldsymbol{x}} \mu_N \cdot \nabla_{\boldsymbol{x}} (\bar{c}_N(t_2,\cdot) - \bar{c}_N(t_1,\cdot)) \, d\boldsymbol{x} dt \right| \\
&\leq C_M \int_{t_1}^{t_2} \|\mu_N(t,\cdot)\|_{H^1(\Omega)} \|\bar{c}_N(t_2,\cdot) - \bar{c}_N(t_1,\cdot)\|_{H^1(\Omega)} dt \\
&\leq C|t_2 - t_1|^{\frac{1}{2}}.
\end{aligned}$$

This estimate shows, that c_N is equicontinuous. Because of Rellich-Kondrachov's theorem, we know $X_m(\Omega) \Subset L_2(\Omega)$ and therefore we apply the theorem of Arzelà-Ascoli and deduce (4.21). In order to prove (4.22), we consider an arbitrary but fixed time $t \in (0,T)$ and choose $1 \leq n \leq N$ as well as $\beta \in [0,1]$, such that $t = \beta n \Delta t + (1-\beta)(n-1)\Delta t$, then we obtain

$$\begin{aligned}
\|\bar{c}_N(t,\cdot) - c_N(t,\cdot)\|_{L_2(\Omega)} &= \|(1-\beta)c^{n-1} + \beta c^n - c^n\|_{L_2(\Omega)} \\
&= (1-\beta)\|c^{n-1} - c^n\|_{L_2(\Omega)} \\
&= C(\Delta t)^{\frac{1}{4}}.
\end{aligned}$$

From Weyl's corollary we get an almost everywhere convergent subsequence $(c_N)_{N\in\mathbb{N}}$, which proves (4.23). From lemma 4.11 we directly deduce (4.24) and (4.25). □

We remark, that due to the uniform a-priori estimates in lemma 4.11 there exists weak, respectively weak-* convergent subsequences denoted without loss of generality with the same index

$$\mu_N \rightharpoonup \mu \qquad \text{in} \quad L_2(0,T;H^1(\Omega)), \tag{4.26}$$

$$\boldsymbol{u}_N \overset{*}{\rightharpoonup} \boldsymbol{u} \qquad \text{in} \quad L_\infty(0,T;\boldsymbol{X}(\Omega)). \tag{4.27}$$

Lemma 4.13. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain satisfying condition* (3.1) *and let assumption (A-1) - (A-7) be satisfied, then it holds*

$$\boldsymbol{u}_N \to \boldsymbol{u} \qquad \text{in} \quad L_2(0,T;\boldsymbol{H}^1(\Omega)).$$

Proof. We consider the weak balance of momentum (4.7), integrate over $(0,T)$ and subtract on both sides $\int_{\Omega_T} \boldsymbol{C}(c_N)\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) \, d\boldsymbol{x}dt$, then we get

$$\begin{aligned}
&\int_{\Omega_T} \boldsymbol{C}(c_N)(\boldsymbol{\varepsilon}(\boldsymbol{u}_N) - \boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) \, d\boldsymbol{x}dt \\
&\quad = \int_{\Omega_T} \boldsymbol{C}(c_N)\overline{\boldsymbol{\varepsilon}}(c_N) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) - \boldsymbol{C}(c_N)\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) \, d\boldsymbol{x}dt + \int_{\Gamma_T^N} \boldsymbol{g} \cdot \boldsymbol{\eta} \, d\boldsymbol{a}_{\boldsymbol{x}} dt.
\end{aligned}$$

We choose as a test function $\boldsymbol{\eta} = \boldsymbol{u}_N - \boldsymbol{u}$, apply assumption (A-5) and Korn's inequality (3.4), therefore we get for the left hand side

$$\int_{\Omega_T} \boldsymbol{C}(c_N)(\boldsymbol{\varepsilon}(\boldsymbol{u}_N) - \boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) \, d\boldsymbol{x}dt \geq \frac{c_C}{c_k} \|\boldsymbol{u}_N - \boldsymbol{u}\|^2_{L_2(0,T;\boldsymbol{H}^1(\Omega))}.$$

In the right hand side we insert zero and calculate

$$\begin{aligned}
&\left|\int_{\Omega_T} \boldsymbol{C}(c_N)\overline{\boldsymbol{\varepsilon}}(c_N) : \boldsymbol{\varepsilon}(\boldsymbol{u}_N - \boldsymbol{u}) - \boldsymbol{C}(c_N)\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u}_N - \boldsymbol{u})\, d\boldsymbol{x}dt + \int_{\Gamma_T^N} \boldsymbol{g}\cdot(\boldsymbol{u}_N - \boldsymbol{u})\, d\boldsymbol{a_x}dt\right| \\
&\leq \left|\int_{\Omega_T} (\boldsymbol{C}(c_N)\overline{\boldsymbol{\varepsilon}}(c_N) - \boldsymbol{C}(c)\overline{\boldsymbol{\varepsilon}}(c)) : \boldsymbol{\varepsilon}(\boldsymbol{u}_N - \boldsymbol{u})\, d\boldsymbol{x}dt\right| + \left|\int_{\Omega_T} \boldsymbol{C}(c)\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u}_N - \boldsymbol{u})\, d\boldsymbol{x}dt\right| \\
&\quad + \left|\int_{\Omega_T} (\boldsymbol{C}(c) - \boldsymbol{C}(c_N))\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u}_N - \boldsymbol{u})\, d\boldsymbol{x}dt\right| + \left|\int_{\Omega_T} \boldsymbol{C}(c)\overline{\boldsymbol{\varepsilon}}(c) : \boldsymbol{\varepsilon}(\boldsymbol{u}_N - \boldsymbol{u})\, d\boldsymbol{x}dt\right| \\
&\quad + \left|\int_{\Gamma_T^N} \boldsymbol{g}\cdot(\boldsymbol{u}_N - \boldsymbol{u})\, d\boldsymbol{a_x}dt\right|.
\end{aligned}$$

Both estimates as well as the boundedness of $(\boldsymbol{u}_N)_{N\in\mathbb{N}}$, the almost everywhere convergence of $(c_N)_{N\in\mathbb{N}}$ given by (4.23) with Lebesgue's convergence theorem and the weak-* convergence of $(\boldsymbol{u}_N)_{N\in\mathbb{N}}$, see (4.27), yield the statement. □

We remark, that due to lemma 4.13 and Weyl's corollary we extract an almost everywhere convergent subsequence denoted with the same index

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_N(t,\boldsymbol{x})) \to \boldsymbol{\varepsilon}(\boldsymbol{u}(t,\boldsymbol{x})) \qquad \text{almost everywhere in } \Omega_T. \tag{4.28}$$

Now we are able to prove, that the above obtained limits $c \in L_\infty(0,T;X_m(\Omega))$, $\mu \in L_2(0,T;H^1(\Omega))$ and $\boldsymbol{u} \in L_2(0,T;\boldsymbol{X}(\Omega))$ provide a weak solution of the generalised Cahn-Larché equation system in the sense of definition 3.8.

Proof of theorem 4.1. **Step 1**: At first we consider the discrete semi-weak diffusion equation (4.20). We integrate over $(0,T)$, apply integration by parts with respect to time and use (4.21) to deduce for the first term of (4.20)

$$\lim_{N\to\infty}\left|\int_{\Omega_T} \dot{\zeta}(c - \bar{c}_N)\, d\boldsymbol{x}dt\right| = 0.$$

In the second term of (4.20) we insert zero and use the fact, that $(\mu_N)_{N\in\mathbb{N}}$ is uniformly bounded and $(c_N)_{N\in\mathbb{N}}$ converge almost everywhere, see (4.23), with assumption (A-3) and Lebesgue's convergence theorem. Furthermore, we apply the weak-* convergence of $(\mu_N)_{N\in\mathbb{N}}$ and obtain

$$\begin{aligned}
&\lim_{N\to\infty}\left|\int_{\Omega_T} \boldsymbol{M}(c_{N-1})\nabla_{\boldsymbol{x}}\mu_N\cdot\nabla_{\boldsymbol{x}}\zeta - \boldsymbol{M}(c)\nabla_{\boldsymbol{x}}\mu\cdot\nabla_{\boldsymbol{x}}\zeta\, d\boldsymbol{x}dt\right| \\
&\quad\leq \lim_{N\to\infty}\Bigg(\left|\int_{\Omega_T} (\boldsymbol{M}(c_{N-1}) - \boldsymbol{M}(c))(\nabla_{\boldsymbol{x}}\mu_N - \nabla_{\boldsymbol{x}}\mu)\cdot\nabla_{\boldsymbol{x}}\zeta\, d\boldsymbol{x}dt\right| \\
&\qquad\qquad + \left|\int_{\Omega_T} \boldsymbol{M}(c)(\nabla_{\boldsymbol{x}}\mu_N - \nabla_{\boldsymbol{x}}\mu)\cdot\nabla_{\boldsymbol{x}}\zeta\, d\boldsymbol{x}dt\right| \\
&\qquad\qquad + \left|\int_{\Omega_T} (\boldsymbol{M}(c_{N-1}) - \boldsymbol{M}(c))(\nabla_{\boldsymbol{x}}\mu_N - \nabla_{\boldsymbol{x}}\mu)\cdot\nabla_{\boldsymbol{x}}\zeta\, d\boldsymbol{x}dt\right|\Bigg) = 0.
\end{aligned}$$

Step 2: In the next step, we consider the discrete semi-weak equation of the chemical potential (4.6), integrate over $(0,T)$ and deduce from the weak convergence of $(\mu_N)_{N\in\mathbb{N}}$ for the left hand side

$$\lim_{N\to\infty}\left|\int_{\Omega_T} (\mu_N - \mu)\xi\, d\boldsymbol{x}dt\right| = 0.$$

As a first step for the right hand side of (4.6), we observe the surface energy and inert zero. Furthermore, by applying the almost everywhere convergence and the boundedness of $(c_N)_{N\in\mathbb{N}}$ with assumption (A-1), Lebesgue's convergence theorem and the weak-* convergence of the mass concentration $(c_N)_{N\in\mathbb{N}}$. We get

$$\begin{aligned}
&\lim_{N\to\infty}\left|\int_{\Omega_T}(\mathbf{\Gamma}(c_N)\nabla_{\boldsymbol{x}}c_N-\mathbf{\Gamma}(c)\nabla_{\boldsymbol{x}}c)\cdot\nabla_{\boldsymbol{x}}\xi\;d\boldsymbol{x}dt\right|\\
&\leq\lim_{N\to\infty}\left(\left|\int_{\Omega_T}(\mathbf{\Gamma}(c_N)-\mathbf{\Gamma}(c))(\nabla_{\boldsymbol{x}}c_N-\nabla_{\boldsymbol{x}}c)\cdot\nabla_{\boldsymbol{x}}\xi\;d\boldsymbol{x}dt\right|\right.\\
&\qquad\left.+\left|\int_{\Omega_T}\mathbf{\Gamma}(c)(\nabla_{\boldsymbol{x}}c_N-\nabla_{\boldsymbol{x}}c)\cdot\nabla_{\boldsymbol{x}}\xi\;d\boldsymbol{x}dt\right|+\left|\int_{\Omega_T}(\mathbf{\Gamma}(c_N)-\mathbf{\Gamma}(c))\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}\xi\;d\boldsymbol{x}dt\right|\right)=0.
\end{aligned}$$

Moreover, we consider the first part ψ_1 of the homogeneous free energy ψ. By assumption (A-2) we have $\psi_1\in C^1(\mathbb{R},\mathbb{R})$ in combination with the almost everywhere convergence of $(c_N)_{N\in\mathbb{N}}$ we get

$$\psi_{1,c}(c_N(t,\boldsymbol{x}))\to\psi_{1,c}(c(t,\boldsymbol{x}))\quad\text{almost everywhere}\quad N\to\infty.$$

Furthermore, let $\varepsilon>0$ arbitrary, we choose a set $\mathcal{A}\in\mathfrak{B}(\Omega_T)$ with the property $\operatorname{meas}(\mathcal{A})\leq\varepsilon$, use assumption (A-2) with $\delta=\varepsilon$ and apply lemma 4.11 statement *(6)*, from which we calculate

$$\begin{aligned}
\int_{\mathcal{A}}|\psi_{1,c}(c_N)|\;d\boldsymbol{x}dt&\leq\int_{\mathcal{A}}\delta\psi_1(c_N)+C_\delta\;d\boldsymbol{x}dt\\
&\leq\delta\sup_{t\in[0,T]}\int_\Omega\psi_1(c_N)\;d\boldsymbol{x}+C_\delta\operatorname{meas}(\mathcal{A})\leq C\varepsilon.
\end{aligned}$$

In view of Vitali's convergence theorem, [13] chapter 6 theorem 5.6, we conclude

$$\lim_{N\to\infty}\left|\int_{\Omega_T}\psi_{1,c}(c_N)\xi-\psi_{1,c}(c)\xi\;d\boldsymbol{x}dt\right|=0.$$

The second part of the free energy ψ_2 is polynomial bounded due to assumption (A-2) and we get by the generalised Lebesgue convergence theorem

$$\lim_{N\to\infty}\left|\int_{\Omega_T}\psi_{2,c}(c_N)\xi-\psi_{2,c}(c)\xi\;d\boldsymbol{x}dt\right|=0.$$

Next, we observe the mechanical part of the chemical potential and we get from assumption (A-5) and (A-6) the estimate $|W_{,c}(c,\boldsymbol{\varepsilon}(\boldsymbol{u}))|\leq C(|c|^2+|\boldsymbol{\varepsilon}(\boldsymbol{u})|^2)$. Therefore we apply the generalised Lebesgue's convergence theorem and yield

$$\lim_{N\to\infty}\left|\int_{\Omega_T}(W_{,c}(c_N,\boldsymbol{\varepsilon}(\boldsymbol{u}_N))-W_{,c}(c,\boldsymbol{\varepsilon}(\boldsymbol{u})))\xi\;d\boldsymbol{x}dt\right|=0.$$

Up to this point, all sequences considered in step 2 are convergent and due to that fact, they are Cauchy sequences, which means that for all $\varepsilon>0$ there exists a $\mathcal{N}\in\mathbb{N}$, such that for all $N,M\geq\mathcal{N}$ hold

$$\begin{aligned}
&\left|\int_{\Omega_T}(\mu_N-\mu_M)\xi\;d\boldsymbol{x}dt\right|\leq\varepsilon,\\
&\left|\int_{\Omega_T}(\mathbf{\Gamma}(c_N)\nabla_{\boldsymbol{x}}c_N-\mathbf{\Gamma}(c_M)\nabla_{\boldsymbol{x}}c_M)\cdot\nabla_{\boldsymbol{x}}\xi\;d\boldsymbol{x}dt\right|\leq\varepsilon,\\
&\left|\int_{\Omega_T}(\psi_{,c}(c_N)-\psi_{,c}(c_M))\xi\;d\boldsymbol{x}dt\right|\leq\varepsilon,\\
&\left|\int_{\Omega_T}(W_{,c}(c_N,\boldsymbol{\varepsilon}(\boldsymbol{u}_N))-W_{,c}(c_M,\boldsymbol{\varepsilon}(\boldsymbol{u}_M)))\xi\;d\boldsymbol{x}dt\right|\leq\varepsilon.
\end{aligned}$$

We use this property in order to show, that the remaining term $\frac{1}{2}\int_{\Omega_T}\mathbf{\Gamma}_{,c}(c_N)\nabla_{\boldsymbol{x}}c_N\cdot\nabla_{\boldsymbol{x}}c_N\;d\boldsymbol{x}dt$ forms a Cauchy sequence. We take the discrete semi-weak equation of the chemical potential (4.6) as an identity and get after integration over $(0,T)$

$$\begin{aligned}
&\frac{1}{2}\left|\int_{\Omega_T}(\mathbf{\Gamma}(c_N)\nabla_{\boldsymbol{x}}c_N\cdot\nabla_{\boldsymbol{x}}c_N-\mathbf{\Gamma}(c_M)\nabla_{\boldsymbol{x}}c_M\cdot\nabla_{\boldsymbol{x}}c_M)\xi\;d\boldsymbol{x}dt\right|\\
&\le\left|\int_{\Omega_T}(\mu_N-\mu_M)\xi\;d\boldsymbol{x}dt\right|+\left|\int_{\Omega_T}(\mathbf{\Gamma}(c_M)\nabla_{\boldsymbol{x}}c_M-\mathbf{\Gamma}(c_N)\nabla_{\boldsymbol{x}}c_N)\cdot\nabla_{\boldsymbol{x}}\xi\;d\boldsymbol{x}dt\right|\\
&\quad+\left|\int_{\Omega_T}(\psi_{,c}(c_M)-\psi_{,c}(c_N))\xi\;d\boldsymbol{x}dt\right|+\left|\int_{\Omega_T}(W_{,c}(c_M,\boldsymbol{\varepsilon}(\boldsymbol{u}_M))-W_{,c}(c_N,\boldsymbol{\varepsilon}(\boldsymbol{u}_N)))\xi\;d\boldsymbol{x}dt\right|\\
&\le 4\varepsilon.
\end{aligned}$$

In view of this result $\lim_{N\to\infty}\frac{1}{2}\int_{\Omega_T}\mathbf{\Gamma}_{,c}(c_N)\nabla_{\boldsymbol{x}}c_N\cdot\nabla_{\boldsymbol{x}}c_N\;d\boldsymbol{x}dt$ exists. Moreover, we have to show the continuity of this term. Therefore, let $\xi\in L_2(0,T;H^1(\Omega))\cap L_\infty(\Omega_T)$ be a test function, then there exist due to [13] chapter 3.4 functions $\xi^+,\xi^-\in L_2(0,T;H^1(\Omega))\cap L_\infty(\Omega_T)$ with the property $\xi^+,\xi^-\ge 0$, such that

$$\xi=\xi^+-\xi^-.$$

Due to that fact, we consider in the following only non-negative test functions. At first we prove the following statement:

$$\lim_{N\to\infty}\int_{\Omega_T}\frac{1}{2}\mathbf{\Gamma}_{,c}(c_N)\nabla_{\boldsymbol{x}}c_N\cdot\nabla_{\boldsymbol{x}}c_N\xi\;d\boldsymbol{x}dt\ge\int_{\Omega_T}\frac{1}{2}\mathbf{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}c\xi\;d\boldsymbol{x}dt.$$

This result follows due to the fact, that we consider non-negative test function $\xi\ge 0$. We take the convexity in $\nabla_{\boldsymbol{x}}c$ into account, insert zeros and apply the almost everywhere convergence of $(c_N)_{N\in\mathbb{N}}$ in combination with the Lebesgue's convergence theorem. Finally we use the weak-* convergence of the mass concentration $(c_N)_{N\in\mathbb{N}}$ in order to get

$$\begin{aligned}
&\lim_{N\to\infty}\int_{\Omega_T}\frac{1}{2}(\mathbf{\Gamma}_{,c}(c_N)\nabla_{\boldsymbol{x}}c_N\cdot\nabla_{\boldsymbol{x}}c_N-\mathbf{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}c)\xi\;d\boldsymbol{x}dt\\
&\quad\ge\lim_{N\to\infty}\int_{\Omega_T}\mathbf{\Gamma}_{,c}(c_N)\nabla_{\boldsymbol{x}}c\cdot(\nabla_{\boldsymbol{x}}c_N-\nabla_{\boldsymbol{x}}c)\xi+\frac{1}{2}(\mathbf{\Gamma}_{,c}(c_N)-\mathbf{\Gamma}_{,c}(c))\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}c\;d\boldsymbol{x}dt\\
&\quad=\lim_{N\to\infty}\Bigg(\int_{\Omega_T}(\mathbf{\Gamma}_{,c}(c_N)-\mathbf{\Gamma}_{,c}(c))\nabla_{\boldsymbol{x}}c\cdot(\nabla_{\boldsymbol{x}}c_N-\nabla_{\boldsymbol{x}}c)\xi\;d\boldsymbol{x}dt\\
&\qquad\quad+\int_{\Omega_T}\mathbf{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c\cdot(\nabla_{\boldsymbol{x}}c_N-\nabla_{\boldsymbol{x}}c)\xi\;d\boldsymbol{x}dt\\
&\qquad\quad+\int_{\Omega_T}\frac{1}{2}(\mathbf{\Gamma}_{,c}(c_N)-\mathbf{\Gamma}_{,c}(c))\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}c\xi\;d\boldsymbol{x}dt\Bigg)=0.
\end{aligned}$$

It remains to show the continuity of this term. We formulate the following statement:

$$\lim_{N\to\infty}\int_{\Omega_T}\frac{1}{2}\mathbf{\Gamma}_{,c}(c_N)\nabla_{\boldsymbol{x}}c_N\cdot\nabla_{\boldsymbol{x}}c_N\xi\;d\boldsymbol{x}dt=\int_{\Omega_T}\frac{1}{2}\mathbf{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}c\;d\boldsymbol{x}dt.$$

In order to prove this statement, we assume that

$$\lim_{N\to\infty}\int_{\Omega_T}\frac{1}{2}\mathbf{\Gamma}_{,c}(c_N)\nabla_{\boldsymbol{x}}c_N\cdot\nabla_{\boldsymbol{x}}c_N\xi\;d\boldsymbol{x}dt\gneqq\int_{\Omega_T}\frac{1}{2}\mathbf{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}c\;d\boldsymbol{x}.$$

Then there exists a constant $\varepsilon_0>0$ and a constant $\mathcal{N}\in\mathbb{N}$, such that for all $M\ge\mathcal{N}$ holds

$$\int_{\Omega_T}\frac{1}{2}\mathbf{\Gamma}_{,c}(c_M)\nabla_{\boldsymbol{x}}c_M\cdot\nabla_{\boldsymbol{x}}c_M\xi-\frac{1}{2}\mathbf{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}c\xi\;d\boldsymbol{x}dt\ge\varepsilon_0>0.$$

Using this fact, then we get for all $M,N\ge\mathcal{N}$ the following estimate

$$\begin{aligned}
0<\varepsilon_0&\le\frac{1}{2}\left|\int_{\Omega_T}\mathbf{\Gamma}_{,c}(c)\nabla_{\boldsymbol{x}}c\cdot\nabla_{\boldsymbol{x}}c\xi-\mathbf{\Gamma}_{,c}(c_N)\nabla_{\boldsymbol{x}}c_N\cdot\nabla_{\boldsymbol{x}}c_N\xi\right|\\
&\le\frac{1}{2}\left|\int_{\Omega_T}\mathbf{\Gamma}_{,c}(c_M)\nabla_{\boldsymbol{x}}c_M\cdot\nabla_{\boldsymbol{x}}c_M\xi-\mathbf{\Gamma}_{,c}(c_N)\nabla_{\boldsymbol{x}}c_N\cdot\nabla_{\boldsymbol{x}}c_N\xi\right|\le\varepsilon.
\end{aligned}$$

This estimate holds for all $\varepsilon > 0$ and therefore the assumption causes a contradiction to the property of a Cauchy sequence.

Step 3: Finally, we consider the discrete semi-weak formulation of the balance of momentum (4.7) integrate over $(0, T)$ and take lemma 4.13 into account in order to get directly

$$\lim_{N\to\infty} \int_{\Omega_T} W_{,\boldsymbol{\varepsilon}}(c_N, \boldsymbol{\varepsilon}(\boldsymbol{u}_N)) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) - W_{,\boldsymbol{\varepsilon}}(c, \boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\, d\boldsymbol{x}dt = 0.$$

This argument finishes the whole existence proof of weak solutions of the strong nonlinear Cahn-Larché equation system. □

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