Universität Stuttgart

Piezoelectricity in multi-layer actuators
Modelling and analysis in two and three dimensions

Winfried Geis, Anna-Margarete Sändig, Gennady Mishuris

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ω	volume charge density	page 1
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Ω_C	domain of the ceramic matrix	page 1
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$\underline{\underline{\boldsymbol{C}}}_M$	material matrix of linear elastic constants for the metal alloy	(1.3)
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$\underline{\underline{\boldsymbol{\varepsilon}}}$	permittivity tensor	(1.2)
$\underline{\boldsymbol{u}}_C$, $\underline{\boldsymbol{u}}_M$	mechanical displacement field vector in the ceramic and the metal	page 1
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$\underline{\underline{\boldsymbol{\sigma}}}$	stress tensor	page 13
$\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$	fourth order tensor of elasticity	page 13
$\underline{\underline{\boldsymbol{\gamma}}}$	linearised strain tensor	page 13
$\underline{\boldsymbol{u}}$	mechanical displacement field vector	page 13
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$\underline{\boldsymbol{E}}_C$	electric field, restricted to Ω_C	(3.4), (3.5), page 14
$\underline{\underline{\boldsymbol{A}}}_C$	piezoelectric material matrix	page 14
$\underline{\underline{\boldsymbol{B}}}$	differential matrix operator	(3.12)
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$\mathcal{D}$	matrix of differential operators	(3.16)
$\underline{\boldsymbol{E}}_M$	electric vector field in the metal domain Ω_M	page 15
Φ_M	electric potential field in the metal domain Ω_M	page 15
λ, μ	Lamé parameters of the metal	(3.8), (3.9)
$\underline{\underline{\boldsymbol{B}}}_p$	reduced differential operator (plane strain assumption)	(3.22)
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$\underline{\boldsymbol{\sigma}}_{C_p}$	stress tensor in the plane strain case, restricted to Ω_C	(3.23)
$\underline{\boldsymbol{D}}_{C_p}$	dielectric displacement vector in the plane strain case, restricted to Ω_C	(3.23)
$\underline{\underline{\boldsymbol{A}}}_{C_p}$	piezoelectric material matrix in the plane strain case	(3.24)
$\nabla_{1,3}$	2D gradient in the anti-plane case	(3.32)
$\gamma_{M_{ap}}$	2D linearised strain tensor in the anti-plane case, restricted to Ω_M	(3.32)
$\mathcal{D}_{ap}$	2D differential operator in the anti-plane strain case	(3.32)
$\underline{\boldsymbol{u}}_{C_{ap}}$	mechanical displacement field vector, restricted to Ω_C in the anti-plane strain case	(3.32)
$\boldsymbol{\sigma}_{C_{ap}}$	stress tensor in the anti-plane strain case, restricted to Ω_C	(3.33)
$\underline{\underline{\boldsymbol{A}}}_{C_{ap}}$	piezoelectric material matrix in the anti-plane strain case	(3.33)
$\underline{\boldsymbol{u}}_{M_{ap}}$	mechanical displacement field vector, restricted to Ω_M in the anti-plane strain case	(3.38)
$\boldsymbol{\sigma}_{M_{ap}}$	stress tensor in the anti-plane strain case, restricted to Ω_M	(3.38)
$\underline{\underline{\boldsymbol{A}}}_{M_{ap}}$	material matrix for the metal in the anti-plane strain case	(3.38)
$\underline{\boldsymbol{\sigma}}_{M_p}$	stress tensor in the plane strain case, restricted to Ω_M	(3.35)
$\underline{\underline{\boldsymbol{C}}}_{M_p}$	material matrix of elastic constants in the plane strain case, restricted to Ω_M	(3.35)
$\underline{\boldsymbol{\gamma}}_{M_p}$	linearised strain tensor in the plane strain case, restricted to Ω_M	(3.35)
$\underline{\underline{\boldsymbol{B}}}_{M_p}$	differential operator in the plane strain case, restricted to Ω_M	(3.35)
$\underline{\boldsymbol{u}}_{M_p}$	mechanical displacement field vector in the plane strain case, restricted to Ω_M	(3.35)
$\underline{\boldsymbol{E}}$	electric vector field	(6.1)
Φ	electric potential field	(6.1)
ε_0	electric field constant	(6.2)
ρ	charge density	(6.2)
$u_{C_i},\ i \in \{1,2,3\}$	component of the mechanical displacement field in the ceramic	(3.14)
$\underline{\boldsymbol{e}}_{x_3}$	unit vector in x_3 direction	(6.6)
∂_n	normal derivative	page 36
Γ_u, Γ_l	upper respectively lower side boundary of an electrode	page 36
Γ_-, Γ_+	left/right boundary of the actuator	page 22
Ⓓ , Ⓝ , Ⓡ	Dirichlet, Neumann and Robin condition	page 4
$\underline{\underline{\boldsymbol{N}}}$, $\underline{\boldsymbol{n}}$	co-normal matrix, normal unit vector	(4.1)

s	absolute value of the face charge density	page 37
Ω_{M-}	Domain, occupied by the negatively charged electrodes	section 6.3
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Γ	set of interfaces	(4.11)
Γ_t	union of the electrode tips $\subset \Gamma$	(4.11)
$\underline{\underline{\boldsymbol{A}}}_M$	material matrix for the metal	(4.46)
$\Gamma^D_{e,-}$	boundary of Ω_C with $\Phi_c = -\Phi_a$	(4.48)
Γ^-, Γ^+	set of interfaces, connected with the negative respectively positive charged side	(4.48)-(4.49)
$\hat{\phi}$	smooth continuation of the non-homogeneous electrical Dirichlet data	page 28
$\underline{\boldsymbol{W}}$	continuation of the Dirichlet data for the extended displacement field vector	page 28
$\tilde{\mathcal{V}}$	Sobolev space	(5.1)
H^1	Sobolev space	definition 5.1
$\tilde{\mathsf{H}}^{-\frac{1}{2}}(\Gamma^*)$	dual space of $\mathsf{H}^{\frac{1}{2}}(\Gamma^*)$	(5.7)
$a, a_M, a_C, a_{C,I}, a_{C,II}$	bilinear and non-linear forms	(5.8)-(5.11)
$\langle\cdot,\cdot\rangle$	dual pairing	page 31
$\mathfrak{A}$	operator, generated by the non-linear form a	page 32

1 Introduction

Actuators are used in injectors for Common-Rail diesel engines. They should show large generative forces while their response time must be very short. Furthermore, the actuator should be controlled in a relatively simple way by an electric field.
These properties can be achieved by using piezoelectric actuators, either as monolithic ceramic pieces or as multi-layer actuators (MLAs) [3] with individual layers and interdigitated electrodes, wired parallel. MLAs require lower voltage for the same amount of actuation in comparison with monolithic pieces. The fabrication of co-fired MLAs is very similar to that of ceramic capacitors and therefore the chances for large scale production at low cost are better than for stacked-disk-actuators.
After a short motivation which illustrates the fabrication processes, the structure of an MLA and its' application in Common-Rail diesel engines, a linear mathematical model of piezoelectricity [7], [10] will be explained. The main point is, that the MLA will be considered as a composite, where the metallic electrodes Ω_M are thin inclusions in the ceramic matrix Ω_C. The resulting quasi-static multi-field equation system of partial differential equations is formulated in the different domains Ω_M and Ω_C, together with appropriate boundary conditions. Because of the quasi-stationary regime, all forces and even the mechanical properties can depend on the time, but in a very slow way. As a special case, this model includes also the stationary problem. Since the only difference between the stationary and quasi-stationary formulation is the time dependent right side, we formulate everything for the stationary case (and thus leave out the time dependent right side and the time-cylinder in the formulation), having in mind, that the modelling is also valid for the quasi-stationary case. The elastic displacement fields in Ω_C and Ω_M are connected by transmission conditions. The external volume loading is assumed to be zero, since the volume force densities $\underline{\boldsymbol{f}}_C$ and $\underline{\boldsymbol{f}}_M$ can be set zero, as the weight has only an evanescent effect and in consequence of the macroscopic model for the ceramic - which is an insulating material - the volume charge density ω also vanishes. The coupled linear partial differential equations read:

$$-\mathcal{D}^\top \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\boldsymbol{u}}_C - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C = \underline{\boldsymbol{0}} \qquad \text{in } \Omega_C, \tag{1.1}$$

$$-\mathrm{div}\,(-\underline{\underline{\boldsymbol{e}}}\,\mathcal{D}\underline{\boldsymbol{u}}_C + \underline{\underline{\boldsymbol{\varepsilon}}}\,\nabla \Phi_C) = 0 \qquad \text{in } \Omega_C, \tag{1.2}$$

$$-\mathcal{D}^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\boldsymbol{u}}_M = \underline{\boldsymbol{0}} \qquad \text{in } \Omega_M. \tag{1.3}$$

$\underline{\underline{\boldsymbol{C}}}_C, \underline{\underline{\boldsymbol{e}}}, \underline{\underline{\boldsymbol{\varepsilon}}}$ and $\underline{\underline{\boldsymbol{C}}}_M$ are tensors, describing the material properties of the piezo-ceramic and the electrodes respectively; the electric potential field in the ceramic is denoted by Φ_C. $\underline{\boldsymbol{u}}_C$ and $\underline{\boldsymbol{u}}_M$ is the displacement field in the ceramic and the electrodes, respectively. Note, that the electric potential field is constant inside the electrodes.
From the technical point of view, the mathematical formulation of this multi-structure-multi-field problem with boundary and transmission conditions becomes rather complex. In particular we have modelled the interaction of the stack actuator with exterior fields as nonlinear boundary conditions. Though we have spent much efforts, to write it in a clear way.
The coupled piezoelectric system of partial differential equations in the electrodes and in the ceramic will be formulated both, in the $3D$ and $2D$ case. The $2D$-model is based on a *plane-strain*-assumption, what exploits the longitudinal piezoelectric effect of the MLA. We derive boundary conditions for the ceramic and metallic part of the actuator; both sub-domains are connected by transmission conditions.
After the classical formulation (chapter 3 and 4), we investigate a weak formulated boundary-

transmission problem in chapter 5. The weak formulation of the presented model can be written:
Find $\underline{\boldsymbol{V}} \in \mathcal{V} \subset \left[\mathsf{H}^1(\Omega)\right]^{n+1}$ $(n \in \{2,3\})$, such that for all $\underline{\boldsymbol{S}} \in \mathcal{V}$ holds:

$$\langle \mathfrak{A}\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} \rangle = \langle \underline{\boldsymbol{F}}, \underline{\boldsymbol{S}} \rangle \qquad \text{in } \Omega.$$

$\underline{\boldsymbol{F}}$ is an external load of a transformed problem with homogeneous Dirichlet data. The fact, that $\mathfrak{A}$ is a strongly monotonous operator, assures existence and uniqueness of the solution $\underline{\boldsymbol{V}}$.

2 The co-fired multi-layer piezoelectric actuator

The different types of piezoelectric multi-layer actuators have the common operating principle: a stack of alternating poled piezoelectric ceramic plates and thin metal films grows linearly (small growth) with the pairwise applied voltage.

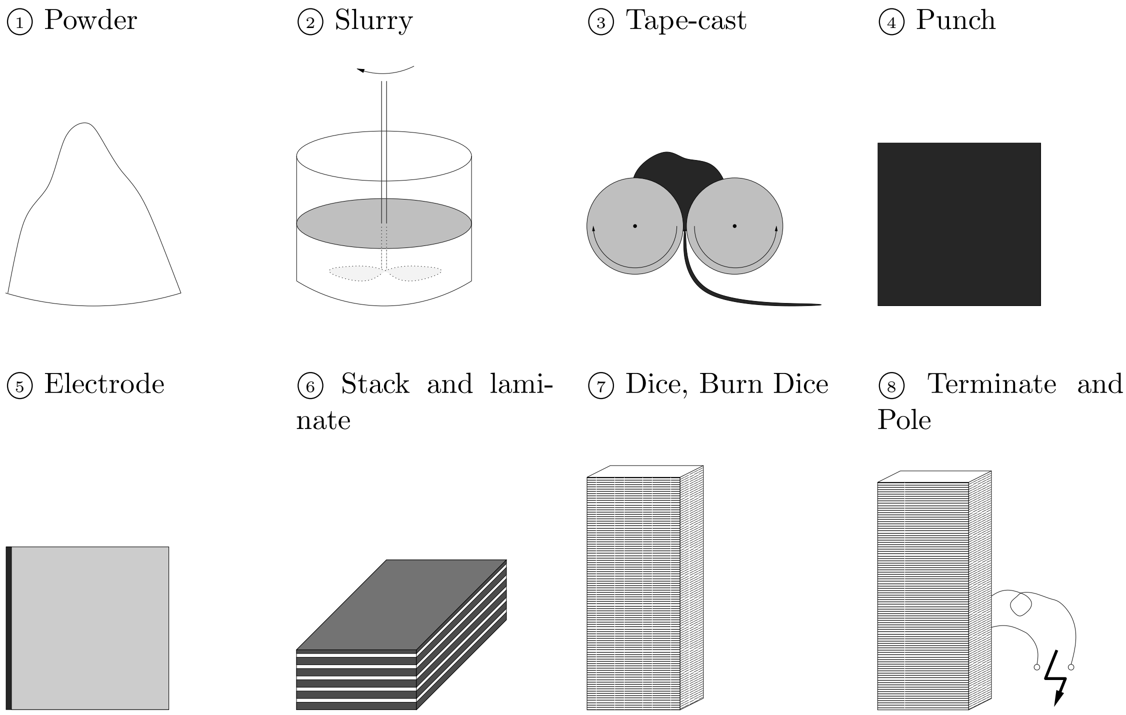


Figure 1: Processing of a co-fired multi-layer actuator [8].

Each of the different designs (stacked disc actuator, co-fired multi-layer actuator with capacitor structure, plate through electrode, slit electrode, interdigitated electrode) have their own application, showing advantages and disadvantages [8]. In this paper, we focus on the so called co-fired multi-layer actuator with the capacitor structure due to its' importance especially for the car industry.

The flow chart in figure 1 shows the fabrication process of the co-fired multi-layer actuator. Directly after sintering the ceramic-electrode dice (therefore we say co-firing: the electrode is also sintered), a high voltage is applied to the interdigitated electrodes, to pole the ceramic in d_{33}-direction (which here is the direction of the x_3-axis). Poling the stack in such a way, dead zones (see figure 2) occur in front of the electrode tips because the electrical field there is not homogeneous like in the interior of an ideal plate condenser. For the considered ceramic material and polarisation, the material is transversely isotropic in x_1- and x_2-direction, with the poling direction x_3 (crystal class $C_{6\nu} = 6mm$ see also [7] for further explanation). The ceramic body of the actuator exhibits two different kinds of piezoelectric effects: the *transverse* piezoelectric effect (i.e. growth in the direction orthogonal to the applied electric field) and the larger *longitudinal* piezoelectric effect, triggered in cause of the arrangement of the

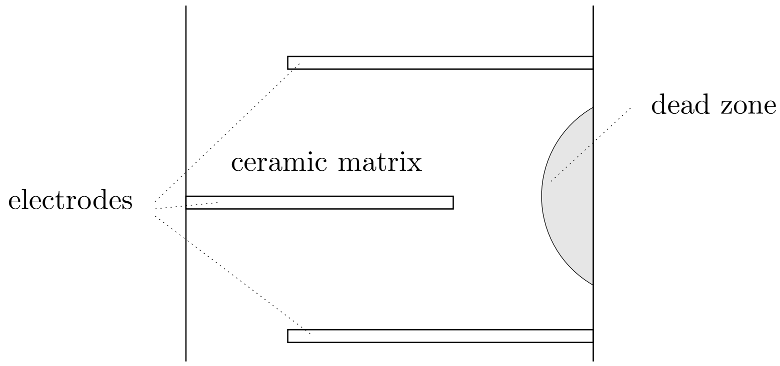


Figure 2: Cross section of a MLA dice. The dead zone, caused by the manufacturing process (co-firing and poling) lies opposite the electrode tip. Here, dead zone means, that this region does not show any piezoelectric behaviour since the poling in the actuator is applied with the help of the thin metal inclusions, which are no plate through electrodes. Thus, the ceramic material between the electrode tip and the opposite side stays without poling.

electrodes, which is implemented in MLAs used e.g. in the injection nozzles of diesel engines (see also figure 3).

To reveal the importance of piezoelectric multi-layer devices for real life applications, e.g. diesel engines, we shortly quote some of the advantages of this technology in comparison with the “classical” electro-mechanic actuator device (EMA):

- Since the piezoelectric property is an inner effect of the solid body (due to the orientation of crystallites), the stack growth takes place nearly instantaneously. Therefore, the MLA has a faster response time than the EMA.
  In the case of diesel injectors, this enables intermediate injections in order to prevent the “nailing sound” of the engine.
- MLAs can handle higher pressures (1600 bar-2000 bar) than magnetic valves ($\approx$ 1200 bar). The higher pressure leads to reduced emissions and a reduction of the fuel consumption.

3 The coupled piezoelectric system of partial differential equations

In this section, we derive in detail the multi-field equation system of partial differential equations (1.1)-(1.3) in the ceramic and metal sub-domains of the actuator. The sub-domains are denoted by Ω_M (set of electrodes) and by Ω_C (ceramic matrix) and it holds: $\overline{\Omega} = \overline{\Omega}_M \cup \overline{\Omega}_C$.

3.1 Hooke’s law

3.1.1 Hooke’s law - general case

Here and in the following, we will only consider linear elastic deformations:

$$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\underline{\underline{\boldsymbol{C}}}}}\,\underline{\underline{\boldsymbol{\gamma}}}\,. \tag{3.1}$$

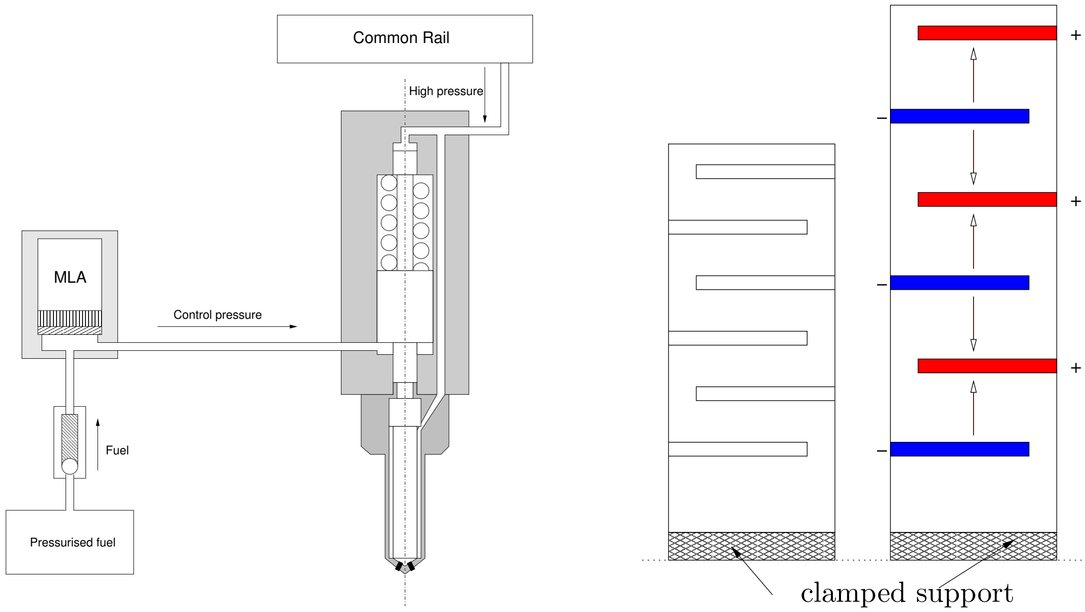


Figure 3: 1) Scheme of an injection nozzle of a common rail engine [4]. The MLA is arranged inside an earthed metal vessel. 2) Mechanical dilatation of a clamped MLA (cross section). In real life applications, the number of the layers is e.g $n = 80$ with an applied voltage of $\Phi_a = \pm 150V$.

We denote by $\underline{\underline{\boldsymbol{\sigma}}}$ the stress tensor; $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$ the fourth-order tensor of elasticity and $\underline{\underline{\boldsymbol{\gamma}}}$ the linearised strain tensor. The stress tensor is written as a symmetric 3×3-Matrix (no rotation in account of the shear stresses)

$$\underline{\underline{\boldsymbol{\sigma}}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}.$$

Since we are only considering small stresses and strains, we use the linearised symmetric strain tensor:

$$\begin{aligned} \underline{\underline{\boldsymbol{\gamma}}} &= \frac{1}{2}\left((\nabla \underline{\boldsymbol{u}})^\top + \nabla \underline{\boldsymbol{u}}\right) \\ &= \begin{pmatrix} \partial_1 u_1 & \frac{1}{2}(\partial_2 u_1 + \partial_1 u_2) & \frac{1}{2}(\partial_3 u_1 + \partial_1 u_3) \\ \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1) & \partial_2 u_2 & \frac{1}{2}(\partial_3 u_2 + \partial_2 u_3) \\ \frac{1}{2}(\partial_1 u_3 + \partial_3 u_1) & \frac{1}{2}(\partial_2 u_3 + \partial_3 u_2) & \partial_3 u_3 \end{pmatrix}, \end{aligned}$$

where

$$\underline{\boldsymbol{u}} := \left(u_1(\underline{\boldsymbol{x}}), u_2(\underline{\boldsymbol{x}}), u_3(\underline{\boldsymbol{x}})\right)^\top$$

is the displacement vector. To write the strain and stress tensors as vectors of 6 independent components, we use an index mapping [10]:

$$\begin{matrix} 11 \mapsto 1, & 22 \mapsto 2, & 33 \mapsto 3, \\ 23/32 \mapsto 4, & 13/31 \mapsto 5, & 12/21 \mapsto 6. \end{matrix} \tag{3.2}$$

We get

$$\underline{\boldsymbol{\sigma}} = \begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6\end{pmatrix}^\top,$$
$$\underline{\boldsymbol{\gamma}} = \begin{pmatrix}\gamma_1 & \gamma_2 & \gamma_3 & 2\gamma_4 & 2\gamma_5 & 2\gamma_6\end{pmatrix}^\top,$$
$$\underline{\underline{\boldsymbol{C}}} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{16} \\ \dots & \dots & \dots & \dots \\ c_{61} & c_{62} & \dots & c_{66} \end{pmatrix}$$

and Hooke's law reads

$$\underline{\boldsymbol{\sigma}} = \underline{\underline{\boldsymbol{C}}}\,\underline{\boldsymbol{\gamma}}.$$

Note, that in general, the components C_{ij} of the elasticity tensor $\underline{\underline{\boldsymbol{C}}}$ are not constant; they can depend on $\underline{\boldsymbol{x}}$ and further interior variables. For piezoelectric materials, an extended Hooke's law will be formulated now. We introduce a generalised displacement vector, defined on the whole actuator-occupied domain Ω:

$$\underline{\boldsymbol{U}} = \left(\underline{\boldsymbol{u}}^\top, \Phi\right)^\top,$$
$$r_{|\Omega_i}\underline{\boldsymbol{U}} = \underline{\boldsymbol{U}}_i = \left(\underline{\boldsymbol{u}}_i^\top, \Phi_i\right)^\top \quad i \in \{C, M\}. \tag{3.3}$$

Here, $r_|$ is the notation for the restriction operator, where the index C denotes the restriction of $\underline{\boldsymbol{U}}$ to the ceramic domain and M the restriction to the metal-occupied domain. $\underline{\boldsymbol{U}}$ is an extension of the mechanical displacement field $\underline{\boldsymbol{u}}$ by the electric potential field Φ (see section 6.1).

3.1.2 Hooke's law for the piezo-ceramic

The constitutive relations for piezoelectric materials are [7]:

$$(\sigma_C)_{ij} = \sum_{k,l=1}^{3} (C_C)_{ijkl}\, \gamma_{C_{kl}} - \sum_{k=1}^{3} e_{kij}\,(E_C)_k\,, \tag{3.4}$$
$$(D_C)_i = \sum_{j=1}^{3} \varepsilon_{ij}\,(E_C)_j + \sum_{j,k=1}^{3} e_{ijk}\,(\gamma_C)_{jk}\,, \tag{3.5}$$

where $\underline{\underline{\boldsymbol{\sigma}}}_C$ is the stress tensor, $\underline{\underline{\boldsymbol{\gamma}}}_C$ the strain tensor, $\underline{\boldsymbol{D}}_C$ the electric flux density (or electric displacement), $\underline{\boldsymbol{E}}_C$ the electric field, $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}_C$ the elasticity tensor in the ceramic, $\underline{\underline{\boldsymbol{\varepsilon}}}$ the permittivity tensor and $\underline{\underline{\underline{\boldsymbol{e}}}}$ the piezoelectric tensor. We underline, that the entries of $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}_C, \underline{\underline{\underline{\boldsymbol{e}}}}$ and $\underline{\underline{\boldsymbol{\varepsilon}}}$ are not constant in general, e.g. due to dead zones in the ceramic (see figure (2)). The relations (3.4) and (3.5) can be shortly written in a matrix-vector-form with the help of the piezoelectric material matrix $\underline{\underline{\boldsymbol{A}}}_C$. To handle with the piezoelectric tensor $\underline{\underline{\underline{\boldsymbol{e}}}}$ in a more comfortable way, we use the Voigt-index mapping (3.2) to reduce the number of indices ($e_{ikm} \mapsto e_{ip}$):

$$\underline{\underline{\boldsymbol{e}}} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix},$$

assuming the above mentioned (page 11) crystal symmetries. Thus we have the extended Hooke's law

$$\begin{pmatrix} \underline{\underline{\boldsymbol{\sigma}}}_C \\ \underline{\boldsymbol{D}}_C \end{pmatrix} = \underline{\underline{\boldsymbol{A}}}_C \begin{pmatrix} \underline{\underline{\boldsymbol{\gamma}}}_C \\ \underline{\boldsymbol{E}}_C \end{pmatrix}, \tag{3.6}$$

where

$$\underline{\underline{\boldsymbol{A}}}_C = \begin{pmatrix} \underline{\underline{\boldsymbol{C}}}_C & -\underline{\underline{\boldsymbol{e}}}^\top \\ \underline{\underline{\boldsymbol{e}}} & \underline{\underline{\boldsymbol{\varepsilon}}} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 & 0 & 0 & -e_{33} \\ 0 & 0 & 0 & c_{44} & 0 & 0 & 0 & -e_{15} & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 & -e_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{c_{11}-c_{12}}{2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{15} & 0 & \varepsilon_{11} & 0 & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 & 0 & \varepsilon_{11} & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix}.$$

For $BaTiO_3$, the values for the matrix entries of $\underline{\underline{\boldsymbol{A}}}_C$ are:

$$\begin{aligned} &c_{11}=150\text{GPa}, && c_{12}=66\text{GPa}, && c_{13}=66\text{GPa}, && c_{33}=146\text{GPa}, \\ &c_{44}=44\text{GPa}, && e_{31}=-4,35\frac{\text{C}}{\text{m}^2}, && e_{33}=17,5\frac{\text{C}}{\text{m}^2}, && e_{15}=11,4\frac{\text{C}}{\text{m}^2}, \\ &\varepsilon_{11}=1115\varepsilon_0, && \varepsilon_{33}=1260\varepsilon_0, && \varepsilon_0=8,85\cdot 10^{-12}\frac{\text{C}}{\text{Nm}^2}. \end{aligned}$$

Here, ε_0 denotes the permittivity of vacuum.

3.1.3 Hooke's law for linear elasticity (metal)

As explained in the appendix, section 6.2 (see figure 5), there is no electric vector field in a well conducting material (therefore the electric potential is constant) and we have a reduced problem (1.3) (compared with (3.14)) of pure elasticity, with $\underline{\boldsymbol{E}}_M = \underline{\boldsymbol{0}}$ and $\Phi_M = \text{const}$, which is physically motivated.
Thus the constitutive relation for the isotropic metal alloy reads:

$$(\sigma_M)_{ij} = \sum_{k,l=1}^{3} (C_M)_{ijkl}\, \gamma_{M_{kl}}, \tag{3.7}$$

where $\underline{\underline{\boldsymbol{\sigma}}}_M$ is the stress tensor, $\underline{\underline{\boldsymbol{\gamma}}}_M$ the strain tensor and $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}_M$ the elasticity tensor. For a non-damaged isotropic silver-palladium alloy (Ag70:Pd30), we have:

$$\underline{\underline{\boldsymbol{C}}}_M = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix},$$

with

$$\lambda \approx 100,0\,\text{GPa}, \tag{3.8}$$
$$\mu \approx 31,7\,\text{GPa}. \tag{3.9}$$

3.2 The 3D force balance equations

3.2.1 Force balance equations in Ω_C

In the quasi-stationary case, the stress tensor $\underline{\underline{\boldsymbol{\sigma}}}_C$ and the electric flux density $\underline{\boldsymbol{D}}_C$ satisfy the corresponding balance equations [7]:

$$\partial_i \left(\sigma_C\right)_{ij} = 0 \qquad \text{in } \Omega_C,\ j = 1, 2, 3, \tag{3.10}$$

$$-\partial_i \left(D_C\right)_i = 0 \qquad \text{in } \Omega_C. \tag{3.11}$$

Note, that we use here the Einstein summation convention.
As mentioned in the introduction, the volume force density $\underline{\boldsymbol{f}}_C$ vanishes, since the influence of the weight is negligible; the charge density ω is zero due to the macroscopic modelling, which describes the insulating nature of the MLA.
Using the matrix differential operator:

$$\underline{\underline{\boldsymbol{B}}} = \begin{pmatrix} \partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 \\ 0 & 0 & \partial_3 & 0 \\ 0 & \partial_3 & \partial_2 & 0 \\ \partial_3 & 0 & \partial_1 & 0 \\ \partial_2 & \partial_1 & 0 & 0 \\ 0 & 0 & 0 & -\partial_1 \\ 0 & 0 & 0 & -\partial_2 \\ 0 & 0 & 0 & -\partial_3 \end{pmatrix} \tag{3.12}$$

and the equation (3.6), equations (3.10) and (3.11) can be rewritten as:

$$-\underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{U}}_C = \underline{\boldsymbol{0}} \qquad \text{in } \Omega_C. \tag{3.13}$$

In detail, the equation system (3.13) with space dependent material parameters reads:

$$-\begin{pmatrix} \partial_1(c_{11}\partial_1)+\partial_3(c_{44}\partial_3) + \partial_2 \frac{c_{11}-c_{12}}{2}\partial_2 & \partial_1(c_{12}\partial_2)+\partial_2\left(\frac{c_{11}-c_{12}}{2}\partial_1\right) & \partial_1(c_{13}\partial_3)+\partial_3(c_{44}\partial_1) & \partial_1(e_{31}\partial_3)+\partial_3(e_{15}\partial_1) \\ \partial_2(c_{12}\partial_1)+\partial_1\left(\frac{c_{11}-c_{12}}{2}\partial_2\right) & \partial_2(c_{11}\partial_2)+\partial_3(c_{44}\partial_3) + \partial_1 \frac{c_{11}-c_{12}}{2}\partial_1 & \partial_2(c_{13}\partial_3)+\partial_3(c_{44}\partial_2) & \partial_2(e_{31}\partial_3)+\partial_3(e_{15}\partial_2) \\ \partial_3(c_{13}\partial_1)+\partial_1(c_{44}\partial_3) & \partial_3(c_{13}\partial_2)+\partial_2(c_{44}\partial_3) & \partial_3(c_{33}\partial_3)+\partial_2(c_{44}\partial_2) + \partial_1(c_{44}\partial_1) & \partial_3(e_{33}\partial_3)+\partial_2(e_{15}\partial_2) + \partial_1(e_{15}\partial_1) \\ -\partial_1(e_{15}\partial_3)-\partial_3(e_{31}\partial_1) & -\partial_2(e_{15}\partial_3)-\partial_3(e_{31}\partial_2) & -\partial_1(e_{15}\partial_1)-\partial_2(e_{15}\partial_2) - \partial_3(e_{33}\partial_3) & \partial_1(\varepsilon_{11}\partial_1)+\partial_2(\varepsilon_{11}\partial_2) + \partial_3(\varepsilon_{33}\partial_3) \end{pmatrix} \begin{pmatrix} u_{C_1} \\ u_{C_2} \\ u_{C_3} \\ \Phi_C \end{pmatrix} = \underline{\boldsymbol{0}}. \tag{3.14}$$

If $\underline{\underline{\boldsymbol{e}}}, \underline{\underline{\boldsymbol{C}}}_C$ and $\underline{\underline{\boldsymbol{\varepsilon}}}$ are constant matrices, then

$$-\begin{pmatrix} c_{11}\partial_1^2+c_{44}\partial_3^2 + \frac{c_{11}-c_{12}}{2}\partial_2^2 & c_{12}\partial_1\partial_2+\frac{c_{11}-c_{12}}{2}\partial_1\partial_2 & c_{13}\partial_1\partial_3+c_{44}\partial_1\partial_3 & e_{15}\partial_1\partial_3+e_{31}\partial_1\partial_3 \\ c_{12}\partial_1\partial_2+\frac{c_{11}-c_{12}}{2}\partial_1\partial_2 & c_{11}\partial_2^2+c_{44}\partial_3^2 + \frac{c_{11}-c_{12}}{2}\partial_1^2 & c_{13}\partial_2\partial_3+c_{44}\partial_2\partial_3 & e_{15}\partial_2\partial_3+e_{31}\partial_2\partial_3 \\ c_{13}\partial_1\partial_3+c_{44}\partial_1\partial_3 & c_{13}\partial_2\partial_3+c_{44}\partial_2\partial_3 & c_{33}\partial_3^2+c_{44}\partial_2^2 + c_{44}\partial_2^2 & e_{15}\partial_1^2+e_{15}\partial_2^2 + e_{33}\partial_3^2 \\ -e_{31}\partial_1\partial_3-e_{15}\partial_1\partial_3 & -e_{31}\partial_2\partial_3-e_{15}\partial_2\partial_3 & -e_{33}\partial_3^2-e_{15}\partial_2^2 - e_{15}\partial_1^2 & \varepsilon_{11}\partial_1^2+\varepsilon_{11}\partial_2^2 + \varepsilon_{33}\partial_3^2 \end{pmatrix} \begin{pmatrix} u_{C_1} \\ u_{C_2} \\ u_{C_3} \\ \Phi_C \end{pmatrix} = \underline{\boldsymbol{0}}. \tag{3.15}$$

Introducing $\mathcal{D}$ as a part of $\underline{\underline{\boldsymbol{B}}}$:

$$\mathcal{D}^\top = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{pmatrix}, \tag{3.16}$$

we get the system (1.1)-(1.2).

3.2.2 Force balance equations in Ω_M

In the quasi-stationary case. the stress tensor $\underline{\underline{\boldsymbol{\sigma}}}_M$ satisfies the balance equation:

$$\partial_i \left(\sigma_M\right)_{ij} = 0 \qquad\qquad \text{in } \Omega_C,\ j = 1,2,3.$$

For the electrode, $\mathcal{D}$ and $\underline{\boldsymbol{u}}_M$ play the role of the former $\underline{\underline{\boldsymbol{B}}}$, defined in (3.12), and $\underline{\boldsymbol{U}}_C$, defined in (3.3). Assuming variable material parameters, the force balance equation reads:

$$-\mathcal{D}^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\boldsymbol{u}}_M = - \begin{pmatrix} \partial_1((\lambda{+}2\mu)\partial_1){+}\partial_3(\mu\partial_3) {+}\partial_2(\mu\partial_2) & \partial_1(\lambda\partial_2){+}\partial_2(\mu\partial_1) & \partial_1(\lambda\partial_3){+}\partial_3(\mu\partial_1) \\ \partial_1(\mu\partial_2){+}\partial_2(\lambda\partial_1) & \partial_2((\lambda{+}2\mu)\partial_2){+}\partial_3(\mu\partial_3) {+}\partial_1(\mu\partial_1) & \partial_2(\lambda\partial_3){+}\partial_3(\mu\partial_2) \\ \partial_3(\lambda\partial_1){+}\partial_1(\mu\partial_3) & \partial_3(\lambda\partial_2){+}\partial_2(\mu\partial_3) & \partial_3((\lambda{+}2\mu)\partial_3){+}\partial_2(\mu\partial_2) {+}\partial_1\mu\partial_1 \end{pmatrix} \underline{\boldsymbol{u}}_M = \underline{\boldsymbol{0}}. \tag{3.17}$$

From now on, we assume, that the material parameters of the electrode are constant. This leads to the simplified formulation:

$$- \begin{pmatrix} (\lambda{+}2\mu)\partial_1^2{+}\mu\partial_3^2 {+}\mu\partial_2^2 & \lambda\partial_1\partial_2{+}\mu\partial_1\partial_2 & (\lambda{+}\mu)\partial_1\partial_3 \\ (\lambda{+}\mu)\partial_1\partial_2 & \mu\partial_3^2{+}\mu\partial_1^2 {+}(\lambda{+}2\mu)\partial_2^2 & (\lambda{+}\mu)\partial_2\partial_3 \\ (\lambda{+}\mu)\partial_1\partial_3 & (\lambda{+}\mu)\partial_2\partial_3 & (\lambda{+}2\mu)\partial_3^2{+}\mu\partial_1^2 {+}\mu\partial_2^2 \end{pmatrix} \underline{\boldsymbol{u}}_M = \underline{\boldsymbol{0}}. \tag{3.18}$$

3.3 The plane strain case, in- and out-plane states

3.3.1 The in-plane case in the ceramic material

As mentioned before (see page 11), we assume, that the material of polarised ceramic is transversely isotropic in x_1, x_2 direction with the poling direction x_3. Considerations of crystal symmetries reduce the number of 21 independent coefficients in the piezoelectric anisotropic material to 10 independent material constants. These ceramic materials show transversely isotropic piezoelectric effects as well as longitudinal piezoelectric effects in d_{33}-direction (the direction of the x_3 axis) . The $x_1 - x_2$ plane is the isotropic plane and, hence, we can concentrate on the $x_1 - x_3$ or the $x_2 - x_3$ plane for further studies of the electro-mechanical phenomena. In this paper we consider the $x_1 - x_3$ plane without loss of generality. In accordance with this, we use the *plane strain* model to reduce the three-dimensional problems to two-dimensional ones.

Statement 3.1 (Plane strain state for the ceramic). *We assume, that the plane strain state is valid. That means:*

$$\gamma_{21} = \gamma_{22} = \gamma_{23} = 0 \qquad in\ \Omega_C, \tag{3.19}$$
$$E_2 = 0 \qquad in\ \Omega_C, \tag{3.20}$$
$$\Phi = \Phi(x_1, x_3),$$
$$u_i = u_i(x_1, x_3) \qquad for\ i = 1, 2, 3. \tag{3.21}$$

Furthermore, we assume, that the material parameters do not depend on x_2:

$$\underline{\underline{\boldsymbol{C}}}_C = \underline{\underline{\boldsymbol{C}}}_C (x_1, x_3),$$
$$\underline{\underline{\boldsymbol{e}}} = \underline{\underline{\boldsymbol{e}}} (x_1, x_3),$$
$$\underline{\underline{\boldsymbol{\varepsilon}}} = \underline{\underline{\boldsymbol{\varepsilon}}} (x_1, x_3).$$

Then, the system of partial differential equations (3.13) reduces for non-constant material parameters $A_{C_{ij}}$ to

$$\begin{aligned}
&-(\partial_1 (c_{11}\partial_1 u_{C_1}) + \partial_3 (c_{44}\partial_3 u_{C_1}) + \partial_1 (c_{13}\partial_3 u_{C_3}) + \partial_3 (c_{44}\partial_1 u_{C_3}) + \partial_1 (e_{31}\partial_3 \Phi_C)\\
&+\partial_3 (e_{15}\partial_1 \Phi_C)) = 0,\\
&-(\partial_3 (c_{13}\partial_1 u_{C_1}) + \partial_1 (c_{44}\partial_3 u_{C_1}) + \partial_3 (c_{33}\partial_3 u_{C_3}) + \partial_1 (c_{44}\partial_1 u_{C_3}) + \partial_3 (e_{33}\partial_3 \Phi_C)\\
&+\partial_1 (e_{15}\partial_1 \Phi_C)) = 0,\\
&-(-\partial_1 (e_{15} (\partial_3 u_{C_1} + \partial_1 u_{C_3})) - \partial_3 (e_{31}\partial_1 u_{C_1}) - \partial_3 (e_{33}\partial_3 u_{C_3}) + \partial_1 (\varepsilon_{11}\partial_1 \Phi_C)\\
&+\partial_3 (\varepsilon_{33}\partial_3 \Phi_C)) = 0.
\end{aligned}$$

For constant material parameters, we obtain the simplified equation system:

$$\begin{aligned}
-\left(c_{11}\partial_1^2 u_{C_1} + c_{44}\left(\partial_3^2 u_{C_1} + \partial_1\partial_3 u_{C_3}\right) + c_{13}\partial_1\partial_3 u_{C_3} + e_{15}\partial_1\partial_3\Phi_C + e_{31}\partial_1\partial_3\Phi_C\right) &= 0\\
-\left(c_{13}\partial_1\partial_3 u_{C_1} + c_{44}\left(\partial_1\partial_3 u_{C_1} + \partial_1^2 u_{C_3}\right) + c_{33}\partial_3^2 u_{C_3} + e_{15}\partial_1^2\Phi_C + e_{33}\partial_3^2\Phi_C\right) &= 0\\
-\left(-e_{31}\partial_1\partial_3 u_{C_1} + e_{15}\left(-\partial_1\partial_3 u_{C_1} - \partial_1^2 u_{C_3}\right) - e_{33}\partial_3^2 u_{C_3} + \varepsilon_{11}\partial_1^2\Phi_C + \varepsilon_{33}\partial_3^2\Phi_C\right) &= 0.
\end{aligned}$$

Remark 3.1. *Note, that in consequence of the geometry of the actuator, E_2 always vanishes - independently of the plane strain assumption.*

Proof of the statement 3.1. From the assumption (3.19) follows, that $\partial_2 u_2 = 0$ and therefore, written with the help of the Voigt index mapping (3.2), the system (3.6) reduces to:

$$\begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{23} \\ 2\gamma_{13} \\ 2\gamma_{12} \\ -E_1 \\ -E_2 \\ -E_3 \end{pmatrix} \rightarrow \begin{pmatrix} \gamma_{11} \\ \gamma_{33} \\ 2\gamma_{13} \\ -E_1 \\ -E_3 \end{pmatrix} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_3 & 0 \\ \partial_3 & \partial_1 & 0 \\ 0 & 0 & -\partial_1 \\ 0 & 0 & -\partial_3 \end{pmatrix} \begin{pmatrix} u_{C_1} \\ u_{C_3} \\ \Phi_C \end{pmatrix} = \underline{\underline{\boldsymbol{B}}}_p \underline{\boldsymbol{U}}_{C_p}, \tag{3.22}$$

where p indicates the plane case. Therefore it holds for the whole system with the material parameters $C_{ij} = C_{ij}(x_1, x_3)$, $e_{ij} = e_{ij}(x_1, x_3)$, $\varepsilon_{ij} = \varepsilon_{ij}(x_1, x_3)$:

$$\begin{pmatrix} \underline{\boldsymbol{\sigma}}_{C_p} \\ \underline{\boldsymbol{D}}_{C_p} \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \\ D_1 \\ D_3 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{13} & 0 & 0 & -e_{31} \\ c_{13} & c_{33} & 0 & 0 & -e_{33} \\ 0 & 0 & c_{44} & -e_{15} & 0 \\ 0 & 0 & e_{15} & \varepsilon_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_3 & 0 \\ \partial_3 & \partial_1 & 0 \\ 0 & 0 & -\partial_1 \\ 0 & 0 & -\partial_3 \end{pmatrix} \begin{pmatrix} u_{C_1} \\ u_{C_3} \\ \Phi_C \end{pmatrix}. \tag{3.23}$$

Non-constant material parameters

Using the force balance equations (3.10), (3.11) and the plane strain presentation of the extended stress vector field $\begin{pmatrix} \underline{\boldsymbol{\sigma}}_p \\ \underline{\boldsymbol{D}}_p \end{pmatrix}$, it follows:

$$-\underline{\underline{\boldsymbol{B}}}_p^\top \begin{pmatrix} \underline{\boldsymbol{\sigma}}_p \\ \underline{\boldsymbol{D}}_p \end{pmatrix} = -\underline{\underline{\boldsymbol{B}}}_p^\top \underline{\underline{\boldsymbol{A}}}_{C_p} \underline{\underline{\boldsymbol{B}}}_p \underline{\boldsymbol{U}}_{C_p} = \underline{\mathbf{0}}. \tag{3.24}$$

At full length written:

$$-\begin{pmatrix} \partial_1 & 0 & \partial_3 & 0 & 0 \\ 0 & \partial_3 & \partial_1 & 0 & 0 \\ 0 & 0 & 0 & -\partial_1 & -\partial_3 \end{pmatrix} \begin{pmatrix} c_{11} & c_{13} & 0 & 0 & -e_{31} \\ c_{13} & c_{33} & 0 & 0 & -e_{33} \\ 0 & 0 & c_{44} & -e_{15} & 0 \\ 0 & 0 & e_{15} & \varepsilon_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_3 & 0 \\ \partial_3 & \partial_1 & 0 \\ 0 & 0 & -\partial_1 \\ 0 & 0 & -\partial_3 \end{pmatrix} \begin{pmatrix} u_{C_1} \\ u_{C_3} \\ \Phi_C \end{pmatrix} =$$

$$-\begin{pmatrix} \partial_1 c_{11} & \partial_1 c_{13} & \partial_3 c_{44} & -\partial_3 e_{15} & -\partial_1 e_{31} \\ \partial_3 c_{13} & \partial_3 c_{33} & \partial_1 c_{44} & -\partial_1 e_{15} & -\partial_3 e_{33} \\ \partial_3 e_{31} & -\partial_3 e_{33} & -\partial_1 e_{15} & -\partial_1 \varepsilon_{11} & -\partial_3 \varepsilon_{33} \end{pmatrix} \begin{pmatrix} \partial_1 u_{C_1} \\ \partial_3 u_{C_3} \\ \partial_3 u_{C_1} + \partial_1 u_{C_3} \\ -\partial_1 \Phi_C \\ -\partial_3 \Phi_C \end{pmatrix} = \underline{\mathbf{0}}. \tag{3.25}$$

The equation system (3.25) for the three unknowns u_{C_1}, u_{C_3}, Φ_C can be written as follows:

$$-\left(\partial_1 \left(c_{11}\partial_1 u_{C_1}\right) + \partial_3 \left(c_{44}\partial_3 u_{C_1}\right) + \partial_1 \left(c_{13}\partial_3 u_{C_3}\right) + \partial_3 \left(c_{44}\partial_1 u_{C_3}\right) + \partial_1 \left(e_{31}\partial_3 \Phi_C\right) + \partial_3 \left(e_{15}\partial_1 \Phi_C\right)\right) = 0, \tag{3.26}$$

$$-\left(\partial_3 \left(c_{13}\partial_1 u_{C_1}\right) + \partial_1 \left(c_{44}\partial_3 u_{C_1}\right) + \partial_3 \left(c_{33}\partial_3 u_{C_3}\right) + \partial_1 \left(c_{44}\partial_1 u_{C_3}\right) + \partial_3 \left(e_{33}\partial_3 \Phi_C\right) + \partial_1 \left(e_{15}\partial_1 \Phi_C\right)\right) = 0, \tag{3.27}$$

$$-\left(-\partial_1 \left(e_{15} \left(\partial_3 u_{C_1} + \partial_1 u_{C_3}\right)\right) - \partial_3 \left(e_{31}\partial_1 u_{C_1}\right) - \partial_3 \left(e_{33}\partial_3 u_{C_3}\right) + \partial_1 \left(\varepsilon_{11}\partial_1 \Phi_C\right) + \partial_3 \left(\varepsilon_{33}\partial_3 \Phi_C\right)\right) = 0. \tag{3.28}$$

Constant material parameters

For constant material parameters, the system of partial differential equations (3.26),(3.27), (3.28) simplifies to:

$$-\left(c_{11}\partial_1^2 u_{C_1} + c_{44}\left(\partial_3^2 u_{C_1} + \partial_1\partial_3 u_{C_3}\right) + c_{13}\partial_1\partial_3 u_{C_3} + e_{15}\partial_1\partial_3 \Phi_C + e_{31}\partial_1\partial_3 \Phi_C\right) = 0, \tag{3.29}$$

$$-\left(c_{13}\partial_1\partial_3 u_{C_1} + c_{44}\left(\partial_1\partial_3 u_{C_1} + \partial_1^2 u_{C_3}\right) + c_{33}\partial_3^2 u_{C_3} + e_{15}\partial_1^2 \Phi_C + e_{33}\partial_3^2 \Phi_C\right) = 0, \tag{3.30}$$

$$-\left(-e_{31}\partial_1\partial_3 u_{C_1} + e_{15}\left(-\partial_1\partial_3 u_{C_1} - \partial_1^2 u_{C_3}\right) - e_{33}\partial_3^2 u_{C_3} + \varepsilon_{11}\partial_1^2 \Phi_C + \varepsilon_{33}\partial_3^2 \Phi_C\right) = 0. \tag{3.31}$$

□

3.3.2 The anti-plane strain state in the ceramic material

Statement 3.2 (Anti-plane strain state for the ceramic). *We consider the anti-plane strain state. That means:*

$$\begin{aligned} \gamma_{11} = \gamma_{33} = \gamma_{13} &= 0 && \text{in } \Omega_C, \\ E_1 = E_3 &= 0 && \text{in } \Omega_C, \\ u_{C_i} &= u_{C_i}(x_1, x_3) && \text{for } i = 1,2,3 \text{ in } \Omega_C. \end{aligned}$$

Furthermore, we assume, that the material parameters do not depend on x_2:

$$\begin{aligned} \underline{\underline{\boldsymbol{C}}}_C &= \underline{\underline{\boldsymbol{C}}}_C (x_1, x_3), \\ \underline{\underline{\boldsymbol{e}}} &= \underline{\underline{\boldsymbol{e}}} (x_1, x_3), \\ \underline{\underline{\boldsymbol{\varepsilon}}} &= \underline{\underline{\boldsymbol{\varepsilon}}} (x_1, x_3). \end{aligned}$$

Then, the system of partial differential equations (3.13) reduces for non-constant material parameters $(A_C)_{ij}$ to:

$$-\partial_3 c_{44} \partial_3 u_{C_2} - \partial_1 \frac{c_{11} - c_{12}}{2} \partial_1 u_{C_2} = 0.$$

Remark 3.2. *Note, that $\gamma_{22} = 0$ as a consequence of $u_{C_2} = u_{C_2}(x_1, x_3)$ and $E_2 = 0$ due to the geometry of the actuator. The anti-plane case will now be indexed by ap.*

Proof of the statement. The anti-plane strain assumptions in statement (3.2) lead to the following reduced system:

$$\nabla_{1,3} u_2 = \gamma_{C_{ap}} = \begin{pmatrix} 2\gamma_{23} \\ 2\gamma_{12} \end{pmatrix} = \underline{\underline{\boldsymbol{B}}}_{ap} \underline{\boldsymbol{u}}_{C_{ap}} = \begin{pmatrix} \partial_3 \\ \partial_1 \end{pmatrix} (u_{C_2}), \tag{3.32}$$

$$\underline{\boldsymbol{\sigma}}_{C_{ap}} = \begin{pmatrix} \sigma_{23} \\ \sigma_{12} \end{pmatrix}, \quad \underline{\underline{\boldsymbol{A}}}_{C_{ap}} = \begin{pmatrix} c_{44} & 0 \\ 0 & \frac{c_{11} - c_{12}}{2} \end{pmatrix}. \tag{3.33}$$

Inserting $\underline{\boldsymbol{\sigma}}_{C_{ap}}$ into the force balance equation (3.10), the resulting system reduces to an equation system for u_2:

$$-\partial_3 c_{44} \partial_3 u_{C_2} - \partial_1 \frac{c_{11} - c_{12}}{2} \partial_1 u_{C_2} = 0. \tag{3.34}$$

□

Note, that as an effect of the anti-plane strain assumption, the electrical components vanish. Therefore it is plausible, that due to the lack of the driving electrical voltage, the trivial solution appears in this model.

3.3.3 The in-plane case in the metal electrodes

As mentioned already before, the elastic constants of the metal are assumed to be constant.

Statement 3.3 (Plane strain state for the metal). *We assume, that the plane strain state is valid. That means:*

$$\gamma_{M_{21}} = \gamma_{M_{22}} = \gamma_{M_{23}} = 0 \qquad \text{in } \Omega_M,$$
$$u_{M_2} = u_{M_2}(x_1, x_3) \qquad \text{in } \Omega_M.$$

Then the system of partial differential equations (3.17) reduces to:

$$-\left((\lambda + 2\mu)\,\partial_1^2 u_{M_1} + \mu\partial_3^2 u_{M_1} + (\mu + \lambda)\,\partial_3\partial_1 u_{M_3}\right) = 0,$$
$$-\left((\lambda + 2\mu)\,\partial_3^2 u_{M_3} + \mu\partial_1^2 u_{M_3} + (\mu + \lambda)\,\partial_3\partial_1 u_{M_1}\right) = 0.$$

Proof of the statement. Analogously to the ceramic case, this assumption reduces the strain, stress and the material matrix to a lower dimension; the plane case is again indexed with p:

$$\underline{\boldsymbol{\sigma}}_{M_p} = \begin{pmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \end{pmatrix}, \quad \underline{\underline{\boldsymbol{C}}}_{M_p} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \underline{\boldsymbol{\gamma}}_{M_p} = \mathcal{D}_p \underline{\boldsymbol{u}}_{M_p} = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_3 \\ \partial_3 & \partial_1 \end{pmatrix} \begin{pmatrix} u_{M_1} \\ u_{M_3} \end{pmatrix}, \tag{3.35}$$

$$\underline{\boldsymbol{\sigma}}_{M_p} = \underline{\underline{\boldsymbol{C}}}_{M_p} \gamma_{M_p} = \underline{\underline{\boldsymbol{C}}}_{M_p} \mathcal{D}_p \underline{\boldsymbol{u}}_{M_p} = \begin{pmatrix} (\lambda + 2\mu)\,\partial_1 u_{M_1} + \lambda\partial_3 u_{M_3} \\ \lambda\partial_1 u_{M_1} + (\lambda + 2\mu)\,\partial_3 u_{M_3} \\ \mu\,(\partial_3 u_{M_1} + \partial_1 u_{M_3}) \end{pmatrix}.$$

The force balance equations read:

$$-\mathcal{D}_p^\top \underline{\underline{\boldsymbol{C}}}_{M_p} \mathcal{D}_p \underline{\boldsymbol{u}}_{M_p} = \mathbf{0},$$

or more detailed written:

$$-\left((\lambda + 2\mu)\,\partial_1^2 u_{M_1} + \mu\partial_3^2 u_{M_1} + (\mu + \lambda)\,\partial_3\partial_1 u_{M_3}\right) = 0, \tag{3.36}$$
$$-\left((\lambda + 2\mu)\,\partial_3^2 u_{M_3} + \mu\partial_1^2 u_{M_3} + (\mu + \lambda)\,\partial_3\partial_1 u_{M_1}\right) = 0. \tag{3.37}$$

□

3.3.4 The anti-plane strain state for the metal matrix

Statement 3.4 (Anti-plane strain state for the metal). *We assume, that the anti-plane strain state is valid. Therefore it holds:*

$$\gamma_{M_{11}} = \gamma_{M_{33}} = \gamma_{M_{13}} = 0 \qquad \text{in } \Omega_M,$$
$$u_{M_2} = u_{M_2}(x_1, x_3) \qquad \text{in } \Omega_M.$$

Then the system of partial differential equations (3.17) reduces to the two-dimensional Laplace-equation:

$$\mathcal{D}_{ap}^\top \underline{\boldsymbol{\sigma}}_{M_{ap}} = \mu\left(\partial_3^2 u_{M_2} + \partial_1^2 u_{M_2}\right) = 0.$$

Proof of the statement 3.4. Due to $\gamma_{22} = 0$, the original three-dimensional system reduces to:

$$\underline{\boldsymbol{\sigma}}_{M_{ap}} := \begin{pmatrix} \sigma_{M_{23}} \\ \sigma_{M_{12}} \end{pmatrix} = \underline{\underline{\boldsymbol{A}}}_{M_{ap}} \mathcal{D}_{ap} \underline{\boldsymbol{u}}_{M_{ap}} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \partial_3 \\ \partial_1 \end{pmatrix} u_{M_2} \tag{3.38}$$
$$= \begin{pmatrix} \mu\partial_3 u_{M_2} \\ \mu\partial_1 u_{M_2} \end{pmatrix}.$$

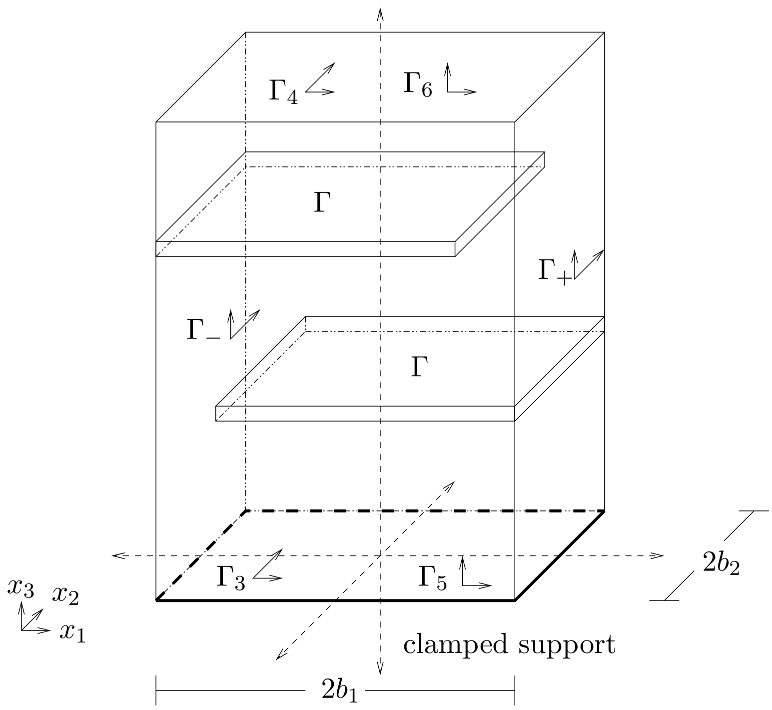
We get the reduced force balance equation:

$$\mathcal{D}_{ap}^{\top}\underline{\boldsymbol{\sigma}}_{M_{ap}} = \mu\left(\partial_3^2 u_{M_2} + \partial_1^2 u_{M_2}\right) = 0. \tag{3.39}$$

□

4 Classical formulation of the boundary-transmission problem

In this section we derive physically reasonable boundary and transmission conditions for the actuator. In accordance with the notation in figure 4, we use the following **notations and general assumptions:**



Γ_-	$x_1 = -b_1$, $-b_2 \le x_2 \le b_2$ $0 \le x_3 \le 2b_3$
Γ_+	$x_1 = +b_1$, $-b_2 \le x_2 \le b_2$ $0 \le x_3 \le 2b_3$
Γ_3	$-b_1 \le x_1 \le b_1$, $-b_2 \le x_2 \le b_2$, $x_3 = 0$,
Γ_4	$-b_1 \le x_1 \le b_1$, $-b_2 \le x_2 \le b_2$, $x_3 = 2b_3$
Γ_5	$-b_1 \le x_1 \le b_1$, $x_2 = -b_2$, $0 \le x_3 \le 2b_3$
Γ_6	$-b_1 \le x_1 \le b_1$, $x_2 = +b_2$, $0 \le x_3 \le 2b_3$

Figure 4: Boundaries and interfaces of the stack actuator of width $2b_1$, measured in x_1 direction, depth $2b_2$ and height $2b_3$.

Stating, that the impressed voltage on Γ_- is $U_- = -\Phi_a$, whereas on Γ_+: $U_+ = +\Phi_a$. Therefore, $r_{|\Gamma_- \cap \partial\Omega_M}\Phi_M = -\Phi_a$ and $r_{|\Gamma_+ \cap \partial\Omega_M}\Phi_M = +\Phi_a$, is constant in all electrodes and metal plates, connected with Γ_- or Γ_+ respectively. As earlier, $\underline{\boldsymbol{u}}_C$ and $\underline{\boldsymbol{u}}_M$ are the mechanical displacement vectors inside the metal and the ceramic, respectively. Neumann boundary conditions are abbreviated by Ⓝ, Dirichlet conditions by Ⓓ and Robin boundary conditions (in a general sense) by Ⓡ. Dependent on the field we are dealing with, Ⓝ, Ⓓ and Ⓡ are indexed by m for the mechanical displacement field and e for the electric potential field.

4.1 Boundary conditions for the ceramic, *3D-case*

Referring to (1.1), (1.2) and (3.6), we use the extended stress tensor in vector notation and the displacement vector, which consists of mechanical and electrical components:

$$\begin{pmatrix} \underline{\boldsymbol{\sigma}}_C \\ \underline{\boldsymbol{D}}_C \end{pmatrix} = \begin{pmatrix} \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\boldsymbol{u}}_C + \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C \\ -\underline{\underline{\boldsymbol{e}}} \mathcal{D} \underline{\boldsymbol{u}}_C + \underline{\underline{\boldsymbol{\varepsilon}}} \nabla \Phi_C \end{pmatrix}.$$

The extended stress vector is given (see (3.4) and (3.5)) in new notation by:

$$\begin{aligned} &Ⓝ_m \qquad \boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\underline{\boldsymbol{N}}}^\top \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\boldsymbol{u}}_C + \underline{\underline{\boldsymbol{N}}}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C, \\ &Ⓝ_e \qquad D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\boldsymbol{n}}^\top \left(\underline{\underline{\boldsymbol{e}}} \mathcal{D} \underline{\boldsymbol{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}} \nabla \Phi_C \right). \end{aligned}$$

Here is:

$$\underline{\underline{\mathbf{N}}} = \begin{pmatrix} \underline{\underline{\boldsymbol{N}}} & \underline{\boldsymbol{0}} \\ \underline{\underline{\boldsymbol{0}}} & -\underline{\boldsymbol{n}} \end{pmatrix} := \begin{pmatrix} n_1 & 0 & 0 & 0 \\ 0 & n_2 & 0 & 0 \\ 0 & 0 & n_3 & 0 \\ 0 & n_3 & n_2 & 0 \\ n_3 & 0 & n_1 & 0 \\ n_2 & n_1 & 0 & 0 \\ 0 & 0 & 0 & -n_1 \\ 0 & 0 & 0 & -n_2 \\ 0 & 0 & 0 & -n_3 \end{pmatrix}, \quad \underline{\boldsymbol{n}} := \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \tag{4.1}$$

n_i, $i \in \{1, 2, 3\}$, denote the components of the exterior normal unit vector $\underline{\boldsymbol{n}}$. Further, the Dirichlet condition can also be split into a mechanical and an electrical part:

$$\begin{aligned} &Ⓓ_m \qquad \underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}_0, \\ &Ⓓ_e \qquad \Phi = \Phi_0, \end{aligned}$$

where $\underline{\boldsymbol{u}}_0$ and Φ_0 are given values.

Boundary conditions on $\Gamma_- \cap \partial\Omega_C$ and $\Gamma_+ \cap \partial\Omega_C$

For the mechanical conditions, we assume, that the stress $\boldsymbol{\sigma}_{C_n}$ vanishes. Due to the applied voltage, the electric potential field satisfies $\Phi_C = \mp\Phi_a$. Note, that we use here the fact, that Γ_- and Γ_+ are vaporised with a thin metal film (without mechanical influence) which is connected on one hand side with the electrodes and on the other hand side with the power supply unit. We assume here, that the mechanical influence of the thin metal film is negligible small and therefore we neglect its' influence on the mechanical displacement field on $\Gamma_\mp$. This means, that the metal film is modelled only in form of electric potential fields on $\Gamma_\mp$.

$$Ⓝ_m \qquad \boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\boldsymbol{0}} \quad \text{on } \partial\Omega_C \cap \Gamma_\mp, \tag{4.2}$$

$$Ⓓ_e \qquad \Phi_C = \mp\Phi_a \quad \text{on } \partial\Omega_C \cap \Gamma_\mp. \tag{4.3}$$

Boundary conditions on Γ_3

This is the clamped boundary part, consisting only of ceramic material. Therefore, the mechanical boundary conditions are Dirichlet conditions:

$$\text{Ⓓ}_m \qquad \underline{\boldsymbol{u}}_C = \underline{\mathbf{0}} \quad \text{on } \Gamma_3. \tag{4.4}$$

First one word about the simplicity of the electrical boundary condition on Γ_3 (and in the following also on Γ_4, Γ_5 and Γ_6): There will be different possibilities to write and approximate the real boundary condition on these boundaries (see appendix, section 6.3 for possible conditions). For example we can choose the following Robin like boundary condition

$$\text{Ⓡ}_e \qquad D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = D_{C_n}(\Phi_C) = -\mathfrak{s}\tanh\left(\Phi_C(\underline{\boldsymbol{x}})\right) \qquad \text{on } \Gamma_3, \tag{4.5}$$

which interpolates $D_{C_n}(\Phi_C)(x_1, x_2, 0)$ on Γ_3 in three curves exactly: $x_1 = \pm b_1$, $\Phi_C(x_1, x_2, 0) = 0$. The interpolation properties are explained in detail in (6.10) and (6.10). Note, that $\mathfrak{s}$ is introduced for rescaling the boundary condition and depends on the face charge density (see (6.8) for further explanations). Since any other monotonously growing and continuous operator with the interpolation properties (6.10) and (6.11) of (4.5) corresponding to the boundary Γ_3 could be used here (we could handle them with the same analysis), we write condition (4.5) shortly as:

$$\text{Ⓡ}_e \qquad D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C)(\underline{\boldsymbol{x}}) = -\mathcal{B}_3(\Phi_C)(\underline{\boldsymbol{x}}) := -\mathfrak{s}\tanh\left(\Phi_C(\underline{\boldsymbol{x}})\right) \qquad \text{on } \Gamma_3. \tag{4.6}$$

The operator $\mathcal{B}_3$ maps from $\mathsf{H}^1(\Omega_C)$ to $\mathsf{H}^1(\Omega_C)$:

$$\mathcal{B}_3 : \Phi_C \mapsto \mathcal{B}(\Phi_C)(\underline{\boldsymbol{x}}) = \tanh(\Phi_C)(\underline{\boldsymbol{x}}).$$

Boundary conditions on Γ_4

This part of the boundary is not clamped. We state, that the mechanical stress components vanish for a free oscillating actuator , since no stresses are transmitted to the surrounding air. The situation for the electrical normal flux density component is the same as on Γ_3. Furthermore, we assume, that

$$\text{Ⓝ}_m \qquad \boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\mathbf{0}} \qquad \text{on } \Gamma_4, \tag{4.7}$$

$$\text{Ⓡ}_e \qquad D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = -\mathcal{B}_4(\Phi_C) := -\mathfrak{s}\tanh\left(\Phi_C(\underline{\boldsymbol{x}})\right) \qquad \text{on } \Gamma_4. \tag{4.8}$$

Boundary conditions on $(\Gamma_5 \cup \Gamma_6) \cap \partial\Omega_C$.

Here, the situation is very similar to that on Γ_4. We assume, that no stresses are transmitted to the external area.
For the electrical normal flux density, we use a similar interpolation as on Γ_3 and Γ_4. Thus we have:

$$\text{Ⓝ}_m \qquad \boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\mathbf{0}} \qquad \text{on } \partial\Omega_C \cap (\Gamma_5 \cup \Gamma_6), \tag{4.9}$$

$$\text{Ⓡ}_e \qquad D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = -\mathcal{B}_{5,6}(\Phi_C) := -\mathfrak{s}\tanh\left(\Phi_C(\underline{\boldsymbol{x}})\right) \qquad \text{on } \partial\Omega_C \cap (\Gamma_5 \cup \Gamma_6) \tag{4.10}$$

Boundary conditions on Γ.

Here and in the following, we denote by Γ the set of interfaces:

$$\Gamma := \partial\Omega_M \cap \partial\Omega_C. \tag{4.11}$$

In consequence of the non-existent counterpart in the metal for the electrical terms in the ceramic, the electrical conditions for the ceramic must be taken as boundary conditions - not as transmission conditions:

$$\text{Ⓓ}_e \qquad \Phi_C = \mp\Phi_a \qquad \text{on } \Gamma. \tag{4.12}$$

4.2 Boundary conditions for the metal, *3D-case*

As metals do not show piezoelectric effects, the electric and elastic fields are not coupled. Therefore, it holds for the stress vector:

$$\text{Ⓝ}_m \qquad \boldsymbol{\sigma}_{M_n} = \underline{\underline{\boldsymbol{N}}}^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\boldsymbol{u}}_M,$$

where $\underline{\underline{\boldsymbol{N}}}$ is the co-normal, described in (4.1)

$$\underline{\underline{\boldsymbol{N}}} := \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \\ 0 & n_3 & n_2 \\ n_3 & 0 & n_1 \\ n_2 & n_1 & 0 \end{pmatrix}. \tag{4.13}$$

The values of the electric potential field in Ω_M are known, but they are not needed for the formulation of the multi-field problem in Ω.

Boundary conditions on $\Gamma_\mp \cap \partial\Omega_M$

We assume, that for the mechanical part, the stresses disappear:

$$\text{Ⓝ}_m \qquad \boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{u}}_M) = \underline{\boldsymbol{0}} \qquad \text{on } \partial\Omega_M \cap \Gamma_\mp. \tag{4.14}$$

Boundary conditions on $(\Gamma_5 \cup \Gamma_6) \cap \partial\Omega_M$

Like in the ceramic part, we assume, that no stresses are transmitted to the exterior domain.

$$\text{Ⓝ}_m \qquad \boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{u}}_M) = \underline{\boldsymbol{0}} \qquad \text{on } \partial\Omega_M \cap (\Gamma_5 \cup \Gamma_6). \tag{4.15}$$

4.3 The transmission conditions on Γ, *3D case*

We assume, that the electrodes and the ceramic matrix are bonded and hence the transmission conditions are:

$$\begin{aligned} \text{Ⓓ}_m \qquad & \underline{\boldsymbol{u}}_C = \underline{\boldsymbol{u}}_M \qquad \text{on } \Gamma, \\ \text{Ⓝ}_m \qquad & \boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{u}}_M) \quad \text{on } \Gamma. \end{aligned} \tag{4.16}$$

Remark 4.1 (Boundary and transmission conditions in the 2D case). *The boundary and transmission conditions in the two-dimensional case are described analogously to the three-dimensional case. Note, that in the 2D model, the boundaries Γ_5 and Γ_6 do not appear any more. For the formulation of the boundary and transmission conditions on the remaining parts, we only have to substitute the 3D quantities with the p-indexed 2D quantities.*

4.4 3D and 2D multi-field problems

Three-dimensional case.

Summarising the above considerations, we formulate now the complete quasi-stationary problem in the three-dimensional case:
Find fields $\underline{\boldsymbol{u}}$ and Φ, such that:

System of partial differential equations

$$-\mathcal{D}^{\top}\underline{\underline{\boldsymbol{C}}}_C\mathcal{D}\underline{\boldsymbol{u}}_C - \mathcal{D}^{\top}\underline{\underline{\boldsymbol{e}}}^{\top}\nabla\Phi_C = \underline{\boldsymbol{0}} \qquad \text{in } \Omega_C, \tag{4.17}$$

$$-\operatorname{div}\left(-\underline{\underline{\boldsymbol{e}}}\mathcal{D}\underline{\boldsymbol{u}}_C + \underline{\underline{\boldsymbol{\varepsilon}}}\nabla\Phi_C\right) = 0 \qquad \text{in } \Omega_C, \tag{4.18}$$

$$-\mathcal{D}^{T}\underline{\underline{\boldsymbol{C}}}_M\mathcal{D}\underline{\boldsymbol{u}}_M = \underline{\boldsymbol{0}} \qquad \text{in } \Omega_M. \tag{4.19}$$

Boundary conditions

$$\boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_{\mp}\cap\partial\Omega_C, \tag{4.20}$$

$$\Phi_C = \mp\Phi_a \qquad \text{on } \Gamma_{\mp}\cup\partial\Omega_M, \tag{4.21}$$

$$\boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{u}}_M) = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_{\mp}\cap\partial\Omega_M, \tag{4.22}$$

$$\underline{\boldsymbol{u}}_C = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_3, \tag{4.23}$$

$$D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) + \mathcal{B}_3(\Phi_C) = 0 \qquad \text{on } \Gamma_3, \tag{4.24}$$

$$\boldsymbol{\sigma}(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_4 \tag{4.25}$$

$$D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) + \mathcal{B}_4(\Phi_C) = 0 \qquad \text{on } \Gamma_4 \tag{4.26}$$

$$\boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\boldsymbol{0}} \qquad \text{on } (\Gamma_5\cup\Gamma_6)\cap\partial\Omega_C, \tag{4.27}$$

$$D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) + \mathcal{B}_{5,6}(\Phi_C) = 0 \qquad \text{on } (\Gamma_5\cup\Gamma_6)\cap\partial\Omega_C \tag{4.28}$$

$$\boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{u}}_M) = \underline{\boldsymbol{0}} \qquad \text{on } (\Gamma_5\cup\Gamma_6)\cap\partial\Omega_M \tag{4.29}$$

Transmission conditions

$$\underline{\boldsymbol{u}}_C = \underline{\boldsymbol{u}}_M \qquad \text{on } \Gamma, \tag{4.30}$$

$$\boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{u}}_M) \qquad \text{on } \Gamma. \tag{4.31}$$

are satisfied. We underline, that the driving forces are expressed by (4.21).

Two-dimensional case (the in-plane case).

Find $\underline{\boldsymbol{u}}_p$ and Φ, such that holds:

System of partial differential equations

$$-\mathcal{D}_p^{\top}\underline{\underline{\boldsymbol{C}}}_{C_p}\mathcal{D}_p\underline{\boldsymbol{u}}_{C_p} - \mathcal{D}_p^{\top}\underline{\underline{\boldsymbol{e}}}_p^{\top}\nabla_p\Phi_{C_p} = \underline{\boldsymbol{0}} \qquad \text{in } \Omega_C, \tag{4.32}$$

$$-\operatorname{div}\left(-\underline{\underline{\boldsymbol{e}}}_p\mathcal{D}_p\underline{\boldsymbol{u}}_{C_p} + \underline{\underline{\boldsymbol{\varepsilon}}}_p\nabla_p\Phi_{C_p}\right) = 0 \qquad \text{in } \Omega_C, \tag{4.33}$$

$$-\mathcal{D}_p^{\top}\underline{\underline{\boldsymbol{C}}}_{M_p}\mathcal{D}_p\underline{\boldsymbol{u}}_{M_p} = \underline{\boldsymbol{0}} \qquad \text{in } \Omega_M. \tag{4.34}$$

The boundary conditions for the $3D$-model hold also for the in-plane-case. Therefore, we have Dirichlet (4.21), (4.23) and Neumann conditions (4.20), (4.24), (4.25), (4.26) for the ceramic; Neumann conditions (4.22) for the metal and transmission conditions (4.30), (4.31).

In all boundary and transmission conditions we have to substitute the $3D$-parameters by the plane-indexed parameters. Note, that we leave out the boundary conditions (4.27)-(4.29).

Two-dimensional case (the anti-plane case).

Due to the theorems 3.2 and 3.3, the anti-plane multi-field problem is reduced to:
Find u_{C_2} and u_{M_2} such that:

System of partial differential equations

$$-\partial_3 c_{44}\partial_3 u_{C_2} - \partial_1\left(\frac{c_{11}-c_{12}}{2}\right)\partial_1 u_{C_2} = 0 \qquad \text{in } \Omega_C, \tag{4.35}$$

$$-\Delta_{13} u_{M_2} = 0 \qquad \text{in } \Omega_M. \tag{4.36}$$

Boundary conditions

$$\sigma_{C_n}(u_{C_2}) = 0 \qquad \text{on } \Gamma_{\mp} \cap \partial\Omega_C, \tag{4.37}$$

$$\sigma_{M_n}(u_{M_2}) = 0 \qquad \text{on } \Gamma_{\mp} \cap \partial\Omega_M, \tag{4.38}$$

$$u_{C_2} = 0 \qquad \text{on } \Gamma_3, \tag{4.39}$$

$$\sigma_{C_n}(u_2) = 0 \qquad \text{on } \Gamma_4. \tag{4.40}$$

Transmission conditions

$$u_{C_2} = u_{M_2} \qquad \text{on } \Gamma, \tag{4.41}$$

$$\sigma_{C_n}(u_{C_2}) = \sigma_{M_n}(u_{M_2}) \qquad \text{on } \Gamma. \tag{4.42}$$

It follows directly, that the solutions u_{C_2} and u_{M_2} of the out-of-plane case vanish. The in-plane case is therefore the only meaningful two-dimensional simplification of the model.

4.5 Transformation to vanishing Dirichlet datum

The stack actuator occupies the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, which consists of the ceramic part Ω_C and the metallic part Ω_M , $\bar{\Omega} = \bar{\Omega}_C \cup \bar{\Omega}_M$ (see figure 4). Ω_C and Ω_M are open domains with Lipschitz-continuous boundaries $\partial\Omega_C$ and $\partial\Omega_M$. Note, that Ω_M is the union of open domains, which are occupied by the electrodes.
In order to simplify the notation, we decompose the boundaries of Ω_M and Ω_C into boundary pieces for the electrical and the mechanical boundary conditions:

$$\partial\Omega_M = \partial\Omega_{M,m} = \Gamma^N_{M,m} \cup \Gamma, \tag{4.43}$$

$$\partial\Omega_C = \partial\Omega_{C,m} = \Gamma^D_{C,m} \cup \Gamma^N_{C,m} \cup \Gamma, \tag{4.44}$$

$$\partial\Omega_C = \partial\Omega_{C,e} = \Gamma^D_{C,e} \cup \Gamma^R_{C,e}. \tag{4.45}$$

Here, the upper index D, N, R denotes boundary pieces with Dirichlet, Neumann and Robin conditions.
To give a clearer, but more abstract formulation of the boundary-transmission problem (4.17)-(4.29), we take now a notation, which will be more practicable in some cases, especially when formulating the boundary-transmission problem in a weak sense (compare (3.6)):

$$\underline{\underline{\boldsymbol{A}}} := \begin{cases} \underline{\underline{\boldsymbol{A}}}_C &= \begin{pmatrix} \underline{\underline{\boldsymbol{C}}}_C & -\underline{\underline{\boldsymbol{e}}}^\top \\ \underline{\underline{\boldsymbol{e}}} & \underline{\underline{\boldsymbol{\varepsilon}}} \end{pmatrix} \quad \text{for } \underline{\boldsymbol{U}}\,|_{\Omega_C}, \\ \underline{\underline{\boldsymbol{A}}}_M &= \begin{pmatrix} \underline{\underline{\boldsymbol{C}}}_M & \underline{\underline{\boldsymbol{0}}} \\ \underline{\underline{\boldsymbol{0}}} & \underline{\underline{\boldsymbol{0}}} \end{pmatrix} \quad \text{for } \underline{\boldsymbol{U}}\,|_{\Omega_M}. \end{cases} \tag{4.46}$$

Thus, we can rewrite the partial differential equation system (4.17)-(4.19) with the help of the operator $\underline{\underline{\boldsymbol{B}}}$ (3.12) as:

$$-\underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{U}} = \underline{\boldsymbol{0}}. \tag{4.47}$$

In order to give a weak formulation of the problem, we have to eliminate the non-homogeneous Dirichlet data. As mentioned before, we have only the system of partial differential equations (4.19) for the displacement field in the metal domain - totally independent of the electric potential field Φ_M. As a consequence, the electric interface condition of the ceramic domain has no counterpart in the metal matrix and we have to treat it as boundary condition of Ω_C:

$$\Phi_C = -\Phi_a \qquad \text{on } \Gamma^D_{e,-} := (\partial\Omega_C \cap \Gamma_-) \cup \Gamma^-, \tag{4.48}$$

$$\Phi_C = \Phi_a \qquad \text{on } \Gamma^D_{e,+} := (\partial\Omega_C \cap \Gamma_+) \cup \Gamma^+, \tag{4.49}$$

where $\Gamma = \Gamma^- \cup \Gamma^+$, Γ^- is the set of interfaces, connected with the negative charged side and Γ^+ is correspondingly defined (see figure 4 for the definition of $\Gamma_\mp$).
We consider a sufficiently smooth continuation $\hat{\Phi}_C \in \mathsf{H}^2(\Omega_C)$ of the non-homogeneous Dirichlet data $\pm\Phi_a$

$$\hat{\Phi}(x) = \begin{cases} \mp\Phi_a & \text{for } x \in \Omega_M \\ \Phi_C(x) & \text{for } x \in \Omega_C, \end{cases} \tag{4.50}$$

such that for $\Phi_C - \hat{\Phi}_C = \Psi_C$ holds (see appendix, section 6.4)

$$\Psi(x) = \begin{cases} 0 & \text{in } \Omega_M \\ \Phi_C(x) - \hat{\Phi}_C(x) & \text{in } \Omega_C. \end{cases} \tag{4.51}$$

Thus we have

$$\Psi_C = 0 \qquad \text{on } \Gamma^D_{C,e}, \tag{4.52}$$

$$\boldsymbol{\sigma}_{C_n}(\Psi_C) = \boldsymbol{\sigma}_{C_n}(\Phi_C) \qquad \text{on } \Gamma. \tag{4.53}$$

Now, we reformulate the original boundary transmission problem (4.17)-(4.31) as follows:
Let $\underline{\boldsymbol{U}}$ be the original solution, $\underline{\boldsymbol{W}} := \begin{pmatrix} \underline{\boldsymbol{0}} \\ \hat{\Phi} \end{pmatrix}$. Find a solution $\underline{\boldsymbol{V}} = \underline{\boldsymbol{U}} - \underline{\boldsymbol{W}} = \begin{pmatrix} \underline{\boldsymbol{u}} \\ \Psi \end{pmatrix}$, which satisfies the non-homogeneous system of partial differential equations:

$$-\underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}} = \underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{W}}$$

and the following boundary and transmission conditions:

Boundary conditions

$$r_{|\Gamma^D_{C,m}} \underline{\boldsymbol{u}}_C = \underline{\mathbf{0}}\,, \tag{4.54}$$

$$r_{|\Gamma^D_{C,e}} \Psi_C = 0, \tag{4.55}$$

$$r_{|\Gamma^N_{M,m}} \boldsymbol{\sigma}_{M_n}\left(\underline{\boldsymbol{u}}_M\right) = \underline{\mathbf{0}}\,, \tag{4.56}$$

$$r_{|\Gamma^N_{C,m}} \boldsymbol{\sigma}_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right) = -\boldsymbol{\sigma}_{C_n}\left(\underline{\mathbf{0}}\,, \hat{\Phi}_C\right), \tag{4.57}$$

$$r_{|\Gamma^R_{C,e}} D_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right) + \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right) = -D_{C_n}\left(\underline{\mathbf{0}}\,, \hat{\Phi}_C\right). \tag{4.58}$$

Transmission conditions

$$r_{|\Gamma} \underline{\boldsymbol{u}}_C = r_{|\Gamma} \underline{\boldsymbol{u}}_M, \tag{4.59}$$

$$r_{|\Gamma} \boldsymbol{\sigma}_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right) = r_{|\Gamma} \boldsymbol{\sigma}_{M_n}\left(\underline{\boldsymbol{u}}_M\right). \qquad \text{see (4.53)} \tag{4.60}$$

Here, $\mathcal{B}$ is defined as:

$$\mathcal{B} := \begin{cases} \mathcal{B}_3 & \text{for } \underline{\boldsymbol{x}} \in \Gamma_3 \\ \mathcal{B}_4 & \text{for } \underline{\boldsymbol{x}} \in \Gamma_4 \\ \mathcal{B}_{5,6} & \text{for } \underline{\boldsymbol{x}} \in (\Gamma_5 \cup \Gamma_6) \cap \partial\Omega_C. \end{cases} \tag{4.61}$$

The boundaries are the following:

$$\begin{aligned} \Gamma^D_{C,m} &= \Gamma_3, & \Gamma^D_{C,e} &= \Gamma_- \cup \Gamma_+ \cup \Gamma, \\ \Gamma^N_{M,m} &= (\Gamma_- \cup \Gamma_+ \cup \Gamma_5 \cup \Gamma_6) \cap \partial\Omega_M, & \Gamma^N_{C,m} &= (\Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \Gamma_- \cup \Gamma_+) \cap \partial\Omega_C, \\ \Gamma^R_{C,e} &= (\Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6) \cap \partial\Omega_C, & \Gamma &= \Gamma^- \cup \Gamma^+. \end{aligned} \tag{4.62}$$

5 Weak formulation of the boundary-transmission problem

In this section, we formulate the boundary and transmission problem in the weak sense. The weak formulation is convenient for FEM computation, on the other side, it is relatively simple to prove existence and uniqueness of the solutions in appropriate Sobolev spaces.

5.1 Notations and formulation of the equivalent weak problems

In order to formulate the boundary-transmission problem and its boundary and transmission conditions in the weak sense, we need Sobolev spaces and the corresponding trace spaces over open parts of the boundaries and interfaces and their dual spaces. We introduce appropriate spaces also for the homogeneous problem, transforming to homogeneous Dirichlet data and introducing bilinear and non-linear forms.

Definition 5.1. *Let Ω be the domain, occupied by the stack actuator. We define:*

$$\mathcal{V} := \left\{ \underline{\boldsymbol{V}} = \left(\begin{smallmatrix} \boldsymbol{v} \\ \Psi \end{smallmatrix}\right) \in \left[\mathsf{H}^1(\Omega)\right]^{n+1}, r_{|\Gamma^D_{C,m}} \underline{\boldsymbol{v}}_C = 0 \text{ and } r_{|\Gamma^D_{C,e}} \Psi_C = 0 \right\}. \tag{5.1}$$

In short words, this means, that we will search solutions in the space $\left[\mathsf{H}^1(\Omega)\right]^{n+1}$ which satisfy the homogeneous Dirichlet boundary conditions.

Definition 5.2. *Let Γ^* be an open and connected part of $\partial\Omega^* \in \{\partial\Omega_C, \partial\Omega_M\}$ and $d \in \mathbb{N}$. We introduce the trace spaces*

$$\left[\mathsf{H}^{\frac{1}{2}}(\Gamma^*)\right]^d := \left\{\underline{\boldsymbol{f}}^* = r_{\big|\Gamma^*}\underline{\boldsymbol{f}} \,:\, \underline{\boldsymbol{f}} \in \left[\mathsf{H}^{\frac{1}{2}}(\partial\Omega^*)\right]^d\right\}, \tag{5.2}$$

$$\tag{5.3}$$

equipped with the norm

$$\|\underline{\boldsymbol{f}}^*\|_{\left[\mathsf{H}^{\frac{1}{2}}(\Gamma^*)\right]^d} := \inf_{\substack{\underline{\boldsymbol{g}} \in \left[\mathsf{H}^{\frac{1}{2}}(\partial\Omega^*)\right]^d \\ \underline{\boldsymbol{f}}^* = r_{\big|\Gamma^*}\underline{\boldsymbol{g}}}} \|\underline{\boldsymbol{g}}\|_{\left[\mathsf{H}^{\frac{1}{2}}(\partial\Omega^*)\right]^d}. \tag{5.4}$$

Furthermore, let

$$\left[\tilde{\mathsf{H}}^{\frac{1}{2}}(\Gamma^*)\right]^d := \left\{\underline{\boldsymbol{f}} \in \left[\mathsf{H}^{\frac{1}{2}}(\partial\Omega^*)\right]^d : \operatorname{supp}\underline{\boldsymbol{f}} \subset \Gamma^*\right\} \tag{5.5}$$

be equipped with the norm

$$\|\underline{\boldsymbol{f}}^*\|_{\left[\tilde{\mathsf{H}}^{\frac{1}{2}}(\Gamma^*)\right]^d} = \left\|\tilde{\underline{\boldsymbol{f}}}^*\right\|_{\left[\mathsf{H}^{\frac{1}{2}}(\partial\Omega^*)\right]^d}, \tag{5.6}$$

where $\tilde{\underline{\boldsymbol{f}}}^*$ is the zero extension of $\underline{\boldsymbol{f}}^*$ onto $\partial\Omega^* \backslash \bar{\Gamma}^*$.
The dual spaces are [5]

$$\begin{aligned} \left(\mathsf{H}^{\frac{1}{2}}(\Gamma^*)\right)' &= \tilde{\mathsf{H}}^{-\frac{1}{2}}(\Gamma^*), \\ \left(\tilde{\mathsf{H}}^{\frac{1}{2}}(\Gamma^*)\right)' &= \mathsf{H}^{-\frac{1}{2}}(\Gamma^*). \end{aligned} \tag{5.7}$$

Remark 5.1. *Here, Γ^* is given by the boundaries*

$$\Gamma^* = \Gamma^l_{k,j} \subseteq \partial\Omega_k,$$

defined in (4.62), with the indices $k \in \{C, M\}$, $j \in \{m, e\}$ and $l \in \{D, N, R\}$.

Remark 5.2. *By definition 5.1 ($\underline{\boldsymbol{V}} \in \left[\mathsf{H}^1(\Omega)\right]^{n+1}$) it follows directly, that $\underline{\boldsymbol{V}}_C - \underline{\boldsymbol{V}}_M = \underline{\mathbf{0}}$ on the interface Γ.*

Since the dimension of the spaces will be clear in the specific case, we will leave out the upper index d, denoting the space dimension. Here, $n \in \{2, 3\}$, which stands for the two- and the three-dimensional case:

$$\mathsf{H}^1(\Omega) \hat{=} \begin{cases} \left[\mathsf{H}^1(\Omega)\right]^{n+1}: & \underline{\boldsymbol{V}} \in \left[\mathsf{H}^1(\Omega)\right]^{n+1} \text{ has the form } \left(\begin{smallmatrix}\underline{\boldsymbol{v}} \\ \Psi\end{smallmatrix}\right), \\ \left[\mathsf{H}^1(\Omega)\right]^n: & \underline{\boldsymbol{v}} \in \left[\mathsf{H}^1(\Omega)\right]^n \text{ is a mechanical displacement vector}, \\ \mathsf{H}^1(\Omega): & \Psi \in \mathsf{H}^1(\Omega) \text{ is a scalar field (electric potential)}. \end{cases}$$

This notation is used also for $\mathcal{V}$ and the definitions of the trace spaces.
Now, we transform the original problem to the weak problem with homogeneous Dirichlet data. This can be done analogously to the classical formulation, introducing an element $\underline{\boldsymbol{W}} \in \mathsf{H}^1(\Omega)$, $\underline{\boldsymbol{W}} = \begin{pmatrix} \boldsymbol{0} \\ \hat{\Phi} \end{pmatrix}$. Now, we formulate the weak problem for $\underline{\boldsymbol{V}} = \underline{\boldsymbol{U}} - \underline{\boldsymbol{W}}$, with vanishing Dirichlet data $\Psi_C = 0$ on $\Gamma^D_{C,e}$ (4.55) and $\underline{\boldsymbol{u}}_C = \boldsymbol{0}$ on $\Gamma^D_{C,m}$ (4.54).

Definition 5.3. *Let $\underline{\boldsymbol{V}} = \begin{pmatrix} \underline{\boldsymbol{v}} \\ \Psi \end{pmatrix} \in \mathcal{V}, \underline{\boldsymbol{S}} = \begin{pmatrix} \underline{\boldsymbol{s}} \\ \chi \end{pmatrix} \in \mathcal{V}$, with the representation (3.3) on Ω_C or Ω_M. We introduce the following forms (see section 3.1 for the notation):*

$$a(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}) := a_C(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}) + a_M(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}})$$

$$a_M(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}) := a_M(\underline{\boldsymbol{V}}_M, \underline{\boldsymbol{S}}_M) := \int_{\Omega_M} \underline{\underline{\boldsymbol{C}}}_M \mathcal{D}\underline{\boldsymbol{v}}_M \mathcal{D}\underline{\boldsymbol{s}}_M \,\mathrm{d}x, = \left\langle \underline{\underline{\boldsymbol{C}}}_M \mathcal{D}\underline{\boldsymbol{v}}_M, \mathcal{D}\underline{\boldsymbol{s}}_M \right\rangle_{\Omega_M} \tag{5.8}$$

$$a_C(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}) := a_C(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C) := a_{C,I}(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}) + a_{C,II}(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}) \tag{5.9}$$

$$a_{C,I}(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C) := \int_{\Omega_C} \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}_C \cdot \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{S}}_C \,\mathrm{d}x = \left\langle \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}_C, \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{S}}_C \right\rangle_{\Omega_C}, \tag{5.10}$$

$$\begin{aligned} a_{C,II}(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C) &:= \int_{\Gamma^R_{C,e}} \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right)\Big|_{\Gamma^R_{C,e}} \cdot \chi_C\Big|_{\Gamma^R_{C,e}} \,\mathrm{d}a_X \\ &= \left\langle \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right), \chi_C \right\rangle_{\Gamma^R_{C,e}}. \end{aligned} \tag{5.11}$$

By $\langle \cdot, \cdot \rangle$, we denote dual pairings, that means, we formulate the weak problem in the distributional sense. Although the integrals in (5.8), (5.10) and (5.11) exist in the usual sense (due to $\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} \in \mathcal{V}$), we will use from now on in most cases the duality notation.

Remark 5.3. *The restrictions*

$$\begin{aligned} &r\big|_{\Gamma^N_{C,m}} \boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{v}}_C, \Psi_C) \in \mathsf{H}^{-\frac{1}{2}}\left(\Gamma^N_{C,m}\right), \\ &r\big|_{\Gamma^N_{M,m}} \boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{v}}_M) \in \mathsf{H}^{-\frac{1}{2}}\left(\Gamma^N_{M,m}\right), \\ &r\big|_{\Gamma^R_{C,e}} D_{C_n}(\Psi_C) \in \mathsf{H}^{-\frac{1}{2}}\left(\Gamma^R_{C,e}\right) \end{aligned}$$

are well defined for elements $\underline{\boldsymbol{V}} \in \mathcal{V}(\Omega)$ [2] which are weak solutions of the system (5.12) hereafter. We leave out the restriction operator in the notation, since it will be clear by the index, with respect to which spaces the dual pairing is formulated:

$$\begin{aligned} \langle \boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{v}}_C, \Psi_C), \underline{\boldsymbol{s}}_C \rangle_{\Gamma^N_{C,m}} &:= \left\langle r\big|_{\Gamma^N_{C,m}} \boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{v}}_C, \Psi_C), r\big|_{\Gamma^N_{C,m}} \underline{\boldsymbol{s}}_C \right\rangle \\ \langle \boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{v}}_M), \underline{\boldsymbol{s}}_M \rangle_{\Gamma^N_{M,m}} &:= \left\langle r\big|_{\Gamma^N_{M,m}} \boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{v}}_M), r\big|_{\Gamma^N_{M,m}} \underline{\boldsymbol{s}}_M \right\rangle \\ \langle D_{C_n}(\underline{\boldsymbol{v}}_C, \Psi_C), \chi_C \rangle_{\Gamma^R_{C,e}} &:= \left\langle r\big|_{\Gamma^R_{C,e}} D_n(\underline{\boldsymbol{v}}_C, \Psi_C), r\big|_{\Gamma^R_{C,e}} \chi_C \right\rangle \end{aligned}$$

Lemma 5.1. *The weak formulation of the system of partial differential equations (see (4.47))*

$$\begin{aligned} -\underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}} &= -\underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{U}} + \underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{W}} \qquad (5.12) \\ &= \underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{W}}, \qquad (5.13) \end{aligned}$$

together with the boundary and transmission conditions (4.54)-(4.58) and (4.59)-(4.60) reads: Find $\underline{\boldsymbol{V}} \in \mathcal{V}$, such that:

$$a\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) = a_C\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C\right) + a_M\left(\underline{\boldsymbol{V}}_M, \underline{\boldsymbol{S}}_M\right) = -a_{C,I}\left(\underline{\boldsymbol{W}}, \underline{\boldsymbol{S}}\right)$$

holds for all $\underline{\boldsymbol{S}} \in \mathcal{V}$.

Proof. Having in mind that $\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} \in \mathsf{H}^1(\Omega)$, scalar multiplication of the classically written system of partial differential equations (5.12) with $\underline{\boldsymbol{S}} \in \mathcal{V}$ and integration of this expression leads to the left hand side:

$$-\int_{\Omega_C \cup \Omega_M} \underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}} \cdot \underline{\boldsymbol{S}} \,\mathrm{d}x.$$

After partial integration, we get:

$$-\int_{\Omega_C \cup \Omega_M} \underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}} \cdot \underline{\boldsymbol{S}} \,\mathrm{d}x = \int_{\Omega_C \cup \Omega_M} \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}} \cdot \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{S}} \,\mathrm{d}x - \int_{\partial\Omega_C \cup \partial\Omega_M} \underline{\underline{\mathbf{N}}}^\top \left(\underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}\right) \cdot \underline{\boldsymbol{S}} \,\mathrm{d}a_x, \tag{5.14}$$

where $\underline{\underline{\mathbf{N}}}$ denotes the co-normal matrix (4.1). Now, we consider the integrals on the domains Ω_C and Ω_M separately and we take into account the boundary and transmission conditions (4.54)-(4.60) for $\underline{\boldsymbol{V}} \in \mathcal{V}$. In this way, we get:

$$\begin{aligned}
&\left\langle \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}_C, \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{S}}_C \right\rangle_{\Omega_C} + \left\langle \underline{\underline{\boldsymbol{A}}}_M \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}_M, \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{S}}_M \right\rangle_{\Omega_M} - \left\langle \underline{\underline{\mathbf{N}}}^\top \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C \right\rangle_{\partial\Omega_C} \\
&- \left\langle \underline{\underline{\mathbf{N}}}^\top \underline{\underline{\boldsymbol{A}}}_M \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}_M, \underline{\boldsymbol{S}}_M \right\rangle_{\partial\Omega_M} \\
&= a_{C,I}\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) + a_M\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) - \left\langle \underline{\underline{\mathbf{N}}}^\top \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C \right\rangle_{\partial\Omega_C} - \left\langle \underline{\underline{\mathbf{N}}}^\top \underline{\underline{\boldsymbol{A}}}_M \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}_M, \underline{\boldsymbol{S}}_M \right\rangle_{\partial\Omega_M} \\
&= a_{C,I}\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) + a_M\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) - \left\langle \boldsymbol{\sigma}_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right), \underline{\boldsymbol{s}}_C \right\rangle_{\Gamma^D_{C,m}} - \left\langle \boldsymbol{\sigma}_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right), \underline{\boldsymbol{s}}_C \right\rangle_{\Gamma^N_{C,m}} \\
&- \left\langle \boldsymbol{\sigma}_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right), \underline{\boldsymbol{s}}_C \right\rangle_{\Gamma} - \left\langle D_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right), \chi_C \right\rangle_{\Gamma^D_{C,e}} - \left\langle D_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right), \chi_C \right\rangle_{\Gamma^R_{C,e}} \\
&- \left\langle \boldsymbol{\sigma}_{M_n}\left(\underline{\boldsymbol{u}}_M\right), \underline{\boldsymbol{s}}_M \right\rangle_{\Gamma^N_{M,m}} - \left\langle \boldsymbol{\sigma}_{M_n}\left(\underline{\boldsymbol{u}}_M\right), \underline{\boldsymbol{s}}_M \right\rangle_{\Gamma} \\
&= a_C\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) + a_M\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) + \left\langle \boldsymbol{\sigma}_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right), \underline{\boldsymbol{s}}_C \right\rangle_{\Gamma^N_{C,m}} + \left\langle D_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right), \chi_C \right\rangle_{\Gamma^R_{C,e}} \\
&= -a_{C,I}\left(\underline{\boldsymbol{W}}, \underline{\boldsymbol{S}}\right) + \left\langle \boldsymbol{\sigma}_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right), \underline{\boldsymbol{s}}_C \right\rangle_{\Gamma^N_{C,m}} + \left\langle D_{C_n}\left(\underline{\boldsymbol{u}}_C, \Psi_C\right), \chi_C \right\rangle_{\Gamma^R_{C,e}}.
\end{aligned} \tag{5.15}$$

Since $\underline{\boldsymbol{S}} \in \mathsf{H}^1(\Omega)$, that means $\underline{\boldsymbol{S}}_M = \underline{\boldsymbol{S}}_C$ on Γ, we get

$$a_C\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) + a_M\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) = -a_{C,I}\left(\underline{\boldsymbol{W}}, \underline{\boldsymbol{S}}\right). \tag{5.16}$$

□

5.2 Existence and uniqueness

5.3 Properties of the form a

The form $a\left(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}\right) = a_C\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C\right) + a_M\left(\underline{\boldsymbol{V}}_M, \underline{\boldsymbol{S}}_M\right)$ generates a nonlinear operator $\mathfrak{A} : \mathcal{V} \rightarrow \mathcal{V}'$, defined for the whole domain Ω which consists of non-linear and bilinear forms for Ω_C and Ω_M, respectively. In order to show existence of the weak solution, we split the above terms into metallic and ceramic parts and verify the Lax-Milgram conditions for the linear metal part and the strong monotonicity, continuity and coercivity of the form a_C, which are corresponding conditions for existence and uniqueness for equations involving nonlinear operators. It follows, that these conditions are satisfied also by the nonlinear form a.

Theorem 5.1. *Let the functions $\mathcal{B}_i$ be monotonous, bounded and continuous. Then $\mathfrak{A}$ is strongly monotone, coercive and continuous and thus existence and uniqueness of the weak solution follows.*

Proof. We recall

$$\langle \mathfrak{A}\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} \rangle = a(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}) = a_M(\underline{\boldsymbol{V}}_M, \underline{\boldsymbol{S}}_M) + a_C(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C).$$

It is obvious, that the linear parts a_M and $a_{C,I}$ satisfy the assertion due to lemma 6.2 (ceramic) and 6.3 (metal) in section 6.5 (ceramic) respectively 6.6 (metal) of the appendix. It remains to consider the nonlinear part $a_{C,II}$ and a_C more detailed.
Since we have the non-linear boundary condition on $\Gamma^R_{C,e}$, the resulting form is no more bilinear and thus the linear theory can not be applied for the whole form a_C:

$$a_C(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C) := a_{C,I}(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}_C}) + \left\langle \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right), \chi_C \right\rangle_{\Gamma^R_{C,e}}.$$

In order to apply the fundamental theorem about monotonous operators, we need to show continuity and strong monotonicity (it follows coercivity) for the operator $\mathfrak{A}_C$ defined via the corresponding form a_C.

Continuity.

By equation (6.12), we know that $a_{C,I}(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C)$ is a continuous bilinear form. Now, we have to show that also the non-linear part $a_{C,II}$ is continuous. That means for a sequence Ψ^n_C with $\|\Psi^n_C - \Psi_C\|_{\mathcal{V}} \to 0$ should hold: $|a_{C,II}(\Psi^n_C, \chi_C) - a_{C,II}(\Psi_C, \chi_C)| \to 0$.

$$\begin{aligned}
|a_{C,II}(\Psi^n_C, \chi_C) - a_{C,II}(\Psi_C, \chi_C)| &= \left| \int_{\Gamma^R_{C,e}} \left(\mathcal{B}\left(\Psi^n_C + \hat{\Phi}_C\right) - \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right)\right) \chi_C \, \mathrm{d}a_x \right| \\
&\leq \left\| \mathcal{B}\left(\Psi^n_C + \hat{\Phi}_C\right) - \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right) \right\|_{L^2(\Gamma^R_{C,e})} \|\chi_C\|_{L^2(\Gamma^R_{C,e})} \\
&\leq c \|\Psi^n_C - \Psi_C\|_{L^2(\Gamma^R_{C,e})} \|\chi_C\|_{L^2(\Gamma^R_{C,e})} \to 0.
\end{aligned}$$

Monotonicity.

$\mathfrak{A}_C$ is strongly monotonous if and only if $\langle \mathfrak{A}_C\underline{\boldsymbol{S}} - \mathfrak{A}_C\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} - \underline{\boldsymbol{V}} \rangle \geq c\|\underline{\boldsymbol{S}} - \underline{\boldsymbol{V}}\|^2_{\mathcal{V}}$ for all $\underline{\boldsymbol{S}}, \underline{\boldsymbol{V}} \in \mathcal{V}$ with $\underline{\boldsymbol{S}} \neq \underline{\boldsymbol{V}}$ and a positive constant c. It holds:

$$\begin{aligned}
&\langle \mathfrak{A}_C\underline{\boldsymbol{S}} - \mathfrak{A}_C\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} - \underline{\boldsymbol{V}} \rangle = \langle \mathfrak{A}_C\underline{\boldsymbol{S}}, \underline{\boldsymbol{S}} \rangle - \langle \mathfrak{A}_C\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} \rangle - \langle \mathfrak{A}_C\underline{\boldsymbol{S}}, \underline{\boldsymbol{V}} \rangle + \langle \mathfrak{A}_C\underline{\boldsymbol{V}}, \underline{\boldsymbol{V}} \rangle \\
&= a_{C,I}(\underline{\boldsymbol{S}}, \underline{\boldsymbol{S}}) + a_{C,I}(\underline{\boldsymbol{V}}, \underline{\boldsymbol{V}}) + \left\langle \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right), \Psi_C \right\rangle_{\Gamma^R_{C,e}} + \left\langle \mathcal{B}\left(\chi_C + \hat{\Phi}_C\right), \chi_C \right\rangle_{\Gamma^R_{C,e}} \\
&- a_{C,I}(\underline{\boldsymbol{S}}, \underline{\boldsymbol{V}}) - a_{C,I}(\underline{\boldsymbol{V}}, \underline{\boldsymbol{S}}) - \left\langle \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right), \chi_C \right\rangle_{\Gamma^R_{C,e}} - \left\langle \mathcal{B}\left(\chi_C + \hat{\Phi}_C\right), \Psi_C \right\rangle_{\Gamma^R_{C,e}} \\
&= a_{C,I}(\underline{\boldsymbol{S}} - \underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} - \underline{\boldsymbol{V}}) + \left\langle \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right), \Psi_C \right\rangle_{\Gamma^R_{C,e}} + \left\langle \mathcal{B}\left(\chi_C + \hat{\Phi}_C\right), \chi_C \right\rangle_{\Gamma^R_{C,e}}
\end{aligned}$$

$$
\begin{aligned}
&- \left\langle \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right), \chi_C \right\rangle_{\Gamma^R_{C,e}} - \left\langle \mathcal{B}\left(\chi_C + \hat{\Phi}_C\right), \Psi_C \right\rangle_{\Gamma^R_{C,e}} \\
&= a_{C,I}\left(\underline{\boldsymbol{S}} - \underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} - \underline{\boldsymbol{V}}\right) + \left\langle \mathcal{B}\left(\Psi_C + \hat{\Phi}_C\right), (\Psi_C - \chi_C) \right\rangle_{\Gamma^R_{C,e}} \\
&+ \left\langle \mathcal{B}\left(\chi_C + \hat{\Phi}_C\right), (\chi_C - \Psi_C) \right\rangle_{\Gamma^R_{C,e}} \\
&= a_{C,I}\left(\underline{\boldsymbol{S}} - \underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} - \underline{\boldsymbol{V}}\right) + \left\langle \mathcal{B}\left(\Psi + \hat{\Phi}_C\right) - \mathcal{B}\left(\chi_C + \hat{\Phi}_C\right), (\Psi_C - \chi_C) \right\rangle_{\Gamma^R_{C,e}}.
\end{aligned}
$$

Since $\mathcal{B}$ is a monotonously growing function, the second term is non negative. Therefore, the operator $\mathfrak{A}_C$ satisfies the inequality

$$
\langle \mathfrak{A}_C \underline{\boldsymbol{S}} - \mathfrak{A}_C \underline{\boldsymbol{V}}, \underline{\boldsymbol{S}} - \underline{\boldsymbol{V}} \rangle \geq c \, \|\underline{\boldsymbol{S}} - \underline{\boldsymbol{V}}\|^2_{\mathcal{V}} > 0,
$$

which means, that $\mathfrak{A}_C$ is a strongly monotonous operator. □

6 Appendix

6.1 The electric field in the electrodes

As one knows from physics, electrical fields $\underline{\boldsymbol{E}}$ are conservative and therefore there exists an electric potential Φ, which is also called *Coulomb potential*, such that

$$\underline{\boldsymbol{E}}\,(\underline{\boldsymbol{x}}\,) = -\nabla\Phi\,(\underline{\boldsymbol{x}}\,)\,. \tag{6.1}$$

The Poisson equation

$$\Delta\Phi\,(\underline{\boldsymbol{x}}\,) = -\frac{1}{\varepsilon_0}\rho(\underline{\boldsymbol{x}}\,) \tag{6.2}$$

is called the *basic equation of electrodynamics*; here, $\rho(\underline{\boldsymbol{x}}\,)$ is a charge density. Its' distribution solutions are well known [9]. We discuss the so called simple layer case, that means $\rho\,(\underline{\boldsymbol{x}}\,) = s\delta_{\mathbb{R}^2}\,(x_3)$, where $s(x) = s$ is the face charge density.

Lemma 6.1. *The solution of the Poisson equation*

$$\Delta\Phi\,(\underline{\boldsymbol{x}}\,) = -\frac{s}{\varepsilon_0}\delta_{\mathbb{R}^2}(x_3)$$

reads:

$$\Phi\,(\underline{\boldsymbol{x}}\,) = -\frac{s}{2\varepsilon_0}\,|x_3| + \text{const.}$$

Furthermore, it holds, that:

$$\nabla\Phi = \underline{\boldsymbol{E}}\,(\underline{\boldsymbol{x}}\,) = \frac{-s}{2\varepsilon_0}\boldsymbol{e}_{x_3}\text{sgn}(x_3) = \frac{-s}{2\varepsilon_0}\boldsymbol{e}_{x_3}\frac{x_3}{|x_3|} \tag{6.3}$$

and

$$\begin{aligned}\left|\left[\!\left[\frac{\partial\Phi}{\partial n}\right]\!\right]\right| &= \left|\frac{\partial\Phi}{\partial n_{u,+}} + \frac{\partial\Phi}{\partial n_{l,-}}\right| = \left|\frac{\partial\Phi}{\partial n_{u,+}} - \frac{\partial\Phi}{\partial n_{l,+}}\right| \\ &= \frac{s}{\varepsilon_0}.\end{aligned}$$

Proof. Since the charge density vanishes outside the $x_1 - x_2$-plane and is constant within it, the potential Φ depends only on the variable x_3. Thus the partial differential equation (6.2) reduces to an ordinary differential equation:

$$\partial_3^2\Phi = -\frac{s}{\varepsilon_0}\delta_{\mathbb{R}^2}\,(x_3)\,. \tag{6.4}$$

From the theory of distributions, it is well known, that the solution Φ of (6.4) (see also [6]) reads:

$$\Phi\,(\underline{\boldsymbol{x}}\,) = -\frac{s}{2\varepsilon_0}\,|x_3| + c_1x_3 + c_2. \tag{6.5}$$

Therefore, for the electrical field follows:

$$\nabla\Phi = \underline{\boldsymbol{E}}\,(\underline{\boldsymbol{x}}\,) = \frac{-s}{2\varepsilon_0}\boldsymbol{e}_{x_3}\text{sgn}(x_3) = \frac{-s}{2\varepsilon_0}\boldsymbol{e}_{x_3}\frac{x_3}{|x_3|}, \tag{6.6}$$

where $\underline{\boldsymbol{e}}_{x_3} = (0,0,1)^\top$ is the normal unit vector to the charged $x_1 - x_2$-plane. Furthermore, we remark, that $\underline{\boldsymbol{E}} \cdot \underline{\boldsymbol{n}} = \nabla\Phi \cdot \underline{\boldsymbol{n}} = \partial_n \Phi$ jumps on the surface of the conductor.

$$\left| \left[\frac{\partial \Phi}{\partial n} \right] \right| = \left| \frac{\partial \Phi}{\partial n_{u,+}} + \frac{\partial \Phi}{\partial n_{l,-}} \right| = \left| \frac{\partial \Phi}{\partial n_{u,+}} - \frac{\partial \Phi}{\partial n_{l,+}} \right| = \frac{s}{\varepsilon_0}. \tag{6.7}$$

The positive sign indicates the normal unit vector in direction of the positive x_3-axis, the negative one in the negative direction of the x_3-axis; the index u indicates the upper side of the thin layer, l the lower side. □

6.2 The electrodes (thin metal plate)

Now, we consider the electrodes as metal plates with a small thickness (figure 5). The charge is located at the upper (Γ_u) and lower (Γ_l) sides of the plate. Γ_u and Γ_l indicate the two simple layers now. $\underline{\boldsymbol{E}}_u$ denotes the electrical field, generated by the charge of the upper side Γ_u of the

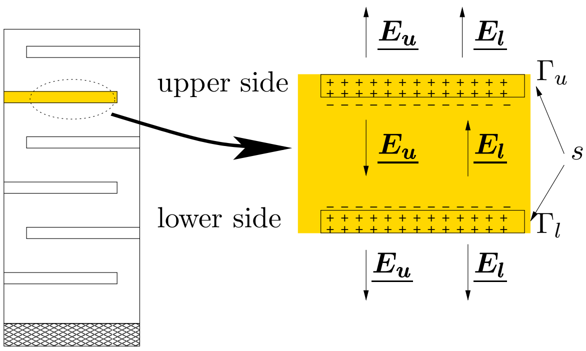


Figure 5: Electrode with positive charge; $\underline{\boldsymbol{E}}_u$ is generated by the upper side of the plate and $\underline{\boldsymbol{E}}_l$ by the lower side.

plate, $\underline{\boldsymbol{E}}_l$ corresponds to the lower side Γ_l. Inside the conductor, the resulting electrostatic field $\underline{\boldsymbol{E}}$ vanishes by superposition of $\underline{\boldsymbol{E}}_u$ and $\underline{\boldsymbol{E}}_l$ (see figure 5). Since $\underline{\boldsymbol{E}} = \nabla\Phi = 0$, it follows, that Φ is constant inside the conductor. On the upper and lower sides of the conductor, the relation (6.7) is valid for the potential fields $\Phi = \Phi_u = \Phi_l$ given by (6.5).

6.3 Boundary conditions on $\Gamma_3, \Gamma_4, \Gamma_5$ and Γ_6

Assuming a large space between the lowest electrode and Γ_3, we could choose approximately:

$$Ⓝ_e \qquad D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = 0, \qquad \text{on } \Gamma_3.$$

This boundary condition does not reflect the real nature of the electric flux density, if the electrode is near the boundary and the voltage is applied up to the lower ends of the boundaries Γ^- and Γ^+. In fact, the electric vector field will be similar to that of a condenser at the edges of the electrode plates. Exact boundary conditions are computed by coupling the electric field in the actuator with the electric field outside the actuator. This leads to a new coupled system of partial differential equations with transmission conditions for the outer and inner electric potential field. In order to avoid these computations, we make a compromise. We know by equation (6.6) of the appendix, the electric flux has the value $\frac{s}{2}$ on the boundary. Here, s denotes the absolute value of the face charge density. We introduce the notation Ω_{M+} for the electrode occupied domain, connected with the positively charged side (Γ_+) and respectively Ω_{M-} for the electrode domain, connected with Γ_-. Furthermore, let

$$\mathfrak{s} := \frac{s}{2\tanh(\Phi_a)}. \tag{6.8}$$

We pose the following Robin-like boundary condition,

$$Ⓡ_e \qquad D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = D_{C_n}(\Phi_C) = -\mathfrak{s}\tanh(\Phi_C(\underline{\boldsymbol{x}})) \qquad \text{on } \Gamma_3, \tag{6.9}$$

which interpolates D_{C_n}, if we denote by b_1 the half width of the stack actuator in x_1 direction. On the edged $x_1 = \pm b_1$, we have

$$D_{C_n}(\Phi_C(\mp b_1, x_2, 0)) = -\frac{s}{2} = \tanh(\Phi_C(\mp b_1, x_2, 0)). \tag{6.10}$$

At the point $y \in \Gamma_3$, where $\Phi_c(y) = 0$, one can realise that

$$D_{C_n}\left(\Phi_C\left(\underline{\boldsymbol{y}}\right)\right) = \underline{\boldsymbol{n}}^\top \underline{\underline{\boldsymbol{\varepsilon}}} \nabla\Phi_C = 0 \quad \text{on } \Gamma_3, \text{ if } \Phi_C\left(\underline{\boldsymbol{y}}\right) = 0. \tag{6.11}$$

We remark, that $s \geq 0$ should be measured.

Analogously, we can assume this boundary condition as an approximation on Γ_4, Γ_5 and Γ_6. To give a justification for (6.11), the simplified problem of an electric field, generated only by the charge $\mp q$ located at the endpoints of Γ_3 in the $2D$ model is considered. The endpoints are (see also figure 6) $(-b_1\, 0)^\top = (\Gamma_3 \cap \Gamma_-)$ and $(b_1\, 0)^\top = (\Gamma_3 \cap \Gamma_+)$. Thus for the outside electric potential field holds

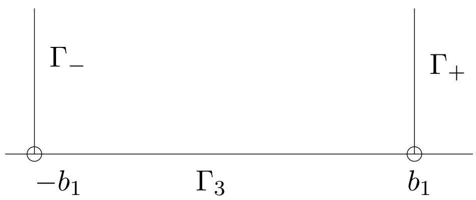


Figure 6: Simplified 2D-model, where the charge resides in the cornerpoints of Γ_3.

$$\Phi(\underline{\boldsymbol{x}}) = -q\frac{1}{\left|\underline{\boldsymbol{x}} - \begin{pmatrix}-b_1\\0\end{pmatrix}\right|} + q\frac{1}{\left|\underline{\boldsymbol{x}} - \begin{pmatrix}b_1\\0\end{pmatrix}\right|}.$$

The electric potential field is assumed to vanish for $y = \underline{\mathbf{0}} \in \Gamma_3$. We demand

$$0 = -q\frac{1}{\left|\underline{\boldsymbol{y}} - \begin{pmatrix}-b_1\\0\end{pmatrix}\right|} + q\frac{1}{\left|\underline{\boldsymbol{y}} - \begin{pmatrix}b_1\\0\end{pmatrix}\right|}$$

$$\left((x_1 - b_1)^2 + x_2^2\right)^{\frac{1}{2}} = \left((x_1 + b_1)^2 + x_2^2\right)^{\frac{1}{2}}.$$

Inserting $\underline{\boldsymbol{y}} = \underline{\mathbf{0}}$ into the equation for the electric vector field, generated by two charges

$$\underline{\boldsymbol{E}}\left(\underline{\boldsymbol{y}}\right) = -q\frac{\underline{\boldsymbol{y}} - \begin{pmatrix}-b_1\\0\end{pmatrix}}{\left|\underline{\boldsymbol{y}} - \begin{pmatrix}-b_1\\0\end{pmatrix}\right|^3} + q\frac{\underline{\boldsymbol{y}} - \begin{pmatrix}b_1\\0\end{pmatrix}}{\left|\underline{\boldsymbol{y}} - \begin{pmatrix}b_1\\0\end{pmatrix}\right|^3},$$

it follows

$$\underline{\boldsymbol{E}}\left(\underline{\mathbf{0}}\right) = \frac{2q}{|b_1|^3}\begin{pmatrix}-b_1\\0\end{pmatrix},$$

$$\underline{\boldsymbol{n}}^\top \underline{\underline{\boldsymbol{\varepsilon}}}\nabla\Phi = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\varepsilon_{11} & 0 & 0\\ 0 & \varepsilon_{11} & 0\\ 0 & 0 & \varepsilon_{33}\end{pmatrix}\begin{pmatrix}-b_1\\0\end{pmatrix} = 0.$$

Note, that extending this model by a general charge distribution $\rho(\underline{\boldsymbol{x}})$ instead of the discrete charges will result in a shifting of the point $\underline{\boldsymbol{y}} \in \Gamma_3$ in x_1-direction.

6.4 A smooth continuation for the electrical Dirichlet data

To introduce a smooth continuation of the non-homogeneous electrical Dirichlet data, we define the closed subsets $\Omega_{\delta-}$ and $\Omega_{\delta+}$ of $\overline{\Omega}_C$ by (see figure 7):

$$\Omega_{\delta-} := \left\{\underline{\boldsymbol{x}} \in \Omega_C : \max\left(|x_1 - y_1|, |x_3 - y_3|\right) \le \frac{\delta}{4}, \forall \underline{\boldsymbol{y}} \in \Gamma^D_{Ce,-}\right\},$$
$$\Omega_{\delta+} := \left\{\underline{\boldsymbol{x}} \in \Omega_C : \max\left(|x_1 - y_1|, |x_3 - y_3|\right) \le \frac{\delta}{4}, \forall \underline{\boldsymbol{y}} \in \Gamma^D_{Ce,+}\right\},$$

where $\delta > 0$ is the minimal distance between $\Gamma^- \cup \Gamma_-$ and $\Gamma^+ \cup \Gamma_+$.
Furthermore let $G_{\delta-} \supset \Omega_{\delta-}$ and $G_{\delta+} \supset \Omega_{\delta+}$ be open domains, such that

$$G_{\delta-} \cap G_{\delta+} = \emptyset.$$

Then, due to [11], page 18, there exists a continuation $\hat{\Phi} \in \mathcal{C}^\infty\left(\mathbb{R}^3\right)$, which equals $\mp\Phi_a$ on $\Omega_{\delta-}$ and $\Omega_{\delta+}$ respectively. This construction yields, that

$$\hat{\Phi}_C = \mp\Phi_a \qquad \text{in a neighbourhood of } \Gamma^D_{C,e},$$
$$\sigma_{C_n}\left(\underline{\mathbf{0}}, \hat{\Phi}_C\right) = -\underline{\underline{\boldsymbol{N}}}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla\hat{\Phi}_C = 0 \qquad \text{on } \Gamma^D_{C,e}.$$

6.5 Coerciveness-properties of the bilinear form $a_{C,I}$

Lemma 6.2. *Let $a_{C,I}\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C\right)$ be the bilinear form over $\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C \in \mathcal{V}$, defined in (5.10). It holds:*

1. *$a_{C,I}\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C\right)$ is $\mathcal{V}$-bounded and thus continuous.*
2. *$a_{C,I}\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C\right)$ is $\mathcal{V}$-elliptic, provided, $\underline{\underline{\boldsymbol{C}}}_C$ and $\underline{\underline{\boldsymbol{\varepsilon}}}$ are positive definite, which holds for standard ceramic materials.*

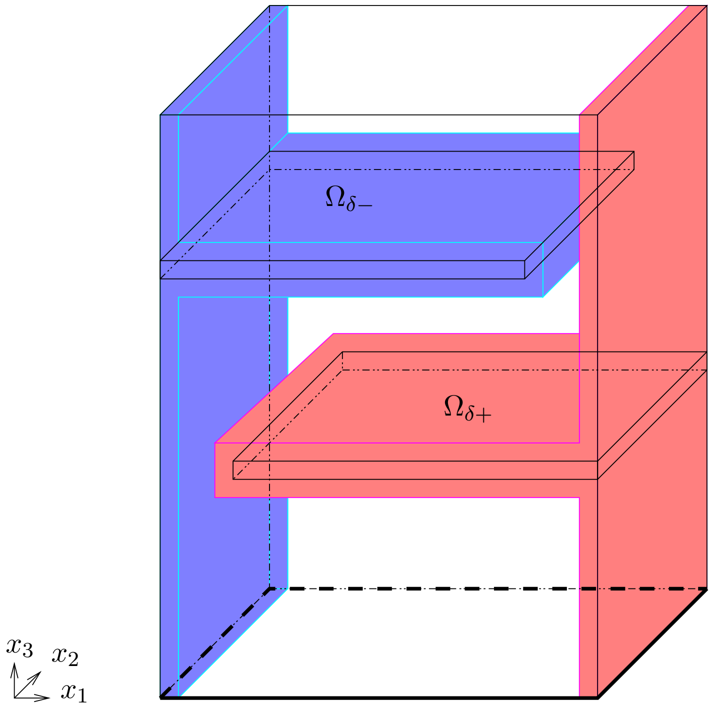


Figure 7: Continuation of the electrical Dirichlet boundary data.

Therefore, the conditions of the Lax-Milgram lemma are satisfied.

Proof of the first assertion. We recall, that $a_{C,I}\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C\right)$ was defined as:

$$\left\langle \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{V}}_C, \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{S}}_C \right\rangle_{\Omega_C} = \int_{\Omega_C} \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{S}}_C \cdot \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{V}}\ \mathrm{d}x.$$

Thus, we get with $\underline{\boldsymbol{S}}_C^\top = (s_{C_1}, s_{C_2}, s_{C_3}, \chi_C)$ and $\underline{\boldsymbol{V}}_C^\top = (v_{C_1}, v_{C_2}, v_{C_3}, \Psi_C)$:

$$\begin{aligned}
\left|a_{C,I}\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{S}}_C\right)\right| &= \left| \int_{\Omega_C} \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{V}}_C \cdot \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{S}}_C\, \mathrm{d}x \right| \\
&\leq \sum_{i=1}^{9} \left| \int_{\Omega_C} \left(\underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{V}}_C\right)_i \left(\underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{S}}_C\right)_i \mathrm{d}x \right| \\
&\leq \sum_{i=1}^{9} \int_{\Omega_C} \left| \left(\underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{V}}_C\right)_i \left(\underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{S}}_C\right)_i \right| \mathrm{d}x \\
&\leq \sum_{i=1}^{9} \left\| \left(\underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{V}}_C\right)_i \right\|_{L^2(\Omega_C)} \left\| \left(\underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{S}}_C\right)_i \right\|_{L^2(\Omega_C)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \max \left|A_{C_{i,j}}\right| (\|\partial_1 s_{C_1}\|_{L^2} \|\partial_1 v_{C_1}\|_{L^2} + \|\partial_2 v_{C_2}\|_{L^2} \|\partial_2 s_{C_2}\|_{L^2} \\
&+ \|\partial_3 v_{C_3}\|_{L^2} \|\partial_3 s_{C_3}\|_{L^2} + \|\partial_2 s_{C_3} + \partial_3 s_{C_2}\|_{L^2} \|\partial_2 v_{C_3} + \partial_3 v_{C_2}\|_{L^2} \\
&+ \|\partial_1 s_{C_3} + \partial_3 s_{C_1}\|_{L^2} \|\partial_1 v_{C_3} + \partial_3 v_{C_1}\|_{L^2} \\
&+ \|\partial_1 s_{C_2} + \partial_2 s_{C_1}\|_{L^2} \|\partial_1 v_{C_2} + \partial_2 v_{C_1}\|_{L^2} \\
&+ \|\partial_1 \Psi_C\|_{L^2} \|\partial_1 \chi_C\|_{L^2} + \|\partial_2 \Psi_C\|_{L^2} \|\partial_2 \chi_C\|_{L^2} + \|\partial_3 \Psi_C\|_{L^2} \|\partial_3 \chi_C\|_{L^2}) \\
&\leq \left(\max \left|A_{C_{i,j}}\right|\right) \left(\sum_{i=1}^{3} \|\partial_1 v_{C_i}\|_{L^2} + \|\partial_2 v_{C_i}\|_{L^2} + \|\partial_3 v_{C_i}\|_{L^2}\right) \\
&\cdot \left(\sum_{i=1}^{3} \|\partial_1 s_{C_i}\|_{L^2} + \|\partial_2 s_{C_i}\|_{L^2} + \|\partial_3 s_{C_i}\|_{L^2}\right) \\
&+ \sum_{i=1}^{3} \left(\|\partial_i \Psi_C\|\right) \sum_{i=1}^{3} \left(\|\partial_i \chi_C\|_{L^2}\right) \qquad (6.12) \\
&\leq c_{I,1} \|\underline{\boldsymbol{V}}_C\|_{\mathcal{V}} \|\underline{\boldsymbol{S}}_C\|_{\mathcal{V}}.
\end{aligned}
$$

□

Proof of the second assertion. We recapitulate, that $\underline{\underline{\boldsymbol{A}}}_C$ has block structure:

$$
\underline{\underline{\boldsymbol{A}}}_C = \begin{pmatrix} \underline{\underline{\boldsymbol{C}}}_C & -\underline{\underline{\boldsymbol{e}}}^\top \\ \underline{\underline{\boldsymbol{e}}} & \underline{\underline{\boldsymbol{\varepsilon}}} \end{pmatrix}
$$

and will be applied to $\left(\underline{\boldsymbol{\gamma}}_C, -\underline{\boldsymbol{E}}_C\right)$. Therefore, $a_{C,I}\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{V}}_C\right)$ can be written as:

$$
\begin{aligned}
a_{C,I}\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{V}}_C\right) &= \int_{\Omega_C} \begin{pmatrix} \underline{\boldsymbol{\gamma}}\left(\underline{\boldsymbol{v}}_C\right) \\ -\underline{\boldsymbol{E}}_C\left(\Psi_C\right) \end{pmatrix}^\top \begin{pmatrix} \underline{\underline{\boldsymbol{C}}}_C & -\underline{\underline{\boldsymbol{e}}}^\top \\ \underline{\underline{\boldsymbol{e}}} & \underline{\underline{\boldsymbol{\varepsilon}}} \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{\gamma}}\left(\underline{\boldsymbol{v}}_C\right) \\ -\underline{\boldsymbol{E}}_C\left(\Psi_C\right) \end{pmatrix} \mathrm{d}x \\
&= \int_{\Omega_C} \underline{\boldsymbol{\gamma}}_C^\top \underline{\underline{\boldsymbol{C}}}_C \underline{\boldsymbol{\gamma}}_C + \underline{\boldsymbol{\gamma}}_C^\top \underline{\underline{\boldsymbol{e}}}^\top \underline{\boldsymbol{E}}_C - \underline{\boldsymbol{E}}_C^\top \underline{\underline{\boldsymbol{e}}} \underline{\boldsymbol{\gamma}}_C + \underline{\boldsymbol{E}}_C^\top \underline{\underline{\boldsymbol{\varepsilon}}} \underline{\boldsymbol{E}}_C \,\mathrm{d}x \\
&= \int_{\Omega_C} \underline{\boldsymbol{\gamma}}_C^\top \underline{\underline{\boldsymbol{C}}}_C \underline{\boldsymbol{\gamma}}_C + \underline{\boldsymbol{E}}_C^\top \underline{\underline{\boldsymbol{\varepsilon}}} \underline{\boldsymbol{E}}_C \,\mathrm{d}x.
\end{aligned}
$$

It follows, that $a_{C,I}\left(\underline{\boldsymbol{V}}_C, \underline{\boldsymbol{V}}_C\right)$ is independent of the piezoelectric tensor $\underline{\underline{\boldsymbol{e}}}$. For the used standard piezoelectric ceramic material, $\underline{\underline{\boldsymbol{C}}}_C$ and $\underline{\underline{\boldsymbol{\varepsilon}}}$ are positive definite. We now split the integral into the mechanical and the electrical term in order to show estimates for each of both summands.

Mechanical part.

$$
\begin{aligned}
\int_{\Omega_C} \underline{\boldsymbol{\gamma}}\left(\underline{\boldsymbol{v}}_C\right)^\top \underline{\underline{\boldsymbol{C}}}_C \underline{\boldsymbol{\gamma}}\left(\underline{\boldsymbol{v}}_C\right) \mathrm{d}x &\geq \lambda_{1m} \int_{\Omega_C} \underline{\boldsymbol{\gamma}}^\top\left(\underline{\boldsymbol{v}}_C\right) \underline{\boldsymbol{\gamma}}\left(\underline{\boldsymbol{v}}_C\right) \mathrm{d}x \\
&= \lambda_{1m} \left\|\underline{\boldsymbol{\gamma}}\left(\underline{\boldsymbol{v}}_C\right)\right\|^2_{[L^2(\Omega_C)]^n},
\end{aligned}
$$

where λ_{1m} is the smallest eigenvalue of $\underline{\underline{\boldsymbol{C}}}_C$. Since the surface measure $\mu\left(\Gamma^D_{C,m}\right) \neq 0$, we can apply Korn's inequality [1]:

$$
\lambda_{1m} \left\|\underline{\boldsymbol{\gamma}}\left(\underline{\boldsymbol{v}}_C\right)\right\|^2_{[L^2(\Omega_C)]^n} \geq c_{1m}\left(\lambda_{1m}, \Omega_C, \Gamma^D_{C,m}\right) \cdot \|\underline{\boldsymbol{v}}_C\|^2_{\mathcal{V}}. \qquad (6.13)
$$

Electrical part.

$$\int_{\Omega_C} \underline{\boldsymbol{E}}_C(\Psi_C)\,\underline{\underline{\boldsymbol{\varepsilon}}}\,\underline{\boldsymbol{E}}_C(\Psi_C)\,\mathrm{d}x \geq \lambda_{1e} \int_{\Omega_C} \underline{\boldsymbol{E}}_C^{\top}(\Psi_C)\cdot\underline{\boldsymbol{E}}_C(\Psi_C)\,\mathrm{d}x$$
$$= \lambda_{1e}\left\|\underline{\boldsymbol{E}}_C(\Psi_C)\right\|^2_{[L^2(\Omega_C)]^n}, \tag{6.14}$$

where λ_{1e} is the smallest eigenvalue of $\underline{\underline{\boldsymbol{\varepsilon}}}$. Here, also $\mu\left(\Gamma^D_{C,e}\right) \neq 0$ holds and we can also apply Korn's inequality:

$$\lambda_{1e}\left\|\underline{\boldsymbol{E}}_C(\Psi_C)\right\|^2_{[L^2(\Omega_C)]^n} \geq c_{1e}\left(\lambda_{1e},\Omega_C,\Gamma^D_{C,el}\right)\cdot\left\|\Psi_C\right\|^2_{\mathcal{V}}. \tag{6.15}$$

Thus, with (6.13) and (6.15), the second assertion follows:

$$\begin{aligned} a_{C,I}\left(\underline{\boldsymbol{V}}_C,\underline{\boldsymbol{V}}_C\right) &\geq \lambda_{1m}\left\|\underline{\boldsymbol{v}}_C\right\|^2_{\mathcal{V}} + \lambda_{1e}\left\|\Psi_C\right\|^2_{\mathcal{V}} \\ &\geq \min\left(\lambda_{1m},\lambda_{1e}\right)\left(\left\|\underline{\boldsymbol{v}}_C\right\|^2_{\mathcal{V}} + \left\|\Psi_C\right\|^2_{\mathcal{V}}\right) \\ &= \min\left(\lambda_{1m},\lambda_{1e}\right)\left\|\underline{\boldsymbol{V}}_C\right\|^2_{\mathcal{V}}. \end{aligned} \tag{6.16}$$

□

6.6 Coerciveness-properties of the bilinear form a_M

Lemma 6.3. *Let $a_M\left(\underline{\boldsymbol{V}}_M,\underline{\boldsymbol{S}}_M\right)$, $\underline{\boldsymbol{V}},\underline{\boldsymbol{S}} \in \mathcal{V}$, be the bilinear form, defined in (5.8). It holds:*

1. *$a_M\left(\underline{\boldsymbol{V}}_M,\underline{\boldsymbol{S}}_M\right)$ is $\mathcal{V}$-bounded and thus continuous.*
2. *$a_M\left(\underline{\boldsymbol{V}}_M,\underline{\boldsymbol{S}}_M\right)$ is $\mathcal{V}$-elliptic, provided, $\underline{\underline{\boldsymbol{C}}}_M$ is positive definite.*

Therefore, the conditions of the Lax-Milgram lemma are satisfied.

Proof of the first assertion.

$$\begin{aligned} \left|a_M\left(\underline{\boldsymbol{V}}_M,\underline{\boldsymbol{S}}_M\right)\right| &= \left|\int_{\Omega_M}\left(\mathcal{D}\underline{\boldsymbol{s}}_M\right)^{\top}\underline{\underline{\boldsymbol{C}}}_M\left(\mathcal{D}\underline{\boldsymbol{v}}_M\right)\mathrm{d}x\right| \\ &\leq \sum_{i=1}^{6}\left|\int_{\Omega_M}\left(\mathcal{D}\underline{\boldsymbol{s}}_M\right)_i^{\top}\left(\underline{\underline{\boldsymbol{C}}}_M\mathcal{D}\underline{\boldsymbol{v}}_M\right)_i\right|\mathrm{d}x \\ &\leq \sum_{i=1}^{6}\int_{\Omega_M}\left|\left(\mathcal{D}\underline{\boldsymbol{s}}_M\right)_i^{\top}\left(\underline{\underline{\boldsymbol{C}}}_M\mathcal{D}\underline{\boldsymbol{v}}_M\right)_i\right|\mathrm{d}x \\ &\leq \sum_{i=1}^{6}\left\|\left(\mathcal{D}\underline{\boldsymbol{s}}_M\right)_i^{\top}\right\|_{L^2(\Omega_M)}\left\|\left(\underline{\underline{\boldsymbol{C}}}_M\mathcal{D}\underline{\boldsymbol{v}}_M\right)_i\right\|_{L^2(\Omega_M)} \\ &\leq \max\left|C_{M_{i,j}}\right|\left(\left\|\partial_1 s_{M_1}\right\|_{L^2}\left\|\partial_1 v_{M_1}\right\|_{L^2} + \left\|\partial_2 s_{M_2}\right\|_{L^2}\left\|\partial_2 v_{M_2}\right\|_{L^2} + \left\|\partial_3 s_{M_3}\right\|_{L^2}\left\|\partial_3 v_{M_3}\right\|_{L^2}\right. \\ &\quad + \left\|\partial_2 s_{M_3} + \partial_3 s_{M_2}\right\|_{L^2}\left\|\partial_2 v_{M_3} + \partial_3 v_{M_2}\right\|_{L^2} + \left\|\partial_1 s_{M_2} + \partial_2 s_{M_1}\right\|_{L^2}\left\|\partial_1 v_{M_2} + \partial_2 v_{M_1}\right\|_{L^2} \\ &\quad \left. + \left\|\partial_1 s_{M_3} + \partial_3 s_{M_1}\right\|_{L^2}\left\|\partial_1 v_{M_3} + \partial_3 v_{M_1}\right\|_{L^2}\right) \\ &\leq \max\left|C_{M_{i,j}}\right|\left(\sum_{i=1}^{3}\left(\left\|\partial_1 s_{M_i}\right\|_{L^2} + \left\|\partial_2 s_{M_i}\right\|_{L^2} + \left\|\partial_3 s_{M_i}\right\|_{L^2}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\cdot \left(\sum_{i=1}^{3} \left(\|\partial_1 v_{M_i}\|_{L^2} + \|\partial_2 v_{M_i}\|_{L^2} + \|\partial_3 v_{M_i}\|_{L^2} \right) \right) \\
&\leq \max \left| C_{M_{i,j}} \right| \left(\sum_{i=1}^{3} \left(\|\partial_1 s_{M_i}\|_{L^2} + \|\partial_2 s_{M_i}\|_{L^2} + \|\partial_3 s_{M_i}\|_{L^2} \right) \right) \\
&\left(\sum_{i=1}^{3} \left(\|\partial_1 v_i\|_{L^2} + \|\partial_2 v_i\|_{L^2} + \|\partial_3 v_{M_i}\|_{L^2} \right) \right) + \sum_{i=1}^{3} \left(\|\partial_i \chi_M\|_{L^2} \right) \sum_{i=1}^{3} \left(\|\partial_i \Psi_M\| \right) \\
&\leq c_{1M} \left\| \underline{\boldsymbol{S}}_M \right\|_{\mathcal{V}_M} \left\| \underline{\boldsymbol{V}}_M \right\|_{\mathcal{V}} .
\end{aligned}
$$

□

Proof of the second assertion. As in the previous subsection, we can show with the help of Korn's inequality, that the bilinear form a_M is $\mathcal{V}$-elliptic. a_M is given in the metal matrix without any electric components. We have:

$$
\begin{aligned}
a_M \left(\underline{\boldsymbol{V}}_M, \underline{\boldsymbol{V}}_M \right) &= \int_{\Omega_M} \left(\mathcal{D} \underline{\boldsymbol{v}}_M \right)^\top \underline{\underline{\boldsymbol{C}}}_M \left(\mathcal{D} \underline{\boldsymbol{v}}_M \right) \,\mathrm{d}x \\
&= \int_{\Omega_M} \underline{\boldsymbol{\gamma}} \left(\underline{\boldsymbol{v}}_M \right)^\top \underline{\underline{\boldsymbol{C}}}_M \underline{\boldsymbol{\gamma}} \left(\underline{\boldsymbol{v}}_M \right) \,\mathrm{d}x \\
&\geq \lambda_{1M} \int_{\Omega_M} \underline{\boldsymbol{\gamma}} \left(\underline{\boldsymbol{v}}_M \right)^\top \underline{\boldsymbol{\gamma}} \left(\underline{\boldsymbol{v}}_M \right) \,\mathrm{d}x,
\end{aligned}
\tag{6.17}
$$

where λ_{1M} is the minimal eigenvalue of $\underline{\underline{\boldsymbol{C}}}_M$. With Korn's inequality follows:

$$
a_M \left(\underline{\boldsymbol{V}}_M, \underline{\boldsymbol{V}}_M \right) \geq \lambda_{1M} \int_{\Omega_M} \underline{\boldsymbol{\gamma}} \left(\underline{\boldsymbol{v}}_M \right)^\top \underline{\boldsymbol{\gamma}} \left(\underline{\boldsymbol{v}}_M \right) \,\mathrm{d}x. \geq c \left\| \underline{\boldsymbol{V}}_M \right\|_{\mathcal{V}}^2 \tag{6.18}
$$

□

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