Universität Stuttgart

A mixed variational formulation and an optimal a priori error estimate for a frictional contact problem in elasto-piezoelectricity

Stefan Hüeber, Andaluzia Matei, Barbara I. Wohlmuth

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Documentation 2005/002

Universität Stuttgart

A mixed variational formulation and an optimal a priori error estimate for a frictional contact problem in elasto-piezoelectricity

Stefan Hüeber, Andaluzia Matei, Barbara I. Wohlmuth

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Documentation 2005/002

Institut für Angewandte Analysis und Numerische Simulation (IANS)
Fakultät Mathematik und Physik
Fachbereich Mathematik
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: `ians-preprints@mathematik.uni-stuttgart.de`
WWW: `http://preprints.ians.uni-stuttgart.de`

ISSN **1611-4176**

A MIXED VARIATIONAL FORMULATION AND AN OPTIMAL A PRIORI ERROR ESTIMATE FOR A FRICTIONAL CONTACT PROBLEM IN ELASTO-PIEZOELECTRICITY*

S. HÜEBER, A. MATEI, B.I. WOHLMUTH

Abstract. We study the frictional contact between an elasto-piezoelectric body and a rigid foundation. Our study is based on a non-symmetric mixed variational formulation involving dual Lagrange multipliers. Using a fixed point technique and the saddle point framework, we verify the well-posedness of the variational problem. Furthermore, we provide optimal a priori error estimates for the displacements, the electric potential and the stress at the contact interface. The results also hold in the multibody case with nonconforming meshes at the contact interface. For this situation a numerical example is given.

Key words. Key words: *frictional contact, dual Lagrange multipliers, mixed formulation, well-posedness, optimal a priori error estimate.*

1. Introduction. Recently, considerable attention has been paid to the analysis of various models in solid mechanics, involving elasto-piezoelectric materials. *Piezoelectricity* is the ability of certain crystals to produce a voltage when subjected to mechanical stress. The word is derived from the Greek piezein, which means to squeeze or press. Piezoelectric materials also show the opposite effect, called *converse piezoelectricity*; i.e., the application of an electrical field creates mechanical stresses (distortion) in the crystal. Because the charges inside the crystal are separated, the applied voltage affects different points within the crystal differently, resulting in the distortion. Many materials exhibit the piezoelectric effect (e.g. ceramics: $BaTiO_3$, $KNbO_3$, $LiNbO_3$, $LiTaO_3$, $BiFeO_3$). In 1880, the brothers Pierre and Jacques Curie predicted and demonstrated piezoelectricity. In 1881, Lippmann deduced mathematically the converse piezoelectricity from fundamental thermodynamic principles and the Curies immediately confirmed the existence of the converse effect. The first mathematical model of an elastic medium taking linear interaction of electric and mechanical fields into account was constructed by W. Voigt, see [Voi10], and more refined models can be found for example in the works of R. Toupin [Tou56, Tou63], R. Mindlin [Min68, Min69, Min72], S. Kalinski and J. Petikiewicz [KP60] and T. Ikeda [Ike90]. A theoretical result in contact mechanics for piezoelectric materials was obtained recently in [SE04], within the framework of variational inequalities.

In this paper, we consider an elasto-piezoelectric body in frictional contact with a rigid foundation. We assume that the contact is bilateral, and we model the friction with Tresca's law. More details concerning the frictional bilateral contact models can be found, e.g., in [DL76, Pan85] and more recently in [HS02, SST04]. For the variational formulation as a variational inequality of the second kind, and for a priori error estimates based on this formulation without Lagrange multipliers, we refer to[HR99, HS02]. Recently, a lot of work on a priori error estimates for the saddle point formulation has been done, we refer to, e.g. [BR03, BHL99, HL02], and the references therein.

Our study is based on a mixed variational formulation with dual Lagrange multipliers. In the numerical tratement of this problem, that is the main aim of this paper, we use a *biorthogonality technique* as in [Woh00, HW03a, HW03b].

The rest of the paper is structured as follows. In Section 2, we present the mechanical model, provide a non-symmetric mixed variational formulation, and we state in Theorem 2.1 the existence, and uniqueness of the weak solution. The prove of this theorem, based on Theorem 3.1, will be provided in Section 3 using a fixed point technique and the abstract saddle point framework. After discretization, in Section 4 we obtain an optimal a priori error estimate of order $h^{\frac{1}{2}+\nu}$, $0 < \nu \leq \frac{1}{2}$, if the solution is regular enough. Finally, in Section 5 we give a numerical example.

To end the introduction, we briefly recall some basic results for saddle point problems, that will be used later. For more details on the saddle point theory, we refer to the textbooks [Bra97, BF91, ET76].

* This work is part of the Research Project *Nonlinear approximation and adaptivity: breaking complexity in numerical modelling and data representation* carried out in the framework of the European Community Program *Improving the Human Potential Network* (HPRN-CT-2002-00286)

Let A and B be two non-empty sets. A pair $(u, \lambda) \in A \times B$ is said to be a *saddle point* of a functional $\mathcal{L} : A \times B \to \mathbb{R}$ on $A \times B$ if and only if

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda), \quad (v, \mu) \in A \times B.$$

Assumption 1.1.

(i) A and B are nonempty, closed, convex subsets of two real Hilbert spaces V and W, respectively;

(ii) For all $\mu \in B$ the mapping $v \to \mathcal{L}(v, \mu)$ is a convex and weakly lower semicontinuous function;

(iii) For all $v \in A$ the mapping $\mu \to \mathcal{L}(v, \mu)$ is a concave and weakly upper semicontinuous function.

The following existence result holds.

Theorem 1.1 *We consider that Assumption 1.1 is satisfied. Let A be bounded or there exists $\mu_0 \in B$ such that*

$$\lim_{\|v\|_V \to \infty, v \in A} \mathcal{L}(v, \mu_0) = \infty,$$

and B be bounded or

$$\lim_{\|\mu\|_W \to \infty, \mu \in B} \inf_{v \in A} \mathcal{L}(v, \mu) = -\infty.$$

Then there exists a saddle point of $\mathcal{L}$ on $A \times B$.

A detailed proof of this result can be found in [ET76].

2. The mechanical problem and its mixed variational formulation. We consider an elasto-piezoelectric body that occupies the bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, in frictional contact with a rigid foundation. For the boundary $\Gamma = \partial\Omega$, we consider two partitions: firstly, let us consider a partition given by the measurable parts Γ_1, Γ_2 and Γ_3, such that meas $\Gamma_1 > 0$ and $\overline{\Gamma_3}$ is a compact subset of $\partial\Omega \backslash \overline{\Gamma_1}$; secondly, a partition given by the measurable parts Γ_a, Γ_b, such that meas $\Gamma_a > 0$. The unit outward normal to Γ is denoted by $\boldsymbol{n}$ and is assumed to be constant on Γ_3, i.e. Γ_3 is a straight line or a face. We associate the body with a rectangular cartesian coordinate system $Ox_1x_2x_3$ such that $\boldsymbol{e}_1 = \boldsymbol{n}_{\Gamma_3}$. We assume that the body is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω, a surface traction of density $\boldsymbol{f}_2$ acts on Γ_2, a surface electric charge of density q_2 acts on Γ_b, and the electric potential vanishes on Γ_a. Moreover, we assume that on Γ_3 the deformable body is in bilateral contact with the rigid foundation. We denote by $\boldsymbol{u}$ the displacement vector, by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ the linearized strain tensor, by $\boldsymbol{\sigma}$ the stress tensor, and by φ the electric potential. The space of second order symmetric tensors on $\mathbb{R}^d$ is denoted by S^d; "$\cdot$" and $|\cdot|$ represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and S^d, respectively, $d = 2, 3$. Thus, for each $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$, $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i$, $|\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}$, and for each $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S^d$, $\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $|\boldsymbol{\tau}| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$. In this section, the indices i and j run between 1 and d and the summation convention over repeated indices is applied.

The equilibrium equations are given by

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{D} = q_0 \qquad \text{in } \Omega, \tag{2.2}$$

where $\boldsymbol{D} = (D_i)$ is the electric displacement field, and q_0 is the volume density of free electric charges. Notice that Div represents the *divergence* operator for tensor valued functions that is $\operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j})$ and div represents the divergence operator for vector valued functions, that is $\operatorname{div} \boldsymbol{D} = (D_{i,i})$.

To describe the behavior of the material, we use the following constitutive law:

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{E}^\top \nabla \varphi \qquad \text{in } \Omega, \tag{2.3}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \beta \nabla \varphi \qquad \text{in } \Omega, \tag{2.4}$$

where $\mathcal{C} = (\mathcal{C}_{ijls})$ is the elastic tensor, $\mathcal{E} = (\mathcal{E}_{ijl})$ is the piezoelectric tensor, and β is the permitivity tensor. We recall that the linearized strain tensor $\boldsymbol{\varepsilon}$ is given by $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$. We use here $\mathcal{E}^\top$ to denote the transpose of the tensor $\mathcal{E}$ given by:

$$\mathcal{E}\,\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^\top \boldsymbol{v}, \qquad \boldsymbol{\sigma} \in S^d,\ \boldsymbol{v} \in \mathbb{R}^d,$$

and we notice that $\mathcal{E}^\top = (\mathcal{E}^\top_{ijl}) = (\mathcal{E}_{lij})$ for all $i, j, l \in \{1, ..., d\}$. Note that (2.3) represents an electro-elastic constitutive law and (2.4) describes a linear dependence of the electric displacement field on the strain and electric fields.

To complete the model, we have to prescribe the mechanic and electric boundary conditions. According to the physical setting, we use

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{2.5}$$

$$\boldsymbol{\sigma}\,\boldsymbol{n} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2, \tag{2.6}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a, \tag{2.7}$$

$$\boldsymbol{D} \cdot \boldsymbol{n} = q_2 \qquad \text{on } \Gamma_b. \tag{2.8}$$

Finally, we describe the frictional bilateral contact using Tresca's law:

$$\begin{cases} u_n = 0,\ |\boldsymbol{\sigma}_\tau| \le g, \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \boldsymbol{u}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| = g \Rightarrow \ \text{there exists } \alpha > 0 \text{ s.t. } \boldsymbol{\sigma}_\tau = -\alpha\,\boldsymbol{u}_\tau \end{cases} \qquad \text{on } \Gamma_3, \tag{2.9}$$

where the constant $g \ge 0$ represents the *friction bound*. When the strict inequality holds, the material point is in the *sticky* zone; when the equality holds, the material point is in the *slippy* zone. The boundary of these zones is unknown a priori.

We note that for each vector field $\boldsymbol{v} \in [H^1(\Omega)]^d$, we use the same symbol $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_n and $\boldsymbol{v}_\tau$ the *normal* and the *tangential* components of $\boldsymbol{v}$ on the boundary, given by $v_n = \boldsymbol{v} \cdot \boldsymbol{n}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_n\,\boldsymbol{n}$. We define, similarly, the *normal* and *tangential* components of the stress on the boundary by the formulas $\sigma_n = (\boldsymbol{\sigma}\boldsymbol{n}) \cdot \boldsymbol{n}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{n} - \sigma_n\,\boldsymbol{n}$.

To resume, we consider the following problem:

Problem 2.1 *Find the displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *and the electric potential field* $\varphi : \Omega \to \mathbb{R}$ *such that* (2.1)–(2.9) *hold.*

In the study of Problem 2.1, we will assume that:

$$\begin{cases} \text{(a) } \mathcal{C} = (\mathcal{C}_{ijls}) : \Omega \times S^d \to S^d, \\ \text{(b) } \mathcal{C}_{ijls} = \mathcal{C}_{ijsl} = \mathcal{C}_{lsij} \in L^\infty(\Omega), \\ \text{(c) there exists } m_{\mathcal{C}} > 0 \text{ such that} \\ \qquad \mathcal{C}_{ijls}\,\varepsilon_{ij}\,\varepsilon_{ls} \ge m_{\mathcal{C}}\,|\varepsilon|^2, \quad \varepsilon \in S^d, \text{ a.e. on } \Omega, \end{cases} \tag{2.10}$$

$$\begin{cases} \text{(a) } \mathcal{E} = (\mathcal{E}_{ijk}) : \Omega \times S^d \to \mathbb{R}^d, \\ \text{(b) } \mathcal{E}_{ijk} = \mathcal{E}_{ikj} \in L^\infty(\Omega), \end{cases} \tag{2.11}$$

$$\begin{cases} \text{(a) } \beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ \text{(b) } \beta_{ij} = \beta_{ji} \in L^\infty(\Omega), \\ \text{(c) there exists } m_\beta > 0 \text{ such that} \\ \qquad \beta_{ij}(x) E_i E_j \ge m_\beta |E|^2, E \in \mathbb{R}^d, \quad \text{a.e. } x \in \Omega, \end{cases} \tag{2.12}$$

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d, \tag{2.13}$$

$$q_0 \in L^2(\Omega), \quad q_2 \in L^2(\Gamma_b). \tag{2.14}$$

Let us introduce the following Hilbert spaces:

$$\begin{aligned} \boldsymbol{V} &:= \left\{ \boldsymbol{v} \in [H^1(\Omega)]^d \mid \boldsymbol{v} = 0 \text{ on } \Gamma_1 \right\}, \\ \boldsymbol{V}_n &:= \{ \boldsymbol{v} \in \boldsymbol{V} \mid v_n = 0 \text{ on } \Gamma_3 \}, \\ \Phi &:= \left\{ \theta \in H^1(\Omega) \mid \theta = 0 \text{ on } \Gamma_a \right\}. \end{aligned}$$

If $\boldsymbol{u}$ and φ are regular functions which satisfy (2.1)-(2.8), then we find

$$\begin{aligned} \int_\Omega \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \nabla\varphi\,dx &= \int_\Omega \boldsymbol{f}_0\,\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2\,\boldsymbol{v}\,ds \\ &\quad + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau\,\boldsymbol{v}_\tau\,ds, \\ -\int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \nabla\theta\,dx + \int_\Omega \beta\nabla\varphi \cdot \nabla\theta\,dx &= -\int_{\Gamma_b} q_2\,\theta\,ds + \int_\Omega q_0\,\theta\,dx, \end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ and $\theta \in \Phi$.

Let us introduce the functional space $\tilde{\boldsymbol{V}} = \boldsymbol{V} \times \Phi$, that is a Hilbert space endowed with the inner product

$$(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}})_{\tilde{V}} := (\boldsymbol{u}, \boldsymbol{v})_{[H^1(\Omega)]^d} + (\varphi, \theta)_{H^1(\Omega)}, \quad \tilde{\boldsymbol{u}} = (\boldsymbol{u}, \varphi),\ \tilde{\boldsymbol{v}} = (\boldsymbol{v}, \theta) \in \tilde{\boldsymbol{V}};$$

the corresponding norm is denoted by $\|\cdot\|_{\tilde{V}}$. Let $a : \tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}} \to \mathbb{R}$ be the bilinear form given by:

$$\begin{aligned} a(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) := &\int_\Omega \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \mathcal{E}\,\boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \nabla\varphi\,dx \\ &- \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \nabla\theta\,dx + \int_\Omega \beta\nabla\varphi \cdot \nabla\theta\,dx. \end{aligned} \tag{2.15}$$

Moreover, using Riesz's representation theorem, we define $\tilde{\boldsymbol{f}} \in \tilde{\boldsymbol{V}}$ such that for all $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}}$,

$$(\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{v}})_{\tilde{V}} := \int_\Omega \boldsymbol{f}_0\,\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2\,\boldsymbol{v}\,da - \int_{\Gamma_b} q_2\,\theta\,da + \int_\Omega q_0\,\theta\,dx.$$

Let $\boldsymbol{M}$ be the dual space of the space $\boldsymbol{W} = [H^{1/2}(\Gamma_3)]^d$. Let us define

$$\boldsymbol{\Lambda} := \left\{ \boldsymbol{\mu} \in \boldsymbol{M} \mid \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3} \le \int_{\Gamma_3} g\,|\boldsymbol{v}|\,ds, \qquad \boldsymbol{v} \in \boldsymbol{V}_n \right\}, \tag{2.16}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_3}$ denotes the duality pairing between $\boldsymbol{M}$ and $\boldsymbol{W}$. We underline that $\boldsymbol{\Lambda}$ is a closed, convex subset of $\boldsymbol{M}$ that contain $\boldsymbol{0}_M$. Furthermore, we introduce a bilinear and continuous form as follows:

$$b : \tilde{\boldsymbol{V}} \times \boldsymbol{M} \to \mathbb{R}, \qquad b(\tilde{\boldsymbol{v}}, \boldsymbol{\mu}) := \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3}. \tag{2.17}$$

We suppose that the stress $\boldsymbol{\sigma}$ is a regular enough function to define $\boldsymbol{\lambda} \in \boldsymbol{M}$ as follows

$$\langle \boldsymbol{\lambda}, \boldsymbol{v} \rangle_{\Gamma_3} := -\int_{\Gamma_3} (\boldsymbol{\sigma}\boldsymbol{n}) \cdot \boldsymbol{v}\,ds, \qquad \boldsymbol{v} \in \boldsymbol{V}.$$

Using (2.1)-(2.8), we get

$$a(\tilde{\boldsymbol{u}},\,\tilde{\boldsymbol{v}}) + b(\tilde{\boldsymbol{v}},\boldsymbol{\lambda}) = (\tilde{\boldsymbol{f}},\tilde{\boldsymbol{v}})_{\tilde{V}}, \qquad \boldsymbol{v} \in \tilde{\boldsymbol{V}}.$$

Using now (2.9) we deduce that

$$\int_{\Gamma_3} (\boldsymbol{\sigma n}) \cdot \boldsymbol{u}\, ds = -\int_{\Gamma_3} g|\boldsymbol{u}| ds$$

and taking into account (2.17) we can write the following equality

$$b(\tilde{\boldsymbol{u}},\boldsymbol{\lambda}) = \int_{\Gamma_3} g|\boldsymbol{u}| ds.$$

On the other hand, keeping in mind (2.16), we get

$$b(\tilde{\boldsymbol{u}},\boldsymbol{\mu}) \leq \int_{\Gamma_3} g|\boldsymbol{u}| ds, \qquad \boldsymbol{\mu} \in \boldsymbol{\Lambda},$$

resulting in $b(\tilde{\boldsymbol{u}},\boldsymbol{\mu}-\boldsymbol{\lambda}) \leq 0$ for all $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$. Thus, we can write the following mixed formulation of Problem 2.1.

Problem 2.2 *Find* $\tilde{\boldsymbol{u}} \in \tilde{\boldsymbol{V}}$ *and* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ *such that*

$$\begin{aligned} a(\tilde{\boldsymbol{u}},\,\tilde{\boldsymbol{v}}) + b(\tilde{\boldsymbol{v}},\boldsymbol{\lambda}) &= (\tilde{\boldsymbol{f}},\tilde{\boldsymbol{v}})_{\tilde{V}}, \quad &\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{V}},\\ b(\tilde{\boldsymbol{u}},\,\boldsymbol{\mu}-\boldsymbol{\lambda}) &\leq 0, &\boldsymbol{\mu} \in \boldsymbol{\Lambda}. \end{aligned}$$

The following result holds.

Theorem 2.1 *Assume that* (2.10)–(2.14) *hold. Then, Problem 2.2 has a unique solution* $(\tilde{\boldsymbol{u}},\,\boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$*. Moreover, if* $(\tilde{\boldsymbol{u}}_1,\,\boldsymbol{\lambda}_1)$ *and* $(\tilde{\boldsymbol{u}}_2,\,\boldsymbol{\lambda}_2)$ *are two solutions of Problem 2.2 for two functions* $\tilde{\boldsymbol{f}}_1$, $\tilde{\boldsymbol{f}}_2 \in \tilde{\boldsymbol{V}}$, *corresponding to two sets of data* $\{\boldsymbol{f}_0,\,\boldsymbol{f}_2,\,q_0,\,q_2\}_1$, *respectively* $\{\boldsymbol{f}_0,\,\boldsymbol{f}_2,\,q_0,\,q_2\}_2$, *then we have the estimate*

$$\|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2\|_{\tilde{V}} + \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_{-1/2,\Gamma_3} \leq C\, \|\tilde{\boldsymbol{f}}_1 - \tilde{\boldsymbol{f}}_2\|_{\tilde{V}},$$

where $C > 0$ *is a constant that depends of* $\mathcal{C}$, $\mathcal{E}$ *and* β.

Remark If $(\tilde{\boldsymbol{u}} = (\boldsymbol{u},\phi),\boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$ is the solution of Problem 2.2, then $\boldsymbol{u} \in \boldsymbol{V}_n$. Indeed, we can easily verify that for $\alpha \in \mathbb{R}$, $\boldsymbol{\lambda} \pm \alpha \boldsymbol{n}_{\Gamma_3}$ are elements of $\boldsymbol{\Lambda}$ and, taking in Problem 2.2 $\boldsymbol{\mu} = \boldsymbol{\lambda} \pm \alpha \boldsymbol{n}_{\Gamma_3}$, we deduce that

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \ \text{ on } \Gamma_3.$$

The proof of Theorem 2.1 that justifies the well-posedness of Problem 2.2, is based on the main result in the next section, Theorem 3.1.

3. Abstract auxiliary results and proof of Theorem 2.1.

Let X and Y be two Hilbert spaces and let us consider two bilinear forms as follows

$$a(\cdot,\,\cdot) : X \times X \to \mathbb{R}, \text{ nonsymmetric}$$

(a) there exists $M_a > 0$ such that

$$|a(u,v)| \leq M_a \|u\|_X \|v\|_X, \qquad u,v \in X, \tag{3.1}$$

(b) there exists $m_a > 0$ such that

$$a(v,v) \geq m_a\, \|v\|_X^2, \qquad v \in X. \tag{3.2}$$

$b(\cdot,\,\cdot) : X \times Y \to \mathrm{I\!R}$,

(c) there exists M_b such that

$$|b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y, \qquad v \in X,\ \mu \in Y, \tag{3.3}$$

(d) there exists $\alpha > 0$ such that

$$\inf_{\mu\in Y,\mu\neq 0} \sup_{v\in X, v\neq 0} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha. \tag{3.4}$$

Let $\Lambda \subset Y$ be a closed, convex set that contains 0_Y. We consider now the following problem:

Problem 3.1 *For a given $f \in X$, find $u \in X$ and $\lambda \in Y$ such that $\lambda \in \Lambda$ and*

$$\begin{aligned} &a(u,v) + b(v,\lambda) = (f,\,v)_X, && v \in X,\\ &b(u,\mu-\lambda) \qquad \leq 0, && \mu \in \Lambda. \end{aligned}$$

We are interested in the following result.

Theorem 3.1 *Let $f \in X$ and assume that* (3.1)–(3.4) *hold. Then, there exists a unique solution of Problem 3.1, $(u,\lambda) \in X \times \Lambda$. Moreover, if $(u_1,\,\lambda_1)$ and $(u_2,\,\lambda_2)$ are two solutions of Problem 3.1, corresponding to two given functions $f_1,\ f_2 \in X$, then we have the estimate*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \leq \frac{\alpha + m_a + 2\,M_a}{\alpha\, m_a} \|f_1 - f_2\|_X. \tag{3.5}$$

The prove of this theorem will be made in several steps. We underline that Problem 3.1 is *not a saddle point problem*, because $a(\cdot,\cdot)$ is *non-symmetric*, but our study reduces to the study of a saddle point problem. The main idea of this proof is to use the results known in the saddle point theory, see, e.g., [Bra97, BF91, ET76, Hcc96], for the symmetric part of $a(\cdot,\cdot)$. Finally, the prove is provided by a fixed point technique. The reader can found a version of this fixed point technique in [KS00], in the framework of the elliptic variational inequalities of the first kind.

Let $a_0(u,\,v)$ and $c(u,\,v)$ be the symmetric, respectively the antisymmetric part of $a(u,\,v)$, that is

$$a_0(u,\,v) := \frac{1}{2}(a(u,\,v) + a(v,\,u)), \quad c(u,v) := \frac{1}{2}(a(u,\,v) - a(v,\,u)).$$

For a given $r \in [0,1]$, we introduce the following bilinear form

$$a_r\,(u,\,v) := a_0(u,\,v) + r\,c(u,\,v), \quad u,\,v \in X, \tag{3.6}$$

as a "perturbation" of $a_0(\cdot,\,\cdot)$. We underline that $a_1(u,\,v) = a(u,\,v)$ and for all $r \in [0,1]$ $a_r(u,\,v)$ is X-elliptic with the same ellipticity-constant m_a. Moreover, the bilinear forms $a_0(\cdot,\,\cdot)$ and $c(\cdot,\,\cdot)$ are continuous with the same continuity-constant M_a. We consider now the following "perturbate" problem.

Problem 3.2 *For a given $f \in X$, find $u \in X$ and $\lambda \in Y$ such that $\lambda \in \Lambda$, and*

$$a_r\,(u,v) + b(v,\lambda) = (f,\,v)_X, \quad v \in X, \tag{3.7}$$

$$b(u,\mu-\lambda) \qquad \leq 0, \qquad \mu \in \Lambda. \tag{3.8}$$

We can prove the following lemma.

Lemma 3.2 *Assume that for every $f \in X$ there exists a unique solution of Problem 3.2, $(u,\,\lambda) \in X \times \Lambda$. If $(u_1,\,\lambda_1)$ and $(u_2,\,\lambda_2)$ are solutions of Problem 3.2 corresponding to two given functions $f_1,\ f_2 \in X$, then* (3.5) *holds.*

Proof. Let us take $f_1,\, f_2 \in X$ and let $(u_1,\, \lambda_1)$ and $(u_2,\, \lambda_2)$ be the corresponding solutions of Problem 3.2. Using (3.7), we can write

$$a_r\,(u_1 - u_2,\, u_1 - u_2) = (f_1 - f_2,\, u_1 - u_2)_X + b(u_1, \lambda_2 - \lambda_1) + b(u_2, \lambda_1 - \lambda_2).$$

Using now (3.8) and taking into account the X-ellipticity of a_r we deduce

$$\|u_1 - u_2\|_X \leq \frac{1}{m_a}\|f_1 - f_2\|_X. \tag{3.9}$$

Moreover, keeping in mind (3.7), we obtain

$$b(v,\, \lambda_1 - \lambda_2) = (f_1 - f_2,\, v)_X + a_r\,(u_2 - u_1,\, v).$$

Using now (3.4) we can write

$$\alpha\,\|\lambda_1 - \lambda_2\|_Y \leq \sup_{v\in X,\, v\neq 0} \frac{b(v,\, \lambda_1 - \lambda_2)}{\|v\|_X} \leq \|f_1 - f_2\|_X + 2\,M_a\|u_1 - u_2\|_X,$$

and from this, taking into account (3.9), we get

$$\|\lambda_1 - \lambda_2\|_Y \leq \frac{m_a + 2\,M_a}{\alpha\, m_a}\,\|f_1 - f_2\|_X. \tag{3.10}$$

Adding now (3.9) and (3.10), we obtain (3.5) that concludes this lemma. □

Lemma 3.3 *Let $\tau \in [0, 1]$. Assume that for every $f \in X$ there exists a unique solution of Problem 3.2 with $r = \tau$, $(u,\, \lambda) \in X \times \Lambda$. Then, for every $f \in X$ there exists a unique solution $(u,\, \lambda)$ of Problem 3.2 with $r \in [\tau, \tau + t_0]$, where*

$$t_0 < \frac{\alpha\, m_a}{M_a\,(\alpha + m_a + 2\,M_a)}. \tag{3.11}$$

Proof. Let $f \in X$ be a given function. Let us define the mapping $\mathcal{T} : X \times \Lambda \to X \times \Lambda$ as follows

$$\mathcal{T}(w,\, \xi) := (u,\, \lambda)$$

if $(u,\, \lambda)$ is the unique solution of the problem

$$\begin{aligned} &a_\tau\,(u, v) + b(v, \lambda) = (F_s,\, v)_X, && v \in X,\\ &b(u, \mu - \lambda) \leq 0, && \mu \in \Lambda, \end{aligned}$$

where

$$(F_s,\, v)_X = (f,\, v)_X - (s - \tau)\, c(w,\, v)$$

and $\tau \leq s \leq \tau + t_0$. Clearly, $\mathcal{T}$ is well defined. Moreover, $\mathcal{T}$ is a contraction. Indeed if we consider two pairs $(w_1,\, \xi_1)$, $(w_2,\, \xi_2) \in X \times Y$, we can write

$$\|\mathcal{T}(w_1,\, \xi_1) - \mathcal{T}(w_2,\, \xi_2)\|_{X\times Y} = \|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y.$$

Using now (3.9) and (3.10), we obtain

$$\|\mathcal{T}(w_1,\, \xi_1) - \mathcal{T}(w_2,\, \xi_2)\|_{X\times Y} \leq \frac{t_0\, M_a\,(\alpha + m_a + 2\,M_a)}{\alpha\, m_a}\|w_1 - w_2\|_X$$

$$\leq \frac{t_0\, M_a\,(\alpha + m_a + 2\,M_a)}{\alpha\, m_a}\|(w_1,\, \xi_1) - (w_2,\, \xi_2)\|_{X\times Y}.$$

Keeping in mind (3.11), we deduce that

$$0 < \frac{t_0\, M_a\, (\alpha + m_a + 2\, M_a)}{\alpha\, m_a} < 1.$$

Using the Banach Fixed Point Theorem, we conclude that $\mathcal{T}$ has a unique fixed point. Let $(u^*,\, \lambda^*)$ be the unique fixed point of the operator $\mathcal{T}$. Using the definition of $\mathcal{T}$, we deduce

$$a_\tau(u^*, v) + b(v, \lambda^*) = (F_s,\, v)_X, \qquad v \in X, \tag{3.12}$$

$$b(u^*, \mu - \lambda^*) \leq 0, \qquad \mu \in \Lambda, \tag{3.13}$$

where

$$(F_s,\, v)_X = (f,\, v)_X - (s - \tau)\, c(u^*,\, v). \tag{3.14}$$

Using now (3.12), (3.13) and (3.14) we deduce that $(u^*,\, \lambda^*)$ is a solution of Problem 3.2 with $r = s$ for $f \in X$. To justify the uniqueness, let us assume that Problem 3.2 with $r = s$ has two solutions $(u_1,\, \lambda_1)$, $(u_2,\, \lambda_2) \in X \times \Lambda$. Consequently, we can write

$$a_s(u_1 - u_2, u_1 - u_2) = b(u_1, \lambda_2 - \lambda_1) + b(u_2, \lambda_1 - \lambda_2),$$

and from this,

$$a_s(u_1 - u_2,\, u_1 - u_2) \leq 0.$$

Taking into account the X-ellipticity of a_s, we find $u_1 = u_2$. Moreover, using (3.4), we deduce that $\lambda_1 = \lambda_2$ that concludes Lemma 3.3. □

Let us consider Problem 3.2 corresponding to $r = 0$:

Problem 3.3 *For a given $f \in X$, find $u \in X$ and $\lambda \in Y$ such that $\lambda \in \Lambda$ and*

$$a_0(u, v) + b(v, \lambda) = (f,\, v)_X, \qquad v \in X,$$

$$b(u, \mu - \lambda) \leq 0, \qquad \mu \in \Lambda.$$

Lemma 3.4 *Assume* (3.1)–(3.4). *Given $f \in X$, there exists a unique solution of Problem 3.3, $(u,\, \lambda) \in X \times \Lambda$.*

Proof. The proof of this lemma, based on the saddle point theory, can be found, e.g., in [Hcc96]. For the convenience of the reader we indicate here the main lines of this proof.

Let $\mathcal{L} : X \times \Lambda \to \mathbb{R}$ be the functional defined as follows:

$$\mathcal{L}(v,\, \mu) := \frac{1}{2} a(v, v) - (f, v)_X + b(v, \mu).$$

Using this definition, an equivalent formulation of Problem 3.3. is the following saddle point problem:

Problem 3.4 *Find $u \in X$ and $\lambda \in \Lambda$ such that*

$$\mathcal{L}(u,\, \mu) \leq \mathcal{L}(u,\, \lambda) \leq \mathcal{L}(v,\, \lambda) \quad v \in X,\, \mu \in \Lambda.$$

Keeping in mind (3.1) we observe that

$$\lim_{\|v\|_X \to \infty, v \in X} \mathcal{L}(v,\, 0) = \infty.$$

Moreover,

$$\lim_{\|\mu\|_Y \to \infty, \mu \in \Lambda} \inf_{v \in X} \mathcal{L}(v,\, \mu) = -\infty. \tag{3.15}$$

Indeed, let μ_0 be an element of Λ and let $u_{\mu_0} \in X$ be the unique solution of the equation

$$a(u_{\mu_0}, v) + b(v, \mu_0) = (f, v)_X, \qquad v \in X. \tag{3.16}$$

Clearly, the following equality holds

$$\inf_{v \in X} \mathcal{L}(v,\, \mu_0) = \frac{1}{2} a(u_{\mu_0}, u_{\mu_0}) - (f, u_{\mu_0})_X + b(u_{\mu_0},\, \mu_0).$$

Substituting $v = u_{\mu_0}$ into (3.16), we get

$$\inf_{v \in X} \mathcal{L}(v,\, \mu_0) \le -\frac{m_a}{2} \|u_{\mu_0}\|_X^2. \tag{3.17}$$

Additionaly, using the inf-sup property of the form $b(\cdot,\, \cdot)$, we deduce that there exists a constant $C > 0$ such that

$$\|\mu_0\|_Y \le C(\|f\|_X + \|u_{\mu_0}\|_X). \tag{3.18}$$

From (3.17) and (3.18), we obtain (3.15). Consequently, all the hypotheseses of Theorem 1.1 are verified that conclude the existence of the solution.

To show the uniqueness of the solution, let us assume that $(u_1,\, \lambda_1)$ and $(u_2,\, \lambda_2) \in X \times \Lambda$ are solutions of Problem 3.3. We can write

$$a(u_1 - u_2, u_1 - u_2) = b(u_1, \lambda_2 - \lambda_1) + b(u_2, \lambda_1 - \lambda_2) \le 0,$$

and from this we deduce that $u_1 = u_2$. In addition, using the inf-sup property of the form $b(\cdot,\, \cdot)$ we obtain $\lambda_1 = \lambda_2$ that concludes Lemma 3.4. □

Applying Lemma 3.3 a finite number of times, we deduce that Problem 3.2 admits a unique solution $(u, \lambda) \in X \times \Lambda$ for $r = 1$. Moreover, using Lemma 3.2, we observe that (3.5) holds, that completes the proof of Theorem 3.1.

Proof of Theorem 2.1. Let us take $X = \tilde{\boldsymbol{V}}$ and $Y = \boldsymbol{M}$. Using (2.10)-(2.12) and (2.15) we deduce that (3.1) and (3.2) hold. On the other hand, keeping in mind (2.17), we find (3.3) and (3.4). In addition, $\boldsymbol{\Lambda}$ introduced in (2.16) is a closed, convex subset of $\boldsymbol{M}$ that contain $\boldsymbol{0}_M$. Consequently, all the hypotheseses in Theorem 3.1 are verified and the proof of Theorem 2.1 is a straightforward application of Theorem 3.1. □

4. Discretization and an optimal a priori error estimate. In this section, we consider the 2D case. Let us assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain and that Γ_1, Γ_3 and Γ_a can be written as union of edges of the triangulation. Furthermore, let us denote by $\boldsymbol{\tau}$ a unit vector such that $\boldsymbol{n} \cdot \boldsymbol{\tau} = 0$. We refer the body to a rectangular cartesian coordinate system Ox_1x_2 such that $\boldsymbol{e}_1 = \boldsymbol{n}_{\Gamma_3}$ and $\boldsymbol{e}_2 = \boldsymbol{\tau}_{\Gamma_3}$. To simplify the writing, everywhere below we will write $\boldsymbol{n}$ and $\boldsymbol{\tau}$ instead of $\boldsymbol{n}_{\Gamma_3}$ and $\boldsymbol{\tau}_{\Gamma_3}$, respectively. To approximate $\tilde{\boldsymbol{V}}$, we use standard conforming finite elements of lowest order on quasi-uniform simplicial triangulations, and we denote by $S_1(\Omega, \mathcal{T}_{h,\Omega})$ the finite element space associated with the shape regular triangulation $\mathcal{T}_{h,\Omega}$. The meshsize h is defined by the maximal diameter of the elements in $\mathcal{T}_{h,\Omega}$. Let us consider the discrete spaces

$$\begin{aligned} \boldsymbol{V}_h \; &:= \Big\{ \boldsymbol{v}_h \in [S_1\,(\Omega,\, \mathcal{T}_{h,\Omega})]^2 \,:\, \boldsymbol{v}_{h|_{\Gamma_1}} = \boldsymbol{0} \Big\} \subset \boldsymbol{V}, \\ (\boldsymbol{V}_h)_n &:= \Big\{ \boldsymbol{v}_h \in \boldsymbol{V}_h \,:\, (\boldsymbol{v}_h)_{n|_{\Gamma_3}} = 0 \Big\} \subset \boldsymbol{V}_n, \\ \Phi_h \; &:= \Big\{ \theta_h \in S_1\,(\Omega,\, \mathcal{T}_{h,\Omega}) \,:\, \theta_{h|_{\Gamma_a}} = 0 \Big\} \subset \Phi. \end{aligned}$$

Let us denote

$$\tilde{\boldsymbol{V}}_h := \boldsymbol{V}_h \times \Phi_h \subset \tilde{\boldsymbol{V}}$$

and

$$\boldsymbol{M}_h := \Big\{ \boldsymbol{\mu}_h \in \boldsymbol{M} \mid \boldsymbol{\mu}_h = \sum_{i=1}^{N_{M_h}} \gamma_i \psi_i \boldsymbol{n} + \sum_{i=1}^{N_{M_h}} \alpha_i \psi_i \boldsymbol{\tau} \Big\},$$

where N_{M_h} is the number of vertices on $\overline{\Gamma_3}$ and for every $i = 1, ..., N_{M_h}$, ψ_i is the i-th. scalar dual basis function of the standard nodal Lagrange finite element basis function and γ_i, α_i are real coefficients. According to [Woh00], we consider the dual basis such that the following biorthogonality relation holds

$$\langle \psi_i, \phi_j \rangle_{\Gamma_3} \;=\; \delta_{ij} \int_{\Gamma_3} \phi_j \, ds, \qquad i, j = 1, \dots, N_{M_h}, \tag{4.1}$$

where ϕ_m, $m = 1, \dots, N_{M_h}$, are the standard scalar nodal basis functions of $S_1(\Omega,\, \mathcal{T}_{h,\Omega})$, restricted to Γ_3. Furthermore, every element $\boldsymbol{v}_h$ of $(\boldsymbol{V}_h)_n$ can be written on Γ_3 as a combination of standard basis functions ϕ_i as follows

$$\boldsymbol{v}_h = \sum_{j=1}^{N_{M_h}} \zeta_j \phi_j \boldsymbol{\tau}, \quad \zeta_j \in \mathbb{R},\; j = 1, ..., N_{M_h}.$$

Defining a *mesh dependent absolute value* of an element $\boldsymbol{v}_h \in (\boldsymbol{V}_h)_n$ by

$$|\boldsymbol{v}_h|_h := \sum_{j=1}^{N_{M_h}} |\zeta_j| \phi_j,$$

we set $\boldsymbol{\Lambda}_h$ as follows

$$\boldsymbol{\Lambda}_h := \Big\{ \boldsymbol{\mu}_h \in \boldsymbol{M}_h \,|\, \langle \boldsymbol{\mu}_h,\, \boldsymbol{v}_h \rangle_{\Gamma_3} \le \int_{\Gamma_3} g\, |\boldsymbol{v}_h|_h \, ds, \quad \boldsymbol{v}_h \in (\boldsymbol{V}_h)_n \Big\}.$$

We now consider the following discrete problem.

Problem 4.1 *Find $\tilde{\boldsymbol{u}}_h \in \tilde{\boldsymbol{V}}_h$ and $\boldsymbol{\lambda}_h \in \boldsymbol{\Lambda}_h$ such that*

$$\begin{aligned} a(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + b(\tilde{\boldsymbol{v}}_h, \boldsymbol{\lambda}_h) &= (\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{v}}_h)_{\tilde{V}}, & \tilde{\boldsymbol{v}}_h \in \tilde{\boldsymbol{V}}_h \\ b(\tilde{\boldsymbol{u}}_h, \boldsymbol{\mu}_h - \boldsymbol{\lambda}_h) &\le 0, & \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h. \end{aligned}$$

Existence and uniqueness of a solution follows from a discrete inf-sup condition for the spaces $\tilde{\boldsymbol{V}}_h$ and $\boldsymbol{M}_h$, see, e.g., [Woh00] and the references therein, and from techniques used in Section 3.

Let us denote by $\mathcal{P}_C := \{p_i \,:\, 1 \le i \le N_{M_h}\}$ the set of vertices on $\overline{\Gamma_3}$.

The following result takes place.

Lemma 4.2 *Let $(\tilde{\boldsymbol{u}} = (\boldsymbol{u}, \varphi), \boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$ be the solution of Problem 2.2 and let $(\tilde{\boldsymbol{u}}_h = (\boldsymbol{u}_h, \varphi_h), \boldsymbol{\lambda}_h) \in \tilde{\boldsymbol{V}}_h \times \boldsymbol{\Lambda}_h$ be the solution of Problem 4.1. Then, the following equalities hold*

$$b(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) = \int_{\Gamma_3} g |\boldsymbol{u}| ds, \tag{4.2}$$

$$b(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) = \int_{\Gamma_3} g |\boldsymbol{u}_h|_h ds. \tag{4.3}$$

Proof. Let us define $\boldsymbol{\mu} \in \boldsymbol{M}$ as follows

$$\langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3} := \int_{\Gamma_3} g \; \mathrm{sgn}(\boldsymbol{u}(s) \cdot \boldsymbol{\tau})\, \boldsymbol{v}(s) \cdot \boldsymbol{\tau} ds, \quad \boldsymbol{v} \in \boldsymbol{V}.$$

Clearly, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$. Using the definiton of $\boldsymbol{\Lambda}$ and taking into account that

$$b(\tilde{\boldsymbol{u}}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \leq 0, \qquad \boldsymbol{\mu} \in \boldsymbol{\Lambda},$$

we can write

$$\begin{aligned}\int_{\Gamma_3} g|\boldsymbol{u}(s)|ds \geq \langle \boldsymbol{\lambda}, \boldsymbol{u}\rangle_{\Gamma_3} \geq \langle \boldsymbol{\mu}, \boldsymbol{u}\rangle_{\Gamma_3} = \\ &= \int_{\Gamma_3} g \ \mathrm{sgn}(\boldsymbol{u}(s)\cdot\boldsymbol{\tau})\, \boldsymbol{u}(s)\cdot\boldsymbol{\tau} ds = \int_{\Gamma_3} g|\boldsymbol{u}(s)\cdot\boldsymbol{\tau}|ds \\ &= \int_{\Gamma_3} g|\boldsymbol{u}(s)|ds.\end{aligned}$$

Keeping in mind (2.17), we deduce (4.2). Let us prove (4.3). Since

$$b(\tilde{\boldsymbol{u}}_h, \boldsymbol{\mu}_h - \boldsymbol{\lambda}_h) \leq 0, \qquad \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h, \tag{4.4}$$

setting $\boldsymbol{\mu}_h = \boldsymbol{\lambda}_h \pm \phi_i \boldsymbol{n}$, $1 \leq i \leq N_{M_h}$, we deduce that

$$\boldsymbol{u}_h \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_3,$$

and thus, $\boldsymbol{u}_h \in (\boldsymbol{V}_h)_n$. Using now the definition of $\boldsymbol{\Lambda}_h$ we get

$$b(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) \leq \int_{\Gamma_3} g|\boldsymbol{u}_h|_h \, ds. \tag{4.5}$$

In order to obtain the inverse inequality, let us consider

$$\boldsymbol{\mu}_h = \sum_{i=1}^{N_{M_h}} g \ \mathrm{sgn}\,(\boldsymbol{u}_h(p_i)\cdot\boldsymbol{\tau})\psi_i \boldsymbol{\tau}$$

and let us write $\boldsymbol{u}_h$ on Γ_3 as a combination of the standard basis functions, in the tangential direction

$$\boldsymbol{u}_h = \sum_{i=1}^{N_{M_h}} (\boldsymbol{u}_h(p_i)\cdot\boldsymbol{\tau})\phi_i \boldsymbol{\tau}.$$

Due to the biorthogonality relation (4.1), we can easily verify that $\boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h$. Furthermore, taking into account (4.4) and (4.1) we find

$$\begin{aligned} b(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) &\geq b\Big(\tilde{\boldsymbol{u}}_h, \sum_{i=1}^{N_{M_h}} g \ \mathrm{sgn}\,(\boldsymbol{u}_h(p_i)\cdot\boldsymbol{\tau})\psi_i \boldsymbol{\tau}\Big) \\ &= \sum_{i=1}^{N_{M_h}} g\,|\boldsymbol{u}_h(p_i)\cdot\boldsymbol{\tau}| \int_{\Gamma_3} \phi_i(s)ds \\ &= \int_{\Gamma_3} g|\boldsymbol{u}_h|_h ds. \end{aligned}$$

Consequently, the following inequality takes place

$$b(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) \geq \int_{\Gamma_3} g|\boldsymbol{u}_h|_h \, ds. \tag{4.6}$$

Using (4.6) and (4.5) we get (4.3), and thus, we conclude Lemma 4.2. □

We will use this result to prove the following lemma.

Lemma 4.3 *Let* $(\tilde{\boldsymbol{u}},\boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$ *be the solution of Problem 2.2 and let* $(\tilde{\boldsymbol{u}}_h,\boldsymbol{\lambda}_h) \in \tilde{\boldsymbol{V}}_h \times \boldsymbol{\Lambda}_h$ *be the solution of Problem 4.1. Then, there exists a positive constant* C *independent of the meshsize* h*, such that for all* $\boldsymbol{v}_h \in \tilde{\boldsymbol{V}}_h$*,* $\boldsymbol{\mu}_h \in \boldsymbol{M}_h$*,*

$$\begin{aligned}\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h\|^2_{\tilde{V}}+\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|^2_{-\frac{1}{2},\Gamma_3} \le C &\left\{\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{v}}_h\|^2_{\tilde{V}}+\|\boldsymbol{\lambda}-\boldsymbol{\mu}_h\|^2_{-\frac{1}{2},\Gamma_3}\right\}\\ &+b(\tilde{\boldsymbol{u}},\boldsymbol{\lambda}_h-\boldsymbol{\lambda}).\end{aligned}$$

Proof. We prove this lemma using similar arguments with those used in [9] for scalar-valued standard Lagrange multipliers. To this end, let us evaluate $a(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h)$. For each $\tilde{\boldsymbol{v}}_h \in \tilde{\boldsymbol{V}}_h$, we can write

$$\begin{aligned}a(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h) &= a(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{v}}_h)+a(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\tilde{\boldsymbol{v}}_h-\tilde{\boldsymbol{u}}_h)\\ &= a(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{v}}_h)-b(\tilde{\boldsymbol{v}}_h-\tilde{\boldsymbol{u}}_h,\boldsymbol{\lambda})+b(\tilde{\boldsymbol{v}}_h-\tilde{\boldsymbol{u}}_h,\boldsymbol{\lambda}_h)\\ &= a(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{v}}_h)-b(\tilde{\boldsymbol{v}}_h-\tilde{\boldsymbol{u}},\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\\ &\quad -b(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\boldsymbol{\lambda}-\boldsymbol{\lambda}_h).\end{aligned}$$

From this evaluation, we find

$$\begin{aligned}\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h\|^2_{\tilde{V}} \le& C\left(\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h\|_{\tilde{V}}+\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{-1/2,\Gamma_3}\right)\|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{v}}_h\|_{\tilde{V}}\\ &-b(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\boldsymbol{\lambda}-\boldsymbol{\lambda}_h).\end{aligned} \tag{4.7}$$

Here and below, we denote by C a positive constant independent of the meshsize, whose value may change from place to place. Using Lemma 4.2, we can write

$$-b(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)=b(\tilde{\boldsymbol{u}},\boldsymbol{\lambda}_h-\boldsymbol{\lambda})+b(\tilde{\boldsymbol{u}}_h,\boldsymbol{\lambda})-\int_{\Gamma_3} g|\boldsymbol{u}_h|_h ds,$$

and from this, using the definiton of $\boldsymbol{\Lambda}$ and $(\boldsymbol{V}_h)_n \subset \boldsymbol{V}_n$ we obtain the following inequality

$$-b(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\le b(\tilde{\boldsymbol{u}},\boldsymbol{\lambda}_h-\boldsymbol{\lambda})+\int_{\Gamma_3} g(|\boldsymbol{u}_h|-|\boldsymbol{u}_h|_h)ds.$$

Furthermore, since $|\boldsymbol{u}_h| \le |\boldsymbol{u}_h|_h$ on Γ_3, the following inequality holds

$$-b(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_h,\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\le b(\tilde{\boldsymbol{u}},\boldsymbol{\lambda}_h-\boldsymbol{\lambda}). \tag{4.8}$$

On the other hand, using the inf-sup property of the form $b(\cdot,\cdot)$ we deduce

$$\begin{aligned}\|\boldsymbol{\mu}_h-\boldsymbol{\lambda}_h\|_{-\frac{1}{2},\Gamma_3} &\le C \sup_{\tilde{\boldsymbol{w}}_h\in\tilde{\boldsymbol{V}}_h}\frac{b(\tilde{\boldsymbol{w}}_h,\boldsymbol{\mu}_h-\boldsymbol{\lambda}_h)}{\|\tilde{\boldsymbol{w}}_h\|_{\tilde{V}}}\\ &= C \sup_{\tilde{\boldsymbol{w}}_h\in\tilde{\boldsymbol{V}}_h}\frac{b(\tilde{\boldsymbol{w}}_h,\boldsymbol{\mu}_h-\boldsymbol{\lambda})+a(\tilde{\boldsymbol{u}}_h-\tilde{\boldsymbol{u}},\tilde{\boldsymbol{w}}_h)}{\|\tilde{\boldsymbol{w}}_h\|_{\tilde{V}}}\\ &\le C(\|\boldsymbol{\mu}_h-\boldsymbol{\lambda}\|_{-\frac{1}{2},\Gamma_3}+\|\tilde{\boldsymbol{u}}_h-\tilde{\boldsymbol{u}}\|_{\tilde{V}}).\end{aligned}$$

Consequently, we can write

$$\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{-\frac{1}{2},\Gamma_3}\le C(\|\boldsymbol{\mu}_h-\boldsymbol{\lambda}\|_{-\frac{1}{2},\Gamma_3}+\|\tilde{\boldsymbol{u}}_h-\tilde{\boldsymbol{u}}\|_{\tilde{V}}). \tag{4.9}$$

Thus, we conclude Lemma 4.3 using (4.7), (4.8) and (4.9). □

To get optimal a priori bounds for the discretization error, we have to consider the residual term $b(\tilde{\boldsymbol{u}},\boldsymbol{\lambda}_h-\boldsymbol{\lambda})$ in more detail. To this end, let us denote $\overline{\gamma_{sl}} := \mathrm{supp}(\boldsymbol{u}_{|\Gamma_3}\cdot\boldsymbol{\tau})$ and $\gamma_{st} := \Gamma_3\backslash\gamma_{sl}$. Furthermore, we need the following assumption.

Assumption 4.1.

- $\overline{\gamma_{st}}$ is a compact subset of $\overline{\Gamma_3}$ such that the number of points in $\mathring{\overline{\gamma}}_{st} \cap \overline{\gamma_{sl}}$ is finite;
- $\mathring{\overline{\gamma}}_{st} = \overline{\gamma_{st}}$.

Let $\mathcal{W}_C := \{w_j \,:\, 1 \le j \le N_w\}$ be the set of points in $\mathring{\overline{\gamma}}_{st} \cap \overline{\gamma_{sl}}$. The minimum distance between the elements in $\mathcal{W}_C$ is denoted by a, i.e., $a := \inf\{|w_j - w_k| \,:\, 1 \le j \ne k \le N_w\}$, where $|\cdot|$ denotes the Euclidean norm. By Assumption 4.1, $N_w < \infty$ and thus $a > 0$. For $h < \frac{a}{2} =: h_0$, we find between two neighbor points in $\mathcal{W}_C$ at least two vertices in $\mathcal{P}_C$.

Let us denote by I_h the standard interpolation operator restricted on Γ_3, i.e.,

$$I_h \boldsymbol{u} = \sum_{i=1}^{N_{M_h}} \boldsymbol{u}(p_i)\phi_i,$$

and let us define the following *modified interpolation operator* by

$$(\tilde{I}_h \boldsymbol{u})(p_i) := \begin{cases} \boldsymbol{u}(p_i) & \text{if} \quad \text{supp}\phi_i \subset \overline{\gamma_{sl}}, \\ \boldsymbol{0} & \text{else}, \end{cases}$$

for each $i = 1, ..., N_{M_h}$.

We underline that, under Assumption 4.1, we can write on Γ_3 the following identities

$$\begin{aligned} |\tilde{I}_h \boldsymbol{u}|_h &= |\tilde{I}_h \boldsymbol{u}|, \\ \text{sgn}(\boldsymbol{u} \cdot \boldsymbol{\tau}) &= \text{sgn}(\tilde{I}_h \boldsymbol{u} \cdot \boldsymbol{\tau}). \end{aligned} \tag{4.10}$$

The following lemma holds.

Lemma 4.4 *Let $(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \boldsymbol{\Lambda}$ be the solution of Problem 2.2 and let $(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) \in \tilde{\boldsymbol{V}}_h \times \boldsymbol{\Lambda}_h$ be the solution of Problem 4.1. Under the additional regularity assumption $\boldsymbol{u} \in \left[H^{\frac{3}{2}+\nu}(\Omega)\right]^2$, $0 < \nu \le \frac{1}{2}$, and Assumption 4.1, we then have the estimate*

$$b(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}_h - \boldsymbol{\lambda}) \le C h^{\frac{1}{2}+\nu} |\boldsymbol{u}|_{\frac{3}{2}+\nu,\Omega} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{-\frac{1}{2},\Gamma_3}$$

for a positive constant C independent of $h < h_0$.

Proof. Let us evaluate $b(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}_h - \boldsymbol{\lambda})$ using the interpolation operators I_h and $\tilde{I}_h$:

$$\begin{aligned} b(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}_h - \boldsymbol{\lambda}) =& \langle \boldsymbol{\lambda}_h - \boldsymbol{\lambda}, \boldsymbol{u} - I_h \boldsymbol{u} \rangle_{\Gamma_3} + \langle \boldsymbol{\lambda}_h - \boldsymbol{\lambda}, I_h \boldsymbol{u} - \tilde{I}_h \boldsymbol{u} \rangle_{\Gamma_3} \\ &+ \langle \boldsymbol{\lambda}_h - \boldsymbol{\lambda}, \tilde{I}_h \boldsymbol{u} \rangle_{\Gamma_3}. \end{aligned} \tag{4.11}$$

For the first term in the right side of the previous equality, we can write

$$\langle \boldsymbol{\lambda}_h - \boldsymbol{\lambda}, \boldsymbol{u} - I_h \boldsymbol{u} \rangle_{\Gamma_3} \le \|\boldsymbol{\lambda}_h - \boldsymbol{\lambda}\|_{-1/2,\Gamma_3} \|\boldsymbol{u} - I_h \boldsymbol{u}\|_{1/2,\Gamma_3}$$

and from this, we get

$$\langle \boldsymbol{\lambda}_h - \boldsymbol{\lambda}, \boldsymbol{u} - I_h \boldsymbol{u} \rangle_{\Gamma_3} \le C \|\boldsymbol{\lambda}_h - \boldsymbol{\lambda}\|_{-1/2,\Gamma_3} h^{1/2+\nu} |\boldsymbol{u}|_{\frac{3}{2}+\nu,\Omega}\,. \tag{4.12}$$

Using the inverse inequality, we deduce

$$\|I_h \boldsymbol{u} - \tilde{I}_h \boldsymbol{u}\|^2_{\frac{1}{2},\Gamma_3} \le \frac{C}{h} \|I_h \boldsymbol{u} - \tilde{I}_h \boldsymbol{u}\|^2_{0,\Gamma_3} \le C \sum_{p_i \in \mathcal{M}_C} (\boldsymbol{u}(p_i) \cdot \boldsymbol{\tau})^2\,,$$

where the set of points $\mathcal{M}_C$ on the contact boundary Γ_3 is defined by

$$\mathcal{M}_C := \big\{p_i \in \mathcal{P}_C \,:\, (\tilde{I}_h \boldsymbol{u})(p_i) \ne (I_h \boldsymbol{u})(p_i)\big\}.$$

Let us introduce the following notation:

$$\begin{aligned} \mathcal{M}^r_C &:= \big\{p_i \in \mathcal{M}_C \,:\, \boldsymbol{u}(p_{i+1}) \cdot \boldsymbol{\tau} = 0\big\}, \\ \mathcal{M}^l_C &:= \big\{p_i \in \mathcal{M}_C \,:\, \boldsymbol{u}(p_{i-1}) \cdot \boldsymbol{\tau} = 0\big\}. \end{aligned}$$

We note that, under Assumption 4.1, we have:

- for each $p_i \in \mathcal{M}_C^r$, there exists a unique element $w_{p_i} \in \mathcal{W}_C$, between p_i and p_{i+1}; in addition, $\boldsymbol{u}\cdot\boldsymbol{\tau} \neq 0$ on $[p_i, w_{p_i}]$ and $\boldsymbol{u}\cdot\boldsymbol{\tau} = 0$ on $(w_{p_i}, p_{i+2}]$;
- for each $p_i \in \mathcal{M}_C^l$ there exists a unique element $w_{p_i} \in \mathcal{W}_C$, between p_{i-1} and p_i; in addition, $\boldsymbol{u}\cdot\boldsymbol{\tau} \neq 0$ on $[w_{p_i}, p_i]$ and $\boldsymbol{u}\cdot\boldsymbol{\tau} = 0$ on $[p_{i-2}, w_{p_i})$.

Let us define $f := \boldsymbol{u}\cdot\boldsymbol{\tau}$ on Γ_3. Clearly, the regularity assumption on $\boldsymbol{u}$ yields $f \in H^{1+\nu}(\Gamma_3)$. Now the Cauchy–Schwarz inequality gives for each point $p_i \in \mathcal{M}_C^l$ the estimate

$$
\begin{aligned}
\big(f(p_i)\big)^2 &= \left(\int_{w_{p_i}}^{p_i} f'(s)\,ds\right)^2 \\
&= \frac{1}{|w_{p_i}-p_{i-2}|^2}\left(\int_{w_{p_i}}^{p_i}\int_{p_{i-2}}^{w_{p_i}} \frac{f'(s)-f'(t)}{|s-t|^{\frac{1+2\nu}{2}}}|s-t|^{\frac{1+2\nu}{2}}\,dtds\right)^2 \\
&\leq \frac{1}{|w_{p_i}-p_{i-2}|^2}\int_{p_{i-2}}^{p_i}\int_{p_{i-2}}^{p_i} \frac{\big(f'(s)-f'(t)\big)^2}{|s-t|^{1+2\nu}}dtds \int_{w_{p_i}}^{p_i}\int_{p_{i-2}}^{w_{p_i}} |s-t|^{1+2\nu}dtds \\
&\leq C\frac{1}{|w_{p_i}-p_{i-2}|^2}|f'|^2_{\nu,[p_{i-2},\,p_i]}\,h^{1+2\nu}\,|p_i-w_{p_i}|\,|w_{p_i}-p_{i-2}| \\
&= C|f'|^2_{\nu,[p_{i-2},\,p_i]}\,h^{1+2\nu}\,\frac{|p_i-w_{p_i}|}{|w_{p_i}-p_{i-2}|} \\
&\leq C|f'|^2_{\nu,[p_{i-2},\,p_i]}\,h^{1+2\nu} \\
&\leq C|f|^2_{1+\nu,[p_{i-2},\,p_i]}\,h^{1+2\nu},
\end{aligned}
$$

where we used the shape regularity of the triangulation. Similarly, for each point $p_i \in \mathcal{M}_C^r$ we get

$$\big(f(p_i)\big)^2 \leq C|f|^2_{1+\nu,[p_i,\,p_{i+2}]}\,h^{1+2\nu}.$$

From this estimates, by summing and using a trace theorem, we get

$$\|I_h\boldsymbol{u}-\tilde{I}_h\boldsymbol{u}\|_{\frac{1}{2},\Gamma_3} \leq C\,h^{\frac{1}{2}+\nu}|\boldsymbol{u}|_{\frac{3}{2}+\nu,\Omega}. \tag{4.13}$$

Finally, we prove that, under Assumption 4.1, the following inequality holds:

$$\langle\boldsymbol{\lambda}_h-\boldsymbol{\lambda},\tilde{I}_h\boldsymbol{u}\rangle_{\Gamma_3} \leq 0. \tag{4.14}$$

To this end, let us consider a constant $\alpha > 0$ such that

$$\operatorname{sgn}(\boldsymbol{u}\cdot\boldsymbol{\tau}) = \operatorname{sgn}(\boldsymbol{u}\cdot\boldsymbol{\tau}-\alpha(\tilde{I}_h\boldsymbol{u})\cdot\boldsymbol{\tau}) \quad \text{on} \quad \Gamma_3.$$

Consequently, under Assumptions 4.1, the following equality takes place on Γ_3:

$$|\boldsymbol{u}| = |\alpha(\tilde{I}_h\boldsymbol{u})| + |\boldsymbol{u}-\alpha(\tilde{I}_h\boldsymbol{u})|.$$

Furthermore, taking into account (4.2) we can write

$$\langle\boldsymbol{\lambda},\boldsymbol{u}\rangle_{\Gamma_3} = \int_{\Gamma_3} g|\alpha(\tilde{I}_h\boldsymbol{u})|ds + \int_{\Gamma_3} g|\boldsymbol{u}-\alpha(\tilde{I}_h\boldsymbol{u})|ds. \tag{4.15}$$

On the other hand,

$$\langle\boldsymbol{\lambda},\boldsymbol{u}\rangle_{\Gamma_3} = \langle\boldsymbol{\lambda},\alpha(\tilde{I}_h\boldsymbol{u})\rangle_{\Gamma_3} + \langle\boldsymbol{\lambda},\boldsymbol{u}-\alpha(\tilde{I}_h\boldsymbol{u})\rangle_{\Gamma_3}, \tag{4.16}$$

and, taking into account the definiton of $\mathbf{\Lambda}$, we have

$$\langle \boldsymbol{\lambda}, \alpha(\tilde{I}_h \boldsymbol{u})\rangle_{\Gamma_3} \leq \int_{\Gamma_3} g|\alpha(\tilde{I}_h \boldsymbol{u})|ds, \tag{4.17}$$

$$\langle \boldsymbol{\lambda}, \boldsymbol{u} - \alpha(\tilde{I}_h \boldsymbol{u})\rangle_{\Gamma_3} \leq \int_{\Gamma_3} g|\boldsymbol{u} - \alpha(\tilde{I}_h \boldsymbol{u})|ds. \tag{4.18}$$

Using (4.15)-(4.18) we deduce that in (4.17) and (4.18) we can write identities. Thus, we have

$$\alpha\langle \boldsymbol{\lambda}, \tilde{I}_h \boldsymbol{u}\rangle_{\Gamma_3} = \langle \boldsymbol{\lambda}, \alpha(\tilde{I}_h \boldsymbol{u})\rangle_{\Gamma_3} = \int_{\Gamma_3} g|\alpha(\tilde{I}_h \boldsymbol{u})|ds = \alpha \int_{\Gamma_3} g|\tilde{I}_h \boldsymbol{u}|ds$$

and from this, using (4.10) we get

$$\langle \boldsymbol{\lambda}, \tilde{I}_h \boldsymbol{u}\rangle_{\Gamma_3} = \int_{\Gamma_3} g|\tilde{I}_h \boldsymbol{u}|_h ds. \tag{4.19}$$

In addition, keeping in mind the definition of $\mathbf{\Lambda}_h$, we can write

$$\langle \boldsymbol{\lambda}_h, \tilde{I}_h \boldsymbol{u}\rangle_{\Gamma_3} \leq \int_{\Gamma_3} g|\tilde{I}_h \boldsymbol{u}|_h ds. \tag{4.20}$$

Using now (4.19) and (4.20) we deduce (4.14). Taking into account (4.11)-(4.14) we conclude Lemma 4.4. □

Due to the approximation property of the spaces $\tilde{\boldsymbol{V}}_h$ and $\boldsymbol{M}_h$ (see, e.g., [2,26,27]), a straightforward consequence of the results obtained in Lemma 4.3 and Lemma 4.4 is the following theorem.

Theorem 4.1 *Let $(\tilde{\boldsymbol{u}}, \boldsymbol{\lambda}) \in \tilde{\boldsymbol{V}} \times \mathbf{\Lambda}$ be the solution of Problem 2.2 and let $(\tilde{\boldsymbol{u}}_h, \boldsymbol{\lambda}_h) \in \tilde{\boldsymbol{V}}_h \times \mathbf{\Lambda}_h$ be the solution of Problem 4.1. Under the additional regularity assumption $\tilde{\boldsymbol{u}} \in [H^{\frac{3}{2}+\nu}(\Omega)]^3$, $0 < \nu \leq \frac{1}{2}$ and Assumption 4.1, we then have the following optimal a priori error estimate*

$$\|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_h\|_{\tilde{V}} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{-\frac{1}{2},\Gamma_3} \leq Ch^{\frac{1}{2}+\nu}|\tilde{\boldsymbol{u}}|_{\frac{3}{2}+\nu,\Omega}$$

for a positive constant C that is independent of the meshsize $h < h_0$.

We remark that the a priori results can be extended to the 3D case. Furthermore we mention that this result can also be obtained in the multibody case using nonconforming meshes at the interface Γ_3. For the necessary techniques we refer to [HL02, HMW, HW03a].

5. Numerical Example. In this last section, we present a numerical example in 2D involving the two deformable bodies case. Therefore we have to choose one of the two bodies playing the role of the slave side Ω_s and the other one playing the role of the master side Ω_m. For the discretization, the Lagrange multiplier $\boldsymbol{\lambda}$ is defined on the triangulation of Γ_3 from the slave side. So, we have to replace the contact conditions (2.9) by

$$\left\{ \begin{array}{l} u_n^s + u_n^m = 0, \ |\boldsymbol{\sigma}_\tau| \leq g, \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \boldsymbol{u}_\tau^s + \boldsymbol{u}_\tau^m = 0, \\ |\boldsymbol{\sigma}_\tau| = g \Rightarrow \text{ there exists } \alpha > 0 \text{ s.t. } \boldsymbol{\sigma}_\tau = -\alpha\left(\boldsymbol{u}_\tau^s + \boldsymbol{u}_\tau^m\right) \end{array} \right. \qquad \text{on } \Gamma_3,$$

where $\boldsymbol{\sigma}_\tau$ is defined by $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma}_\tau^s = \boldsymbol{\sigma}_\tau^m$. We use the superscripts s and m to indicate that the value is related to the domain Ω_s or Ω_m, respectively. Our implementation is based on the finite element toolbox UG, see [BBJ$^+$97]. To solve the nonlinear multibody contact problem we use a primal-dual active set strategy in combination with an optimal multigrid method. For details we refer to [HIK03, HMW, HW03b].

We consider the problem depicted in the left picture of Figure 5.1. To fix the geometry we set the three points P_1, P_2 and P_3 equal to $P_1 = (0, 1.5)$, $P_2 = (-1, 0)$ and $P_3 = (1, 0)$. The radius r of the upper halfdisc is set to be $r = 1$ and for the angle ϕ we chosse $\phi = \pi/2$. The upper body

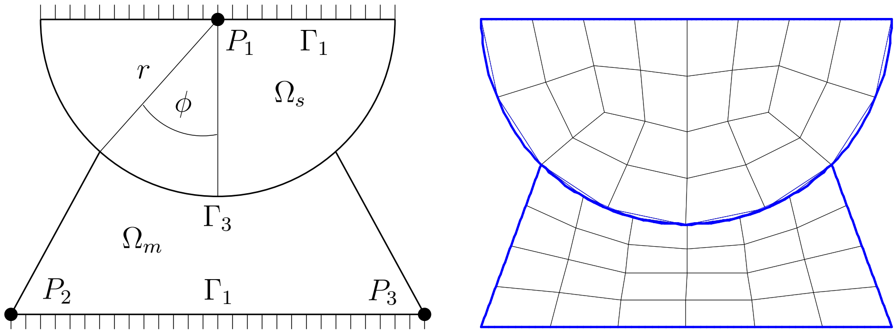

FIGURE 5.1. *Problem definition (left) and the grid on level 1 (right).*

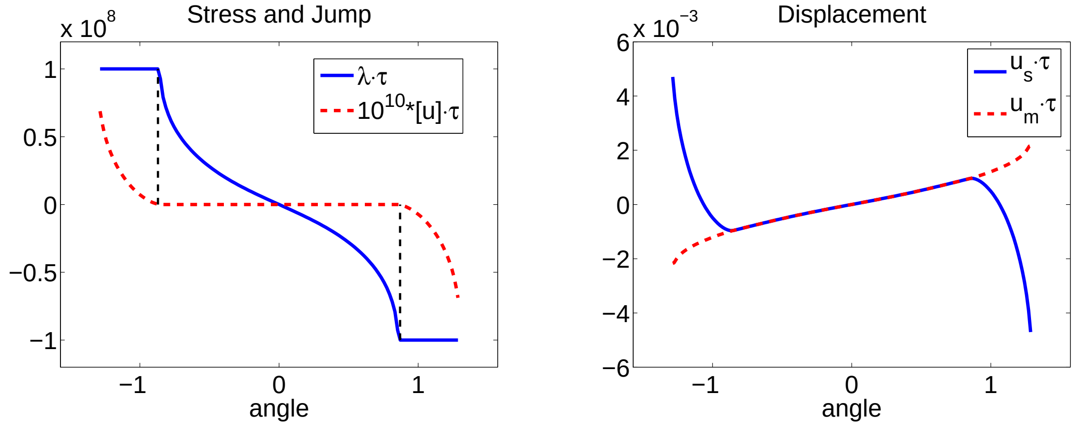

FIGURE 5.2. *Contact stress $\boldsymbol{\lambda}\cdot\boldsymbol{\tau}$ in tangential direction and the amplified jump of the displacement in tangential direction $[\boldsymbol{u}]\cdot\boldsymbol{\tau}$ (left picture) and the displacements in tangential direction on the slave and the master side $\boldsymbol{u}_s\cdot\boldsymbol{\tau}$ and $\boldsymbol{u}_m\cdot\boldsymbol{\tau}$ on the boundary region Γ_3 (right picture).*

plays the role of the slave side Ω_s. We assume that only the lower body Ω_m is a piezoelectric one. Here we use as material the ceramic $BaTiO_3$. The parameters can be found in [Qui01]. For the upper body Ω_s, we use a linear elastic material law given by

$$\boldsymbol{\sigma} = \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{Id} + 2\mu\boldsymbol{\varepsilon},$$

where λ and μ are the Lamé parameters given by the relations $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$ with Young's modulus E and Poisson's number ν. $\mathbf{Id}$ is the identity matrix in $\mathbb{R}^{2\times 2}$. On the upper domain, we set $E = 10^{10}$ and $\nu = 0.3$.

We fix the lower body Ω_m on the bottom with homogeneous Dirichlet data $\boldsymbol{u} = \mathbf{0}$. At the top of the upper domain Ω_s, we set the Dirichlet value for the displacement to be $\boldsymbol{u} = (0, -0.05(1-x^2))$. On the remaining parts of the boundary regions, except the contact zone Γ_3, we assume homogeneous Neumann boundary conditions $\boldsymbol{\sigma}\boldsymbol{n} = 0$. For the boundary condition for the electric potenzial on Ω_m we choose $\Gamma_a = \Gamma_3$ and set $\varphi = 0$ on Γ_a. On all the remaining boundary Γ_b of Ω_m, we assume $\boldsymbol{D}\cdot\boldsymbol{n} = 0$. For the friction bound in the contact condition (2.9), we use $g = 10^8$. The picture in the right of Figure 5.1 shows the grid on level 1. We use bilinear finite elements on quadrilaterals. Figure 5.2 shows in the left picture the tangential part of the Lagrange multiplier $\boldsymbol{\lambda}\cdot\boldsymbol{\tau} = -\boldsymbol{\sigma}_\tau$ and the amplified tangential part of the jump of the displacement $[\boldsymbol{u}]\cdot\boldsymbol{\tau} := \boldsymbol{u}_\tau^s + \boldsymbol{u}_\tau^m$. In the right picture the tangential part $\boldsymbol{u}_s\cdot\boldsymbol{\tau}$ and $\boldsymbol{u}_m\cdot\boldsymbol{\tau}$ on Γ_3 of the displacement is presented.

Figure 5.3 shows the distribution of the electric potential φ at the bottom of the lower domain Ω_m. In a last test we consider the example given before for two more different values for the

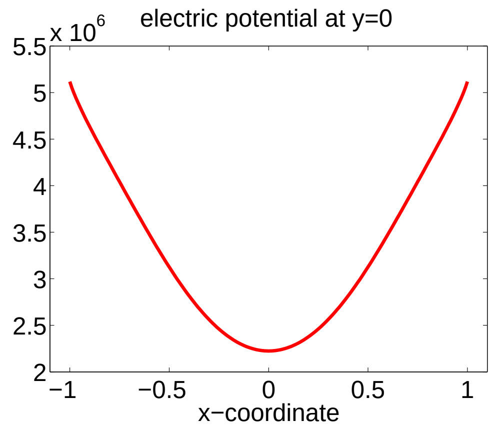

FIGURE 5.3. *Electric potential at the bottom of* Ω_m.

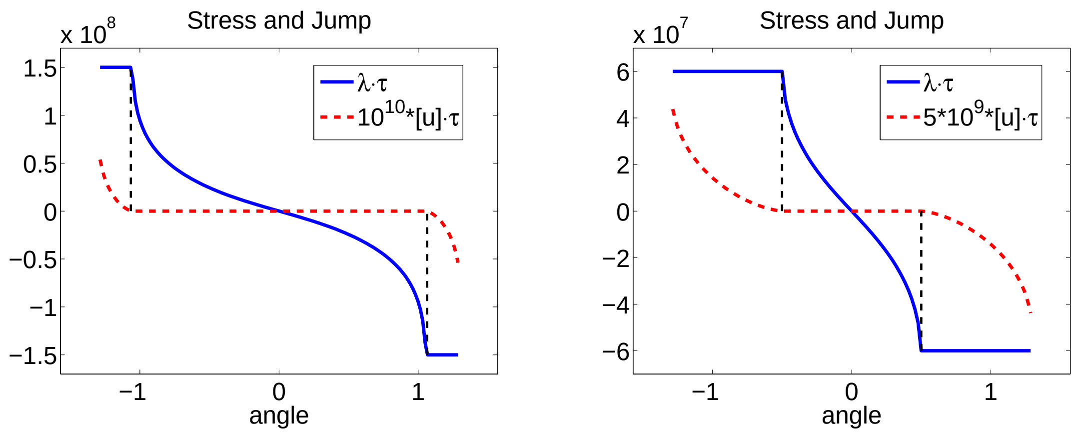

FIGURE 5.4. *Contact stress in tangential direction and the amplified jump of the displacement in tangential direction for* $g = 1.5 \times 10^8$ *(left picture) and* $g = 0.6 \times 10^8$ *(right picture).*

friction bounds g. The results are presented in Figure 5.4. In the left picture of Figure 5.4 we set the friction bound equal to $g = 1.5 \times 10^8$ and in the right we set $g = 0.6 \times 10^8$.

REFERENCES

[BBJ+97] P. Bastian, K. Birken, K. Johannsen, S. Lang, N. Neuß, H. Rentz–Reichert, and C. Wieners. UG – a flexible software toolbox for solving partial differential equations. *Computing and Visualization in Science*, 1:27–40, 1997.

[BF91] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods.* Springer-Verlag, New York, 1991.

[BHL99] F. Ben Belgacem, P. Hild, and P. Laborde. Extension of the mortar finite element method to a variational inequality modeling unilateral contact. *Mathematical Models and Methods in Applied Sciences*, 9:287–303, 1999.

[BR03] F. Ben Belgacem and Y. Renard. Hybrid finite element methods for the Signorini problem. *Mathematics of Computation*, 72(243):1117–1145, 2003.

[Bra97] D. Braess. *Finite Elements.* Cambridge University Press, 1997.

[DL76] G. Duvout and J.-L. Lions. *Inequalities in Mechanics and Physics.* Springer, Berlin, 1976.

[ET76] I. Ekeland and R. Temam. *Convex analysis and variational problems (Studies in mathematics and its applications; 1).* North-Holland Publ. Comp., 1976.

[Hcc96] J. Haslinger, I. Hlaváček, and J. Nečas. Numerical methods for unilateral problems in solid mechanics. In J.-L. Lions P. Ciarlet, editor, *Handbook of Numerical Analysis*, volume IV, pages 313–485. North-Holland, Amsterdam, 1996.

[HIK03] M. Hintermüller, K. Ito, and K. Kunisch. The primal-dual active set strategy as a semi–smooth Newton method. *SIAM Journal on Optimization*, 13(3):865–888, 2003.

[HL02] P. Hild and P. Laborde. Quadratic finite element methods for unilateral contact problems. *Applied*

Numerical Mathematics, 41:410–421, 2002.
[HMW] S. Hüeber, A. Matei, and B. Wohlmuth. A mixed variational formulation and a priori error estimates for a frictional antiplane contact problem. In preparation.
[HR99] W. Han and B.D. Reddy. *Plasticity - Mathematical Theory and Analysis*. Springer, 1999.
[HS02] W. Han and M. Sofonea. *Quasistatic contact problems in viscoelasticity and viscoplasticity*. Studies in Advanced Mathematics. American Mathematical Society, International Press, 2002.
[HW03a] S. Hüeber and B. Wohlmuth. An optimal a priori error estimate for non-linear multibody contact problems. Technical Report 39, Universität Stuttgart, SFB 404, 2003. To appear in SIAM Journal on Numerical Analysis.
[HW03b] S. Hüeber and B. Wohlmuth. A primal-dual active set strategy for non–linear multibody contact problems. Technical Report 40, Universität Stuttgart, SFB 404, 2003. To appear in Computer Methods in Applied Mechanics and Engineering.
[Ike90] T. Ikeda. *Fundamentals of Piezoelectricity*. Oxford University Press, Oxford, 1990.
[KP60] S. Kalinski and J. Petikiewicz. Equations of motion coupled with the field of temperature in magnetic field involving mechanical and electromagnetic relaxations for anisotropic bodies. In *Proc. Vibr. Probl.*, pages 17–28, 1960.
[KS00] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications*. SIAM, 2000.
[Min68] R.D. Mindlin. Polarisation gradient in elastic dielectrics. *Internat. J. Solids Structures*, 4:637–663, 1968.
[Min69] R.D. Mindlin. Continuum and lattice theories of influence of electromechanical coupling on capacitance of thin dielectric films. *Internat. J. Solids Structures*, 4:1197–1213, 1969.
[Min72] R.D. Mindlin. Elasticity, piezoelasticity and crystal lattice dynamics. *J. Elasticity*, 4(2):217–280, 1972.
[Pan85] P.D. Panagiotopoulos. *Inequality problems in Mechanics and Applications*. Birkhäuser, Boston, 1985.
[Qui01] Q.H. Quin. *Fracture mechanics of piezoelectric materials*. WIT Press, Southampton, Boston, 2001.
[SE04] M. Sofonea and El-H. Essoufi. A piezoelectric contact problem with slip dependent coefficient of friction. *Mathematical Modelling and Analysis*, 9:229–242, 2004.
[SST04] M. Shillor, M. Sofonea, and J.J Telega. *Models and Variational Analysis of Quasistatic Contact*, volume 655. Springer, Berlin Heidelberg, 2004.
[Tou56] R.A. Toupin. The elastic dielectrics. *J. Rat. Mech. Analysis*, pages 849–915, 1956.
[Tou63] R.A. Toupin. A dynamical theory of elastic dielectrics. *Internat. J. Engrg. Sci.*, pages 101–126, 1963.
[Voi10] W. Voigt. *Lehrbuch der Kristall-Physik*. Teubner, Leipzig, 1910.
[Woh00] B. Wohlmuth. A mortar finite element method using dual spaces for the Lagrange multiplier. *SIAM Journal on Numerical Analysis*, 38:989–1012, 2000.

Stefan H¨ueber
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: hueeber@ians.uni-stuttgart.de

Andaluzia Matei
A.I. Cuza street 13
200585 Craiova
Romania
E-Mail: AndaluziaMatei@k.ro

Barbara I. Wohlmuth
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: wohlmuth@ians.uni-stuttgart.de

Erschienene Preprints ab Nummer 2005/001

Komplette Liste: `http://preprints.ians.uni-stuttgart.de`

2005/001 *Nicaise, S., Sändig, A.-M.*: Dynamical crack propagation in a 2D elastic body. The out-of plane state.

2005/002 *S. Hüeber, A. Matei, B.I. Wohlmuth*: A mixed variational formulation and an optimal a priori error estimate for a frictional contact problem in elasto-piezoelectricity