Dynamical crack propagation in a 2D elastic body
The out-of plane state

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Abstract

Already in 1920 Griffith has formulated an energy balance criterium for quasi-static crack propagation in brittle elastic materials. A generalized energy criterium is used nowadays in mechanics in order to predict how a running crack will grow [8, 11, 7]. We discuss this situation in a rigorous mathematical way for the out-of plane state. This model is described by a two-dimensional scalar wave equation in a bounded domain with a running crack together with boundary and initial conditions. The weak and strong solvability of the problem will be studied and the crack-tip singularities will be derived under the assumption that the crack is straight and moves tangentially. Using the energy balance law, we arrive at an equation of motion for the crack tip.

1 Introduction

Simply spoken, the topic of fracture mechanics is to study why and how materials break. Cracks can be catastrophic in complicated large structures as bridges, oil platforms, aeroplanes, trains as well as in small structures as electronic devices, actuators and sensors. There is a huge number of papers and books on fracture mechanics written by mechanicians, physicists, engineers and mathematicians. We refer here only to the books [8, 11], where dynamic fracture mechanics is worked out.

In this paper we investigate mathematically the behaviour of a linear elastic body with a running crack under the influence of a wave. Reducing the three-dimensional wave model for a linear elastic, isotropic and homogeneous body to a two-dimensional one we get an in-plane model for plane elastic waves and a out-of-plane model for shear waves. Here, we investigate the simpler out-of-plane state as a model. The extension of our method to the more interesting in-plane state seems to be possible and will be done in future.

Our main goals are: the description of the behaviour of the elastic fields near the running crack tip and the derivation of the equation of motion of the crack tip. To derive the singular crack tip fields, we transform the actual configuration (a noncylindrical space-time domain) into the reference configuration (a cylindrical space-time domain) like in [6]. For this purpose we assume there is a family of mappings $y = F_t(x) = x + h(t)\theta(x)$ which maps the reference configuration $\Omega = \Omega_0$ into the actual configuration Ω_t. Roughly speaking, $h(t)$ describes the motion of the crack tip. Performing the above change of variables we get a wave equation with time-dependent coefficients and lower order terms in Ω_0.

Using functional analysis arguments we study the weak and strong solvability of the transformed initial boundary value problem under some realistic assumptions on h and θ. For appropriate initial data and loadings we prove that the displacement field $u = u(x,t)$ admits the decomposition:

$$
\begin{aligned}
u(x,t) &= u_R(x,t) + k(t)\eta(x)\sqrt{\frac{r}{2\alpha(t)}}sgn(\varphi)\sqrt{\sqrt{\cos^2\varphi + \alpha^2(t)\sin^2\varphi} - \cos\varphi}, \\
\alpha(t) &= \sqrt{1 - \frac{(h'(t))^2}{c^2}},
\end{aligned}
$$

where c denotes the shear wave speed, η is a cut-off function, $k(t)$ is the time dependent stress intensity factor and (r,φ) are the polar coordinates with respect to the crack tip in Ω_0.

To derive the equation of motion of the crack tip, we show that the rate of the dissipative energy $\dot{D}(t)$ at time t can be expressed as

$$\dot{D}(t) = \frac{h'(t)c^2k^2(t)\pi(\alpha(t)-1)}{4\alpha(t)}.$$

This relation is obtained inserting the above decomposition of u into a generalized Griffith energy balance criterion.

To our knowledge the proper analysis of the above model as well as the derivation of the equation of motion of the crack tip are new. Furthermore, in contrast to papers in mechanics [7, 8, 11], we consider a bounded domain of arbitrary shape, which renders our analysis complicated.
The paper is organized as follows: In section 2, we give the weak and strong formulation of the problem in the reference configuration. Section 3 is concerned with the existence and uniqueness results of the weak formulation of the problem; these are based on a new abstract setting adapted from [5]. In section 4, we investigate the strong formulation and show that the solution admits a decomposition into a regular part and a singular one. This decomposition is obtained with the help of an appropriated change of variables. Finally in section 5, we use the energy balance law and the singular decomposition of the solution in order to arrive at an equation of motion for the crack tip.

2 The wave equation in a cracked domain

Let Ω' be a fixed plane domain with a smooth boundary Γ which contains exactly one crack σ emerging at one point of Γ, the other extremety (which is inside Ω') is denoted by 0 and is called the cracktip of σ. We remark, that it is also possible to consider cracks in the interior of Ω'. In such a case the crack propagation can be studied separately for both crack tips. In this paper a crack is always supposed to be a C^2 non self-intersecting curve with a finite length. For the sake of shortness we denote $\Omega = \Omega' \setminus \sigma$. For a fixed $T > 0$, we assume a family of mappings

$$F_t = Id + h(t)\theta, \forall t \in [0, T]$$

is given, where Id is the identity mapping of $\mathbf{R}^2$, $h \in C^3([0, T])$ satisfies $h(0) = 0$ and $\theta \in (C^2(\bar{\Omega}))^2$ such that $\theta \equiv 0$ in a neighborhood of $\partial\Omega'$.

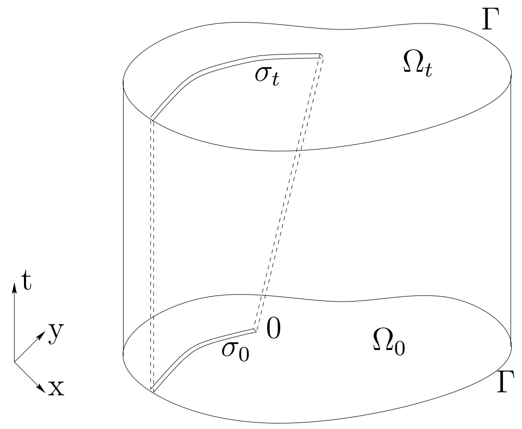

Figure 1: noncylindrical domain Q

For all $t \in [0, T]$ we define $\Omega_t = F_t(\Omega)$, and $\sigma_t = F_t(\sigma)$, where $\sigma = \sigma_0$,see Fig.1. We assume that F_t is a diffeomorphism from $\Omega = \Omega_0$ onto Ω_t and

$$\det \nabla F_t = \det(I + h\nabla\theta) \geq D_0 > 0 \quad \text{for} \quad (x, t) \in \bar{\Omega} \times [0, T] \tag{1}$$

for some positive constant D_0 independent of x and t, where

$$\nabla\theta = \begin{pmatrix} \partial_1\theta_1 & \partial_2\theta_1 \\ \partial_1\theta_2 & \partial_2\theta_2 \end{pmatrix}$$

is the Jacobian matrix. Clearly, if T is small enough, these conditions hold. Elastic waves are often modeled by the linear Navier-Lamé equation system in a three-dimensional space-domain:

$$\rho u_{tt} - (\mu \Delta u + (\lambda + \mu)\ \mathbf{grad}(\mathrm{div}\ u)) = \tilde{f} \tag{2}$$

where $u(x,t) = u = (u_1, u_2, u_3)^T$ is the displacement field, ρ the mass-density, λ, μ the Lamé coefficients and $\tilde{f}(x,t) = \tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)^T$ is the density vector of the volume forces. We study a simple model for the out-of plane state, that means we assume that $u_1 = u_2 = 0, u_3 = u$ and $\partial_3 u_3 = 0$. Then the system (2) reduces to a scalar wave equation

$$u_{tt} - c^2 \Delta u = f, \tag{3}$$

where $f = \frac{\tilde{f}_3}{\rho}$, $c^2 = \frac{\mu}{\rho}$. Here $c = \sqrt{\frac{\mu}{\rho}}$ is the speed of the shear waves which is quite large (for steel $c \approx 3200$ $[\frac{m}{s}]$ or for glass $c \approx 3300$ $[\frac{m}{s}]$, see [11]). We will study an initial-boundary value problem for the wave equation (3) in a cracked plane domain whose precise formulation reads:
Find $u = u(x,t)$ such that

$$\left.\begin{array}{l} \partial_t^2 u - c^2 \Delta u = f \text{ in } Q := \cup_{t=0}^{T} \Omega_t, \\ \partial_n u = 0 \text{ on } \cup_{t=0}^{T} \sigma_t, \\ \partial_n u = q \text{ on } \Sigma_N := \Gamma_N \times (0,T), \\ u = 0 \text{ on } \Sigma_D := \Gamma_D \times (0,T), \\ u(0) = u_0,\ \partial_t u(0) = u_1 \text{ in } \Omega, \end{array}\right\} \tag{4}$$

where $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \cap \Gamma_N = \emptyset$.
On Γ_N, $\partial_n u = \nabla u \cdot n$ means the outward normal derivative while on σ_t it means the normal derivative in one fixed normal direction (therefore $\partial_n u = 0$ on σ_t means that the normal derivative from above and from below are both zero).
Since problem (4) is set in a noncylindrical domain with nonsmooth plane sections, standard arguments (see e.g. [16, 5]) cannot be applied to get existence, uniqueness and regularity results. Moreover there is a large literature on weak formulation of the wave equation in noncylindrical domains (see for instance the recent results of [3] and the references cited there), unfortunately our problem does not enter in this framework. Since strong solutions are of interest for our future analysis of Griffith's criterion we shall consider the existence and uniqueness of both kind of solutions. But, at first we transform the problem into the reference configuration.

2.1 Strong formulation in the reference configuration

We perform a change of variables in each Ω_t in order to transform the noncylindrical domain Q into a cylindrical one, as a consequence we get a wave equation with time dependent coefficients. This last one will be still modified and then analyzed using results on time-dependent wave equations [12, 13] (see also [1, §3.2]). More precisely, we start from (4) and perform the change of variables

$$y = F_t(x) = x + h(t)\theta(x), \quad x \in \Omega, \quad y \in \Omega_t, \tag{5}$$

which maps Ω to Ω_t. Setting

$$\begin{aligned} u(y,t) &= u(F_t(x),t) =: v(x,t), \\ f(y,t) &= f(F_t(x),t) =: g(x,t) \end{aligned} \tag{6}$$

we get the following Lemma, assuming that all derivatives exist at least in the distributional sense.

Lemma 2.1 *The change of variables (5) leads to the following transformed problem in the reference configuration Ω :*

$$v_{tt} + \mathcal{A}_0(t)v + \mathcal{A}_1(t)v + \mathcal{B}(t)v_t = g \text{ in } \Omega \times (0,T), \tag{7}$$

$$(\nabla F_t)^{-\top}\nabla v \cdot (\nabla F_t)^{-\top} n = 0 \text{ on } \sigma \times (0,T), \tag{8}$$

$$c^2 \partial_n v = c^2 q \text{ on } \Sigma_N, \tag{9}$$

$$v = 0 \text{ on } \Sigma_D, \tag{10}$$

$$v(x,0) = u_0(x) \text{ in } \Omega, \tag{11}$$

$$v_t(x,0) = u_1(x) + h'(0)\theta \cdot \nabla v(x,0) = v_1(x) \text{ in } \Omega, \tag{12}$$

where

$$\begin{aligned} \mathcal{A}_0(t)v = & -\frac{c^2}{\det(\nabla F_t)} \operatorname{div}\left(\det(\nabla F_t)(\nabla F_t)^{-1}(\nabla F_t)^{-\top}\nabla v\right) \\ & + h'^2\theta \cdot (\nabla F_t)^{-\top}\nabla\left(\theta \cdot (\nabla F_t)^{-\top}\nabla v\right), \\ \mathcal{A}_1(t)v = & \, h'^2\theta \cdot (\nabla F_t)^{-\top}(\nabla\theta)^{\top}(\nabla F_t)^{-\top}\nabla v \\ & - h''\theta \cdot (\nabla F_t)^{-\top}\nabla v, \\ \mathcal{B}(t)v_t = & -2h'\theta \cdot (\nabla F_t)^{-\top}\nabla v_t. \end{aligned}$$

Proof:
We apply the chain rule:

$$\nabla_y u = (\nabla F_t)^{-\top} \nabla_x v \tag{13}$$

$$\Delta_y u = \operatorname{div}_y \nabla_y u = \frac{1}{\det \nabla F_t(x)} \operatorname{div}_x \left[\det \nabla F_t (\nabla F_t)^{-1} (\nabla F_t)^{-\top} \nabla_x v\right] \tag{14}$$

$$\partial_t u = \partial_t v - h'\theta \cdot (\nabla F_t)^{-\top} \nabla_x v \tag{15}$$

$$\partial_{tt} u = \partial_{tt} v - h''\theta \cdot (\nabla F_t)^{-\top} \nabla_x v + h'^2 \theta \cdot (\nabla F_t)^{-\top} (\nabla \theta)^\top (\nabla F_t)^{-\top} \nabla_x v \tag{16}$$

$$- 2h'\theta \cdot (\nabla F_t)^{-\top} \nabla_x v_t + h'^2 \theta \cdot (\nabla F_t)^{-\top} \nabla_x \left[\theta \cdot (\nabla F_t)^{-\top} \nabla_x v\right] \tag{17}$$

$$\frac{\partial u}{\partial n_y} = \nabla_y u \cdot n_y = (\nabla F_t)^{-\top} \nabla_x v \cdot \frac{(\nabla F_t)^{-\top} n_x}{|(\nabla F_t)^{-\top} n_x|} \tag{18}$$

Inserting these relations into (4) we get the conclusion. ■

2.2 Weak formulation in the reference configuration

Besides the strong formulation of the transformed problem (7) to (12) we derive its weak formulation. To this end we multiply the equation (7) by a test function $w \in H^1_D(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ and integrate on Ω. For the second term after integration by parts and the application of Leibniz's rule, we get

$$\begin{aligned}
\int_\Omega \mathcal{A}_0(t) v w \, dx &= -c^2 \int_\Omega \operatorname{div}\left(\det(\nabla F_t)(\nabla F_t)^{-1}(\nabla F_t)^{-\top} \nabla v\right) \frac{w}{\det(\nabla F_t)} \, dx \\
&\quad + \int_\Omega h'^2 \theta \cdot \left((\nabla F_t)^{-\top} \nabla \left(\theta \cdot (\nabla F_t)^{-\top} \nabla v\right) w\right) \, dx \\
&= c^2 \int_\Omega (\nabla F_t)^{-\top} \nabla v \cdot (\nabla F_t)^{-\top} \nabla w \, dx \\
&\quad - c^2 \int_\Omega (\nabla F_t)^{-\top} \nabla v \cdot (\nabla F_t)^{-\top} \nabla (\det(\nabla F_t)) (\det(\nabla F_t))^{-1} w \, dx \\
&\quad - c^2 \int_{\Gamma_N} q w \, ds \\
&\quad - \int_\Omega h'^2 (\theta \cdot (\nabla F_t)^{-\top} \nabla v)(\theta \cdot (\nabla F_t)^{-\top} \nabla w) \, dx \\
&\quad - \int_\Omega h'^2 (\theta \cdot (\nabla F_t)^{-\top} \nabla v) \operatorname{div}\left((\nabla F_t)^{-1} \theta\right) w \, dx \\
&\quad + \int_{\partial\Omega} h'^2 (\theta \cdot (\nabla F_t)^{-\top} \nabla v)(\theta \cdot (\nabla F_t)^{-\top} n) w \, ds
\end{aligned}$$

The remaining terms are:

$$\int_\Omega \mathcal{A}_1(t)vw\,dx = \int_\Omega (h'^2\theta\cdot(\nabla F_t)^{-\top}(\nabla\theta)^\top(\nabla F_t)^{-\top}\nabla vw - h''\theta\cdot(\nabla F_t)^{-\top}\nabla vw)\,dx,$$

$$\int_\Omega \mathcal{B}(t)v_t w\,dx = -\int_\Omega 2h'\theta\cdot(\nabla F_t)^{-\top}\nabla\partial_t vw\,dx.$$

For this last one, we need the follwing lemma:

Lemma 2.2 *Let h and θ satisfy (1) and*

$$\theta\cdot(\nabla F_t)^{-\top}n = 0 \text{ on } \sigma. \tag{19}$$

Then the next identity holds for all $w_1, w \in H^1(\Omega)$

$$\int_\Omega \theta\cdot(\nabla F_t)^{-\top}\nabla w_1 w\,dx = -\int_\Omega w_1 w\,\mathrm{div}((\nabla F_t)^{-1}\theta)\,dx - \int_\Omega \theta\cdot(\nabla F_t)^{-\top}\nabla w w_1\,dx.$$

Proof: Green's formula leads to

$$\int_\Omega (\theta\cdot(\nabla F_t)^{-\top}\nabla w_1)w\,dx = -\int_\Omega \sum_{i,j=1,2} w\partial_j(\theta_i(\nabla F_t)^{-\top}_{ij}w_1)\,dx + \int_\sigma (\theta\cdot(\nabla F_t)^{-\top}n)w_1 w\,ds.$$

By (19) the boundary term disappears and by Leibniz's rule we arrive at the requested identity. ■

Remark 2.3 The assumption (19) means that the crack growths tangentially to the crack tip and is therefore quite realistic.

Arranging these expressions according to the order of differentation and using the above Lemma, we get the following weak formulation:
For $g \in L^2(0,T,L^2(\Omega)), q \in \tilde{H}^{1/2}(\Gamma_N\times(0,T)), u_0 \in H^1_D(\Omega), u_1 \in L^2(\Omega)$ find $v \in L^2(0,T;H^1_D(\Omega))$ with $v' \in L^2(0,T;L^2(\Omega)), v'' \in L^2(0,T;H^1_D(\Omega)')$ such that

$$\left.\begin{array}{l} \int_\Omega v_{tt}w\,dx + a_0(t,v,w) + a_1(t,v,w) + b(t,v_t,w) \\ +r(t,v,w) = \int_\Omega gw dx + c^2\int_{\Gamma_N} qw\,ds, \forall w \in H^1_D(\Omega), \\ v(x,0) = u_0(x) \text{ in } \Omega, \\ v_t(x,0) = u_1(x) + h'\theta\cdot\nabla u_0(x) \text{ in } \Omega. \end{array}\right\} \tag{20}$$

Here and below we set

$$\begin{aligned}
a_0(t,v,w) &= \int_\Omega \big(c^2(\nabla F_t)^{-\top}\nabla v\cdot(\nabla F_t)^{-\top}\nabla w \\
&- h'^2(\theta\cdot(\nabla F_t)^{-\top}\nabla v)(\theta\cdot(\nabla F_t)^{-\top}\nabla w)\big)\,dx, && (21)\\
a_1(t,v,w) &= \int_\Omega m\cdot(\nabla F_t)^{-\top}\nabla v w dx, && (22)\\
b(t,v_t,w) &= 2h'(t)\int_\Omega v_t w \mathrm{div}((\nabla F_t)^{-1}\theta)\,dx \\
&+ 2h'(t)\int_\Omega \theta\cdot(\nabla F_t)^{-\top}\nabla w v_t\,dx, && (23)\\
r(t,v,w) &= \int_\sigma h'^2\left[\theta\cdot(\nabla F_t)^{-\top}\nabla v\right]\left[\theta\cdot(\nabla F_t)^{-\top}n\right] w ds, && (24)
\end{aligned}$$

where the vector m is defined by

$$\begin{aligned}
m &= -\frac{c^2}{\det(\nabla F_t)}(\nabla F_t)^{-\top}\nabla(\det(\nabla F_t)) - (h')^2\,\mathrm{div}((\nabla F_t)^{-1}\theta)\theta \\
&+ (h')^2\nabla\theta(\nabla F_t)^{-1})\theta - h''\theta.
\end{aligned}$$

We recall that

$$\tilde{H}^{1/2}(\Gamma_N\times(0,T)) = \{u\in H^{1/2}(\Gamma\times(0,T)) : \operatorname{supp} u\subset\overline{\Gamma_N}\times(0,T)\}.$$

3 Existence results for the weak formulation

We start with an abstract result that will be applied in a second step to our weak formulation.

3.1 An abstract existence result

In this subsection we prove an abstract result in a Hilbert space setting that will be used later on to prove an existence and uniqueness result for our system (20). To our knowledge this abstract existence result does not exist in the literature. For that purpose let us consider the following setting similar to the one in section 18.5 in [5]: Let V and H be two real and separable Hilbert spaces such that V is continuously embedded into H with respective inner products and norms denoted by $((\cdot,\cdot)), ||\cdot||$ and $(\cdot,\cdot), |\cdot|$. For a fixed $T>0$,we consider two families of bilinear forms $\{a(t;\cdot,\cdot)\}_{t\in[0,T]}$ and $\{b(t;\cdot,\cdot)\}_{t\in[0,T]}$ continuous on $V\times V$ and on $H\times V$ respectively

and fulfilling the following properties:
1. The bilinear form a admits the splitting

$$a(t; u, v) = a_0(t; u, v) + a_1(t; u, v), \forall u, v \in V, \tag{25}$$

where a_0 is the principal part and a_1 a remainder. We assume:
The bilinear form a_0 is symmetric, weakly coercive in the sense that there exist two nonnegative constants λ and α independent of t such that

$$a_0(t; u, u) \geq \alpha ||u||^2 - \lambda |u|^2, \forall u \in V, \tag{26}$$

and non negative

$$a_0(t; u, u) \geq 0, \forall u \in V. \tag{27}$$

Furthermore, a_0 is continuously differentiable with respect to $t \in [0, T]$, in particular it holds

$$a_0'(t; u, u) \leq c\|u\|^2, \forall u \in V. \tag{28}$$

The bilinear form a_1 is continuous with respect to $t \in [0, T]$ and satisfies

$$|a_1(t; u, v)| \leq c_1 ||u|| \cdot |v|, \forall u, v \in V, \tag{29}$$

for some $c_1 > 0$ independent of t.
2. The bilinear form b is continuous with respect to $t \in [0, T]$, satisfies the estimates

$$|b(t; u, v) \leq c_2 ||u|| \cdot |v|, \forall u, v \in V, \tag{30}$$

$$b(t; u, u) \geq -c_3 |u|^2, \forall u \in V, \tag{31}$$

for some $c_2, c_3 > 0$ independent of t, and for any $t \in [0, T]$ and any $v \in V$, $b(t, \cdot, v)$ is weakly continuous in H, i.e., if u_n is weakly convergent to u in H as $n \to \infty$, then

$$b(t, u_n, v) \to b(t, u, v) \text{ in } \mathbf{R}.$$

The difference with the setting of section 18.5 of [5] consists in the assumptions on the bilinear form b.
Now we may formulate our abstract problem and its existence and uniqueness result:

Theorem 3.1 *Let $u_0 \in V$, $u_1 \in H$ and $f \in L^2(0, T; H)$. Then there exists a unique solution $u \in L^2(0, T; V) \cap H^1(0, T; H)$ of*

$$\begin{cases} \frac{d}{dt}(u'(\cdot), v) + a(\cdot; u(\cdot), v) + b(\cdot; u'(\cdot), v) = (f(\cdot), v), \forall v \in V \text{ in the sense of } \mathcal{D}'(0, T), \\ u(0) = u_0, \ u'(0) = u_1, \end{cases} \tag{32}$$

where $u' = \frac{du}{dt}$. Moreover, u satifies $u'' \in L^2(0, T; V')$.

Proof: The proof relies on a perturbation argument. Indeed, for any $\epsilon > 0$ we introduce the bilinear form $b_\epsilon(t; \cdot, \cdot)$ as follows

$$b_\epsilon(t; u, v) = \epsilon((u, v)) + b(t; u, v), \forall u, v \in V.$$

By the assumptions on b, the bilinear form b_ϵ satisfies the assumptions of section 18.5 of [5] (with $b_0(t; u, v) = \epsilon((u, v))$ and $b_1 = b$). By Theorem 18.5.1 of [5], there exists a unique solution $u_\epsilon \in H^1(0, T; V)$ of

$$\begin{cases} \frac{d}{dt}(u'_\epsilon(\cdot), v) + a(\cdot; u_\epsilon(\cdot), v) + b_\epsilon(\cdot; u'_\epsilon(\cdot), v) = (f(\cdot), v), \forall v \in V \text{ in the sense of } \mathcal{D}'(0, T), \\ u_\epsilon(0) = u_0,\ u'_\epsilon(0) = u_1. \end{cases} \tag{33}$$

The energy $X_\epsilon(t)$ of the above system at time t is defined by (see [5], p.680)

$$X_\epsilon(t) = (u'_\epsilon(t), u'_\epsilon(t)) + a_0(t; u_\epsilon(t), u_\epsilon(t)) + 2\epsilon \int_0^t ((u'_\epsilon(s), u'_\epsilon(s)))\, ds. \tag{34}$$

Note that $X_\epsilon(0) = (u_1, u_1) + a_0(0; u_0, u_0)$ is independent of ϵ.
By the identity of energy (18.5.83) of [5] we have

$$\begin{aligned} X_\epsilon(t) &= X_\epsilon(0) + \int_0^t a'_0(s; u_\epsilon(s), u_\epsilon(s))\, ds \\ &- 2\int_0^t a_1(s; u_\epsilon(s), u'_\epsilon(s))\, ds - 2\int_0^t b(s; u'_\epsilon(s), u'_\epsilon(s))\, ds \\ &+ 2\int_0^t (f(s), u'_\epsilon(s))\, ds, \end{aligned} \tag{35}$$

By the assumptions (26),(28), (29) and (31), the identity (35) implies that

$$\begin{aligned} X_\epsilon(t) &\leq X_\epsilon(0) + C\int_0^t (a_0(s; u_\epsilon(s), u_\epsilon(s)) + |u_\epsilon(s)|^2)\, ds \\ &+ c_1 \int_0^t ||u_\epsilon(s)|| \cdot |u'_\epsilon(s)|\, ds \\ &+ 2c_3 \int_0^t |u'_\epsilon(s)|^2\, ds \\ &+ 2\int_0^t |f(s)| \cdot |u'_\epsilon(s)|\, ds, \end{aligned}$$

for some $C > 0$ independent of t. The identity $u_\epsilon(t) = \int_0^t u'_\epsilon(s)\, ds + u_0$ implies

$$|u_\epsilon(t)|^2 \leq 2t \int_0^t |u'_\epsilon(s)|^2\, ds + 2|u_0|^2. \tag{36}$$

Consequently the above inequality may be transformed into

$$X_\epsilon(t) \le CX_\epsilon(0) + C\int_0^T |f(s)|^2\,ds + C\int_0^t X_\epsilon(s)\,ds,$$

for some $C > 0$ independent of t. By Gronwall's lemma, we conclude

$$X_\epsilon(t) \le C(X_\epsilon(0) + \int_0^T |f(s)|^2\,ds)te^{Ct}, \forall t \in [0,T].$$

This means that the energy $X_\epsilon(t)$ is uniformly bounded in $[0,T]$ and applying the inequality (27) to (34), we obtain

$$|u_\epsilon'(t)|^2 + \epsilon\int_0^t ||u_\epsilon'(s)||^2\,ds \le K, \forall t \in [0,T].$$

This estimate, the weak coerciveness of a_0 and the inequality (36) lead to

$$|u_\epsilon'(t)|^2 + ||u_\epsilon(t)||^2 + \epsilon\int_0^t ||u_\epsilon'(s)||^2\,ds \le K, \forall t \in [0,T], \tag{37}$$

for some positive constant K independent of t.
This estimate and the fact that Hilbert spaces are weakly sequentially compact imply that there exists a subsequence of (u_ϵ), still denote by (u_ϵ) for the sake of shorthness, such that for $\epsilon \to 0$

$$u_\epsilon \to u \text{ weakly in } H^1(0,T;H), \tag{38}$$

$$u_\epsilon \to u \text{ weakly in } L^2(0,T;V). \tag{39}$$

We will now show that u is the solution of (32). For that purpose fix arbitrary $v \in V$ and $\varphi \in \mathcal{D}(0,T)$. Then the first identity of (33) is equivalent to

$$\int_0^T (-(u_\epsilon'(t),v)\varphi'(t) + a(t;u_\epsilon(t),v)\varphi(t) \tag{40}$$

$$+b(t;u_\epsilon'(t),v)\varphi(t) + \epsilon((u_\epsilon(t),v))\varphi(t))\,dt = \int_0^T (f(t),v)\varphi(t)\,dt.$$

By (37) we may write

$$\begin{aligned}\int_0^T \epsilon((u_\epsilon(t),v))\varphi(t)\,dt &\le \epsilon^{1/2}\int_0^T \epsilon^{1/2}||u_\epsilon(t)||\cdot||v||\varphi(t)\,dt \\ &\le \epsilon^{1/2}||v||\left(\int_0^T \epsilon||u_\epsilon(t)||^2\,dt\right)^{1/2}\left(\int_0^T \varphi(t)^2\,dt\right)^{1/2} \\ &\le \epsilon^{1/2}K||v||\left(\int_0^T \varphi(t)^2\,dt\right)^{1/2}.\end{aligned}$$

As the right-hand side of this inequality tends to zero as ϵ goes to zero, we have shown that

$$\int_0^T \epsilon((u_\epsilon(t), v))\varphi(t)\, dt \to 0 \text{ as } \epsilon \to 0.$$

Passing to the limit on $\epsilon \to 0$ in the identity (40), using the above property as well as (38) and (39) we obtain

$$\int_0^T (-(u'(t), v)\varphi'(t) + a(t; u(t), v)\varphi(t) + b(t; u'(t), v)\varphi(t))\, dt = \int_0^T (f(t), v)\varphi(t)\, dt,$$

which is nothing else than the first identity of (32).
It remains to show the regularity $u'' \in L^2(0, T; V')$: But the above identity implies for any $\varphi \in \mathcal{D}(0, T; V)$ that

$$\begin{aligned} \langle u'', \varphi \rangle &= -\int_0^T (u'(t), \varphi'(t))\, dt \\ &= \int_0^T [-a(t; u(t), \varphi(t)) - b(t; u'(t), \varphi(t)) + (f(t), \varphi(t))]\, dt. \end{aligned}$$

Denoting this right-hand side by $l(\varphi)$, by the assumptions on a, b, we get

$$|l(\varphi)| \leq C \left(\int_0^T ||\varphi(t)||^2\, dt \right)^{1/2}.$$

The announced regularity follows from this estimate and the above identity since $(L^2(0, T; V))' = L^2(0, T; V')$. ■

3.2 Existence result for the weak formulation

We use the abstract theory of the previous subsection with $V = H^1_D(\Omega)$, $H = L^2(\Omega)$ and the bilinear forms a_0, a_1 and b as described in subsection 2.2. Therefore, we need to check that a_0, a_1 and b satisfy the assumptions from subsection 3.1. We only check the nontrivial properties, namely the weak coerciveness and non negativeness of a_0 and the estimates (30) and (31) for b, the other properties being direct consequences of the properties on h and θ.
We first investigate the weak $H^1_D(\Omega)$-coerciveness of a_0.

Lemma 3.2 *If*

$$c^2 - h'(t)^2 \|\theta\|^2_{E^2} \geq \gamma_0 > 0 \quad \forall t \in (0, T), \forall x \in \Omega, \tag{41}$$

then

$$a_0(t, v, v) \geq c_0 |v|^2_{H^1(\Omega)} \quad \forall v \in H^1(\Omega), \tag{42}$$

for some $c_0 > 0$ independent of t. Furthermore if meas $D \neq 0$, then

$$a_0(t, v, v) \geq c_0 \|v\|^2_{H^1(\Omega)} \quad \forall v \in H^1_D(\Omega).$$

Proof: Denoting shortly $\vec{a} = (\nabla F_t)^{-\top} \nabla v$ we have

$$a_0(t, v, v) = \int_\Omega \left(c^2 \vec{a} \cdot \vec{a} - h'^2 (\theta \cdot \vec{a})(\theta \cdot \vec{a}) \right) dx.$$

Since by discrete Cauchy-Schwarz's inequality

$$c^2 \vec{a} \cdot \vec{a} - h'^2 (\theta \cdot \vec{a})^2 \geq c^2 \vec{a} \cdot \vec{a} - h'^2 (\theta \cdot \theta)(\vec{a} \cdot \vec{a}) = (c^2 - h'^2 \|\theta\|^2_{E_2}) \|\vec{a}\|^2_{E_2},$$

it follows

$$a_0(t, v, v) \geq \gamma_0 \int_\Omega (\nabla F_t)^{-1} (\nabla F_t)^{-\top} \nabla v \cdot \nabla v \, dx.$$

Using that $(\nabla F_t)^{-1} (\nabla F_t)^{-\top}$ is symmetric and positive definite, the first assertion follows.
The second assertion follows from the estimate (42) and Poincaré-Friedrichs' inequality. ■

Remark 3.3 The assumption (41) is very reasonable assuming that $\|\theta\|^2_{E^2} = 1$, since experiments (cf. [21, **?**, 11]) show that $\max h'(t)^2 \approx c^2/2$.

Let us finish by the properties on b. First we remark that the weak continuity property follows directly from the definition of b, for the other properties, we use the next Lemma.

Lemma 3.4 *If θ satisfies (19), then the bilinear form b defined by (23) satisfies (30) and (31).*

Proof: By Lemma 2.2, for u, v in V, we have

$$b(t; u, v) = -2h'(t) \int_\Omega \theta \cdot (\nabla F_t)^{-\top} \nabla u v \, dx,$$

and the first assertion follows from Cauchy-Schwarz's inequality.
For the second assertion, again Lemma 2.2 yields the identity

$$\int_\Omega (\theta \cdot (\nabla F_t)^{-\top} \nabla v) v \, dx = -\frac{1}{2} \int_\Omega |v|^2 \operatorname{div}((\nabla F_t)^{-1} \theta) \, dx, \forall v \in V,$$

or equivalently

$$b(t; v, v) = h'(t) \int_\Omega |v|^2 \mathrm{div}((\nabla F_t)^{-1}\theta)\, dx, \forall v \in V.$$

Therefore there exists $C_0 > 0$ such that

$$b(t; v, v) \geq -C_0 \int_\Omega |v|^2\, dx, \forall v \in V.$$

■

As a consequence of the above lemmas we conclude the

Theorem 3.5 *Assume that (1), (19) and (41) hold. Then for* $g \in L^2(0, T, L^2(\Omega)), q \in \tilde{H}^{1/2}(\Gamma_N \times (0, T)), u_0 \in H_D^1(\Omega), u_1 \in L^2(\Omega)$ *problem (20) has a unique solution* $v \in L^2(0, T; H_D^1(\Omega))$ *with* $v' \in L^2(0, T; L^2(\Omega)), v'' \in L^2(0, T; H_D^1(\Omega)')$.

Proof: We remark that the assumption (19) guarantees that $r(t; u, v) = 0$. Moreover a standard trace theorem [9] yields: there is an element $l \in H^2(\Omega \times (0, T))$ with a small support included in a neighborhood of $\Sigma = \Sigma_N \cup \Sigma_D$ such that

$$\begin{aligned} &l = 0 \text{ on } \Sigma_D, \\ &\partial_n l = q \text{ on } \Sigma_N, \partial_n l = 0 \text{ on } \sigma. \end{aligned}$$

Then $\tilde{v} = v - l$ is solution of (20) with homogeneous Neumann boundary condition and data $\tilde{u}_0, \tilde{u}_1$ and $\tilde{f}$ with the same regularity as before. This new problem (20) enters now in the framework of the abstract setting of subsection 3.1. Since the assumptions on the bilinear forms were checked in the two previous lemmas, Theorem 3.1 allows to conclude the existence of a unique solution v of (20) with the announced regularity. ■

4 Existence results for the strong formulation

The operator $-\mathcal{A}_0(t)$ is a second order operator with time dependent coefficients which is strongly elliptic if (41) holds (note further that $\mathcal{A}_0(t) = -\Delta, \mathcal{A}_1(t) = \mathcal{B}(t) = 0$ outside the support of θ). So we may expect existence and uniqueness of a strong solution to (7) to (12) using results from functional analysis. For that purpose we use the standard argument of reduction of order by introducing

$$V = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} v \\ \partial_t v \end{pmatrix}.$$

With this notation, (7) is equivalent to the first order evolution equation

$$\partial_t V(t) + A(t)V(t) = F \text{ in } [0,T], \tag{43}$$

where

$$\begin{aligned} F &= \begin{pmatrix} 0 \\ g \end{pmatrix}, \\ A(t)\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} &= \begin{pmatrix} -v_1 \\ \mathcal{A}_0(t)v_0 + \mathcal{A}_1(t)v_0 + \mathcal{B}(t)v_1 \end{pmatrix}. \end{aligned}$$

A general theory of equation of type (43) has been developed using semigroup theory [12, 13, 20]. The easiest way to prove existence and uniqueness results is to show that the triplet $\{A, X, Y\}$ forms a CD-system (see [12, 13] for the details): This means that X and Y are two real separable Banach spaces such that Y is continuously and densely included into X and that the following properties are valid:
(i) $A = \{A(t)\}_{t\in[0,T]}$ is a stable family of generators of strongly continuous semigroups on X, with stability constants M, β,
(ii) The domain $D(A(t)) = Y$ of $A(t)$ is independent of t, for all $t \in [0,T]$,
(iii) $\partial_t A \in L^\infty_\star([0,T], B(Y,X))$,
where $L^\infty_\star([0,T], B(Y,X))$ is the space of equivalence classes of essentially bounded, strongly measurable functions from $[0,T]$ into $B(Y,X)$.
In our setting we take

$$X = H^1_D(\Omega) \times L^2(\Omega),$$

equipped with the standard norm. But if we stay in the previous formulation the condition (ii) is not satisfied in general since the singularity of $\mathcal{A}_0$ near the crack tip depends on t (see below). In order to be able to apply the above theory we again transform our problem using an appropriate change of variables.
To simplify our exposition we now assume that σ is straight in a small neighborhood V of the crack tip 0 and that the crack growths tangentially. In other words, we assume that

$$F_t(x) = x + h(t)\begin{pmatrix} \eta(r) \\ 0 \end{pmatrix},$$

where $\eta = \eta(r)$ is a smooth cut-off function such that $\eta \equiv 1$ in a neighborhood of the crack tip 0 with a support included into V and as usual, $r = |x - 0| = |x|$ is the distance from x to the crack tip. This means that the vector function θ is here given by

$$\theta(x) = \eta(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{44}$$

In this case we have the simple expressions:

$$\nabla\theta = \begin{pmatrix} \partial_1\eta & \partial_2\eta \\ 0 & 0 \end{pmatrix},$$

$$\det(\nabla F_t) = 1 + h\partial_1\eta,$$

$$\theta\cdot(\nabla F_t)^{-\top}\nabla v = \frac{\eta}{1+h\partial_1\eta}\partial_1 v.$$

And consequently the operators $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{B}$ simplify to

$$\begin{aligned} \mathcal{A}_0(t)v &= -\frac{c^2}{1+h\partial_1\eta}\operatorname{div}[\frac{1}{1+h\partial_1\eta}\nabla v - h\begin{pmatrix} \partial_2\eta\partial_2 v \\ \partial_2\eta\partial_1 v - 2\partial_1\eta\partial_2 v \end{pmatrix} \\ &\quad -h^2\begin{pmatrix} -(\partial_2\eta)^2\partial_1 v + \partial_1\eta\partial_2\eta\partial_2 v \\ \partial_2\eta\partial_1\eta\partial_1 v - (\partial_1\eta)^2\partial_2 v \end{pmatrix}] + \frac{h'^2\eta}{1+h\partial_1\eta}\partial_1(\frac{\eta}{1+h\partial_1\eta}\partial_1 v), \\ \mathcal{A}_1(t)v &= h'^2\eta\partial_1\eta\partial_1 v\frac{1}{(1+h\partial_1\eta)^2} - \frac{h''\eta}{1+h\partial_1\eta}\partial_1 v, \\ \mathcal{B}(t)v_t &= -\frac{2h'\eta}{1+h\partial_1\eta}\partial_1 v_t. \end{aligned}$$

With this choice of θ, we also notice that the condition (19) holds.
¿From now on we assume that (41) holds, which here reduces to

$$\exists\alpha_0 > 0 : 1 - \frac{h'(t)^2}{c^2} \geq \alpha_0^2 > 0, \forall t \in (0,T). \tag{45}$$

Without loss of generality we may assume that $\bar{B}(0,1)$ is included inside Ω and that the set $W := \{x \in \Omega : \eta(x) = 1\}$ contains $\bar{B}(0,1)$ (otherwise we only have to rescale the domain Ω). We further fix the x_1-axis of coordinates tangent to σ and oriented "outside" σ, i. e., the half-line $x_1 \leq 0$ contains $\sigma \cap V$.
Now we remark that on the set W the operator $\mathcal{A}_0$ reduces to

$$\mathcal{A}_0 = -c^2(\alpha(t)^2\partial_1^2 + \partial_2^2),$$

where $\alpha(t)^2 = 1 - \frac{h'(t)^2}{c^2}$. This means that the change of variables $z_1 = \frac{x_1}{\alpha(t)}, z_2 = x_2$ would transform the operator $\mathcal{A}_0$ on W into the Laplace operator. This would suggest that this change of variables will guarantee the assumption (ii). Unfortunately the above change of variables is global and transforms Ω into a time-dependent domain and therefore (ii) cannot hold anymore. This means that we have to take the above change of variables near the crack tip and modify it far away. Namely we perform the following change of variables:

$$z_1 = q(x,t), z_2 = x_2, t = t, \tag{46}$$

where q is defined as follows:

$$q(x,t)=\frac{x_1}{d(x,t)}, \tag{47}$$

where

$$d(x,t)=\alpha(t)\kappa(x)+(1-\kappa(x))\alpha_0,$$

and κ is a cut-off function defined by

$$\kappa(x)=\begin{cases} 1 & \text{if } |x|\le 1/2,\\ \lambda_0(2|x|-1) & \text{if } 1/2\le |x|\le 1,\\ 0 & \text{if } |x|\ge 1,\end{cases}$$

where $\lambda_0(\hat r)=(\hat r-1)^2(2\hat r+1)$.
Due to the fact that $\alpha(t)\ge\alpha_0$ one easily checks that for any fixed x_2,t, $q(\cdot,x_2,t)$ is strictly increasing and therefore injective. Consequently for any fixed t, the change of variables (46) induces a diffeomorphism G_t between Ω and $\tilde\Omega$, $\tilde\Omega$ being defined by

$$\tilde\Omega:=\{(\frac{y_1}{\alpha_0},y_2):(y_1,y_2)\in\Omega\}.$$

Indeed for $(y_1,y_2)\in\Omega\backslash B(0,1)$, this is direct since $G_t(y_1,y_2)=(\frac{y_1}{\alpha_0},y_2)$. On the other hand for $(y_1,y_2)\in\Omega\cap B(0,1)$, let us show that there exists $(x_1,x_2)\in\Omega\cap B(0,1)$ such that $(\frac{y_1}{\alpha_0},y_2)=G_t(x_1,x_2)$. By the definition of G_t, we clearly have $x_2=y_2$. Now fixing $x_2\in(-1,1)$, by the fact that $q(x_1,x_2,t)$ is strictly increasing in $x_1\in(-\sqrt{1-x_2^2},\sqrt{1-x_2^2})$, its range is the interval $(q(-\sqrt{1-x_2^2},x_2,t),q(\sqrt{1-x_2^2},x_2,t))=(-\frac{\sqrt{1-x_2^2}}{\alpha_0},\frac{\sqrt{1-x_2^2}}{\alpha_0})$. Since y_1 belongs to $(-\sqrt{1-x_2^2},\sqrt{1-x_2^2})$, we deduce the existence of $x_1\in(-\sqrt{1-x_2^2},\sqrt{1-x_2^2})$ such that $\frac{y_1}{\alpha_0}=q(x_1,x_2,t)$.
These arguments also show that the crack $\tilde\sigma$ of $\tilde\Omega$ is given by

$$\tilde\sigma:=\{(\frac{y_1}{\alpha_0},y_2):(y_1,y_2)\in\sigma\},$$

and that it is straight in a neighbourhood of $(0,0)$.
Let us notice that neither $\tilde\Omega$ nor $\tilde\sigma$ depend on t.
Setting

$$w(z,t):=v(x,t), \tag{48}$$

the chain rule yields

$$\begin{aligned} v_{tt}+\mathcal{A}_0v+\mathcal{A}_1v+\mathcal{B}v_t &= w_{tt}+(-c^2\alpha(t)^2(\partial_1 q)^2+(\partial_t q)^2)\partial^2_{z_1}w-c^2\partial^2_{z_2}w\\ &+ (\partial_t^2 q-c^2\alpha(t)^2\partial_1^2 q)\partial_{z_1}w+\partial_t q\partial^2_{z_1 t}w\\ &- h''\partial_1 q\partial_{z_1}w\\ &- 2h'\left(\partial_t q\partial_1 q\partial^2_{z_1}w+\partial^2_{t1}q\partial_{z_1}w+\partial_1 q\partial^2_{tz_1}w\right)\ \text{on } \bar B(0,1)\times(0,T).\end{aligned}$$

The principal part of the right-hand side without second order mixed derivatives (in space-time) is

$$w_{tt} + \mathcal{A}_p w,$$

where the operator $\mathcal{A}_p$ is defined by

$$\mathcal{A}_p := (-c^2\alpha(t)^2(\partial_1 q)^2 + (\partial_t q)^2 - 2h'\partial_t q\partial_1 q)\partial^2_{z_1} w - c^2\partial^2_{z_2} w.$$

Let us look whether the spatial part $\mathcal{A}_p$ is strongly elliptic.

Lemma 4.1 *If there exists $0 < \gamma_0 < c^2$ (independent of t) such that*

$$|h''| \leq c^2\alpha_0 \min\{2\alpha_0[-1 + \sqrt{1 + \alpha_0^2 \frac{1-\frac{\gamma_0}{c^2}}{1-\alpha_0^2}}], -(4-3\alpha_0) + \sqrt{(4-3\alpha_0)^2 + \alpha_0^4 \frac{1-\frac{\gamma_0}{c^2}}{1-\alpha_0^2}}\}, \tag{49}$$

then the operator $\mathcal{A}_p$ is strongly elliptic on $Q := B(0, \alpha_0^{-1})$ (with a constant of ellipticity independent of t).

Proof: We distinguish between the case $|x| < 1/2$ and the case $1/2 < |x| < 1$.
1. If $|x| < 1/2$, then

$$-c^2\alpha(t)^2(\partial_1 q)^2 + (\partial_t q)^2 - 2h'\partial_t q\partial_1 q = -c^2 + (\alpha(t))^{-4}(\alpha'(t))^2 x_1^2 - 2h'(\alpha(t))^{-3}\alpha'(t)x_1.$$

The strong ellipticity will be satisfied if there exists a constant γ_0 such that

$$(\alpha(t))^{-4}(\alpha'(t))^2 x_1^2 - 2h'(\alpha(t))^{-3}\alpha'(t)x_1 \leq c^2 - \gamma_0.$$

As $|x_1| \leq 1/2$ and deriving α, the above estimate holds if

$$\frac{(h')^2(h'')^2}{4c^4\alpha^6} + \frac{(h')^2|h''|}{c^2\alpha^4} \leq c^2 - \gamma_0.$$

Using the assumption (41), we see that this inequality will hold if

$$\frac{(h'')^2}{4c^4\alpha_0^6} + \frac{|h''|}{c^2\alpha_0^4} \leq \frac{1-\frac{\gamma_0}{c^2}}{1-\alpha_0^2},$$

which follows from (49) by regarding the above inequality as a second order inequality in $X = \frac{|h''|}{c^2}$.
2. For $1/2 < |x| < 1$, we similarly have the ellipticity if

$$-c^2(\alpha(t))^2(\partial_1 q)^2 + \frac{x_1^2(\alpha'(t))^2(\lambda_0(2x_1-1))^2}{d^4} + \frac{2h'x_1\alpha'(t)\lambda_0(2|x|-1)}{d^2}(\partial_1 q) \leq -\gamma_0.$$

Since

$$\frac{4-3\alpha_0}{d^2} \geq \partial_1 q \geq d^{-1} \geq \alpha^{-1},$$

and $d \geq \alpha_0$, $0 \leq \lambda_0(2|x|-1) \leq 1$, the above estimate holds if

$$\frac{(\alpha')^2}{\alpha_0^4} + \frac{|h'||\alpha'(t)|(4-3\alpha_0)}{\alpha_0^4} \leq c^2 - \gamma_0.$$

As before using (41), and deriving α, the above estimate will hold if

$$\frac{(h'')^2}{c^4\alpha_0^6} + 2\frac{|h''|(4-3\alpha_0)}{c^2\alpha_0^5} \leq \frac{1-\frac{\gamma_0}{c^2}}{1-\alpha_0^2},$$

which also follows from (49). ■

Remark 4.2 The condition (49) is relatively weak for h'' since in practice c is large and $\alpha_0^2 \sim 1/2$. This condition on h'' comes from the above change of variables (46), another choice could give a weaker condition. ■

Now we go back to our wave equation (7)-(12) with homogeneous Neumann boundary conditions (i.e. q is supposed to be zero, the reduction to this case is made as in Theorem 3.5) and transform it with the help of the change of variables (46) and of unknown (48). We then get

$$\begin{aligned}
w_{tt} + \tilde{\mathcal{A}}_0(t)w + \tilde{\mathcal{A}}_1(t)w + \tilde{\mathcal{B}}(t)w_t = g(G_t^{-1}(\cdot),t) &= \tilde{g} \text{ in } \tilde{\Omega}\times(0,T), && (50)\\
\partial_{\tilde{n}} w &= 0 \text{ on } \tilde{\sigma}\times(0,T), && (51)\\
\partial_{\tilde{n}} w &= 0 \text{ on } \tilde{\Sigma}_N, && (52)\\
w &= 0 \text{ on } \tilde{\Sigma}_D, && (53)\\
w(z,0) = u_0(G_0^{-1}(z)) &= \tilde{w}_0(z) \text{ in } \tilde{\Omega}, && (54)\\
w_t(z,0) = v_1(G_0^{-1}(z)) - \partial_t q(G_0^{-1}(z),0)\partial_{z_1}\tilde{w}_0(z) &= \tilde{w}_1(z) \text{ in } \tilde{\Omega}, && (55)
\end{aligned}$$

where G_t^{-1} is the inverse mapping of (46) for the space variables and $\tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{B}}$ are given by

$$\begin{aligned}
\tilde{\mathcal{A}}_0(t)w &= \chi_Q \mathcal{A}_p w + (1-\chi_Q)\mathcal{A}_0(t,\alpha_0^{-1}\partial_{z_1},\partial_{z_2})w,\\
\tilde{\mathcal{A}}_1(t)w &= \chi_Q(\partial_t^2 q - c^2\alpha(t)^2\partial_1^2 q - h''\partial_1 q - 2h'\partial_{t1}^2 q)\partial_{z_1} w + (1-\chi_Q)\mathcal{A}_1(t,\alpha_0^{-1}\partial_{z_1},\partial_{z_2})w,\\
\tilde{\mathcal{B}}(t)w_t &= \chi_Q(\partial_t q - 2h'\partial_1 q)\partial_{z_1} w_t + (1-\chi_Q)\mathcal{B}_1(t,\alpha_0^{-1}\partial_{z_1},\partial_{z_2})w_t\\
&= \frac{\eta}{1+h\partial_1\eta}(\partial_t q - 2h'\partial_1 q)\partial_{z_1} w_t,
\end{aligned}$$

$\partial_{\tilde{n}} w$ means the conormal derivative of w with respect to $\tilde{\mathcal{A}}_0$ and χ_Q is the characteristic function of the set Q. Note that the splitting of the operators into the sum of factors of χ_Q and of $1-\chi_Q$ is purely artificial since the factors of χ_Q and $1-\chi_Q$ are identical in a neighborhood of ∂Q. This splitting is used to underline the behaviour of the operators near the crack tip.
Its vectorial form is then: The vector function

$$W = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} w \\ \partial_t w \end{pmatrix}$$

satifies the first order evolution equation

$$\partial_t W(t) + \tilde{A}(t) W(t) = \tilde{F} \text{ in } [0,T], \tag{56}$$

where

$$\begin{aligned} \tilde{F} &= \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}, \\ \tilde{A}(t) \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} &= \begin{pmatrix} -w_1 \\ \tilde{\mathcal{A}}_0(t) w_0 + \tilde{\mathcal{A}}_1(t) w_0 + \tilde{\mathcal{B}}(t) w_1 \end{pmatrix}. \end{aligned}$$

In this new setting, we are able to check the hypothesis (ii): We here take $X = H^1_D(\tilde{\Omega}) \times L^2(\tilde{\Omega})$ and from the definition of $\tilde{A}(t)$, we see that

$$D(\tilde{A}(t)) = D(\tilde{\mathcal{A}}(t)) \times H^1_D(\tilde{\Omega}),$$

where $\tilde{\mathcal{A}}(t)$ is defined as follows: w belongs to $D(\tilde{\mathcal{A}}(t))$ if and only if $w \in H^1_D(\tilde{\Omega})$ and satisfies

$$\tilde{\mathcal{A}}(t) w := \tilde{\mathcal{A}}_1(t) w + \tilde{\mathcal{A}}_0(t) w \in L^2(\tilde{\Omega}),$$

as well as

$$\left.\begin{array}{l} \partial_{\tilde{n}} w = 0 \text{ on } \tilde{\sigma}, \\ \partial_{\tilde{n}} w = 0 \text{ on } \tilde{\Gamma}_N, \\ w = 0 \text{ on } \tilde{\Gamma}_D. \end{array}\right\} \tag{57}$$

From well known results [14, 17, 10, 19, 15, 4], the domain $D(\tilde{\mathcal{A}}(t))$ is the direct sum of $H^2(\tilde{\Omega})$ with a singular function induced by the crack tip and which is determined by the principal part of $\tilde{\mathcal{A}}(t)$ frozen at the crack tip (here 0). From the expression of $\tilde{\mathcal{A}}$, this principal part frozen at zero is $\mathcal{A}_p(z_1 = z_2 = 0)$ and is equal to

$$\mathcal{A}_p(z_1 = z_2 = 0) = -c^2(\partial^2_{z_1} + \partial^2_{z_2}).$$

Consequently by regularity results on domains with a crack (see for instance [14, 10, 4, 19, 15]), $w \in D(\tilde{\mathcal{A}}(t))$ admits the following decomposition:

$$w = w_R + k\tilde{\eta}S_N, \tag{58}$$

where $w_R \in H^2(\tilde{\Omega})$ is the regular part of w, $k \in \mathbf{R}$ is the so-called stress intensity factor of w, $\tilde{\eta}$ is a cut-off function equal to 1 in a neighborhood of 0 and zero outside another neighborhood, and finally S_N is the Neumann singular function related to the operator $-\Delta$ in the z-coordinates, which is given by

$$S_N(z) = \sqrt{r_2}\sin(\phi_2/2), \tag{59}$$

where (r_2, ϕ_2) are polar coordinates centred at 0 of the Cartesian coordinates (z_1, z_2) such that the half-lines $\phi_2 = -\pi$ and $\phi_2 = \pi$ contain the negative z_1-axis.
All together we have proved the

Lemma 4.3 *Under the above assumptions, the operator $\tilde{\mathcal{A}}(t)$ has a domain independent of t given by*

$$D(\tilde{\mathcal{A}}(t)) = \mathcal{Y}, \forall t \in [0, T],$$

where

$$\mathcal{Y} = \{w \in H^2(\tilde{\Omega}) \text{ satisfying } (57)\} \oplus (\tilde{\eta}S_N).$$

Consequently $\tilde{A}(t)$ satisfies $D(\tilde{A}(t)) = Y = \mathcal{Y} \times H^1_D(\tilde{\Omega})$ is independent of t, for all $t \in [0, T]$.

$\mathcal{Y}$ is a Banach space equipped with the norm

$$||w||_{\mathcal{Y}} = ||w_R||_{2,\tilde{\Omega}} + |k|,$$

for all $w \in \mathcal{Y}$ which admits the unique decomposition

$$w = w_R + k\tilde{\eta}S_N,$$

with $w_R \in H^2(\tilde{\Omega})$ and $k \in \mathbf{R}$. Since S_N belongs to $H^1(\tilde{\Omega})$ we have the continuous embedding of $\mathcal{Y}$ into $H^1_D(\tilde{\Omega})$, the density of $\mathcal{Y}$ into $H^1_D(\tilde{\Omega})$ being direct since $H^2(\tilde{\Omega})$ is dense in $H^1(\tilde{\Omega})$.
To check the assumption (i), we use the variable norm technique (see Proposition 1.1 in [12]) which consists in showing that there exists a sequence of norms $|\cdot|_t$ on X depending continuously on t in the following sense

$$|x|_t \leq e^{c_0|t-s|}|x|_s, \forall x \in X, s, t \in [0, T], \tag{60}$$

for some $c_0 > 0$ and that there exists a real number $\beta \geq 0$ such that for all $\lambda > \beta$, $\lambda I + \tilde{A}(t)$ is invertible and

$$|\lambda x + \tilde{A}(t)x|_t \geq (\lambda - \beta)|x|_t, \forall x \in D(\tilde{A}(t)), \forall t \in [0, T]. \tag{61}$$

If such properties hold then Proposition 1.1 in [12] shows that the assumption (i) holds.
Let us introduce the inner product (one easily checks that it is actually an inner product)

$$\left(\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right)_t := \tilde{a}_0(t, v_0, w_0) + (v_0, w_0) + (v_1, w_1), \tag{62}$$

where $(\cdot, \cdot)$ means here and below the $L^2(\tilde{\Omega})$-inner product and $\tilde{a}_0$ is defined by

$$\begin{aligned} \tilde{a}_0(t, v, w) &= -\int_Q ((-c^2\alpha(t)^2(\partial_1 q)^2 + (\partial_t q)^2 - 2h'\partial_t q \partial_1 q)\partial_{z_1} v \partial_{z_1} w - c^2 \partial_{z_2} v \partial_{z_2} w)\, dz \\ &+ \int_{\tilde{\Omega}\setminus Q} \Big[\frac{c^2 - (h')^2\eta^2}{\alpha_0^2} \partial_{z_1} v \partial_{z_1} w + c^2 \left(\partial_{z_2} v + h(\partial_1 \eta \partial_{z_2} v - \frac{\partial_2 \eta}{\alpha_0} \partial_{z_1} v) \right) \cdot \\ &\qquad \left(\partial_{z_2} w + h(\partial_1 \eta \partial_{z_2} w - \frac{\partial_2 \eta}{\alpha_0} \partial_{z_1} w) \right) \Big] (1 + h\partial_1 \eta)^{-2}\, dz. \end{aligned}$$

As before the subdivision between Q and $\tilde{\Omega} \setminus Q$ is artificial since the integrand are the same in a neighborhood of their common boundary. This subdivision is used here to deduce easily the coerciveness of $\tilde{a}_0$.
The bilinear form $\tilde{a}_0$ corresponds to the principal part of $\tilde{\mathcal{A}}_0$, namely by integration by parts (in the distributional sense), we get

$$\int_{\tilde{\Omega}} \tilde{\mathcal{A}}_0(t) v w\, dz = \tilde{a}_0(t, v, w) + \tilde{r}(t, v, w), \forall v \in D(\tilde{\mathcal{A}}(t)), w \in \mathcal{D}(\tilde{\Omega}), \tag{63}$$

where the remainder satisfies

$$|\tilde{r}(t, v, w)| \leq C|v|_{1,\tilde{\Omega}} \|w\|_{0,\tilde{\Omega}}, \forall v, w \in H^1(\tilde{\Omega}), \tag{64}$$

for some $C > 0$ independent of t.
The norm $|\cdot|_t$ is simply induced by the above inner product (62):

$$\left| \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right|_t^2 := \left(\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right)_t.$$

Lemma 4.4 *Under the above assumptions, there exists a positive constant* κ *such that for all* $t \in [0,T]$*, it holds*

$$\left| \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right|_t^2 \geq \kappa(\|v_0\|^2_{1,\tilde{\Omega}} + \|v_1\|^2_{0,\tilde{\Omega}}), \forall \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in X. \tag{65}$$

Consequently (60) holds for some $c_0 > 0$*.*

Proof: Let us first show how (65) yields (60). Indeed for a fixed $\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$ in X we have

$$\frac{d}{dt}\left| \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right|_t^2 = \frac{d}{dt}\tilde{a}_0(t, v_0, v_0).$$

By Leibniz's rule and Cauchy-Schwarz's inequality there exists $c_1 > 0$ such that

$$\frac{d}{dt}\tilde{a}_0(t, v_0, v_0) \leq c_1 |v_0|^2_{1,\tilde{\Omega}}.$$

This inequality and the estimate (65) in the above identity yields a positive constant $c_0 > 0$ such that

$$\frac{d}{dt}\left| \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right|_t^2 \leq 2c_0 \left| \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right|_t^2, \forall t \in [0,T].$$

By Gronwall's lemma we conclude that (60) holds.
The estimate (65) follows from

$$\tilde{a}_0(t, v, v) \geq \kappa |v|^2_{1,\tilde{\Omega}}, \forall v \in H^1(\tilde{\Omega}), \tag{66}$$

this estimate being a direct consequence of Lemmas 3.2 and 4.1. ■

To show the property (61) we first state the following technical lemma which is proved exactly as Lemma 3.4.

Lemma 4.5 *If* θ *is of the form (44), then there exists* $C_1 > 0$ *independent of* t *such that*

$$\int_{\tilde{\Omega}} \tilde{\mathcal{B}}(t) w w \, dz \geq -C_1 \int_{\tilde{\Omega}} |w|^2 \, dz, \forall w \in H^1(\tilde{\Omega}). \tag{67}$$

We are now ready to prove (61), namely we have the

Lemma 4.6 *Under the above assumptions, there exists* $\beta \geq 0$ *such that for all* $\lambda > \beta$*,* $\lambda I + \tilde{A}(t)$ *is invertible and (61) holds.*

Proof: We first prove the invertibility property. This is equivalent to show that for all $\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$ in X there exists a unique $\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$ in $D(\tilde{A}(t))$ such that

$$(\lambda I + \tilde{A}(t)) \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

¿From the definition of $\tilde{A}(t)$ we equivalently have

$$v_1 = \lambda v_0 - f_0, \tag{68}$$

$$\tilde{\mathcal{A}}_0(t)v_0 + \tilde{\mathcal{A}}_1(t)v_0 + \lambda^2 v_0 + \lambda\tilde{\mathcal{B}}(t)v_0 = h, \tag{69}$$

where

$$h = f_1 + \lambda f_0 + \tilde{\mathcal{B}}(t)f_0$$

belongs to $L^2(\tilde{\Omega})$. We now show the existence and uniqueness of a solution $v_0 \in D(\tilde{\mathcal{A}}(t))$ of (69). By the definition of $\tilde{\mathcal{A}}(t)$ problem (69) is equivalent to

$$a_\lambda(t, v_0, w) = (h, w), \forall w \in H^1_D(\tilde{\Omega}), \tag{70}$$

where we have set

$$a_\lambda(t, v_0, w) = \tilde{a}_0(t, v_0, w) + \lambda^2(v_0, w) + r_\lambda(t, v_0, w),$$
$$r_\lambda(t, v, w) = \tilde{r}(t, v, w) + \lambda(\tilde{\mathcal{B}}(t)v, w) + (\tilde{\mathcal{A}}_1(t)v, w).$$

Problem (70) will have a unique solution $v_0 \in H^1_D(\tilde{\Omega})$ by Lax-Milgram's lemma if the bilinear form a_λ is coercive on $H^1_D(\tilde{\Omega})$, i.e., if there exists a positive constant α such that

$$a_\lambda(t, v, v) = \tilde{a}_0(t, v, v) + \lambda^2(v, v) + r_\lambda(t, v, v) \geq \alpha\|v\|^2_{1,\tilde{\Omega}}, \forall v \in H^1_D(\tilde{\Omega}). \tag{71}$$

By Lemma 4.4 we have

$$\tilde{a}_0(t, v, v) \geq \kappa|v|^2_{1,\tilde{\Omega}},$$

while by Lemma 4.5, the estimate (64) and the definition of $\tilde{\mathcal{A}}_1$ (using the smoothness of h and θ), there exists $C_2 > 0$ (independent of t) such that

$$r_\lambda(t, v, v) \geq -\lambda C_1\|v\|^2_{0,\tilde{\Omega}} - C_2|v|_{1,\tilde{\Omega}}\|v\|_{0,\tilde{\Omega}} \geq -(\lambda C_1 + \frac{C_2}{2\epsilon})\|v\|^2_{0,\tilde{\Omega}} - \frac{C_2\epsilon}{2}|v|^2_{1,\tilde{\Omega}}.$$

These inequalities show that

$$a_\lambda(t, v, v) \geq (\kappa - \frac{C_2\epsilon}{2})|v|^2_{1,\tilde{\Omega}} + [\lambda^2 - (\lambda C_1 + \frac{C_2}{2\epsilon})]\|v\|^2_{0,\tilde{\Omega}},$$

for all $\epsilon > 0$. Choosing $\epsilon = \frac{\kappa}{C_2}$, we obtain

$$a_\lambda(t, v, v) \geq \frac{\kappa}{2}|v|^2_{1,\tilde{\Omega}} + [\lambda^2 - (\lambda C_1 + \frac{C_2^2}{2\kappa})]\|v\|^2_{0,\tilde{\Omega}}.$$

Since $\lambda^2 - (\lambda C_1 + \frac{C_2^2}{2\kappa})$ is quadratic in λ, there exists $\beta > 0$ such that for all $\lambda > \beta$,

$$\lambda^2 - (\lambda C_1 + \frac{C_2^2}{2\kappa}) > \frac{\kappa}{2},$$

and we deduce the coerciveness property (71) for $\lambda > \beta$.
Once problem (70) has a unique solution $v_0 \in H^1_D(\tilde{\Omega})$ we deduce that v_0 belongs to $D(\tilde{\mathcal{A}}(t))$ since

$$h - (\tilde{\mathcal{A}}_1(t)v_0 + \lambda^2 v_0 + \lambda\tilde{\mathcal{B}}(t)v_0)$$

belongs to $L^2(\tilde{\Omega})$.
Let us now prove (61): Fix $\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$ in $D(\tilde{A}(t))$ then from the definition of $\tilde{A}(t)$ and the inner product $(\cdot, \cdot)_t$ we have

$$\left((\lambda I + \tilde{A}(t))\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)_t = \tilde{a}_0(t, -v_1 + \lambda v_0, v_0) + (-v_1 + \lambda v_0, v_0)$$
$$+(\tilde{\mathcal{A}}_0(t)v_0 + \tilde{\mathcal{A}}_1(t)v_0 + \tilde{\mathcal{B}}(t)v_1 + \lambda v_1, v_1).$$

By the identity (63) we get

$$\left((\lambda I + \tilde{A}(t))\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)_t = \lambda(\tilde{a}_0(t, v_0, v_0) + \|v_0\|^2_{0,\Omega} + \|v_1\|^2_{0,\Omega})$$
$$+\tilde{r}(t, v_0, v_1) - (v_1, v_0) + (\tilde{\mathcal{A}}_1(t)v_0 + \tilde{\mathcal{B}}(t)v_1, v_1).$$

Lemma 4.5 and the estimate (64) lead to

$$\begin{aligned}\left((\lambda I + \tilde{A}(t))\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)_t &\geq \lambda(a(t, v_0, v_0) + \|v_0\|^2_{0,\Omega} + \|v_1\|^2_{0,\Omega}) \\ &- \gamma(\|v_0\|^2_{1,\tilde{\Omega}} + \|v_1\|^2_{0,\tilde{\Omega}}),\end{aligned}$$

for some $\gamma > 0$ (independent of t). This leads to (61) thanks to Lemma 4.4. ■
It remains to check the hypothesis (iii). From the definition of $\tilde{A}(t)$ we have

$$\partial_t\tilde{A}(t)\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_t\mathcal{A}_0(t)v_0 + \partial_t\tilde{\mathcal{A}}_1(t)v_0 + \partial_t\tilde{\mathcal{B}}(t)v_1 \end{pmatrix}.$$

Only the term $\partial_t \mathcal{A}_0(t)v_0$ requires a careful analysis since v_0, v_1 are in $H^1(\tilde{\Omega})$ and in $\partial_t \tilde{\mathcal{A}}_1(t)$ and $\partial_t \tilde{\mathcal{B}}(t)$ only first order derivatives in z are involved. Now for the term $\partial_t \mathcal{A}_0(t)v_0$, v_0 is in H^2 far from the crack tip therefore we only need to consider $\partial_t \mathcal{A}_p(t)(\hat{\eta}v_0)$ where $\hat{\eta}$ is a cut-off function with support in $B(0, 1/2)$. Using the definition of $\mathcal{A}_p$ we get

$$\partial_t \mathcal{A}_p(t)(\hat{\eta}v_0) = \{\partial_t[(\alpha(t))^{-2}(\alpha'(t))^2]z_1^2 - 2\partial_t[h'(\alpha(t))^{-2}\alpha'(t)]z_1\}\partial^2_{z_1}(\hat{\eta}v_0).$$

As $\partial^2_{z_1}(\hat{\eta}v_0)$ behaves like $r_2^{-3/2}$ and $z_1 r_2^{-3/2}$ belongs to $L^2(\tilde{\Omega})$, we may conclude that

$$||\partial_t \mathcal{A}_p(t)(\hat{\eta}v_0)||_{0,\tilde{\Omega}} \leq C||v_0||_{\mathcal{Y}},$$

for some $C > 0$ independent of t. Consequently

$$\left\| \partial_t \tilde{A}(t) \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right\|_X \leq C_1 \left[\|v_0\|_{\mathcal{Y}} + \|v_1\|_{1,\tilde{\Omega}} \right],$$

for some $C_1 > 0$ independent of t, which proves the hypothesis (iii).
In summary we have checked that the triplet $\{\tilde{A}, X, Y\}$ forms a CD-system and by Theorem 1.2 of [13] we deduce the

Theorem 4.7 *Let us assume that θ is of the form (44) and that h satisfies (45) and (49). Then for $\tilde{g} \in Lip([0, T]; L^2(\tilde{\Omega}))$, $\tilde{w}_0 \in \mathcal{Y}$ and $\tilde{w}_1 \in H^1_D(\tilde{\Omega})$, there exists a unique (strong) solution $w \in C([0, T]; \mathcal{Y}) \cap C^1([0, T]; H^1_D(\tilde{\Omega})) \cap C^2([0, T]; L^2(\tilde{\Omega}))$ of (50) to (55).*

Note that by the successive change of unknowns (6) and (48) the above theorem yields a unique strong solution u of (4) for appropriate data. Namely for all $t \geq 0$, let us set

$$\mathcal{Y}_t = \{v \in H^2(\Omega) \text{ satisfying (8) and (10)}\} \oplus (\eta S_N(\frac{x_1}{\alpha(t)}, x_2)).$$

Then we obtain the main result of this section:

Theorem 4.8 *Let us assume that θ is of the form (44) and that h satisfies (45) and (49). Assume that $f \circ F_t \in Lip([0, T]; L^2(\Omega))$, $q \in \tilde{H}^{1/2}(\Gamma_N \times (0, T))$ and initial data u_0 and u_1 as follows: $u_0 \in \mathcal{Y}_0$ of the form*

$$u_0(x) = u_{R0}(x) + k\eta(x)S_N(\frac{x_1}{\alpha(0)}, x_2), \tag{72}$$

with $u_{R0} \in H^2(\Omega)$, $k \in \mathbf{R}$ *and* $u_1 \in H^1_D(\Omega)$. *Then there exists a unique strong solution* u *of (4) such that* $u(F_t(\cdot),\cdot) \in C([0,T];\mathcal{Y}_t) \cap C^1([0,T];H^1_D(\Omega)) \cap C^2([0,T];L^2(\Omega))$ *and admits the decomposition*

$$u(y,t) = v_R(F_t^{-1}(y),t) + k(t)\tilde{\eta}(F_t^{-1}(y))S_N(\frac{y_1 - h(t)}{\alpha(t)}, y_2), \tag{73}$$

with $v_R \in C([0,T];H^2(\Omega))$ *and the stress intensity function* k *satisfies* $k \in C([0,T])$ *and* $k(0) = k$.

Proof: As usual we may reduce our analysis to the case $q = 0$. For data as in the statement of the Theorem, we readily check that $\tilde{g}, \tilde{w}_0$ and $\tilde{w}_1$ defined in (50) to (55) satisfy the assumptions of Theorem 4.7. Consequently a unique strong solution w of (50) to (55) exists. By the definition of $\mathcal{Y}$, w admits the decomposition

$$w(z,t) = w_R(z,t) + k(t)\eta(z)S_N(z),$$

with $w_R \in C([0,T];H^2(\tilde{\Omega}))$ and $k \in C([0,T])$. Using the change of unknown (48), v belongs to $C^1([0,T];H^1_D(\Omega)) \cap C^2([0,T];L^2(\Omega))$ and admits the splitting

$$v(x,t) = w_R(q(x,t),x_2,t) + k(t)\eta(q(x,t),x_2)S_N(q(x,t),x_2).$$

As $S_N(q(x,t),x_2)$ is regular far from the crack tip, the above splitting is equivalent to

$$v(x,t) = v_R(x,t) + k(t)\eta(x)S_N(\frac{x_1}{\alpha(t)}, x_2), \tag{74}$$

with $v_R \in C([0,T];H^2(\Omega))$ and $k \in C([0,T])$, which also implies that $v(\cdot,t)$ belongs to $C([0,T];\mathcal{Y}_t)$.
This decomposition of v and the change of unknowns (6) yield the conclusion. ∎

5 Griffith criterion

In the quasi-stationary case crack growth processes in brittle materials are often studied using Irwin's critical stress intensity factor criterion or Griffith's critical energy release rate criterion. In the dynamical case a generalized Griffith's energy balance criterion is used, which leads to an equation of motion of the crack tip [8]. More precisely, the rate of the total energy of problem (4) at time t is given by

$$\hat{\Pi}(t) = \dot{U}(t) + \dot{K}(t) + \dot{D}(t), \tag{75}$$

where

$$\dot{U}(t) = \dot{E}(t) - \hat{A}(t).$$

For a solution $u = u(y,t)$ of problem (4) with $g = c^2\frac{\partial u}{\partial n}$ on Γ_N holds:

$$\dot{E}(t) = \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t} c^2|\nabla_y u(t,y)|^2\,dy$$

is the rate of the elastic energy;

$$\hat{A}(t) = \int\limits_{\Omega_t} f u_t\,dy + \int\limits_{\Gamma_N} g u_t\,ds,$$

denotes the rate of the external energy and

$$\dot{K}(t) = \frac{d}{dt}\int\limits_{\Omega_t}\frac{1}{2}(u_t)^2\,dy$$

is the rate of the kinetic energy. D is the dissipative energy and is the sum of all irreversibles energies such as the surface free energy or fracture energy, plastic work and viscous dissipation. We set $\dot{D} = \frac{d}{dt}D$.
An energy balance holds if $\hat{\Pi}(t) = 0$, that means

$$\dot{D}(t) = -\dot{E}(t) + \hat{A}(t) - \dot{K}(t). \tag{76}$$

The problem is now: Derive from (76) an equation of motion of the crack tip. If we assume that the crack is straight and it is growing tangentially, then θ is given by (44), and it remains to find a relation from (76) which gives us $h(t)$ or the crack speed $h'(t)$ respectively.
With the help of Reynolds'-transport theorem we can show the following lemma in the actual configuration.

Lemma 5.1 *The identity holds:*

$$\hat{A}(t) - \dot{E}(t) - \dot{K}(t) = -\frac{1}{2}\int_{\Omega_t}\mathrm{div}_y\left[(u_t^2 + c^2|\nabla u|^2)\frac{dy(t)}{dt}\right]\,dy \tag{77}$$

where $y = F_t(x) = x + h(t)\theta(x)$ *and* θ *given by (44).*

Proof: We start with the wave equation in the actual configuration

$$u_{tt} - c^2\Delta u = f. \tag{78}$$

Multiplication with u_t and integration on Ω_t yields

$$\begin{aligned}\int_{\Omega_t}\left(\frac{1}{2}\frac{d}{dt}\left(u_t^2\right)+c^2\nabla u\cdot\nabla u_t\right)\,dy &= \frac{1}{2}\int_{\Omega_t}\left(\frac{d}{dt}(u_t)^2+c^2\frac{d}{dt}\left(\nabla u\right)^2\right)\,dy\\ &= \int_{\Omega_t} f u_t\,dy+\int_{\partial\Omega_t} c^2\frac{\partial u}{\partial n}u_t\,ds\\ &= \int_{\Omega_t} f u_t\,dy+\int_{\Gamma_N} g u_t\,ds,\end{aligned}$$

with $g=c^2\frac{\partial u}{\partial n}$ on Γ_N. It follows from Reynolds'-transport theorem that

$$\begin{aligned}\dot{E}(t)+\dot{K}(t) &= \frac{d}{dt}\frac{1}{2}\int_{\Omega_t}\left(u_t^2+c^2|\nabla u|^2\right)\,dy\\ &= \frac{1}{2}\int_{\Omega_t}\frac{d}{dt}\left(u_t^2+c^2|\nabla u|^2\right)\,dy+\frac{1}{2}\int_{\Omega_t}\operatorname{div}\left(u_t^2c^2|\nabla u|^2\frac{dy}{dt}\right)\,dy\\ &= \hat{A}+\frac{1}{2}\int_{\Omega_t}\operatorname{div}\left(u_t^2c^2|\nabla u|^2\frac{dy}{dt}\right)\,dy.\end{aligned}$$

The relation (77) follows. ■

Now, we calculate the right hand side of (77), transforming this integral into the reference domain Ω.

Lemma 5.2 *For the right hand side of (77) it holds:*

$$\begin{aligned}I &= -\frac{1}{2}\int_{\Omega_t}\operatorname{div}_y\left([u_t^2+c^2|\nabla u|^2]\frac{dy(t)}{dt}\right)\,dy\\ &= h'(t)\lim_{\delta\to 0}\delta\int_0^{\pi}\cos\phi[(\partial_t v_S(\delta,\phi,t)-h'(t)\partial_1 v_S(\delta,\phi,t))^2]\,d\phi\\ &\quad+\lim_{\delta\to 0}\delta\int_0^{\pi}\cos\phi[c^2(\partial_1 v_S(\delta,\phi,t))^2+(\partial_2 v_S(\delta,\phi,t))^2)]\,d\phi, \qquad (79)\end{aligned}$$

where $y=F_t(x)=x+h(t)\theta(x)$, $v=v(x,t)=u(y,t)=v(r,\phi,t)$, $x=(r\cos\phi,r\sin\phi)^\top$ *and* $v(x,t)=v_R(x,t)+k(t)\eta(x)S_N(\frac{x_1}{\alpha(t)},x_2)=v_R+v_S$, *compare (74).*

Proof: We transform the integral on the actual configuration Ω_t to the reference configuration recalling that

$$y=\begin{pmatrix}x_1\\x_2\end{pmatrix}+h(t)\begin{pmatrix}\eta(x)\\0\end{pmatrix},$$

where η is a cut-off function with support in a neighborhood of the crack tip. We get

$$I = -\frac{1}{2}\int_\Omega \partial_{x_1}\left(\left[\partial_t v - \frac{h'\eta\partial_1 v}{1+h\partial_1\eta}\right]^2 h'(t)\eta\right)\, dx$$
$$-\frac{1}{2}\int_\Omega \partial_{x_1}\left(\frac{c^2}{(1+h\partial_1\eta)^2}\left((\partial_1 v)^2 + (-\partial_1 v h\partial_2\eta + \partial_2 v(1+h\partial_1\eta))^2\right) h'(t)\eta\right)\, dx.$$

For a fixed $\delta > 0$ we consider a circle $B_\delta(0)$ with center in the crack tip and radius δ and introduce the domain $\Omega \setminus B_\delta(0) = \Omega_\delta$. In Ω_δ Gauss' formula is applicable. Due to the asymptotic expansion (74) of v

$$v(x,t) = v_R(x,t) + k(t)\eta(x)S_N(\frac{x_1}{\alpha(t)}, x_2) = v_R + v_S$$

we have:

$$I = -\lim_{\delta\to 0}\frac{1}{2}\int_{\Omega_\delta} \partial_{x_1}\left(\left[\partial_t v - \frac{h'\eta\partial_1 v}{1+h\partial_1\eta}\right]^2 h'(t)\eta\right)\, dx$$
$$-\lim_{\delta\to 0}\frac{1}{2}\int_{\Omega_\delta} \partial_{x_1}\left(\frac{c^2}{(1+h\partial_1\eta)^2}\left((\partial_1 v)^2 + (-\partial_1 v h\partial_2\eta + \partial_2 v(1+h\partial_1\eta))^2\right) h'(t)\eta\right)\, dx$$
$$= -\lim_{\delta\to 0}\frac{1}{2}\int_{\partial B_\delta} h'(t)(\partial_t v_S - h'\partial_1 v_S)^2 + c^2((\partial_1 v_S)^2 + (\partial_2 v_S)^2)n_1\, ds$$
$$= \lim_{\delta\to 0}\frac{1}{2}h'(t)\int_{-\pi}^{\pi} \cos\phi\left[(\partial_t v_S(\delta,\phi,t) - h'(\partial_1 v_S(\delta,\phi,t))^2\right]\, d\phi$$
$$+ \lim_{\delta\to 0}\frac{1}{2}h'(t)\int_{-\pi}^{\pi} \cos\phi\left[c^2((\partial_1 v_S(\delta,\phi,t))^2 + (\partial_2 v_S(\delta,\phi,t))^2)\right]\, d\phi. \qquad (80)$$

Note, that we have used that the limit value of the integral vanishes for the regular part v_R of v, that η vanishes on $\partial\Omega\backslash\sigma_0$, and that the first component n_1 of the normal unit vector vanishes on the crack σ_0. The integral in (80) is even and therefore the assertion (79) follows. ■

Now, we calculate explicitely the right hand side of (77). The result is formulated in the follwing lemma:

Lemma 5.3 *For the right hand side of (77) it holds*

$$
\begin{aligned}
I &= -\frac{1}{2}\int_{\Omega_t} \mathrm{div}_y \left([u_t^2 + c^2|\nabla u|^2] \frac{dy(t)}{dt} \right) dy \\
&= h'(t)k^2(t)c^2 \frac{\pi}{4}\frac{(\alpha-1)}{\alpha}
\end{aligned}
\tag{81}
$$

and therefore the identity is valid

$$
\dot{D}(t) = \hat{A}(t) - \dot{E}(t) - \dot{K}(t) = h'(t)k^2(t)c^2\frac{\pi}{4}\frac{(\alpha-1)}{\alpha}, \tag{82}
$$

where

$$
\alpha = \sqrt{1 - \frac{h'^2}{c^2}}. \tag{83}
$$

Proof: We use the fact, compare (74), that for small δ

$$
v_S(x,t) = k(t)S_N(z,t),
$$

where $z = (z_1, z_2) = (\frac{x_1}{\alpha(t)}, x_2)$ and $S_N(z,t) = \sqrt{r_z}\sin\left(\frac{\phi_z}{2}\right)$. Let us transform $S_N(z,t)$ in the (x_1, x_2) - coodinates. We have

$$
\begin{aligned}
\sin\left(\frac{\phi_z}{2}\right) &= \sqrt{\frac{1}{2}}\sqrt{1-\cos\phi_z}, \quad \cos\phi_z = \frac{z_1}{r_z} = \frac{x_1}{\alpha(t)r_z} = \frac{r(x)\cos\phi}{\alpha(t)r_z}, \\
r_z^2 &= z_1^2 + z_2^2 = \frac{x_1^2}{\alpha(t)^2} + x_2^2 = r^2\left(\frac{\cos^2\phi}{\alpha(t)^2} + \sin^2\phi\right), \\
r_z &= \frac{r}{\alpha(t)}\sqrt{\cos^2\phi + \alpha(t)^2\sin^2\phi}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S_N(z,t) &= r_z^{\frac{1}{2}}\sin\frac{\phi_z}{2} = \frac{r^{\frac{1}{2}}}{\alpha(t)^{\frac{1}{2}}}(\cos^2\phi + \alpha^2\sin^2\phi)^{\frac{1}{4}}\sqrt{\frac{1}{2}}\sqrt{1-\cos\phi} \\
&= \frac{r^{\frac{1}{2}}}{\alpha(t)^{\frac{1}{2}}}\sqrt{\frac{1}{2}}\sqrt{\sqrt{\cos^2\phi+\alpha^2\sin^2\phi}\left(1 - \frac{\cos\phi}{\sqrt{\cos^2\phi+\alpha^2\sin^2\phi}}\right)} \\
&= \sqrt{\frac{r}{\alpha(t)}}\sqrt{\frac{1}{2}}\sqrt{\sqrt{\cos^2\phi+\alpha^2(t)\sin^2\phi} - \cos\phi} \\
&= \sqrt{\frac{r}{2\alpha(t)}}v_s(\phi,t) = \hat{v}_s(r,\phi,t),
\end{aligned}
\tag{84}
$$

where

$$v_s = v_s(\phi, t) = \sqrt{\sqrt{\cos^2\phi + \alpha^2(t)\sin^2\phi} - \cos\phi}. \tag{85}$$

Inserting (84) into (79), we see that

$$\lim_{\delta\to 0} \delta \int_0^\pi \cos\phi[\partial_t v_S(\delta, \phi, t)^2 - 2h'(t)\partial_1 v_S(\delta, \phi, t)\partial_t v_S(\delta, \phi, t)]\, d\phi = 0.$$

It remains to calculate

$$\begin{aligned} I &= h'(t)k^2(t) \lim_{\delta\to 0} \delta \int_0^\pi \cos\phi \left[(h'(t)^2 + c^2)(\partial_1 v_s(\delta, \phi, t))^2 + c^2(\partial_2 v_s(\delta, \phi, t)^2)\right]\, d\phi, \\ &= h'(t)k^2(t)c^2 \lim_{\delta\to 0} \delta \int_0^\pi \cos\phi \left[(2-\alpha^2)(\partial_1 \hat{v})^2 + (\partial_2 \hat{v})^2\right]\, d\phi. \end{aligned}$$

Since

$$\begin{aligned} (\partial_1 \hat{v})^2 &= \left[\frac{\partial \hat{v}}{\partial r}\cos\phi - \frac{1}{r}\frac{\partial \hat{v}}{\partial \phi}\sin\phi\right]^2 = \frac{1}{2\alpha r}\left[\frac{1}{2}v_s\cos\phi - \frac{\partial v_s}{\partial \phi}\sin\phi\right]^2, \\ (\partial_2 \hat{v})^2 &= \left[\frac{\partial \hat{v}}{\partial r}\sin\phi + \frac{1}{r}\frac{\partial \hat{v}}{\partial \phi}\cos\phi\right]^2 = \frac{1}{2\alpha r}\left[\frac{1}{2}v_s\sin\phi + \frac{\partial v_s}{\partial \phi}\cos\phi\right]^2 \end{aligned}$$

we have

$$\begin{aligned} &\lim_{\delta\to 0} \delta \int_0^\pi \cos\phi \left[(2-\alpha^2)(\partial_1 \hat{v})^2 + (\partial_2 \hat{v})^2\right]\, d\phi, \\ &= \frac{1}{2\alpha}\int_0^\pi \left[\frac{1}{4}v_s^2(1-\alpha^2)\cos^3\phi + \frac{1}{4}v_s^2\cos\phi + \left(\frac{\partial v_s}{\partial \phi}\right)^2(\alpha^2-1)\cos^3\phi\right] \\ &\quad + \frac{1}{2\alpha}\int_0^\pi \left[(2-\alpha^2)\left(\frac{\partial v_s}{\partial \phi}\right)^2\cos\phi + (\alpha^2-1)\cos^2\phi\sin\phi v_s\frac{\partial v_s}{\partial \phi}\right]\, d\phi \end{aligned} \tag{86}$$

Furthermore, we easily check that

$$
\begin{aligned}
J &= \int_0^\pi \cos^2\phi \sin\phi v_s \frac{\partial v_s}{\partial \phi}\, d\phi \\
&= -\int_0^\pi \left[-2\cos\phi \sin^2\phi v_s + \cos^3\phi v_s + \cos^2\phi \sin\phi \frac{\partial v_s}{\partial \phi} \right] v_s\, d\phi, \\
&= \int_0^\pi (2\cos\phi(1-\cos^2\phi) v_s^2 - \cos^3\phi v_s^2) - J, \\
J &= \int_0^\pi (\cos\phi v_s^2 - \cos^3\phi v_s^2 - \frac{1}{2}\cos^3\phi v_s^2)\, d\phi, \\
&= \int_0^\pi (\cos\phi v_s^2 - \frac{3}{2}\cos^3\phi v_s^2)\, d\phi.
\end{aligned}
$$

The expressions v_s^2 and $\left(\frac{\partial v_s}{\partial \phi}\right)^2$ read:

$$
\begin{aligned}
(v_s)^2 &= \sqrt{\cos^2\phi + \alpha^2 \sin^2\phi} - \cos\phi = \text{ symmetric part} - \cos\phi, \\
\left(\frac{\partial v_s}{\partial \phi}\right)^2 &= \frac{1}{4(\cos^2\phi + \alpha^2\sin^2\phi)} \left[\sqrt{\cos^2\phi + \alpha^2\sin^2\phi}((\alpha^2-1)\cos^2\phi + 1) \right] \\
&\quad + \frac{1}{4}\cos\phi \left(1 + \frac{\alpha^2 - 1}{\cos^2\phi + \alpha^2\sin^2\phi)} \right) \\
&= \text{symmetric part} \quad + \frac{1}{4}\cos\phi \left(1 + \frac{\alpha^2 - 1}{\cos^2\phi + \alpha^2\sin^2\phi)} \right)
\end{aligned}
$$

Both terms consist of a symmetric and an odd part with respect to $\frac{\pi}{2}$ and the odd parts contribute to the integral I only. Inserting the odd parts in our integral we get the formula (81) after some elementary integrations. ∎

The equation of motion

If the rate of the dissipative energy $\dot{D}(t)$ is known, then the identities (82,83) lead to an ordinary differential equation for the unknown $h(t)$, called equation of motion of the crack tip,

$$
\dot{D}(t) = h'(t) k^2(t) c^2 \frac{\pi}{4} \frac{(\sqrt{1 - \frac{h'^2}{c^2}} - 1)}{\sqrt{1 - \frac{h'^2}{c^2}}} = G(h, h') h'(t). \tag{87}
$$

In the plane case [8] this quantity is given by

$$
\dot{D}(t) = 2d\gamma(h, h')h'(t),
$$

where d denotes the thickness of the plate and γ describes a material property, which can be determined only by experiments. If $h'(t) \neq 0$, then the equation of motion together with the initial condition reads

$$G(h,h') = 2d\gamma(h,h') = k^2(t)c^2 \frac{\pi}{4} \frac{(\sqrt{1-\frac{h'^2}{c^2}}-1)}{\sqrt{1-\frac{h'^2}{c^2}}} = k^2(t,h,h')c^2 \frac{\pi}{4} \frac{(\sqrt{1-\frac{h'^2}{c^2}}-1)}{\sqrt{1-\frac{h'^2}{c^2}}},$$
$$h(0) = 0. \tag{88}$$

Even if γ is constant, the solution of the initial problem for the nonlinear ordinary differential equation (88) cannot be calculated explicitely, since the dynamic stress intensity factor $k(t)$ depends on h and h'.
A numerical solution could be computed by iterative procedures,but this reqired a careful analysis of the evolution of the stress intensity factor $k(t) = k(t,h,h')$.

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