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# Limit analysis method for some static problems of Continuum Mechanics[1]


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**Abstract:** The limit analysis method for an estimation of mechanical and electrical durability for non-linear elastic and dielectric solids, respectively, is examined. The appropriate initial and dual Limit Analysis Problems (LAPs) are formulated. It is demonstrated that the initial LAPs need a relaxation, but the appropriate fully relaxed problems have no clear physical interpretation. On the other hand, the dual LAPs have a clear physical interpretation. The general initial and dual LAPs for quasi-static problems of Continuum Mechanics are formulated. The appropriate 2-D finite-element approximations and numerical solutions are presented.


# Introduction

Investigation of the elastostatic and electrostatic boundary-value problems (BVPs) is of particular interest in both theory and practice. The current research is motivated by significance and practical interests in Mechanical and Electrical Engineering.

The porous rubber-like materials working in water or oil are described by strain-energy functions or elastic potentials having the linear growth in the modulus of distortion tensor. For such materials, within the framework of the variational formulation the existence of the limit static load (such external static forces with no solution of the appropriate BVP)


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and discontinuous maps with jumps of the sliding type was proven in [3-5,7-10]. From the physical point of view these effects are treated as a loss of the global stability, i.e as the destruction of a solid.

For dielectrics in powerful electric fields the essentially non-linear phenomena of polarization saturation and powerful growth of the electric current must be taken into account. As a result, within the framework of the variational formulation, the existence of the limit electrostatic load (such external charges with no solution of the appropriate BVP) was pointed out in [11,12]. From the physical point of view this effect is treated as a loss of electrostatic balance or global stability, i.e. as the dielectric breakdown.

For an estimation of the limit states of non-linear elastic and dielectric solids the appropriate original variational limit analysis problems (LAPs) are formulated. But from the mathematical point of view LAPs need a relaxation because their solutions belong to the space of discontinuous functions with bounded variation [1,23,29].

Unfortunately no clear physical interpretation can be provided for the fully relaxed elastostatic and electrostatic LAPs. Therefore, the original partial relaxation of LAPs is proposed [8,9,11,12]. The relaxation is based on the special discontinuous finite-element approximation (FEA). But after relaxation the appropriate finite dimension LAPs become ill-conditioned and thus need special preconditioned numerical methods as, for example, presented in [5,6].

On the other hand, using methods of the duality theory, the dual elastostatic and electrostatic LAPs are formulated [13-16]. They have a clear physical interpretation and from the mathematical point of view are fully correct. After the standard piecewise linear continuous FEA they are transformed into problems of mathematical programming with linear limitations as equalities. These finite dimension problems are effectively solved by the standard method of gradient projection, which is easily adapted for parallel computations.

The numerical results show that the proposed limit analysis method has a qualitative advantage over classical techniques of estimation of the global stability of elastic and dielectric solids.

# 1 Limit analysis method in elastostatics

Let a body in the undeformed reference configuration occupy a bounded domain $\Omega \subset \mathbb{R}^3$ with the Lipschitz boundary. In the deformed configuration each point $\mathbf{x} \in \overline{\Omega}$ moves into the position $\mathbf{X}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \mathbf{x} \in \mathbb{R}^3$, where $\mathbf{X}$ and $\mathbf{u}$ are the map and displacement, respectively. Here and in what follows Lagrangian's co-ordinates are used. We consider locally invertible and orientation-preserving maps $\mathbf{X} : \overline{\Omega} \to \mathbb{R}^3$ with gradient (*the distortion tensor*) $\mathbf{Q} = \nabla\mathbf{X} : \Omega \to \mathbb{M}^3$ such that $\det\mathbf{Q} > 0$ in $\Omega$ [18, 27], where $\nabla = \partial/\partial\mathbf{x}$ and the symbol $\mathbb{M}^3$ denotes the space of real 3×3 matrices.

The finite deformation of materials is described by the energy pair $(\mathbf{Q}, \mathbf{\Sigma})$, where $\mathbf{\Sigma} = \{\Sigma_i^\alpha\}$ is the first non-symmetric Piola–Kirchhoff stress tensor [18, 25, 27, 30]. It is known that the Cauchy stress tensor $\boldsymbol{\sigma}$ has the components $\sigma^{\alpha\beta} = (\det\mathbf{Q})^{-1}\Sigma_i^\alpha Q_i^\beta$. Here and in what follows the Roman lower and Greek upper indices correspond to the reference and deformed configurations, respectively, and over repeated indices the summation rule applies.

Elastic materials are characterized by the scalar strain-energy function or *elastic potential* $\Phi : \Omega \times \mathbb{M}^3 \to \mathbb{R}_+$ such that $\Sigma_i^\alpha = \partial\Phi(\mathbf{x},\mathbf{Q})/\partial Q_i^\alpha$ for every $\mathbf{Q} \in \mathbb{M}^3$ and almost every $\mathbf{x} \in \Omega$, and $\Phi(\mathbf{x},\mathbf{I}) = 0$, where $\mathbf{I}$ is the second-order identity tensor. If a material is incompressible, then $\det\mathbf{Q} = 1$, but for a compressible material $\Phi(\mathbf{x},\mathbf{Q}) \to +\infty$ as $\det\mathbf{Q} \to +0$.

We consider the following boundary-value problem. The quasi-static forces acting on the body are: a bulk force with density $\mathbf{f}$ in $\Omega$, a surface force with density $\mathbf{F}$ on a portion $\Gamma^2$ of the boundary, and a portion $\Gamma^1$ of the boundary is fixed, i.e. $\mathbf{u} \equiv \mathbf{0}$ on $\Gamma^1$, where $\Gamma^1 \cup \Gamma^2 = \partial\Omega$, $\Gamma^1 \cap \Gamma^2 = \emptyset$ and $|\Gamma^1| > 0$. Here and what follows the symbol $|U|$ denotes the Lebesgue measure of the appropriate open set $U$.

For hyperelastic materials the elastostatic BVP is formulated as the following variational problem [18]:

$$\mathbf{u}^* = \arg\inf\{\Pi(\mathbf{u}) - A(\mathbf{u}) :\ \mathbf{u} \in V\} \ , \tag{1.1}$$

$$\Pi(\mathbf{u}) = \int\limits_{\Omega} \Phi(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x}) + \mathbf{I})\, d\Omega \ ,$$

$$A(\mathbf{u}) = \int\limits_{\Omega} \langle \mathbf{f}, \mathbf{u}\rangle(\mathbf{x})\, d\Omega + \int\limits_{\Gamma^2} \langle \mathbf{F}, \mathbf{u}\rangle(\mathbf{x})\, d\gamma \ , \qquad \langle \mathbf{g}, \mathbf{u}\rangle(\mathbf{x}) = \int\limits_{\mathbf{0}}^{\mathbf{u}(\mathbf{x})} g^\alpha(\mathbf{x}, \mathbf{v})\, dv^\alpha \ .$$

Here $V = \{\mathbf{u} : \overline{\Omega} \to \mathbb{R}^3;\ \mathbf{u}(\mathbf{x}) = \mathbf{0},\ \mathbf{x} \in \Gamma^1\}$ is the set of kinematically admissible displacements, $\langle *, \mathbf{u}\rangle$ is the specific and $A(\mathbf{u})$ is the full work of the external forces on the displacement $\mathbf{u}$. It must be noted that even for "dead" forces, i.e. $\mathbf{f}, \mathbf{F} = \mathrm{const}\,(\mathbf{u}, \nabla\mathbf{u})$, the specific work has the form of $\langle \mathbf{g}, \mathbf{u}\rangle(\mathbf{x}) = g^\alpha(\mathbf{x})\, u^\alpha(\mathbf{x})$ only in Descartes's coordinates [4].

## 1.1 Initial elastostatic LAP

The porous rubber-like materials working in water or oil are described by elastic potentials having the linear growth in the modulus of the distortion tensor [2].

**Definition 1.1** *An elastic material has the ideal saturation if for any constant $C > 0$ and every matrix $\mathbf{Q} \in \mathbb{M}^3$ with $|\mathbf{Cof\,Q}| \le C$ and $C^{-1} \le \det\mathbf{Q} \le C$, the elastic potential satisfies the following estimation from above:*

$$\Phi(\mathbf{x}, \mathbf{Q}) \le \lambda(\mathbf{x})\, |\mathbf{Q} - \mathbf{I}| \ ,$$

*where $\lambda > 0$ is the elastic saturation.*

Here and in what follows $\mathbf{Cof\,Q}$ is the cofactor matrix for $\mathbf{Q}$ and $|\mathbf{Q}| = (Q_i^\alpha Q_i^\alpha)^{1/2}$. In what follows, we consider a homogeneous elastic medium for which $\lambda = \mathrm{const}\,(\mathbf{x})$.

According to the general theory [22, 29], for potentials of linear growth the set of admissible displacements is the following subspace:

$$V = \left\{ \mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^3) : \ \mathbf{u}(\mathbf{x}) = 0, \ \mathbf{x} \in \Gamma^1 \right\} \ . \tag{1.2}$$

We remind the definition of the limit static load [3]. For this reason we introduce the set of admissible "dead" external forces for which the functional in the problrem (1.1) is bounded from below on $V$ and, therefore, a solution of the problem exists:

$$B = \left\{ (\mathbf{f}, \mathbf{F}) \in L^\infty(\Omega, \mathbb{R}^3) \times L^\infty(\Gamma^2, \mathbb{R}^3) : \ \inf\{\Pi(\mathbf{u}) - A(\mathbf{u}) : \mathbf{u} \in V\} > -\infty \right\} \ .$$

The set is non-empty because for small external forces the problem (1.1) is transformed into the classical variational problem of linear elasticity [25, 27] which always has a solution [18].

For arbitrary external forces $(\mathbf{f}_0, \mathbf{F}_0) \in B$ we examine the sequence of forces which are proportional to the real parameter $t \geq 0$.

**Definition 1.2** *A number $t_* \geq 0$ is defined as the limit parameter of static loading and $(t_*\mathbf{f}_0, t_*\mathbf{F}_0)$ is the limit static load, if $(t\mathbf{f}_0, t\mathbf{F}_0) \in B$ for $0 \leq t \leq t_*$ and $(t\mathbf{f}_0, t\mathbf{F}_0) \notin B$ for $t > t_*$.*

For the ideal saturating material, the analysis of the global stability of a solid is the investigation of the set of positive parameters $t$, for which the one-parametric functional

$$I_t(\mathbf{u}) = \Pi(\mathbf{u}) - t\, A(\mathbf{u})$$

is bounded from below on the set (1.2). The following basic result was proven in [3]:

**Theorem 1.1** *The finite limit parameter of static loading exists such that $t_* \leq t_+$, where $t_+$ is the solution of the Limit Analysis Problem*

$$t_+ = \lambda \inf \left\{ \int_\Omega |\nabla \mathbf{u}(\mathbf{x})| \, d\Omega : \ \mathbf{u} \in V, \ A(\mathbf{u}) = 1 \right\} \ . \tag{1.3}$$

*Proof.* From the Definition 1.1 it follows that

$$\inf\{\, \Pi(\mathbf{u}) : \ \mathbf{u} \in V \,\} \leq t_+$$

for every $\mathbf{u}$ with $A(\mathbf{u}) = 1$. Then for arbitrary $t \geq t_+$ the functional $I_t(\mathbf{u})$ is unbounded from below on the set $V$, that means $t_* \leq t_+$ ■

From the Definition 1.2 it follows that for $t_+ < 1$ the elastostatic variational problem (1.1) has no solution. From the physical point of view this effect is treated as a loss of the global stability or destruction of a solid. Therefore, *the limit analysis problem* (1.3) is the main problem for an estimation of the mechanical durability of non-linear elastic solids.

## 1.2 Example of the limit static load

We consider the following problem. Suppose that a circular hole of length $l$ and radius $b$ ($l \gg b$) is drilled through an absolutely rigid body and in the hole an absolutely rigid rod of smaller radius $a$ is placed co-axially. The free space of the hole is filled with a deformable material that is attached rigidly to the entire length of the rod and to the inside surface of the hole. The rod is pushed statically along the axis by a given force $F_z$.

We assume that the filler is subject to the Bartenev-Hazanovich elastic potential [2]

$$\Phi(\mathbf{Q}) = 2\mu \, (\nu_1(\mathbf{Q}) + \nu_2(\mathbf{Q}) + \nu_3(\mathbf{Q}) - 3) \ , \tag{1.4}$$

$$\det \mathbf{Q} = \nu_1\nu_2\nu_3 = 1 \ ,$$

where $\mu > 0$ is the shear modulus under small deformations and $\nu_k(\mathbf{Q}) > 0$ ($k = 1, 2, 3$) are the singular values of $\mathbf{Q}$ (i.e., the eigenvalues of the matrix $\{Q_i^\alpha Q_j^\alpha\}^{1/2}$). It is easily seen that this potential has the ideal suturation, i.e. the linear growth in the modulus of the distortion tensor because $\Phi(\mathbf{Q}) \le 2\sqrt{3}\mu\, |\mathbf{Q} - \mathbf{I}|$ for every $\mathbf{Q} \in \mathbb{M}^3$ with $\det \mathbf{Q} = 1$.

If the filler is homogeneous and isotropic, then, in view of the axial symmetry of the problem, its incompressible deformed configuration can be described within the framework of the model of an antiplanar shear [25] in the reference redused cylindrical co-ordinates by the following map:

$$\mathbf{X}(\rho, \varphi, z) = \mathbf{x}(a\rho, \varphi, az + aw(\rho)) \ ,$$

$$\mathbf{Q} = \nabla \mathbf{X} = \begin{pmatrix} 1 & 0 & w' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

where $\rho \in [1, \eta]$ and $\eta = b/a$ is the geometric parameter.

For the elastic potential (1.4) the variational problem (1.1) assumes the form

$$w_* = \arg \inf\{I(w) : \ w \in V\} \ , \tag{1.5}$$

where

$$I(w) = \int_1^\eta \left(4 + w'^2\right)^{1/2} \rho \, d\rho - t\, w(1) \ ,$$

$$V = \left\{w \in W^{1,1}(1, \eta) : \ w(\eta) = 0\right\}$$

and $t = F_z/(4\pi\mu a l) > 0$.

In view of the convexity of the problem (1.5) in both the functional and the set of admissible functions, a local extremal (if it exists) coincides with a global minimizer [20, 22].

From the condition of stationarity of $I(w)$ we find the local extremal

$$w_*(\rho) = 2t \, \left[\cosh^{-1}(\eta/t) - \cosh^{-1}(\rho/t)\right] \ ,$$

which exists only for $0 < t \le 1$.

By taking $t > 1$ and the sequence of *admissible* displacements having the simplest form $w_m \equiv m(\eta-\rho)^m/(\eta-1)^m$ $(m \in \mathbb{N})$, we can see easily that the energy functional is unbounded from below on $V$, i.e. $I(w_m) \to -\infty$ as $m \to \infty$. This phenomenon shows that for $t > 1$ the variational problem (1.5) has no solution. Moreover, from the Definition 1.2 it follows that $t_* = 1$. The general theorem about existence of the limit static load for hyperelastic materials with ideal saturation was proven by the author in [3].

For elastic potential (1.4) the limit analysis problem (1.3) assumes the form

$$t_+ = \inf \left\{ \int_1^\eta |w'|\rho\, d\rho : \ w \in V,\ w(1) = 1 \right\} . \tag{1.6}$$

Using the equality $|a| = \sup\{ab : \ |b| \leq 1\}$ we can rewrite the problem (1.6) in the following way:

$$t_+ = \inf \{ \sup ( L(v,w) : \ v \in V^* ) : \ w \in V,\ w(1) = 1 \} ,$$

$$L(v,w) = \int_1^\eta v w' \rho\, d\rho , \qquad V^* = \{ v \in L^\infty(1,\eta) : \ |v| \leq 1 \} .$$

For bilinear functional $L(v,w)$ on $L^\infty(1,\eta) \times W^{1,1}(1,\eta)$ the classical equality

$$\inf_w \sup_v L(v,w) = \sup_v \inf_w L(v,w)$$

is fulfilled [20], therefore, we receive

$$t_+ = \sup\{ J(v) : \ v \in V^* \} ,$$

where the dual functional

$$J(v) = -v(1) - \sup \left\{ \int_1^\eta (\rho v)' w\, d\rho : \ w \in V,\ w(1) = 1 \right\}$$

is proper (i.e. $J(v) \not\equiv -\infty$) on $V^*$ if the dual function has the form $v(\rho) = C/\rho$ for $\rho \in [1,\eta]$ with an indefinite constant $C$. As a result, we receive

$$t_+ = \sup\{ -C : \ |C| \leq 1 \} = 1 .$$

It is easily seen that for the *admissible* minimizing sequence of the simplest form $w_m(\rho) = (\eta-\rho)^m/(\eta-1)^m$ $(m \in \mathbb{N})$ the infimum of the functional in the problem (1.6) is reached on the discontinuous function because $w_m(\rho) \to 1 - H(\rho - 1) \notin V$ for almost every $\rho \in (1,\eta)$, where $H$ is the Heaviside function of bounded variation [1,23,29].

## 1.3 Fully relaxed and dual elastostatic LAP

As it follows from the previous example, the solution of the elastostatic LAP (1.3) belongs to the space of discontinuous vector-functions of bounded variation $BV(\Omega, \mathbb{R}^3) \supset W^{1,1}(\Omega, \mathbb{R}^3)$. This space consists of functions $\mathbf{u} \in L^1(\Omega, \mathbb{R}^3)$ with bounded full variation

$$\int_\Omega |\partial \mathbf{u}| = \sup \left\{ \int_\Omega (\nabla \cdot \mathbf{\Sigma}) \cdot \mathbf{u}\, d\Omega : \ \mathbf{\Sigma} \in C_0^1(\Omega, \mathbb{M}^3), \ |\mathbf{\Sigma}| \leq 1 \right\} ,$$

where $\partial \mathbf{u}$ is the gradient of the vector-function $\mathbf{u}$ in the sense of the distribution theory [1,23,29] such that $\int_\Omega |\partial \mathbf{u}| = \int_\Omega |\nabla \mathbf{u}(\mathbf{x})|\, d\Omega$ for every $\mathbf{u} \in W^{1,1}(\Omega, \mathbb{M}^3)$. Here and in what follows the point denotes the scalar product in $\mathbb{R}^3$.

The space $BV(\Omega, \mathbb{R}^3)$ with the norm $\|\mathbf{u}\|_{BV} = \|\mathbf{u}\|_1 + \int_\Omega |\partial \mathbf{u}|$ is the Banach weakly closed space [1,23,29], therefore, the mathematically correct and fully relaxed elastostatic LAP has the following form:

$$t_+ = \lambda \inf \left\{ \int_\Omega |\partial \mathbf{u}| : \ \mathbf{u} \in BV(\Omega, \mathbb{R}^3), \ \mathbf{u}_{|_{\Gamma^1}} = 0, \ A(\mathbf{u}) = 1 \right\} . \tag{1.7}$$

Unfortunately, at present this problem has no clear physical interpretation.

On the other hand, we can construct the dual elastostatic LAP using methods from the duality theory [20]. Thus, we introduce the set of admissible non-symmetric Piola–Kirchhoff stress tensors equilibrating the given external forces in the weak sense [22]:

$$G = \left\{ \mathbf{\Sigma} \in L^\infty(\Omega, \mathbb{M}^3) : \ \int_\Omega (\nabla \cdot \mathbf{\Sigma} + \mathbf{f}) \cdot \mathbf{u}\, d\Omega = 0, \right.$$

$$\left. \int_{\Gamma^2} (\mathbf{n} \cdot \mathbf{\Sigma} - \mathbf{F}) \cdot \mathbf{u}\, d\gamma = 0, \ \forall \mathbf{u} \in V \right\} . \tag{1.8}$$

Here and in what follows $\mathbf{n}$ is the unit normal vector on the portion $\Gamma^2$ of the boundary.

The dual elastostatic LAP is formulated as the problem

$$\tau_+ = \inf \{ \|\mathbf{\Sigma}\|_\infty : \ \mathbf{\Sigma} \in G \} , \tag{1.9}$$

where $\|\mathbf{\Sigma}\|_\infty = \inf\{ \sup(|\mathbf{\Sigma}(\mathbf{x})| : \ \mathbf{x} \in \Omega \setminus \omega) : \ |\omega| = 0 \}$. For the initial and dual elastostatic LAPs the following main result has been proved recently in [14,15]:

**Theorem 1.2** *For solutions of the problems* (1.3) *and* (1.9) *the relation* $t_+ \tau_+ = \lambda$ *is fulfilled.*

*Proof.* It is easily verified that the dual elastostatic LAP (1.9) is equivalent to the problem

$$\tau_+^{-1} = \sup\{\,\nu > 0:\ \mathbf{\Sigma} \in G,\ \|\nu\mathbf{\Sigma}\|_\infty \le 1\,\}\ . \tag{1.10}$$

For every value $\nu > 0$ and tensor-field $\mathbf{\Sigma} \in G$ the following equality is true:

$$\nu = \nu + \nu \inf \left\{ \int\limits_{\Gamma^2} (\mathbf{n}\cdot\mathbf{\Sigma} - \mathbf{F})\cdot\mathbf{u}\, d\gamma - \int\limits_{\Omega} (\nabla\cdot\mathbf{\Sigma} + \mathbf{f})\cdot\mathbf{u}\, d\Omega:\ \mathbf{u} \in V \right\}\ .$$

After integration by parts and taking into account the boundary condition on $\Gamma^1$, we find

$$\nu = \inf \left\{ \int\limits_{\Omega} \nu\mathbf{\Sigma}\cdot\cdot\nabla\mathbf{u}\, d\Omega:\ \mathbf{u} \in V,\ A(\mathbf{u}) = 1 \right\}\ , \tag{1.11}$$

where $\mathbf{\Sigma}\cdot\cdot\mathbf{Q} = \Sigma_i^\alpha Q_i^\alpha$ is the double scalar product of second-order tensors [25].

We introduce on $L^\infty(\Omega, \mathbb{M}^3)\times W^{1,1}(\Omega, \mathbb{R}^3)$ the bilinear functional

$$L(\mathbf{\Sigma}, \mathbf{u}) = \int\limits_{\Omega} \mathbf{\Sigma}\cdot\cdot\nabla\mathbf{u}\, d\Omega\ ,$$

then from (1.11) it follows that the problem (1.10) has the form

$$\tau_+^{-1} = \sup\{\,\inf(\,L(\mathbf{\Sigma}, \mathbf{u}):\ \mathbf{u} \in V,\ A(\mathbf{u}) = 1\,):\ \mathbf{\Sigma} \in G,\ \|\mathbf{\Sigma}\|_\infty \le 1\,\}\ .$$

For the bilinear functional $L(\mathbf{\Sigma}, \mathbf{u})$ the classical equality

$$\sup_{\mathbf{\Sigma}} \inf_{\mathbf{u}} L(\mathbf{\Sigma}, \mathbf{u}) = \inf_{\mathbf{u}} \sup_{\mathbf{\Sigma}} L(\mathbf{\Sigma}, \mathbf{u})$$

is fulfilled [20], therefore,

$$\tau_+^{-1} = \inf\{\,\sup(L(\mathbf{\Sigma}, \mathbf{u}):\ \mathbf{\Sigma} \in G,\ \|\mathbf{\Sigma}\|_\infty \le 1):\ \mathbf{u} \in V,\ A(\mathbf{u}) = 1\,\}\ .$$

For every matrix $\mathbf{A}, \mathbf{B} \in \mathbb{M}^3$ the relation $|\mathbf{A}| = \sup\{\mathbf{A}\cdot\cdot\mathbf{B}:\ |\mathbf{B}| \le 1\}$ is true. As a result, for every $\mathbf{u} \in V$ we have the equality completing the proof:

$$\sup\{L(\mathbf{\Sigma}, \mathbf{u}):\ \mathbf{\Sigma} \in G,\ \|\mathbf{\Sigma}\|_\infty \le 1\} = \int\limits_{\Omega} |\nabla\mathbf{u}(\mathbf{x})|\, d\Omega \ \blacksquare$$

So, the estimation of the mechanical durability of a homogeneous non-linear elastic solid reduces to finding an admissible non-symmetric Piola–Kirchhoff stress tensor of minimal intensiveness equilibrating the given external forces. If $\tau_+ > \lambda$ then the elastostatic BVP has no solution.

# 2 Limit analysis method in electrostatics

We remind that the electric state of a medium in a given domain $\Omega \subset \mathbb{R}^3$ is characterized by the bulk and surface density of charges and by vectors of electric field intensity $\mathbf{E} = \{E_i\} \in \mathbb{R}^3$, electric induction $\mathbf{D} = \{D_i\} \in \mathbb{R}^3$ and electric current density $\mathbf{J} = \{J_i\} \in \mathbb{R}^3$. Vector $\mathbf{D}$ is introduced by the relation $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, where $\varepsilon_0 \approx 8.85 \cdot 10^{-12}$ is the electric permittivity of vacuum and $\mathbf{P} \in \mathbb{R}^3$ is the vector of polarization density [21,24,26,28]. For the electric field intensity the scalar electrostatic potential $u$ is introduced such that $\mathbf{E}(\mathbf{x}) = -\nabla u(\mathbf{x})$ for almost every $\mathbf{x} \in \Omega$.

The polarization and ionization properties of the dielectrical medium are discribed by two constitutive relations $\mathbf{D} = \hat{\mathbf{D}}(\mathbf{x}, \mathbf{E}) : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{J} = \hat{\mathbf{J}}(\mathbf{x}, \mathbf{E}) : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ [21,24,26,28]. In practice the relations $D_i = \varepsilon_{ij}(\mathbf{x}, \mathbf{E}) E_j$ and $J_i = \sigma_{ij}(\mathbf{x}, \mathbf{E}) E_j$ are used, where $\{\varepsilon_{ij}\}$ and $\{\sigma_{ij}\}$ are the symmetric second-order tensors of dielectric permittivity and conductivity, respectively. For an isotropic medium $\varepsilon_{ij} = \varepsilon \delta_{ij}$ and $\sigma_{ij} = \sigma \delta_{ij}$, where $\varepsilon = \varepsilon(\mathbf{x}, |\mathbf{E}|)$ and $\sigma = \sigma(\mathbf{x}, |\mathbf{E}|)$ are scalar functions and $\delta_{ij}$ is the Kronecker symbol. For a homogeneous medium $\{\varepsilon_{ij}\}, \{\sigma_{ij}\} = \text{const}\,(\mathbf{x})$.

In weak electric fields the conductivity current in a dielectrical medium is practically absent and the simplest linear constitutive relation $\mathbf{E} \mapsto \mathbf{D}$ is used [21,24,28]. As a result, for the solution of the appropriate *linear* BVPs various effective analytical and numerical methods have been worked out, for example, in [26].

The classical method of estimation of the dielectric breakdown is based on the *point criteria.* Namely, it is assumed that the dielectric breakdown is began if $\max\{|\mathbf{E}(\mathbf{x})| : \mathbf{x} \in \Omega\} \geq E_0$, where $\mathbf{E}$ is the solution of the linear electrostatic BVP and $E_0 > 0$ is the critical value, which is measured in physical experiments on thin films in a homogeneous electric field [26,28]. Unfortunately, for dielectrics with a complex shape in nonhomogeneous electric fields this method introduces a large error.

We consider the following boundary-value problem. Let a dielectric occupy a bounded domain $\Omega \in \mathbb{R}^3$ with the Lipschitz boundary. Let the following quasi-static electric charges act on the dielectric: a bulk charge with density $\rho$ in $\Omega$, a surface charge with density $g$ on a portion $\Gamma^2$ of the boundary, and a portion $\Gamma^1$ of the boundary is grounded, i.e. $u \equiv 0$ on $\Gamma^1$. Here $\Gamma^1 \cup \Gamma^2 = \partial\Omega$, $\Gamma^1 \cap \Gamma^2 = \emptyset$ and $|\Gamma^1| > 0$. Point charges are absent.

In electrostatics it is assumed that an external source of the electric field compensates the work of the electric current in the dielectric. In accordance with the classical Thomson principle the true electrostatic potential is the solution of the following variational problem:

$$u_* = \arg\inf \{ \Pi(u) - A(u) : \ u \in V \} \ , \tag{2.1}$$

$$\Pi(u) = \int_\Omega \Phi(\mathbf{x}, \nabla u(\mathbf{x}))\, d\Omega \,, \qquad A(u) = \int_\Omega \rho u \, d\Omega + \int_{\Gamma^2} g u \, d\gamma \ ,$$

where $V = \left\{ u : \overline{\Omega} \to \mathbb{R};\ u(\mathbf{x}) = 0,\ \mathbf{x} \in \Gamma^1 \right\}$ is the set of admissible electrostatic potentials, $\Phi$ is the specific and $\Pi(u)$ is the full potential energy of the electric field, $A(u)$ is the work of an external source on a transference of charges from infinity to $\Omega$.

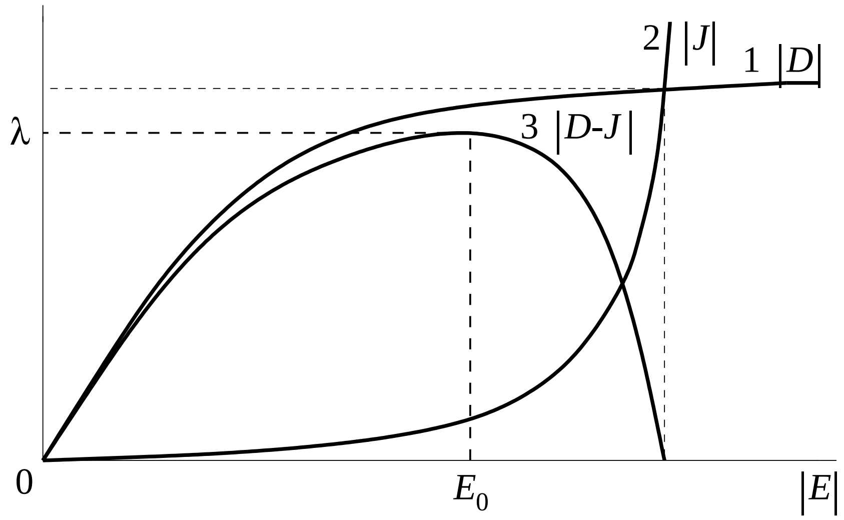


**Figure 1.** Experimental (lines 1,2) and effective (line 3) constitutive relations.

In compliance with the Thompson and Joule–Lenz laws the function $\Phi(\mathbf{x}, \mathbf{E})$ is calculated as

$$\Phi(\mathbf{x}, \mathbf{E}) = E_i \int_0^1 \left[ \hat{D}_i(\mathbf{x}, p\mathbf{E}) - \hat{J}_i(\mathbf{x}, p\mathbf{E}) \right] \, dp \, .$$

## 2.1 Initial electrostatic LAP

In powerful electric fields the essentially *nonlinear* phenomena of polarization saturation ($|\mathbf{P}| \leq P_* < +\infty$) and powerful growth of the electric current ($\partial|\mathbf{J}|/\partial|\mathbf{E}| \gg \varepsilon_0$) must be taken into account [19,21,28].

In Figure 1 experimental constitutive relations $|\mathbf{E}| \mapsto |\mathbf{D}|$ (line 1) and $|\mathbf{E}| \mapsto |\mathbf{J}|$ (line 2) for real isotropic dielectrical media are presented [19,21,28]. The appropriate function of *effective induction* is shown by the line 3. It is easily seen that for every dielectrical medium the scalar $\lambda > 0$ always exists such that for every $\mathbf{E} \in \mathbb{R}^3$ and almost every $\mathbf{x} \in \Omega$ the following estimation is true:

$$\Phi(\mathbf{x}, \mathbf{E}) \leq \lambda(\mathbf{x}) |\mathbf{E}| \, , \tag{2.2}$$

where

$$\lambda(\mathbf{x}) = \max \left\{ \, |\hat{\mathbf{D}}(\mathbf{x}, \mathbf{E}) - \hat{\mathbf{J}}(\mathbf{x}, \mathbf{E})| : \; \mathbf{E} \in \mathbb{R}^3 \, \right\} \, .$$

From the physical point of view $\lambda$ is *the electric saturation.* In what follows, we consider a homogeneous dielectrical medium for which $\lambda = \text{const}\,(\mathbf{x})$.

From the estimation (2.2) it follows [22,29] that the set of admissible electrostatic potentials is defined as the following subspace:

$$V = \left\{ u \in W^{1,1}(\Omega) : \; u(\mathbf{x}) = 0, \; \mathbf{x} \in \Gamma^1 \right\} \, . \tag{2.3}$$

From the mathematical point of view the variational problem (2.1) can have no solution because the functional $\Pi(u) - A(u)$ can be unbounded from below on the set $V$. In particular, after the point $E_0$ (see Figure 1) the full potential energy of the electric field $\Pi(u)$ has growth in $\|u\|_{1,1}$ *less than linear*. But the work of the electric field on the external charges $A(u)$ always has the *linear* growth in $\|u\|_{1,1}$. As a result, for an admissible minimizing sequence $\{u_m\} \subset V$ with $\|u_m\|_{1,1} \to \infty$ we have $\Pi(u) - A(u) \to -\infty$ as $m \to \infty$ (details see in [12]), i.e. the electrostatic variational problem (2.1) is non-correct. Namely, the limit electrostatic load exists, i.e. external charges $(\rho, g)$ with no solution of the problem (2.1). From the physical point of view this effect can be treated as the dielectric breakdown because it corresponds to a loss of the global electrostatic balance between an external source of charges and the dielectrical body. Here we have the full analogy with some problems of global stability and fracture in Mechanics of Solids [3,4,8,9,30].

For a definition of the limit electrostatic load we introduce the set of admissible external charges for which the functional in problem (2.1) is bounded from below on $V$ and, therefore, a solution of this problem exists:

$$B = \left\{ \, (\rho, g) \in L^\infty(\Omega) \times L^\infty(\Gamma^2) : \; \inf \left( \Pi(u) - A(u) : u \in V \, \right) > -\infty \, \right\} \, .$$

The set is non-empty because for small external charges the problem (2.1) is transformed into the classical variational problem of linear electrostatics, which always has a solution [19].

For arbitrary external charges $(\rho_0, g_0) \in B$ we examine the sequence of charges, which are proportional to the real parameter $t \geq 0$.

**Definition 2.1** *A number $t_* \geq 0$ is defined as the limit parameter of electrostatic loading and $(t_*\rho_0, t_*g_0)$ is the limit electrostatic load, if $(t\rho_0, tg_0) \in B$ for $0 \leq t \leq t_*$ and $(t\rho_0, tg_0) \notin B$ for $t > t_*$.*

As a result, the analysis of the global electrostatic balance in a dielectic comes to the investigation of the set of positive parameters $t$ for which the one-parametric functional

$$I_t(u) = \Pi(u) - t\, A(u)$$

is bounded from below on the set of admissible electrostatic potentials $V$. The following basic result was proven in [12]:

**Theorem 2.1** *The finite limit parameter of electrostatic loading exists such that $t_* \leq t_+$, where $t_+$ is the solution of the Limit Analysis Problem*

$$t_+ = \lambda \inf \left\{ \int_\Omega |\nabla u(\mathbf{x})| \, d\Omega : \; u \in V, \; A(u) = 1 \right\} . \tag{2.4}$$

From the Definition 2.1 it follows that for $t_+ < 1$ the electrostatic variational problem (2.1) has no solution. This phenomenon corresponds to the dielectric breakdown. Therefore, the limit analysis problem (2.4) is the main problem for an estimation of the electric breakdown for dielectrics of complex shape in powerful nonhomogeneous electric fields that closes one of the modern fundamental problems [21,24,28].

## 2.2 Example of the limit electrostatic load

For illustration of the existence of the limit electrostatic load we examine the simplest problem about the spheric condenser which is formed by two concentric conducting spheres. The inside sphere of the radius $a$ is charged by $Q > 0$ and the outside sphere of the radius $b$ is grounded. The space between spheres is filled by a homogeneous and isotropic dielectric which is described by the following effective constitutive relation:

$$\mathbf{D} = \lambda\varphi(|\mathbf{E}|)\frac{\mathbf{E}}{|\mathbf{E}|} , \qquad |\varphi| \le 1$$

with the model material function $\varphi(p) = p$ for $0 \le p < 1$ and $\varphi(p) = 1/p$ for $p \ge 1$. In this case the electrostatic BVP has the form

$$\frac{d}{dr}\left[ r^2\varphi\left(|u'(r)|\right)\right] = 0 , \qquad r \in (a, b) ,$$

$$\lambda\varphi\left(|u'(a)|\right) = g , \qquad u(b) = 0$$

with the classical solution

$$u(r) = ga^2\left(\frac{1}{r} - \frac{1}{b}\right) ,$$

which exists only for $g \le \lambda$, where $g = Q/(4\pi a^2) > 0$.

The appropriate electrostatic variational problem has the form

$$u_* = \arg\inf\left\{ I(u) : \ u \in V \right\} ,$$

$$I(u) = \lambda\int_a^b \Phi\left(|u'(r)|\right) r^2 dr - ga^2u(a) , \qquad \Phi(p) = \int_0^p \varphi(q)\, dq ,$$

$$V = \left\{u \in W^{1,1}(a, b) : \ u(b) = 0\right\} .$$

It is easily verified that for every external charge $g > \lambda$ the electrostatic variational problem has no solution because for the simplest *admissible* minimizing sequence $u_m(r) = m(b - r)^m/(b - a)^m$ $(m \in \mathbb{N})$ the energy functional $I(u_m) \to -\infty$ as $m \to \infty$.

The appropriate reduced electrostatic LAP has the form

$$t_+ = \inf\left\{ I(u) : \ u \in V, \ u(a) = 1 \right\} , \tag{2.5}$$

$$I(u) = a^{-2}\int_a^b |u'(r)|r^2\, dr .$$

Using the equality $|a| = \sup\{ab : \ |b| \le 1\}$ we can rewrite the problem (2.5) in the following way:

$$t_+ = \inf\left\{ \sup\left( L(v, u) : \ v \in V^* \right) : \ u \in V, \ u(a) = 1 \right\} ,$$

$$L(v,u) = a^{-2} \int_a^b v u' r^2 \, dr \ , \qquad V^* = \{ v \in L^\infty(a,b) : \ |v| \le 1 \} \ .$$

For bilinear functional $L(v,u)$ on $L^\infty(a,b) \times W^{1,1}(a,b)$ the classical equality

$$\inf_u \sup_v L(v,u) = \sup_v \inf_u L(v,u)$$

is fulfilled [20], therefore, we receive

$$t_+ = \sup\{ J(v) : \ v \in V^* \} \ ,$$

where the dual functional

$$J(v) = -a^{-2} \left[ v(a) + \sup \left\{ \int_a^b (r^2 v)' u \, dr : \ u \in V, \ u(a) = 1 \right\} \right]$$

is proper (i.e. $J(v) \not\equiv -\infty$) on $V^*$ if the dual function has the form $v(\rho) = C/r^2$ for $r \in [a,b]$ with an indefinite constant $C$. As a result, we receive

$$t_+ = \sup\{ -C : \ |C| \le 1 \} = 1 \ .$$

It is easily seen that for the *admissible* minimizing sequence of the simplest form $u_m(r) = (b-r)^m/(b-a)^m$ ($m \in \mathbb{N}$) the infimum of the functional $I(u)$ in the problem (2.5) is reached on the discontinuous function because $u_m(r) \to 1 - H(r-a) \notin V$ for almost every $r \in (a,b)$, where $H$ is the Heaviside function of bounded variation [1,23,29].

## 2.3 Fully relaxed and dual electrostatic LAP

As it follows from the previous example, the solution of the electrostatic LAP (2.4) belongs to the space of discontinuous scalar-functions of bounded variation $BV(\Omega) \supset W^{1,1}(\Omega)$. This space consists of functions $u \in L^1(\Omega)$ with bounded full variation

$$\int_\Omega |\partial u| = \sup \left\{ \int_\Omega (\nabla \cdot \mathbf{D}) \, u \, d\Omega : \ \mathbf{D} \in C_0^1(\Omega, \mathbb{R}^3), \ |\mathbf{D}| \le 1 \right\} \ ,$$

where $\partial u$ is the gradient of the function $u$ in the sense of the distribution theory [1,23,29] such that $\int_\Omega |\partial u| = \int_\Omega |\nabla u(x)| \, d\Omega$ for every $u \in W^{1,1}(\Omega)$.

The space $BV(\Omega)$ with the norm $\|u\|_{BV} = \|u\|_1 + \int_\Omega |\partial u|$ is the Banach weakly closed space [1,23,29], therefore, the mathematically correct and fully relaxed electrostatic LAP has the following form:

$$t_+ = \lambda \inf \left\{ \int_\Omega |\partial u| : \ u \in BV(\Omega), \ u_{|_{\Gamma^1}} = 0, \ A(u) = 1 \right\} \ . \tag{2.6}$$

Unfortunately, at present this problem has no clear physical interpretation.

We construct here the dual electrostatic LAP. Thus, we introduce the set of admissible effective electric inductions equilibrating the electric field of external charges in the weak sense [22]:

$$G = \left\{ \mathbf{D} \in L^\infty(\Omega, \mathbb{R}^3) : \int_\Omega (\nabla \cdot \mathbf{D} + \rho) u \, d\Omega = 0, \right.$$

$$\left. \int_{\Gamma^2} (\mathbf{n} \cdot \mathbf{D} - g) u \, d\gamma = 0, \ \forall u \in V \right\} . \tag{2.7}$$

The dual electrostatic LAP is formulated as the problem

$$\tau_+ = \inf \left\{ \|\mathbf{D}\|_\infty : \ \mathbf{D} \in G \right\} . \tag{2.8}$$

For the initial and dual electrostatic LAPs the following main result has been proved recently in [13,16]:

**Theorem 2.2** *For solutions of the problems* (2.4) *and* (2.8) *the relation* $t_+ \tau_+ = \lambda$ *is fulfilled.*

So, the estimation of the electric breakdown for a homogeneous dielectric is equivalent to finding an effective electric induction of minimal intensiveness equilibrating the electric field of external charges. If $\tau_+ > \lambda$ then the electrostatic BVP has no solution.

# 3 FEA and numerical solution of 2-D initial and dual LAPs

Here we consider the general LAP for quasi-static problems of Continuum Mechanics. Namely, let a body occupy a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ with the fixed topology and the Lipschitz boundary $\partial\Omega = \Gamma^1 \cup \Gamma^2$, where $\Gamma^1 \cap \Gamma^2 = \emptyset$ and $|\Gamma^1| > 0$.

Let a physical state of a homogeneous continuum medium in $\Omega$ be described by the field $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^m$ $(m \geq 1)$ with the zero trace on $\Gamma^1$.

Let a work of external quasi-static influences $(\mathbf{f}, \mathbf{g}) \in L^\infty(\Omega, \mathbb{R}^m) \times L^\infty(\Gamma^2, \mathbb{R}^m)$ on the field $\mathbf{u}$ be described by the linear functional

$$L(\mathbf{u}) = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, d\Omega + \int_{\Gamma^2} \mathbf{g}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, d\gamma \, . \tag{3.1}$$

Here and what follows we use the reduced external influences $\mathbf{f} = \mathbf{f}_0 / \lambda$ and $\mathbf{g} = \mathbf{g}_0 / \lambda$, where $(\mathbf{f}_0, \mathbf{g}_0)$ are the physical external influences and $\lambda$ is the physical saturation.

**The initial LAP** is formulated in the following form:

$$t_+ = \inf \left\{ \int\limits_\Omega |\nabla \mathbf{u}(\mathbf{x})| \, d\Omega : \ \mathbf{u} \in V, \ L(\mathbf{u}) = 1 \right\} , \tag{3.2}$$

$$V = \left\{ \mathbf{u} \in W^{1,1}(\Omega, \mathbb{R}^m) : \ \mathbf{u}_{|_{\Gamma^1}} = \mathbf{0} \right\} .$$

**The dual LAP** has the form

$$\tau_+ = \inf \left\{ \|\mathbf{S}\|_\infty : \ \mathbf{S} \in G \right\} , \tag{3.3}$$

$$G = \left\{ \mathbf{S} \in L^\infty(\Omega, \mathbb{M}^{n\times m}) : \ \int\limits_\Omega (\nabla{\cdot}\mathbf{S} + \mathbf{f}){\cdot}\mathbf{u} \, d\Omega = 0, \ \int\limits_{\Gamma^2} (\mathbf{n}{\cdot}\mathbf{S} - \mathbf{g}){\cdot}\mathbf{u} \, d\gamma = 0, \ \forall \mathbf{u} \in V \right\} ,$$

where symbol $\mathbb{M}^{n\times m}$ denotes the space of real $n{\times}m$ matrices.

For initial and dual LAPs the following main result is true:

**Theorem 3.1** *For solutions of the problems* (3.1) *and* (3.2) *the main relation* $t_+\tau_+ = 1$ *is fulfilled.*

For simplifying we examine here FEAs of 2-D initial and dual LAPs, i.e. the case $n = 2$. Namely, let the given bounded domain $\Omega \subset \mathbb{R}^2$ be approximated by the set of domains $\Omega_h = \cup T_h^i$ such that $|\Omega \backslash \Omega_h| \to 0$ and $|\partial\Omega \backslash \partial\Omega_h| \to 0$ as $h \to +0$, where $h$ is the characteristic step of approximation [17]. Every 2-D FEA is described by the set of triangles $\{T_h^i\}_{i=1}^{N_t}$, the set of nodes $\{\mathbf{x}^\alpha\}_{\alpha=1}^{N_x}$ and the set of ribs $R_h = \left\{[\mathbf{x}^\alpha, \mathbf{x}^\beta]\right\}$ including inside and boundary ones.

For the given external influences the standard piecewise linear continuous FEAs ($\mathbf{f}_h$,$\mathbf{g}_h$) are used [17]. In this case the linear functional $L(\mathbf{u})$ from (3.1) is approximated by

$$L_h(\mathbf{u}) = \int\limits_{\Omega_h} \mathbf{f}_h(\mathbf{x}){\cdot}\mathbf{u}(\mathbf{x}) \, d\Omega + \int\limits_{\Gamma_h^2} \mathbf{g}_h(\mathbf{x}){\cdot}\mathbf{u}(\mathbf{x}) \, d\gamma \, . \tag{3.4}$$

## 3.1 FEA and partial relaxation of the initial LAP

We remind that the initial LAP (3.2) needs a relaxation. The main idea of a relaxation consists of the following [29]. Let $B$ be the Banach weakly unclosed space with norm $\|\cdot\|$ and $I(u)$ be the coercive on $B$ functional, i.e. $I(u) \to +\infty$ as $\|u\| \to \infty$. The solution of the standard minimization problem $inf\{I(u) : \ u \in B\}$ can be absent. But a minimizing sequence $\{u_k\} \subset B$ can exist such that $u_k \to u_0$ almost everywhere and $I(u_k) \to I_0 < +\infty$ as $k \to \infty$, where the limit element $u_0 \notin B$. In this case we can construct a continuation $\overline{I}$ of the functional $I$ into a wider class of functions $\overline{B} \supset B$ such that $\overline{I}(u_0) = I_0$ and $\overline{I}(u) = I(u)$ for every $u \in B$.

For variational problems with the multiple integral functional of linear growth the appropriate space $\overline{B}$ is the $BV$ (or $BD$) space of functions with bounded variation (or deformation) [1, 23, 29]. Unfortunately, the fully relaxed initial LAP

$$t_+ = \inf \left\{ \int_\Omega |\partial \mathbf{u}| : \ \mathbf{u} \in BV(\Omega, \mathbb{R}^m), \ \mathbf{u}_{|_{\Gamma^1}} = \mathbf{0}, \ L(\mathbf{u}) = 1 \right\} ,$$

$$\int_\Omega |\partial \mathbf{u}| = \sup \left\{ \int_\Omega (\nabla \cdot \mathbf{S}) \cdot \mathbf{u} \, d\Omega : \ \mathbf{S} \in C_0^1(\Omega, \mathbb{M}^{n\times m}), \ |\mathbf{S}| \leq 1 \right\} .$$

has no clear physical interpretation.

Therefore, for the initial LAP (3.2) we propose *the partial relaxation* which is based on the special discontinuous FEA with functions having jumps of the first type.

**Example 3.1.** For illustration of the main idea of the partial relaxation we examine the simplest initial LAP (1.6) from the Section 1.2:

$$t_+ = \inf \left\{ \int_1^\eta |w'| \rho \, d\rho : \ w \in W^{1,1}(1,\eta), \ w(1) = 1, \ w(\eta) = 0 \right\} .$$

We remind that the infimum $t_+ = 1$ is reached on the discontinuous function $w_* = 1 - H(\rho - 1) \notin V$, where $H \in BV(1, \eta)$ is the Heaviside function.

The concept of a generalized solution and its mathematical and physical justification for variational problems with multiple integral functionals of the linear growth are given in many publications (see, for example, References in [1, 23, 29]).

The generalized solution of the considered problem coincides with the solution of the following extended or partially relaxed variational problem:

$$\hat{t}_+ = \inf \left\{ \int_1^\eta |w'| \rho \, d\rho + |w(1) - 1| + \eta |w(\eta)| : \ w \in W^{1,1}(1,\eta) \right\} ,$$

where the penalty items are motivated by the classical De Giorgi problem [20]. It is easily verified that the infimum $\hat{t}_+ = 1$ is reached on the discontinuous function $w_*$ too.

### 3.1.1 Case of the scalar-field $(m = 1)$

For the scalar-field the following piecewise linear discontinuous approximation is used

$$u_h(\mathbf{x}) = U^i_\alpha \Lambda^i_\alpha(\mathbf{x}) \quad (\alpha = \overline{1, N_x}; \quad i = \overline{1, N_t}) ,$$

where $U^i_\alpha$ is the scalar-field in the node $\mathbf{x}^\alpha$ of the triangle $T^i_h$ (see Figure 2), $\Lambda^i_\alpha : \Omega_h \to \mathbb{R}$ is the piecewise linear discontinuous basic function such that $supp\,(\Lambda^i_\alpha) = T^i_h$ and $\Lambda^i_\alpha(x^\beta) = \delta_{\alpha\beta}$ $(\alpha, \beta = \overline{1, N_x}; \ i = \overline{1, N_t})$.

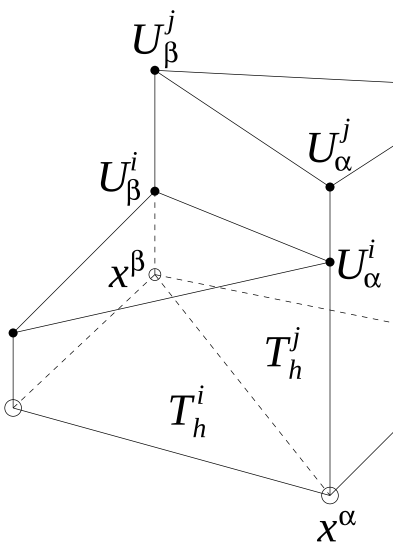


**Figure 2.** The piecewise linear discontinuous approximation of the scalar-field in 2-D initial LAP.

In this case the set $V \subset W^{1,1}(\Omega)$ is approximated by the set $V_h \subset BV(\Omega_h)$ which is isomorphous to the space $\mathbb{R}^{3N_t}$. As a result, we have the special discontinuous FEA with functions having jumps of the first type in all inside nodes and nodes on the portion $\Gamma^1_h$ of the boundary $\partial\Omega_h$.

The partially relaxated 2-D initial LAP (3.2) for the scalar-field has the following form:

$$t_h = \min \left\{ \int\limits_{\Omega_h} |\nabla u(\mathbf{x})| \, d\Omega + P_h(u) : \; u \in V_h, \; L_h(u) = 1 \right\} , \tag{3.5}$$

$$P_h(u) = \sum_{R_h} \int\limits_{\mathbf{x}^\alpha}^{\mathbf{x}^\beta} \left| u^i(\mathbf{x}) - u^j(\mathbf{x}) \right| \, d\gamma \; ,$$

where indeces $i$ and $j$ correspond to the scalar-field on the triangles $T^i_h$ and $T^j_h$ having the rib $[\mathbf{x}^\alpha, \mathbf{x}^\beta] \in R_h$ as common (see Figure 2). For ribs on $\Gamma^1_h$ we suppose $u^j(\mathbf{x}) \equiv 0$. Note that for every *admissible continuous* scalar-field $u \in C(\Omega_h)$ the penalty item $P_h(u) = 0$.

### 3.1.2 Case of the plane vector-field ($m = 2$)

We remind that the domain topology is fixed, therefore, admissible jumps of the vector-field in the initial LAP (3.2) can have the only sliding type [4,5,7,8]. Therefore, for the plane vector-field the following piecewise linear discontinuous approximation is used

$$\mathbf{u}_h(\mathbf{x}) = \mathbf{U}_{\alpha\beta}\Phi_{\alpha\beta}(\mathbf{x}) \quad (\alpha, \beta = \overline{1, N_x}; \quad [\mathbf{x}^\alpha, \mathbf{x}^\beta] \in R_h) \, ,$$

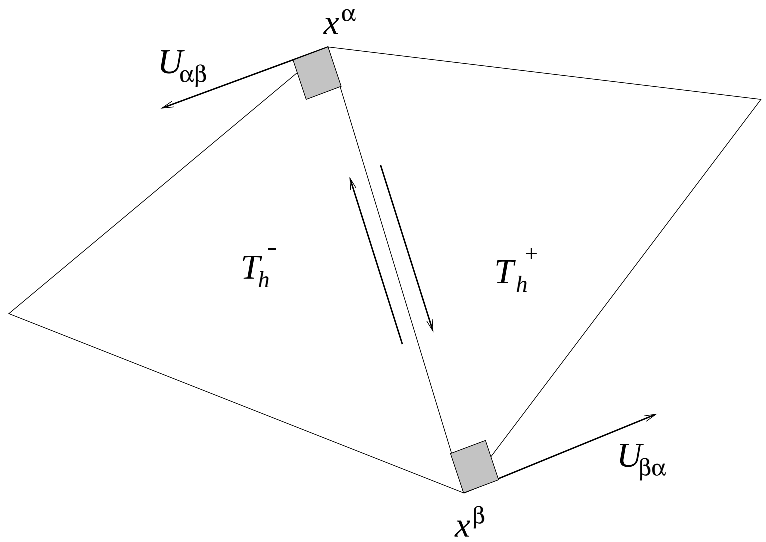


**Figure 3.** The piecewise linear discontinuous approximation of the vector-field in 2-D initial LAP.

where $\mathbf{U}_{\alpha\beta}$ is the component of the vector-field in the node $\mathbf{x}^\alpha$ which is perpendicular to the rib $[\mathbf{x}^\alpha, \mathbf{x}^\beta]$ (see Figure 3), $\Phi_{\alpha\beta} : \Omega_h \to \mathbb{R}$ is the piecewise linear discontinuous basic function such that $\Phi_{\alpha\beta}(x^\gamma) = \delta_{\alpha\gamma}$ $(\alpha, \beta, \gamma = \overline{1, N_x})$ and $\Phi_{\alpha\beta} \neq \Phi_{\beta\alpha}$. The *supp* $(\Phi_{\alpha\beta})$ = *supp* $(\Phi_{\beta\alpha})$ consists of two triangles $\left\{T_h^-, T_h^+\right\}$ having the common rib $[\mathbf{x}^\alpha, \mathbf{x}^\beta]$. If a rib $[\mathbf{x}^\alpha, \mathbf{x}^\beta] \in \Gamma_h^1$ then the *supp* $(\Phi_{\alpha\beta})$ consists of the only triangle.

As a result, the set $V \subset W^{1,1}(\Omega, \mathbb{R}^2)$ is approximated by the set $V_h \subset BV(\Omega_h, \mathbb{R}^2)$ which is isomorphous to the space $\mathbb{R}^{2N_r}$, where $N_r$ is the number of ribs in the set $R_h$.

The described FEA possesses the following properties. The component of the approximated plane vector-field, which is perpendicular to an appropriate rib, is continuous; but the tangent projection on this rib has a finite jump. As a result, we have the special discontinuous FEA with functions having jumps of the sliding type along all inside ribs and ribs on the portion $\Gamma_h^1$ of the boundary $\partial\Omega_h$.

The partially relaxed 2-D initial LAP (3.2) for the plane vector-field has the form

$$t_h = \min \left\{ \int\limits_{\Omega_h} |\nabla \mathbf{u}(\mathbf{x})| \, d\Omega + P_h(\mathbf{u}) : \; \mathbf{u} \in V_h, \; L_h(\mathbf{u}) = 1 \right\} , \tag{3.6}$$

$$P_h(\mathbf{u}) = \sum_{R_h} \int\limits_{\mathbf{x}^\alpha}^{\mathbf{x}^\beta} \left| \mathbf{u}_\tau^+(\mathbf{x}) - \mathbf{u}_\tau^-(\mathbf{x}) \right| \, d\gamma \; ,$$

where indeces "+" and "−" correspond to the vector-field on the triangles $T_h^+$ and $T_h^-$ having the common rib $[\mathbf{x}^\alpha, \mathbf{x}^\beta]$ (see Figure 3), index $\tau$ corresponds to the tangent projection of the vector-field on this rib, and for ribs on $\Gamma_h^1$ we suppose $\mathbf{u}_\tau^-(\mathbf{x}) \equiv \mathbf{0}$. As in the previous case, the penalty item equals to zero for every admissible continuous vector-field.

## 3.2 FEA of the dual LAP

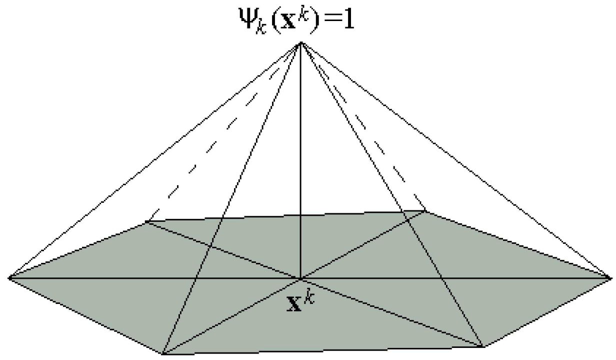


**Figure 4.** The piecewise linear continuous basic function for the approximation of the addmissible field in 2-D dual LAP.

For the admissible field in the dual LAP (3.3) the standard piecewise linear continuous approximation is used [17]:

$$\mathbf{S}_h(x) = \mathbf{S}^k \Psi_k(\mathbf{x}) \quad (k = \overline{1, N_x}) \ ,$$

where $\mathbf{S}^k \in \mathbb{M}^{2\times m}$ $(m \geq 1)$ is the dual field in the node $\mathbf{x}^k$, $\Psi_k : \Omega_h \to \mathbb{R}$ is the standard piecewise linear continuous basic function such that $\Psi_k(\mathbf{x}^r) = \delta_{kr}$ $(k, r = \overline{1, N_x})$. The *supp* $(\Psi_k)$ consists of triangles having the node $\mathbf{x}^k$ as common (see Figure 4).

The set of admissible dual fields is approximated by the set

$$G_h = \big\{ \mathbf{S}^k \in \mathbb{M}^{2\times m} : \ k = \overline{1, N_x}; \ \mathbf{S}^k \cdot \nabla \Phi_k(\mathbf{x}) + \mathbf{f}_h(\mathbf{x}) = \mathbf{0}, \ \mathbf{x} \in \Omega_h;$$

$$\mathbf{n}_h(\mathbf{x}^k) \cdot \mathbf{S}^k = \mathbf{g}_h(\mathbf{x}^k), \ \mathbf{x}^k \in \Gamma^2_h \big\} \ , \tag{3.7}$$

which is the *convex set with linear boundaries* in the space of global variables $\mathbb{R}^{2mN_x}$. We remind that here for external influences and normal boundary vector the standard peiswise liniear continuous FEAs $(\mathbf{f}_h, \mathbf{g}_h)$ and $\mathbf{n}_h$, respectively, are used

As a result, the dual LAP (3.3) is approximated by the problem of mathematical programming with linear limitations as equalities

$$\tau_h = \min \big\{ \max \big( |\mathbf{S}^k| : \ k = \overline{1, N_x} \big) : \ \mathbf{S}^k \in G_h \ \big\} \ . \tag{3.8}$$

If the number of nodes on $\Gamma^2_h$ equals to $N_2$, then the number of free variables in the problem (3.8) equals to $2mN_x - (mN_t + N_2)$. It is easily seen that the minimal number of variables equals to 2 which is reached for the domain coinciding with the triangle because in this case $N_x = 3$, $N_t = 1$ and $N_2 = 3$.

## 3.3 Numerical solution of the initial and dual LAPs

From the computational point of view the function in the partially relaxed initial LAP (3.5) or (3.6) is singular because it has no classical derivative. Therefore, the simplest regularization of the modulus $|z| \approx (z^2+\varepsilon^2)^{1/2}$ with *the regularization parameter* $\varepsilon \ll 1$ is proposed.

By the necessary condition of stationarity the problems (3.5) and (3.6) transform into non-linear systems of algebraic equations which can be *ill-conditioned* [5, 6]. The main cause of this phenomenon consists of the following: the global stiffness matrix has lines with significantly different factors if minimizing field has a large gradient or jumps of the first type. For the regularization parameter $\varepsilon \ll 1$ this situation is more difficult. Therefore, for the numerical solution of the partially relaxed 2-D initial LAP, the decomposition method of adaptive block relaxation can be used [5, 6]. It is practically disregards the condition number of the global stiffness matrix, but it has a slow convergence.

On the other hand, the object function in the discrete dual LAP (3.8) is the linear combination of convex hypercones in the space $\mathbb{R}^{2mN_x}$. Therefore, due to linearity of limitations in the set of admissible fields $G_h$ from (3.7) this finite dimensional problem is effectively solved by the standard *method of gradient projection* which is easily adapted for parallel computations.

**Example 3.2.** In the numerical experiment the following axial symmetric initial LAP (3.2) for the scalar-field was considered:

$$t_+ = \inf\{I(u):\ u \in V,\ u(\rho,1) \equiv 1\}\ , \tag{3.9}$$

$$I(u) = 2\int_0^1\int_0^1 \left[\eta^2\left(\frac{\partial u}{\partial \rho}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]^{1/2} \rho d\rho dz\ ,$$

$$V = \left\{u \in W^{1,1}(\Omega):\ u(\rho,0)\equiv 0,\ \frac{\partial u}{\partial z}(\rho,0)\equiv 0,\ \frac{\partial u}{\partial \rho}(0,z)\equiv 0\right\}\ ,$$

where $\eta > 0$ is the geometric parameter and $\rho \in [0,1]$, $\varphi \in [0,2\pi)$, $z \in [0,1]$ are the reduced cylindrical co-ordinates. Here $\Omega = (0,1)\times(0,1)$ and $\Gamma^2 = \{\rho \in [0,1], z = 1\}$.

This problem is motivated by the electrostatic BVP for an isotropic and homogeneous dielectric having the form of a finite round rod with the radius of section $a$ and length $2l$. In this case $\eta = l/a$. The small round blocks of dielectric are covered by conductors having charges $\pm Q$. It is easily verified that the breakdown charge $Q_* \le \pi a l \lambda\, t_+$ [28], where $\lambda$ is the electric saturation from (2.2) (see Figure 1).

According to the convexity of domain and axial symmetry of the problem (3.9), the minimizer can have a jump of the first type on the only line $z = 1$. Therefore, the following partial relaxation of the problem (3.9) was used

$$t_+ = \inf\{I(u) + P(u):\ u \in V\}\ , \qquad P(u) = 2\int_0^1 |u(\rho,1) - 1|\, \rho d\rho\ .$$

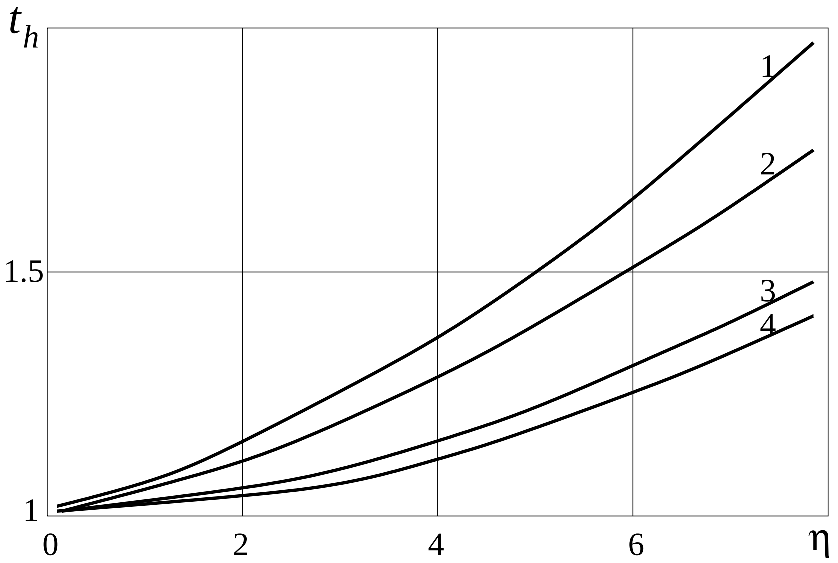


**Figure 5.** The experimental relations between the geometrical parameter $\eta$ and the limit parameter of loading $t_h$ for different FEAs.

In the computer experiment the uniform $N \times N$ triangulation of the domain $\Omega$ and the regularization parameter $\varepsilon = 10^{-2}$ were used.

In Figure 5 the experimental relations $\eta \mapsto t_h$ are shown. Lines 1, 2 and 3 correspond to the continuous FEA with $N = 10$, $N = 20$ and $N = 40$, respectively. Line 4 corresponds to the discontinuous FEA with $N = 10$. It is easily seen that continuous solutions converge to the discontinuous solution with increase of domain's triangulation. But the appropriate finite dimension problems were ill-conditioned for large $N$ and the iteration procedure had a slow convergence. The decrease of the regularization parameter $\varepsilon$ until $10^{-3}$ practically does not improve either continuous or discontinuous solutions.

From the Theorem 3.1 we have $t_+ = 1/\tau_+$ , where the parameter $\tau_+$ is the solution of the appropriate dual LAP (3.3) on the following set:

$$G = \left\{ \mathbf{S} \in L^\infty(\Omega, \mathbb{R}^2) : \int\limits_\Omega (\nabla \cdot \mathbf{S}) u \, d\Omega = 0, \ \forall u \in V, \ \ S_z \equiv 1 \ \text{ on } \ \Gamma^2 \right\} .$$

For the uniform $N \times N$ triangulation of the domain $\Omega$, the problem of mathematical programming (3.8) was solved for $2(N+1)^2$ variables satisfying $(2N^2 + N + 1)$ linear limitations as equalities, i.e. for $(3N+1)$ free variables. The experimental results for the parameter $\eta = 0.1$ are shown in Table 1. It is easily seen that $\tau_h \searrow \tau_+ = 1$ as $h \to +0$ that fully coincides with the theoretical result for the plane condenser [28].

**Table 1**

| $N$ | 5 | 10 | 20 | 40 | 80 |
|---|---|---|---|---|---|
| $\tau_h$ | 2.17 | 1.83 | 1.36 | 1.06 | 1.01 |

**Example 3.3.** The numerical analysis of the initial and dual LAPs for the vector-field was based on the solution of the following BVP: a finite round rod is axial symmetrically stretched in the test machine of the rigid type by a given axial force $F_z$. In this case, the map is described by the following relation in the reference reduced cylindrical co-ordinates

$$\mathbf{X}(\rho,\varphi,z) = \mathbf{x}\left(a\rho + a\,r(\rho,z),\varphi, lz + lw(\rho,z)\right)\ ,$$

$$\mathbf{C}(r,w) = \nabla\mathbf{u} = \begin{pmatrix} \partial r/\partial\rho & 0 & \eta\,\partial w/\partial\rho \\ 0 & r/\rho & 0 \\ \eta^{-1}\,\partial r/\partial z & 0 & \partial w/\partial z \end{pmatrix}\ ,$$

where $\rho\in[0,1]$, $\varphi\in[0,2\pi)$, $z\in[0,1]$; $a$ and $l$ are the radius of section and semilength of the rod, respectively; $\eta = l/a$ is the geometrical parameter. Here $\Omega = (0,1)\times(0,1)$ and $\Gamma^2 = \{\rho\in[0,1], z=1\}$.

For the limit stretching force $F_z^*$ the estimation from above $F_z^* \le \pi a^2\lambda\, t_+$ is true, where $\lambda$ is the elastic saturation and the parameter $t_+$ is the solution of the following initial LAP:

$$t_+ = \inf\left\{ 2\int_0^1\int_0^1 |\mathbf{C}(r,w)|\,\rho d\rho dz:\ (r,w)\in V,\ w(\rho,1)\equiv 1 \right\}\ , \tag{3.10}$$

$$V = \left\{ (r,w)\in\left(W^{1,1}(\Omega)\right)^2:\ r(0,z)\equiv 0,\ \frac{\partial r}{\partial z}(\rho,0)\equiv 0,\ w(\rho,0)\equiv 0,\ r(\rho,1)\equiv 0 \right\}\ .$$

According to the convexity of domain, axial symmetry of the problem (3.10) and continuity of the axial component of dispalcement, the minimizer can have a jump of the sliding type along the only line $z=1$. This jump is defined by a finite jump of the function $r(\rho,1)$. Therefore, in the set $V$ the condition $r(\rho,1)\equiv 0$ is ignored, but in the minimized functional the following penal item is used:

$$P(r) = 2\eta^{-1}\int_0^1 |r(\rho,1)|\,\rho\,d\rho\ .$$

In the computer experiment the uniform $N\times N$ triangulation of the domain $\Omega$ and the regularization parameter $\varepsilon = 10^{-2}$ were used. It was numerically shown that continuous solutions converge to the discontinuous solution with increase of domain's triangulation. But the appropriate finite dimension problems were ill-conditioned for large $N$ and the iteration procedure had a slow convergence. The decrease of the regularization parameter $\varepsilon$ until $10^{-3}$ practically does not improve either continuous or discontinuous solutions.

From the Theorem 3.1 it follows that $t_+ = 1/\tau_+$, where the parameter $\tau_+$ is the solution of the appropriate dual LAP (3.3) on the following set of admissible tensor-fields:

$$G = \left\{ \mathbf{S}\in L^\infty(\Omega,\mathbb{M}^{2\times 2}):\ \int_\Omega (\nabla\cdot\mathbf{S})\cdot\mathbf{u}\,d\Omega = 0,\ \forall\mathbf{u}\in V,\ S_{zz}\equiv 1\ \text{ on }\ \Gamma^2 \right\}\ .$$

For the uniform $N \times N$ triangulation of the domain $\Omega$, the problem of mathematical programming (3.8) was solved for $4(N+1)^2$ variables satisfying $(2N^2+N+1)$ linear limitations as equalities, i.e. for $(2N^2+7N+3)$ free variables. In the Table 2 the experimental results for the geometric parameter $\eta = 2$ are shown. It is easily seen that $\tau_h \searrow 1.5$ with increase of $N$ what is coincides with numerical results presented in [14].

**Table 2**

| $N$ | 5 | 10 | 20 | 40 | 80 |
|---|---|---|---|---|---|
| $\tau_h$ | 3.62 | 2.72 | 1.76 | 1.62 | 1.51 |

# Conclusions

The limit analysis method in elastostatics and electrostatics was considered. The appropriate initial and dual Limit Analysis Problems (LAPs) were formulated. Within the framework of this method the global stability of non-linear elastic and dielectric solids is estimated that is very important for practical problems in Mechanical and Electrical Engineering.

It was demonstrated that the initial LAPs need a relaxation, but the appropriate fully relaxed problems have no clear physical interpretation. On the other hand, the dual LAPs have a clear physical interpretation.

The general initial and dual LAPs for quasi-static problems of Continuum Mechanics were formulated. For the 2-D initial LAP the partial relaxation based on the discontinuous FEA was proposed, and for the dual LAP the standard piecewise linear continuous FEA was used. The appropriate numerical solutions were examined.

The presented analytical and numerical results are new. They are of practical interest, but more theoretical and experimental research is desirable. For example, the Limit Analysis Method can be used for the design of electric isolators or rubber-like constructions as well as in the appropriate shape optimization problems.

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