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Abstract We investigate three–dimensional transmission problems related to the interaction of metallic and piezoelectric ceramic bodies. We give a mathematical formulation of the physical problem when the metallic and ceramic sub-domains are bonded along some proper parts of their boundaries. The corresponding nonclassical mixed boundary-transmission problem is reduced by potential methods to an equivalent strongly elliptic system of pseudodifferential equations on manifolds with boundary. We investigate the solvability of this system in different function spaces. On the basis of these results we prove uniqueness and existence theorems for the original boundary-transmission problem. We study also the regularity of the electrical and mechanical fields near the curves where the boundary conditions change and where the interfaces intersect the exterior boundary. The electrical and mechanical fields can be decomposed into singular and more regular terms near these curves. A power of the distance from a reference point to the corresponding edge-curves occures in the singular terms and describes the regularity explicitly. We compute these complex-valued exponents and demonstrate their dependence on the material parameters.

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1 Introduction

There is a growing interest in the investigation of mathematical models of an elastic medium which take into account the influence of different physical fields such as thermal, electric, magnetic and other ones. An impetus for such studies was the creation of new materials possessing properties which are not characteristic for usual elastic bodies. Among these are piezoelectric materials.

The phenomenon of piezoelectricity is of great importance. It is essentially applied in many electro-mechanical converters (transducers) and in the so-called "smart materials" transforming mechanical loadings into electric effects and vice versa. These properties are widely used in electronics, acoustics, measuring and controlling devices. In particular, stack actuators are used in injectors for common-rail engines as vaporizers and valves.

W. Voigt [Vo1] was the first who constructed a linear mathematical model of an elastic medium taking the interaction of electric and mechanical fields into account and derived the corresponding system of differential equations. In their works R. Toupin, R. Mindlin, L. Knopoff, S. Kaliski and J. Petikiewicz suggested new, more refined models of an elastic medium, where a polarization vector occurs [To1], [To2], [Mi2], [Mi3]. Furthermore, effects caused by thermal [Mi1] and magnetic fields [Kn1], [KP1] (for details see also [No1], [Pa1]) and hysteresis effects are considered [Ka].

In this paper we study the following problem:

Given is a three-dimensional composite consisting of a piezo-electric matrix with metallic inclusions (*electrodes*)*. Derive a linear model* (*neglecting the hysteresis effects*) *for the interaction of the elastic and electrical fields and perform a rigorous mathematical analysis by potential methods.*

The main difficulty in modelling was to find appropriate boundary and transmission conditions for the composed body and to formulate them in an efficient way. At the end we got linear systems of partial differential equations in the metallic and ceramic parts coupled by transmission conditions and endowed with mixed boundary conditions. For simplicity we have concentrated to the static case. The mathematical analysis includes the study of existence, uniqueness and regularity of the resulting elliptic boundary-transmission problem assuming the metallic and ceramic materials occupy smooth domains.

It is well known, that stress singularities appear near zones, where the boundary conditions change and where the interfaces intersect the exterior boundary. The detailed theoretical description and the numerical computation of these stress singularities to our composed complex problem are challenging. Thus, we have to take into account the composed structure and the diversity of the fields in the ceramic and metallic part.

There are different methods to handle the solvability, regularity and stress singularities of the boundary-transmission problem. One possibility is to use variational methods combined with Mellin techniques. To do this the Mellin technique in domains with edges, developed by V.A. Kondratjev [Kon], V.G.Maz'ya & B.A.Plamenevski [MP], S.A.Nazarov & B.A Plamenevski [NP] for boundary value problems is to transfer to our more complicated boundary-transmission problem (see [NS1, NS2, NS3]).

In this paper we apply potential methods which lead to boundary integral (pseudo-differential) equations. The solutions will be constructed with the help of an indirect boundary integral equations (BIE) method, writing them as layer potentials in the ceramic and metallic parts with unknown densities. The densities are to determine in such a way, that the interface and boundary conditions are satisfied. The solvability and regularity of the resulting boundary-integral equations are analyzed in Sobolev-Slobodetski (W_p^s), Bessel potential (H_p^s), and Besov ($B_{p,t}^s$) spaces. The results for the original problem follow from the representation of the solution by boundary inegrals. Due to stress singularities near curves (smooth edges) where the boundary conditions change and the interfaces intersect the smooth exterior boundary there are restrictions to s and p. These restrictions are related to the distribution of the eigenvalues of the symbol matrices of the corresponding pseudo-differential boundary operators (cf. [NCS1], [BC1] [Ck1]-[Ck4]).
The paper is organized as follows: In section 2 we derive a boundary-transmission problem in appropriate function spaces for the composed body consisting of metallic and piezoelectric ceramic parts. In section 3 we summarize some known properties on potential operators and prove the invertibility of pseudo-differential operators acting on the boundaries of the metallic and ceramic sub-domains. Section 4 is the main part of this paper. Here the original transmission problem is reduced to the system of pseudodifferential equations involving boundary operators acting on the interface Γ_1 and the Dirichlet part Γ of the exterior boundary (see Figure 1). Their principal homogeneous symbol matrices yield information on the existence and regularity of the solution fields. In particular, in Theorem 4.5, the global C^α-regularity results are shown with some $\alpha \in (\,0,\,\frac{1}{2}\,)$ depending on the eigenvalues of these symbol matrices. Note, that these eigenvalues actually define the singularity exponents for the first order derivatives of solutions and they depend on the material parameters. We compute these complex-valued exponents of the distance from a reference point to the edge-curves $\partial\Gamma_1$ and $\partial\Gamma$ and demonstrate their dependence on the material parameters.
We note that in the paper we develop the boundary integral equations method mainly for smooth domains. However, our approach and all the results related to the existence and uniqueness of solutions in the Sobolev W_2^1 space remain valid also for the above described composite structures with piecewise smooth Lipschitz boundaries (see Subsection 3.4). In comparison with the smooth one, there appear additional singularities of solutions at corner and edge points. Unfortunately, by the boundary integral equations technique applied here it is not possible to get explicitly the singularity exponents for such polyhedral domains.
We remark that the computation of these singular terms by other methods (Mellin transform, weak formulation of a quadratic eigenvalue problem, finite element methods) is challenging too, due to the complexity of the boundary transmission problem.

2 Formulation of the boundary-transmission problem

Let Ω_1 and Ω be bounded non-intersecting domains of the three-dimensional Euclidean space $\mathbb{R}^3$ with C^∞-smooth boundaries $\partial\Omega$ and $\partial\Omega_1$, respectively. Moreover, let $\partial\Omega$ and $\partial\Omega_1$ have a nonempty intersection $\overline{\Gamma}_1$ with a positive measure, i.e., $\partial\Omega \cap \partial\Omega_1 = \overline{\Gamma}_1$.
We set $S_1 := \partial\Omega_1 \setminus \overline{\Gamma}_1$ and $S_2^* := \partial\Omega \setminus \overline{\Gamma}_1$. Further, we denote by Γ some open, nonempty, proper sub-manifold of S_2^* and let $S := S_2^* \setminus \overline{\Gamma}$. Thus, we have the following decomposition of the boundary surfaces (see Figure 1)

$$\partial\Omega = \overline{\Gamma}_1 \cup \overline{S} \cup \overline{\Gamma}, \quad \partial\Omega_1 = \overline{\Gamma}_1 \cup \overline{S}_1.$$

Throughout the paper, for simplicity, we assume that

$$\partial\Omega_1,\ \partial\Omega,\ \partial S_1,\ \partial\Gamma_1,\ \partial\Gamma,\ \partial S \in C^\infty, \quad \text{and} \quad \partial\Omega_1 \cap \overline{\Gamma} = \varnothing\,.$$

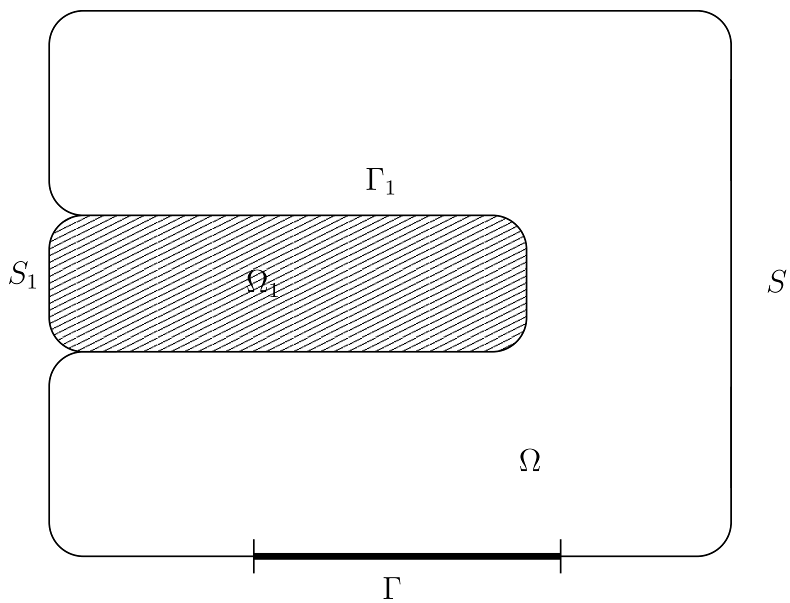

Figure 1: Composed body: on S - the Neumann type conditions for mechanical and electical fields, on S_1 - the Neumann type conditions for mechanical fields, on Γ_1 - mechanical transmission conditions and electical Dirichlet condition, on Γ - mechanical and electical Dirichlet conditions.

Let Ω_1 be occupied by an isotropic or anisotropic homogeneous elastic (metallic) medium and Ω be filled by an anisotropic homogeneous piezoelectric (ceramic) medium. These two bodies interact to each other along the subsurface Γ_1. In the "metallic" domain Ω_1 we have a usual three-dimensional elastic field described by the displacement vector $u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top$, while in the piezoelectric domain Ω we have a four-dimensional physical field described by the displacement vector $u = (u_1, u_2, u_3)^\top$ and by the electric potential $u_4 := \varphi$. Here and throughout the paper the superscript $\top$ denotes transposition. The physical problem under consideration is described by strongly elliptic systems of linear partial differential equations in the corresponding elastic and piezoelastic domains with appropriate boundary and transmission conditions on S, S_1, Γ, and Γ_1 (see Subsection 2.4).

2.1 The elastic field equations for the metal

The basic equations of the linear elastostatics for homogeneous anisotropic media read

$$c^{(1)}_{ijlk}\,\partial_i\,\partial_l\,u^{(1)}_k(x) + X^{(1)}_j(x) = 0 \ \text{ in } \ \Omega_1, \quad j = 1,2,3, \tag{2.1}$$

or in matrix form

$$A^{(1)}(\partial_x)\,u^{(1)}(x) + X^{(1)}(x) = 0 \ \text{ in } \ \Omega_1, \tag{2.2}$$

where $u^{(1)} = (u^{(1)}_1, u^{(1)}_2, u^{(1)}_3)^\top$ is the displacement vector in Ω_1, $X^{(1)} = (X^{(1)}_1, X^{(1)}_2, X^{(1)}_3)^\top$ is a given mass force density in Ω_1, $\partial_x = \partial = (\partial_1, \partial_2, \partial_3)$, $A^{(1)}(\partial_x)$ is the matrix differential operator

$$A^{(1)}(\partial_x) = \Big[A^{(1)}_{jk}(\partial_x)\Big]_{3\times 3}\,, \quad A^{(1)}_{jk}(\partial_x) = c^{(1)}_{ijlk}\,\partial_i\,\partial_l\,, \ \ \partial_l = \frac{\partial}{\partial x_l},$$

$c^{(1)}_{ijlk}$ are elastic constants satisfying the symmetry conditions

$$c^{(1)}_{ijlk} = c^{(1)}_{lkij} = c^{(1)}_{ijkl}, \quad i,j,l,k = 1,2,3.$$

Throughout the paper we employ the Einstein summation convention.
We assume that

$$c^{(1)}_{ijlk}\,\xi_{ij}\,\xi_{lk} \geq \delta_1\,\xi_{lk}\,\xi_{lk} \quad \text{for all} \quad \xi_{lk} = \xi_{kl} \tag{2.3}$$

with some positive constant δ_1 depending only on the elastic constants. In particular, this relation implies that the density of potential energy in Ω_1

$$E^{(1)}\left(u^{(1)}, u^{(1)}\right) = c^{(1)}_{ijlk}\,s^{(1)}_{ij}\left(u^{(1)}\right)\,s^{(1)}_{lk}\left(u^{(1)}\right) \tag{2.4}$$

is positive definite with respect to the symmetric components of the strain tensor

$$s^{(1)}_{lk}\left(u^{(1)}\right) = \frac{1}{2}\left(\partial_l\,u^{(1)}_k + \partial_k\,u^{(1)}_l\right).$$

Note that the stress tensor $\sigma^{(1)}_{kj}\left(u^{(1)}\right)$ and the strain tensor $s^{(1)}_{kj}\left(u^{(1)}\right)$ are related by Hooke's law,

$$\sigma^{(1)}_{ij}\left(u^{(1)}\right) = c^{(1)}_{ijlk}\,s^{(1)}_{lk}\left(u^{(1)}\right).$$

It is evident that the equations (2.2) correspond to the statical equilibrium of a body which is described by the relations

$$\partial_i\sigma^{(1)}_{ij}\left(u^{(1)}\right) + X^{(1)}_j = 0, \quad j = 1,2,3. \tag{2.5}$$

By $T^{(1)}(\partial_x, n(x))u^{(1)}(x)$ we denote the stress vector acting on $\partial\Omega_1$ associated with the unit normal vector $n = (n_1, n_2, n_3)$,

$$\left[T^{(1)}(\partial_x, n(x))u^{(1)}(x) \right]_j := \sigma_{ij}^{(1)} \left(u^{(1)}\right) n_i(x) = c_{ijlk}^{(1)} n_i(x) \partial_l u_k^{(1)}(x), \quad j = 1, 2, 3.$$

The *stress operator* on the boundary $\partial\Omega_1$ is defined by

$$T^{(1)}(\partial_x, n(x)) := \left[T_{jk}^{(1)}(\partial_x, n(x)) \right]_{3\times 3}, \quad T_{jk}^{(1)}(\partial_x, n(x)) = c_{ijlk}^{(1)} n_i(x) \partial_l,$$

where n is the unit normal vector to $\partial\Omega_1$.
From the symmetry properties of the coefficients $c_{ijlk}^{(1)}$ and the positive definiteness of the quadratic form (2.3) it follows that $A^{(1)}(\partial_x)$ is a strongly elliptic, formally self-adjoint differential operator. Therefore, for any real $\xi \in \mathbb{R}^3$ and any complex $\eta \in \mathbb{C}^3$ there holds the inequality

$$A^{(1)}(\xi)\, \eta \cdot \eta \geq \delta_2 \, |\xi|^2 \, |\eta|^2$$

with a positive constant δ_2 depending only on the elastic constants.
Throughout the paper the symbol $a \cdot b$ denotes the usual scalar product of two (in general, complex) vectors $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, i.e., $a \cdot b = a_i \, \overline{b_i}$, where the over-bar denotes the complex conjugation.

2.2 The piezoelastic field equations for the ceramics

The piezoelastic field $U = (u, \varphi)^\top =: (u_1, u_2, u_3, u_4)^\top$ in Ω is described by the displacement vector $u = (u_1, u_2, u_3)^\top$ and the electric potential $\varphi := u_4$. It is given by the linear system of statics for a homogeneous anisotropic piezoelectric medium

$$\begin{aligned} &c_{ijlk} \, \partial_i \, \partial_l \, u_k(x) + e_{pjq} \, \partial_p \, \partial_q \, \varphi(x) + X_j(x) = 0, \quad j = 1, 2, 3, \\ &-e_{plq} \, \partial_p \, \partial_q \, u_l(x) + \varepsilon_{pq} \, \partial_p \, \partial_q \, \varphi(x) + X_4(x) = 0, \end{aligned} \tag{2.6}$$

or in the matrix form (see, e.g., [No1])

$$A(\partial_x) \, U(x) + \widetilde{X}(x) = 0 \;\text{ in }\; \Omega, \tag{2.7}$$

where $\widetilde{X} = (X, X_4)^\top$, $X = (X_1, X_2, X_3)^\top$ is a mass force dencity, while $-X_4$ is a charge density (see (2.14)-(2.15)), and

$$A(\partial_x) = [A_{jk}(\partial_x)]_{4\times 4}\,, \tag{2.8}$$

$$\begin{aligned} &A_{jk}(\partial_x) = c_{ijlk} \, \partial_i \, \partial_l && \text{for} \quad j, k = 1, 2, 3, \\ &A_{j4}(\partial_x) = -A_{4j}(\partial_x) = e_{pjq} \, \partial_p \, \partial_q && \text{for} \quad j = 1, 2, 3, \\ &A_{44}(\partial_x) = \varepsilon_{pq} \, \partial_p \, \partial_q. \end{aligned}$$

Here c_{ijlk}, e_{kjl}, and ε_{lk} are the elastic, piezoelectric, and dielectric (permittivity) constants satisfying the symmetry conditions:

$$c_{ijlk} = c_{lkij} = c_{ijkl}, \quad e_{kjl} = e_{klj}, \quad \varepsilon_{lk} = \varepsilon_{kl}, \quad i, j, l, k = 1, 2, 3.$$

Moreover, we assume that for arbitrary real $\xi_{ij} = \xi_{ji}$ and ξ_j

$$\begin{aligned} & c_{ijlk}\, \xi_{ij}\, \xi_{lk} \geq \delta_3\, \xi_{lk}\, \xi_{lk} \quad \text{for all} \quad \xi_{lk} = \xi_{kl}, \\ & \varepsilon_{lk} \xi_l \xi_k \geq \delta_4\, \xi_l\, \xi_l, \end{aligned} \tag{2.9}$$

where δ_3 and δ_4 are some positive constants depending only on the elastic, piezoelectric and dielectric constants.
It can be easily seen that the operator $A(\partial_x)$ is strongly elliptic but not formally self-adjoint. Note that the following inequality

$$\Re(A(\xi)\, \eta \,\cdot\, \eta) \geq \delta_5\, |\xi|^2\, |\eta|^2 \tag{2.10}$$

holds for arbitrary real $\xi \in \mathbb{R}^3$ and complex $\eta \in \mathbb{C}^4$ with a positive constant δ_5 depending only on the elastic, piezoelectric, and dielectric constants.
Denote by $A^*(\partial_x)$ the 4×4 matrix differential operator formally adjoint to $A(\partial_x)$. Clearly,

$$A^*(\partial_x) = \big[A^*_{jk}(\partial_x) \big]_{4\times 4} = [A_{kj}(-\partial_x)]_{4\times 4} = [A_{kj}(\partial_x)]_{4\times 4}\ , \tag{2.11}$$

$$\begin{aligned} & A^*_{jk}(\partial_x) = A_{jk}(\partial_x) = c_{ijlk}\, \partial_i\, \partial_l && \text{for} \quad j, k = 1, 2, 3, \\ & A^*_{j4}(\partial_x) = -A_{j4}(\partial_x) = -e_{pjq}\, \partial_p\, \partial_q && \text{for} \quad j = 1, 2, 3, \\ & A^*_{4j}(\partial_x) = -A_{4j}(\partial_x) = e_{pjq}\, \partial_p\, \partial_q && \text{for} \quad j = 1, 2, 3, \\ & A^*_{44}(\partial_x) = A_{44}(\partial_x) = \varepsilon_{pq}\, \partial_p\, \partial_q. \end{aligned} \tag{2.12}$$

We remind that in the theory of elasticity of piezoelectric bodies the stress tensor $\sigma_{ij}\,(u, \varphi)$, the components of electric displacement vector $D\,(u, \varphi)$, and the electric field vector $E = (E_1, E_2, E_3)^\top$ have the form

$$\begin{aligned} & \sigma_{ij}\,(u, \varphi) = c_{ijlk}\; s_{lk}\,(u) + e_{kij}\; \partial_k \varphi, \quad s_{lk}\,(u) = \tfrac{1}{2}\,(\partial_l u_k + \partial_k u_l)\,, \\ & D_i\,(u, \varphi) = e_{ikl}\; s_{kl}\,(u) - \varepsilon_{ik}\; \partial_k \varphi, \\ & E_i = -\partial_i \varphi, \qquad i, j = 1, 2, 3. \end{aligned} \tag{2.13}$$

The system (2.7) is equivalent to the mechanical static equilibrium equations

$$\partial_i \sigma_{ij} + X_j = 0, \quad j = 1, 2, 3, \tag{2.14}$$

and the static electric field equation

$$-\partial_i D_i + X_4 = 0. \tag{2.15}$$

The components of the three-dimensional mechanical stress vector acting on a surface with the normal $n = (n_1, n_2, n_3)$ read as follows

$$\sigma_{ij}\, n_i = c_{ijlk}\, n_i\, \partial_l u_k + e_{lij}\, n_i\, \partial_l \varphi \;\text{ for }\; j = 1, 2, 3, \tag{2.16}$$

while the normal component of the electric displacement vector (with opposite sign) is

$$-D_i\, n_i \;= -e_{ikl}\, n_i\, \partial_l u_k + \varepsilon_{il}\, n_i\, \partial_l \varphi. \tag{2.17}$$

Let us introduce the following matrix operator associated with the operator $A(\partial_x)$

$$\mathcal{T}(\partial, n) = ||\ \mathcal{T}_{jk}(\partial, n)\ ||_{4\times 4}, \tag{2.18}$$

where

$$\begin{aligned}
&\mathcal{T}_{jk}(\partial, n) = c_{ijlk}\, n_i\, \partial_l && \text{for} \quad 1 \le j, k \le 3,\\
&\mathcal{T}_{j4}(\partial, n) = e_{lij}\, n_i\, \partial_l && \text{for} \quad 1 \le j \le 3,\\
&\mathcal{T}_{4k}(\partial, n) = -e_{ikl}\, n_i\, \partial_l && \text{for} \quad 1 \le k \le 3,\\
&\mathcal{T}_{44}(\partial, n) = \varepsilon_{il}\, n_i\, \partial_l .
\end{aligned} \tag{2.19}$$

For $U = (u, \varphi)^\top$ we have (cf. (2.16) and (2.17))

$$\mathcal{T}(\partial, n)\, U = [\, (\sigma_{ij}\, n_i,\ -D_i\, n_i)_{1\times 4}\,]^\top. \tag{2.20}$$

We introduce also the following boundary differential matrix operator associated to the operator $A^*(\partial_x)$

$$\widetilde{\mathcal{T}}(\partial, n) = ||\ \widetilde{\mathcal{T}}_{jk}(\partial, n)\ ||_{4\times 4}, \tag{2.21}$$

where

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{jk}(\partial, n) = \mathcal{T}_{jk}(\partial, n) = c_{ijlk}\, n_i\, \partial_l && \text{for} \quad 1 \le j, k \le 3,\\
&\widetilde{\mathcal{T}}_{j4}(\partial, n) = -\mathcal{T}_{j4}(\partial, n) = -e_{lij}\, n_i\, \partial_l && \text{for} \quad 1 \le j \le 3,\\
&\widetilde{\mathcal{T}}_{4k}(\partial, n) = -\mathcal{T}_{4k}(\partial, n) = e_{ikl}\, n_i\, \partial_l && \text{for} \quad 1 \le k \le 3,\\
&\widetilde{\mathcal{T}}_{44}(\partial, n) = \mathcal{T}_{44}(\partial, n) = \varepsilon_{il}\, n_i\, \partial_l .
\end{aligned} \tag{2.22}$$

2.3 Green's formulae

To avoid complications related to the directions of normal vectors on the contact surfaces from now on we assume that the normal vector to $\partial\Omega_1$ is directed outward, while on $\partial\Omega$ it is directed inward.

Here we recall the well-known Green's formulae for the operators $A^{(1)}(\partial)$ and $A(\partial)$ in the metallic domain Ω_1 and the ceramic domain Ω, respectively (see, e.g., [KGBB1], [BG1]):

$$\int_{\Omega_1} \left[A^{(1)}(\partial)u^{(1)} \cdot v^{(1)} + E^{(1)}\left(u^{(1)}, v^{(1)}\right) \right] dx$$

$$= \int_{\partial\Omega_1} \{T^{(1)}(\partial,n)u^{(1)}\}^+ \cdot \{v^{(1)}\}^+ \, dS, \tag{2.23}$$

$$\int_{\Omega_1} \left[A^{(1)}(\partial)u^{(1)} \cdot v^{(1)} - u^{(1)} \cdot A^{(1)}(\partial)v^{(1)} \right] dx$$

$$= \int_{\partial\Omega_1} \left[\{T^{(1)}(\partial,n)u^{(1)}\}^+ \cdot \{v^{(1)}\}^+ - \{u^{(1)}\}^+ \cdot \{T^{(1)}(\partial,n)v^{(1)}\}^+ \right] dS, \tag{2.24}$$

$$\int_{\Omega} \left[A(\partial)U \cdot V + E(U,V) \right] dx = -\int_{\partial\Omega} \left[\{\mathcal{T}(\partial,n)U\}^+ \cdot \{V\}^+ \right] dS, \tag{2.25}$$

$$\int_{\Omega} \left[A(\partial)U \cdot V - U \cdot A^*(\partial)V \right] dx$$

$$= -\int_{\partial\Omega} \Big[\{\mathcal{T}(\partial,n)U\}^+ \cdot \{V\}^+ - \{U\}^+ \cdot \{\widetilde{\mathcal{T}}(\partial,n)V\}^+ \Big] dS, \tag{2.26}$$

where we assume that $u^{(1)}, v^{(1)} \in \left[C^2(\overline{\Omega}_1)\right]^3$ and $U = (u_1, u_2, u_3, \varphi)^\top, V = (v_1, v_2, v_3, \psi)^\top \in \left[C^2(\overline{\Omega})\right]^4$,

$$E^{(1)}\left(u^{(1)}, v^{(1)}\right) = c^{(1)}_{ijlk}\, \partial_i u^{(1)}_j \, \overline{\partial_l v^{(1)}_k}, \tag{2.27}$$

$$E(U,V) = c_{ijlk}\, \partial_i u_j \, \overline{\partial_l v_k} + e_{pjq}\, \partial_p \varphi \, \overline{\partial_q v_j} - e_{pjq}\, \partial_q u_j \, \overline{\partial_p \psi} + \varepsilon_{pq}\, \partial_p \varphi \, \overline{\partial_q \psi}. \tag{2.28}$$

The symbol $\{\,\cdot\,\}^+$ denotes the interior one-sided limit on $\partial\Omega$ (respectively $\partial\Omega_1$) from Ω (respectively Ω_1). Similarly, $\{\,\cdot\,\}^-$ denotes the exterior one-sided limit on $\partial\Omega$ (respectively $\partial\Omega_1$) from the exterior of Ω (respectively Ω_1).
We remark that the above Green's formulae (2.23) and (2.25) by a standard limiting procedure can be generalized to Lipschitz domains and to vector-functions

$$u^{(1)} \in \left[W^1_p(\Omega_1)\right]^3, \quad v^{(1)} \in \left[W^1_{p'}(\Omega_1)\right]^3, \quad U \in \left[W^1_p(\Omega)\right]^4, \quad V \in \left[W^1_{p'}(\Omega)\right]^4,$$

whose second order distributional derivatives satisfy the inclusions

$$A^{(1)}(\partial)u^{(1)} \in [L_p(\Omega_1)]^3, \quad A(\partial)U \in [L_p(\Omega)]^4, \quad \frac{1}{p} + \frac{1}{p'} = 1\,.$$

Moreover, in addition, if $A^{(1)}(\partial)v^{(1)} \in \left[L_{p'}(\Omega_1)\right]^3, \quad A^*(\partial)V \in \left[L_{p'}(\Omega)\right]^4$, then formulae (2.24) and (2.26) hold true as well (for details see [Ne1], [MMP1], [Gao1]).

2.4 Formulation of the boundary-transmission problem

Throughout the paper L_p, W_p^r, H_p^s, and $B_{p,q}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) denote the well-known Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g., [Tr1], [Tr2], [LiMa1]). Sometimes we will use the abbreviations $W_2^r = W^r$, $H_2^s = H^s$. We recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. We use also the notation $W_p^t := B_{p,p}^t$ for negative non-integer t.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a smooth sub-manifold $\mathcal{M} \subset \mathcal{M}_0$ we denote by $\widetilde{H}_p^s(\mathcal{M})$ and $\widetilde{B}_{p,q}^s(\mathcal{M})$ the subspaces of $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$\widetilde{H}_p^s(\mathcal{M}) = \{g : g \in H_p^s(\mathcal{M}_0),\ \mathrm{supp}\, g \subset \overline{\mathcal{M}}\},$$

$$\widetilde{B}_{p,q}^s(\mathcal{M}) = \{g : g \in B_{p,q}^s(\mathcal{M}_0),\ \mathrm{supp}\, g \subset \overline{\mathcal{M}}\},$$

while $H_p^s(\mathcal{M})$ and $B_{p,q}^s(\mathcal{M})$ denote the spaces of restrictions on $\mathcal{M}$ of functions from $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$H_p^s(\mathcal{M}) = \{r_{\mathcal{M}} f :\ f \in H_p^s(\mathcal{M}_0)\},$$

$$B_{p,q}^s(\mathcal{M}) = \{r_{\mathcal{M}} f\ :\ f \in B_{p,q}^s(\mathcal{M}_0)\},$$

where $r_{\mathcal{M}}$ is the restriction operator on $\mathcal{M}$.

Now, we come back to our boundary-transmission problem, restricting the fields to the metallic and ceramic sub-domains, denoted by $u^{(1)}$ and U. The problem reads:

Find vector-functions

$$u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top\ :\ \Omega_1 \to \mathbb{R}^3 \quad \text{and} \quad U = (u_1, u_2, u_3, \varphi)^\top\ :\ \Omega \to \mathbb{R}^4$$

belonging to the spaces $[W_p^1(\Omega_1)]^3$ and $[W_p^1(\Omega)]^4$, respectively, such that

$$\big[\, A^{(1)}(\partial_x)\, u^{(1)}\,\big]_j = 0 \quad \text{in} \quad \Omega_1,\ \ j = 1,2,3, \tag{2.29}$$

$$[\, A(\partial_x)\, U\,]_k = 0 \quad \text{in} \quad \Omega,\ \ k = 1,2,3,4, \tag{2.30}$$

$$r_{S_1} \big\{ [\, T^{(1)}\, u^{(1)}\,]_j \big\}^+ = F_j^{(1)} \quad \text{on} \quad S_1,\ \ j = 1,2,3, \tag{2.31}$$

$$r_S \{ [\, \mathcal{T}\, U\,]_k \}^+ = F_k \quad \text{on} \quad S,\ \ k = 1,2,3,4, \tag{2.32}$$

$$r_\Gamma \{u_k\}^+ = f_k \quad \text{on} \quad \Gamma,\ \ k = 1,2,3,4, \tag{2.33}$$

$$r_{\Gamma_1} \{u_j^{(1)}\}^+ - r_{\Gamma_1} \{u_j\}^+ = g_j \quad \text{on} \quad \Gamma_1,\ \ j = 1,2,3, \tag{2.34}$$

$$r_{\Gamma_1} \big\{ [\, T^{(1)}\, u^{(1)}\,]_j \big\}^+ - r_{\Gamma_1} \{ [\, \mathcal{T} U\,]_j \}^+ = G_j \quad \text{on} \quad \Gamma_1,\ \ j = 1,2,3, \tag{2.35}$$

$$r_{\Gamma_1} \{\varphi\}^+ = g_4 \quad \text{on} \quad \Gamma_1, \tag{2.36}$$

where

$$
\begin{aligned}
&F_k \in B_{p,p}^{-1/p}(S), \qquad f_k \in B_{p,p}^{1/p'}(\Gamma), \qquad g_k \in B_{p,p}^{1/p'}(\Gamma_1), \quad k=1,2,3,4, \\
&F_j^{(1)} \in B_{p,p}^{-1/p}(S_1), \quad G_j \in B_{p,p}^{-1/p}(\Gamma_1), \quad j=1,2,3, \qquad \frac{1}{p'}+\frac{1}{p}=1, \quad 1<p<\infty.
\end{aligned}
\tag{2.37}
$$

Note that the functions $F_j^{(1)}$, F_j, and G_j $(j=1,2,3)$ have to satisfy some compatibility conditions. Namely, for any extension $\widehat{F}_j^{(1)} \in B_{p,p}^{-1/p}(\overline{S}_1 \cup \overline{\Gamma}_1)$ of $F_j^{(1)}$ from S_1 onto $\overline{S}_1 \cup \overline{\Gamma}_1$ and for any extension $\widehat{F}_j \in B_{p,p}^{-1/p}(\overline{S} \cup \overline{\Gamma}_1)$ of F_j from S onto $\overline{S} \cup \overline{\Gamma}_1$, the following inclusions have to be fulfilled

$$
G_j - [\, r_{\Gamma_1} \widehat{F}_j^{(1)} - r_{\Gamma_1} \widehat{F}_j \,] \in \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_1), \quad j=1,2,3. \tag{2.38}
$$

In the classical (continuous) setting these inclusions correspond to the natural compatibility conditions

$$
G_j(x) - [\, F_j^{(1)}(x) - F_j(x) \,] = 0 \text{for all} x \in \partial\Gamma_1, \quad j=1,2,3.
$$

We set

$$
\begin{aligned}
&F = (F_1, F_2, F_3, F_4)^\top \in [\, B_{p,p}^{-1/p}(S) \,]^4, \quad f = (f_1, f_2, f_3, f_4)^\top \in [\, B_{p,p}^{1/p'}(\Gamma) \,]^4, \\
&g = (g_1, g_2, g_3, g_4)^\top \in [\, B_{p,p}^{1/p'}(\Gamma_1) \,]^4, \qquad F^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})^\top \in [\, B_{p,p}^{-1/p}(S_1) \,]^3, \\
&G = (G_1, G_2, G_3)^\top \in [\, B_{p,p}^{-1/p}(\Gamma_1) \,]^3.
\end{aligned}
\tag{2.39}
$$

A pair $(u^{(1)}, U) \in [W_p^1(\Omega_1)]^3 \times [W_p^1(\Omega)]^4$ will be called a solution to the boundary-transmission problem (2.29)-(2.36).

The differential equations (2.29) and (2.30) describe mechanical and electro-mechanical equilibrium state in Ω_1 and Ω without loading and are understood in the distributional sense, in general. We remark that $u^{(1)} \in [W_p^1(\Omega_1)]^3 \cap [C^\infty(\Omega_1)]^3$ and $U \in [W_p^1(\Omega)]^4 \cap [C^\infty(\Omega)]^4$ due to the ellipticity of the corresponding differential operators (in fact, $u^{(1)}$ and U are real analytic vectors in Ω_1 and Ω, respectively).

The Dirichlet-type conditions (2.33), (2.34), and (2.36) involving boundary limiting values of the vectors $u^{(1)}$ and U are understood in the usual trace sense, while the Neumann-type conditions (2.31), (2.32) and (2.35) involving boundary limiting values of the vectors $T^{(1)} u^{(1)}$ and $\mathcal{T} U$ are understood in the functional sense defined by the relations (related to Green's formulae)

$$
\big\langle \{T^{(1)}(\partial, n) u^{(1)}\}^+, \{v^{(1)}\}^+ \big\rangle_{\partial\Omega_1} := \int_\Omega E^{(1)} \big(u^{(1)}, v^{(1)}\big)\, dx \quad \forall\, v^{(1)} \in [W_{p'}^1(\Omega_1)]^3, \tag{2.40}
$$

$$
\big\langle \{\mathcal{T}(\partial, n) U\}^+ ,\, \{V\}^+ \big\rangle_{\partial\Omega} := -\int_\Omega E(U, V) dx \quad \forall\, V \in [W_{p'}^1(\Omega)]^4, \tag{2.41}
$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega_1}$ (respectively $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$) denotes the duality between the spaces $[B^{-1/p}_{p,p}(\partial\Omega_1)]^3$ and $[B^{1/p}_{p',p'}(\partial\Omega_1)]^3$ (respectively $[B^{-1/p}_{p,p}(\partial\Omega)]^4$ and $[B^{1/p}_{p',p'}(\partial\Omega)]^4$) which extends the usual L_2 scalar product.
It can be shown (see, e.g., [Mc1], Ch. 4, Lemma 4.3) that the functionals ("generalized traces") $\{T^{(1)}(\partial,n)u^{(1)}\}^+ \in [B^{-1/p}_{p,p}(\partial\Omega_1)]^3$ and $\{\mathcal{T}(\partial,n)U\}^+ \in [B^{-1/p}_{p,p}(\partial\Omega)]^4$ are correctly determined by the above relations.
Now, we prove the following uniqueness theorem.

THEOREM 2.1 *The homogeneous boundary-transmission problem* (2.29)-(2.36) ($F^{(1)}_j = 0,\ F_k = 0,\ f_k = 0,\ g_k = 0,\ G_j = 0$) *has only the trivial solution in the space* $[W^1_2(\Omega_1)]^3 \times [W^1_2(\Omega)]^4$, *provided* $\operatorname{meas}\Gamma > 0$.

Proof. Let a pair $\big(u^{(1)}, U\big) \in [W^1_2(\Omega_1)]^3 \times [W^1_2(\Omega)]^4$ be a solution to the homogeneous boundary-transmission problem (2.29)-(2.36).
Green's formulae (2.23) and (2.25) with $v^{(1)} = u^{(1)}$ and $V = U$ then imply

$$\int\limits_{\Omega_1} E^{(1)}\big(u^{(1)}, u^{(1)}\big)\,dx = \big\langle\, \{T^{(1)}(\partial,n)u^{(1)}\}^+ \,,\, \{u^{(1)}\}^+ \big\rangle_{\partial\Omega_1}\,, \tag{2.42}$$

$$\int\limits_{\Omega} E(U,U)dx = -\big\langle\, \{\mathcal{T}(\partial,n)U\}^+ \,,\, \{U\}^+ \big\rangle_{\partial\Omega}\,, \tag{2.43}$$

where $u^{(1)} = (u^{(1)}_1, u^{(1)}_2, u^{(1)}_3)^\top$ and $U = (u_1, u_2, u_3, \varphi)^\top = (u, \varphi)^\top$ are real vectors and therefore

$$E^{(1)}\big(u^{(1)}, u^{(1)}\big) = c^{(1)}_{ijlk}\,\partial_i u^{(1)}_j\,\partial_l u^{(1)}_k, \tag{2.44}$$

$$E(U,U) = c_{ijlk}\,\partial_i u_j\,\partial_l u_k + \varepsilon_{pq}\,\partial_p\varphi\,\partial_q\varphi \tag{2.45}$$

due to (2.27) and (2.28).
Taking into account the homogeneous boundary and transmission conditions (2.31)-(2.36) we derive from (2.44) and (2.45)

$$\int\limits_{\Omega_1} E^{(1)}\big(u^{(1)}, u^{(1)}\big)\ dx + \int\limits_{\Omega} E(U,U)\,dx = 0.$$

Due to the inequalities (2.3) and (2.9) we conclude that $E^{(1)}\big(u^{(1)}, u^{(1)}\big) = 0$ and $E(U,U) = 0$, and consequently

$$\begin{aligned}
&\partial_j\,u^{(1)}_i + \partial_i\,u^{(1)}_j = 0 \quad &&\text{in}\quad \Omega_1,\quad i,j = 1,2,3,\\
&\partial_j\,u_i + \partial_i\,u_j = 0 &&\text{in}\quad \Omega,\quad i,j = 1,2,3,\\
&\partial_i\varphi = 0 &&\text{in}\quad \Omega,\quad i = 1,2,3.
\end{aligned}$$

We get by standard arguments

$$u^{(1)} = a^{(1)} \times x + b^{(1)} \quad \text{in} \quad \Omega_1,$$
$$u = a \times x + b, \;\; \varphi = c \quad \text{in} \quad \Omega,$$

where $a^{(1)}, b^{(1)}, a$, and b are arbitrary three-dimensional constant vectors, and c is a scalar constant, the symbol "$\times$" denotes the cross product of vectors in $\mathbb{R}^3$. The homogeneous Dirichlet conditions on Γ yield that $a = b = 0$ and $c = 0$. Thus $U = 0$ in Ω. With the help of the homogeneous transmission conditions (2.34) on Γ_1 we get that $a^{(1)} = b^{(1)} = 0$, whence $u^{(1)} = 0$ in Ω_1 follows. This completes the proof. □
The similar uniqueness theorem for $p \neq 2$ will be proved later in Subsection 4.2 (see Theorem 4.4).

3 Properties of potentials

Here, we establish basic properties of the layer potentials and certain boundary integral (pseudodifferential) operators generated by them. We recall also some necessary information concerning the theory of pseudo-differential equations on manifolds with boundary. These results are crucial to develop the potential method to the boundary-transmission problem (2.29)-(2.36) and prove the corresponding existence and regularity results for solutions in different function spaces.

3.1 Fundamental solutions and integral representations

Denote by $H^{(1)}(\cdot) = [\, H^{(1)}_{kj}(\cdot)\,]_{3\times 3}$ and $H(\cdot) = [\, H_{kj}(\cdot)\,]_{4\times 4}$ the homogeneous (of order -1) fundamental matrix-functions of the differential operators $A^{(1)}(\partial_x)$ and $A(\partial_x)$, respectively (for details, see [Jo1], [Na1], [Ck4], [BG1], [Mc1] and references therein). It is well known that these matrices can be written in the form

$$\begin{aligned} H^{(1)}(x) &= \mathcal{F}^{-1}_{\xi'\to x'}\Big(\pm \frac{1}{2\pi} \int\limits_{\ell^{\pm}} \big[A^{(1)}(i\xi', i\tau)\big]^{-1} e^{-i\tau x_3}\, d\tau \Big) \\ &= -\frac{1}{8\pi^2\,|x|} \int\limits_0^{2\pi} \big[A^{(1)}(\Lambda\,\eta)\big]^{-1}\, d\theta, \end{aligned} \tag{3.1}$$

$$\begin{aligned} H(x) &= \mathcal{F}^{-1}_{\xi'\to x'}\Big(\pm \frac{1}{2\pi} \int\limits_{\ell^{\pm}} (A(i\xi', i\tau))^{-1} e^{-i\tau x_3}\, d\tau \Big) \\ &= -\frac{1}{8\pi^2\,|x|} \int\limits_0^{2\pi} [A(\Lambda\,\eta)]^{-1}\, d\theta, \end{aligned} \tag{3.2}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform, $x=(x_1,x_2,x_3)$, $x'=(x_1,x_2)$, $\xi'=(\xi_1,\xi_2)$, the sign "$-$" corresponds to the case $x_3>0$, and the sign "$+$" to the case $x_3<0$; ℓ^+ (respect. ℓ^-) is a closed simple contour in the half-plane $\Im\tau>0$ (respect. $\Im\tau<0$) orientated counterclockwise (respectively clockwise) and enveloping all the roots of the corresponding polynomials $\det A^{(1)}(i\xi',i\tau)$ and $\det A(i\xi',i\tau)$ with respect to τ with positive (respectively negative) imaginary parts; here $\Lambda=[\Lambda_{kj}]_{3\times 3}$ is an orthogonal matrix associated with x and possessing the property $\Lambda^\top x=(0,0,|x|)^\top$, and $\eta=(\cos\theta,\sin\theta,0)^\top$.
We recall that these matrices are real, have the singularity $O(|x|^{-1})$ in a neighbourhood of the origin and at infinity decay as $O(|x|^{-1})$. Moreover,

$$H^{(1)}(x-y)=H^{(1)}(y-x)=[H^{(1)}(x-y)]^\top,\quad H(x-y)=H(y-x),$$

$$H_{kj}(x-y)=H_{jk}(x-y)\ \text{ for }\ 1\le j,k\le 3,$$

$$H_{4j}(x-y)=-H_{j4}(x-y)\ \text{ for }\ 1\le j\le 3.$$

Note that

$$A^{(1)}(\partial_x)H^{(1)}(x-y)=\delta(x-y)\,I_3,$$

$$A(\partial_x)H(x-y)=\delta(x-y)\,I_4,$$

$$A^*(\partial_x)[H(x-y)]^\top=\delta(x-y)\,I_4,$$

where I_m stands for the $m\times m$ unit matrix, and $\delta(\cdot)$ denotes Dirac's delta function.
With the help of Green's formulae (2.24) and (2.26) we can derive the following general integral representations of arbitrary regular vectors $u^{(1)}\in[C^2(\overline{\Omega}_1)]^3$ and $U\in[C^2(\overline{\Omega})]^4$ by means of surface and Newtonian type potentials

$$\begin{aligned} u^{(1)}(x) &= \int\limits_{\Omega_1} H^{(1)}(x-y)\,A^{(1)}(\partial_y)u^{(1)}(y)\,dy \\ &\quad+\int\limits_{\partial\Omega_1}[\,T^{(1)}(\partial_y,n(y))H^{(1)}(y-x)\,]^\top\,\{\,u^{(1)}(y)\,\}^+\,d_yS \\ &\quad-\int\limits_{\partial\Omega_1} H^{(1)}(x-y)\,\{\,T^{(1)}(\partial_y,n(y))u^{(1)}(y)\,\}^+d_yS,\qquad x\in\Omega_1, \end{aligned} \tag{3.3}$$

$$\begin{aligned} U(x) &= \int\limits_{\Omega} H(x-y)\,A(\partial_y)U\,dy \\ &\quad-\int\limits_{\partial\Omega}[\,\widetilde{\mathcal{T}}(\partial_y,n(y))H^\top(y-x)\,]^\top\,\{\,U(y)\,\}^+\,d_yS \\ &\quad+\int\limits_{\partial\Omega} H(x-y)\,\{\,\mathcal{T}(\partial_y,n(y))U(y)\,\}^+d_yS,\qquad x\in\Omega. \end{aligned} \tag{3.4}$$

Note that due to our agreement the normal vector n to $\partial\Omega_1$ is directed outward, while on $\partial\Omega$ it is directed inward. Moreover, the right-hand side expressions in (3.3) and (3.4) vanish if x belongs to the exterior domains, i.e., $x \in \mathbb{R}^3 \setminus \overline{\Omega}_1$ or $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, respectively.
For Lipschitz domains Ω_1 and Ω these formulae can be extended to the spaces $[W^1(\Omega_1)]^3$ and $[W^1(\Omega)]^4$ with $A^{(1)}(\partial_y)u^{(1)} \in [L_2(\Omega_1)]^3$ and $A(\partial_y)U \in [L_2(\Omega)]^4$ by a standard limiting procedure (for details see, e.g., [LiMa1], [CW1], [Mc1]).
In the next section we will study some properties of these potentials which will afterwards be applied in our analysis.

3.2 Layer potentials

Let us introduce the single and double layer potentials corresponding to the operators $A^{(1)}(\partial_x)$ and $A(\partial_x)$:

$$V^{(1)}(h^{(1)})(x) = \int\limits_{\partial\Omega_1} H^{(1)}(x-y)\, h^{(1)}(y)\, d_yS, \quad x \notin \partial\Omega_1, \tag{3.5}$$

$$W^{(1)}(h^{(1)})(x) = \int\limits_{\partial\Omega_1} [\, T^{(1)}(\partial_y, n(y)) H^{(1)}(y-x)\,]^\top\, h^{(1)}(y)\, d_yS, \quad x \notin \partial\Omega_1, \tag{3.6}$$

$$V(h)(x) = \int\limits_{\partial\Omega} H(x-y)\, h(y)\, d_yS, \quad x \notin \partial\Omega, \tag{3.7}$$

$$W(h)(x) = \int\limits_{\partial\Omega} [\, \widetilde{\mathcal{T}}(\partial_y, n(y)) H^\top(y-x)\,]^\top\, h(y)\, d_yS, \quad x \notin \partial\Omega, \tag{3.8}$$

where $h^{(1)} = (h_1^{(1)}, h_2^{(1)}, h_3^{(1)})^\top$ and $h = (h_1, h_2, h_3, h_4)^\top$ are densities of the potentials.
For the readers convenience, here we collect some results concerning these layer potentials and the corresponding boundary operators needed in subsequent analysis.
We recall that $\partial\Omega,\ \partial\Omega_1 \in C^\infty$.

THEOREM 3.1 [Se1], [DNS1], [DNS2], [NCS1], [BC1] *Let $1 < p < \infty$, $1 \le t \le \infty$, and $s \in \mathbb{R}$. The operators*

$$\begin{aligned} V^{(1)} &: [\, B^s_{p,p}(\partial\Omega_1)\,]^3 \to [\, H_p^{s+1+\frac{1}{p}}(\Omega_1)\,]^3, & W^{(1)} &: [\, B^s_{p,p}(\partial\Omega_1)\,]^3 \to [\, H_p^{s+\frac{1}{p}}(\Omega_1)\,]^3, \\ &: [\, B^s_{p,t}(\partial\Omega_1)\,]^3 \to [\, B_{p,t}^{s+1+\frac{1}{p}}(\Omega_1)\,]^3, & &: [\, B^s_{p,t}(\partial\Omega_1)\,]^3 \to [\, B_{p,t}^{s+\frac{1}{p}}(\Omega_1)\,]^3, \\ V &: [\, B^s_{p,p}(\partial\Omega)\,]^4 \to [\, H_p^{s+1+\frac{1}{p}}(\Omega)\,]^4, & W &: [\, B^s_{p,p}(\partial\Omega)\,]^4 \to [\, H_p^{s+\frac{1}{p}}(\Omega)\,]^4, \\ &: [\, B^s_{p,t}(\partial\Omega)\,]^4 \to [\, B_{p,t}^{s+1+\frac{1}{p}}(\Omega)\,]^4, & &: [\, B^s_{p,t}(\partial\Omega)\,]^4 \to [\, B_{p,t}^{s+\frac{1}{p}}(\Omega)\,]^4 \end{aligned}$$

are continuous.

For the boundary integral (pseudodifferential) operators generated by the layer potentials we will employ the following notation:

$$\mathcal{H}^{(1)}(h^{(1)})(x) := \int\limits_{\partial\Omega_1} H^{(1)}(x-y)\, h^{(1)}(y)\, d_yS, \quad x \in \partial\Omega_1,$$

$$\mathcal{K}^{(1)}(h^{(1)})(x) := \int\limits_{\partial\Omega_1} [\, T^{(1)}(\partial_x, n(x)) H^{(1)}(x-y)\,]\, h^{(1)}(y)\, d_yS, \quad x \in \partial\Omega_1,$$

$$\mathcal{K}^{(1)*}(h^{(1)})(x) := \int\limits_{\partial\Omega_1} [\, T^{(1)}(\partial_y, n(y)) H^{(1)}(y-x)\,]^\top\, h^{(1)}(y)\, d_yS, \quad x \in \partial\Omega_1,$$

$$\mathcal{L}^{(1)}(h^{(1)})(x) := \{\, T^{(1)}(\partial_x, n(x)) W^{(1)}(h^{(1)})(x)\,\}^\pm, \quad x \in \partial\Omega_1,$$

$$\mathcal{H}(h)(x) := \int\limits_{\partial\Omega} H(x-y)\, h(y)\, d_yS, \quad x \in \partial\Omega,$$

$$\mathcal{K}(h)(x) := \int\limits_{\partial\Omega} [\, \mathcal{T}(\partial_x, n(x)) H(x-y)\,]\, h(y)\, d_yS, \quad x \in \partial\Omega$$

$$\widetilde{\mathcal{K}}^*(h)(x) := \int\limits_{\partial\Omega} [\, \widetilde{\mathcal{T}}(\partial_y, n(y)) H^\top(y-x)\,]^\top\, h(y)\, d_yS, \quad x \in \partial\Omega,$$

$$\mathcal{L}(h)(x) := \{\, \mathcal{T}(\partial_x, n(x)) W(h)(x)\,\}^\pm, \quad x \in \partial\Omega.$$

THEOREM 3.2 [DNS1], [DNS2], [NCS1], [BC1] *Let* $1 < p < \infty$, $1 \le t \le \infty$,

$$h^{(1)} \in [\, B_{p,t}^{-\frac{1}{p}}(\partial\Omega_1)\,]^3, \quad g^{(1)} \in [\, B_{p,t}^{1-\frac{1}{p}}(\partial\Omega_1)\,]^3, \quad h \in [\, B_{p,t}^{-\frac{1}{p}}(\partial\Omega)\,]^4, \quad g \in [\, B_{p,t}^{1-\frac{1}{p}}(\partial\Omega)\,]^4.$$

Then

$$\{V^{(1)}(h^{(1)})\}^+ = \{V^{(1)}(h^{(1)})\}^- = \mathcal{H}^{(1)}\, h^{(1)} \;\; on \;\; \partial\Omega_1,$$

$$\{T^{(1)}(\partial, n) V^{(1)}(h^{(1)})\}^\pm = [\, \mp\, 2^{-1} I_3 + \mathcal{K}^{(1)}\,]\, h^{(1)} \;\; on \;\; \partial\Omega_1,$$

$$\{W^{(1)}(g^{(1)})\}^\pm = [\, \pm\, 2^{-1} I_3 + \mathcal{K}^{(1)*}\,]\, g^{(1)} \;\; on \;\; \partial\Omega_1,$$

$$\{V(h)\}^+ = \{V(h)\}^- = \mathcal{H}\, h \;\; on \;\; \partial\Omega,$$

$$\{\mathcal{T}(\partial, n) V(h)\}^\pm = [\, \pm\, 2^{-1} I_4 + \mathcal{K}\,]\, h, \;\; on \;\; \partial\Omega,$$

$$\{W(g)\}^\pm = [\, \mp\, 2^{-1} I_4 + \widetilde{\mathcal{K}}^*\,]\, g \;\; on \;\; \partial\Omega,$$

where I_m *stands for the* $m \times m$ *unit matrix.*

Note that the boundary operators in classical linear elasticity $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(1)*}$ are formally mutually adjoint singular integral operators of normal type with zero indices, while $\mathcal{H}^{(1)}$ and $\mathcal{L}^{(1)}$ are formally self-adjoint pseudodifferential elliptic operators (with positive definite principal symbol matrices) of order -1 and 1, respectively (for details see [Na1], [Ck4], [DNS1], [DNS2], [CW1], [NCS1]).
In contrast to the classical case, the layer boundary operators in the piezoelastic case $\mathcal{H}$ and $\mathcal{L}$ are strongly elliptic, but not formally self-adjoint, pseudodifferential operators (with strongly elliptic principal symbol matrices) of order -1 and 1. The operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}^*$ are singular integral operators of normal type with zero indices, but they are not formally mutually adjoint(for details see [BG1], [BC1], [BCD1]). It can easily be shown that the corresponding adjoint operators $\mathcal{K}^*$ and $\widetilde{\mathcal{K}}$ are generated by the potentials related to the adjoint operator $A^*(\partial_x)$:

$$\widetilde{V}(h)(x) = \int\limits_{\partial\Omega_1} H^\top(x-y)\, h(y)\, d_yS,$$

$$\widetilde{W}(h)(x) = \int\limits_{\partial\Omega} [\, \mathcal{T}(\partial_y, n(y)) H(y-x)\,]^\top\, h(y)\, d_yS;$$

in particular,

$$\mathcal{K}^*(h)(x) := \int\limits_{\partial\Omega} [\, \mathcal{T}(\partial_y, n(y)) H(y-x)\,]^\top\, h(y)\, d_yS, \quad x \in \partial\Omega,$$

$$\widetilde{\mathcal{K}}(h)(x) := \int\limits_{\partial\Omega} [\, \widetilde{\mathcal{T}}(\partial_x, n(x)) H^\top(x-y)\,]\, h(y)\, d_yS, \quad x \in \partial\Omega.$$

The operators $\mathcal{L}^{(1)}$ and $\mathcal{L}$ are well defined and have the following properties.

LEMMA 3.3 *Let $1 < p < \infty$, $1 \le t \le \infty$, and*

$$h^{(1)} \in [\, B_{p,t}^{1-\frac{1}{p}}(\partial\Omega_1)\,]^3, \quad h \in [\, B_{p,t}^{1-\frac{1}{p}}(\partial\Omega)\,]^4.$$

Then

$$\{\, T^{(1)}(\partial, n) W^{(1)}(h^{(1)})\,\}^+ = \{\, T^{(1)}(\partial, n) W^{(1)}(h^{(1)})\,\}^- \quad \text{on } \partial\Omega_1$$

and

$$\{\, \mathcal{T}(\partial, n) W(h)\,\}^+ = \{\, \mathcal{T}(\partial, n) W(h)\,\}^- \quad \text{on } \partial\Omega.$$

Proof. We prove the second relation.
Let $W(x) := W(h)(x)$ be a double layer potential with sufficiently smooth density h. By the integral representation formulae in the domains Ω and $\mathbb{R}^3 \setminus \overline{\Omega}$ we have:

$$
\begin{aligned}
-\int\limits_{\partial\Omega} [\,\widetilde{\mathcal{T}}(\partial_y, n(y)) H^\top(y-x)\,]^\top \; \{W(y)\}^+ \; d_yS \qquad\qquad & \\
+\int\limits_{\partial\Omega} H(x-y)\,\{\,\mathcal{T}(\partial_y, n(y)) W(y)\,\}^+ d_yS \; &= \; \begin{cases} W(x), & x\in\Omega, \\ 0, & x\in\mathbb{R}^3\setminus\overline{\Omega}, \end{cases} \\
\int\limits_{\partial\Omega} [\,\widetilde{\mathcal{T}}(\partial_y, n(y)) H^\top(y-x)\,]^\top \; \{W(y)\}^- \; d_yS \qquad\qquad & \\
-\int\limits_{\partial\Omega} H(x-y)\;\{\,\mathcal{T}(\partial_y, n(y)) W(y)\,\}^- \, d_yS \; &= \; \begin{cases} 0, & x\in\Omega, \\ W(x), & x\in\mathbb{R}^3\setminus\overline{\Omega}. \end{cases}
\end{aligned}
$$

By adding termwise these equalities and applying the jump relations for the double layer potential $W(x) := W(h)(x)$ we get

$$
\begin{aligned}
W(x) \; &= \; \int\limits_{\partial\Omega} H(x-y)\,[\{\,\mathcal{T}(\partial_y, n(y)) W(y)\,\}^+ - \{\,\mathcal{T}(\partial_y, n(y)) W(y)\,\}^-]\,dS \\
&\quad + \int\limits_{\partial\Omega} [\,\widetilde{\mathcal{T}}(\partial_y, n(y)) H^\top(y-x)\,]^\top \, h(y)\,dS, \quad x\in\Omega\cup[\,\mathbb{R}^3\setminus\overline{\Omega}\,].
\end{aligned}
$$

By (3.8)

$$
\int\limits_{\partial\Omega} H(x-y)\;\big[\{\,\mathcal{T}(\partial_y, n(y)) W(y)\,\}^+ - \{\,\mathcal{T}(\partial_y, n(y)) W(y)\,\}^-\big]\;dS = 0, \quad x\in\Omega\cup[\,\mathbb{R}^3\setminus\overline{\Omega}\,]\,,
$$

which shows that the single layer potential $V(g)$ with the density $g := \{\,\mathcal{T}(\partial_y, n(y)) W(y)\,\}^+ - \{\,\mathcal{T}(\partial_y, n(y)) W(y)\,\}^-$ vanishes in Ω and $\mathbb{R}^3\setminus\overline{\Omega}$. Therefore $\{\mathcal{T}V(g)\}^+ = 0$ and $\{\mathcal{T}V(g)\}^- = 0$. Then due to the jump relation for the single layer potential $\{\mathcal{T}V(g)\}^+ - \{\mathcal{T}V(g)\}^- = g$ (see Theorem 3.2) it follows that $g = 0$. Thus the theorem holds for smooth densities.
By standard limiting and duality arguments this result can be extended to the Bessel potential and Besov spaces. □
The following mapping properties of the above introduced boundary operators are well known.

THEOREM 3.4 [Se1], [DNS1], [DNS2], [BC1], [Na1] *Let $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbb{R}$. The operators*

$$\begin{aligned}
\mathcal{H}^{(1)} \ &: \ [\, H_p^s(\partial\Omega_1)\,]^3 \to [\, H_p^{s+1}(\partial\Omega_1)\,]^3, \\
&: \ [\, B_{p,t}^s(\partial\Omega_1)\,]^3 \to [\, B_{p,t}^{s+1}(\partial\Omega_1)\,]^3, \\
\mathcal{K}^{(1)}, \mathcal{K}^{(1)*} \ &: \ [\, H_p^s(\partial\Omega_1)\,]^3 \to [\, H_p^s(\partial\Omega_1)\,]^3, \\
&: \ [\, B_{p,t}^s(\partial\Omega_1)\,]^3 \to [\, B_{p,t}^s(\partial\Omega_1)\,]^3, \\
\mathcal{L}^{(1)} \ &: \ [\, H_p^{s+1}(\partial\Omega_1)\,]^3 \to [\, H_p^s(\partial\Omega_1)\,]^3, \\
&: \ [\, B_{p,t}^{s+1}(\partial\Omega_1)\,]^3 \to [\, B_{p,t}^s(\partial\Omega_1)\,]^3, \\
\mathcal{H} \ &: \ [\, H_p^s(\partial\Omega)\,]^4 \to [\, H_p^{s+1}(\partial\Omega)\,]^4, \\
&: \ [\, B_{p,t}^s(\partial\Omega)\,]^4 \to [\, B_{p,t}^{s+1}(\partial\Omega)\,]^4, \\
\mathcal{K}, \mathcal{K}^*, \widetilde{\mathcal{K}}^*, \widetilde{\mathcal{K}} \ &: \ [\, H_p^s(\partial\Omega)\,]^4 \to [\, H_p^s(\partial\Omega)\,]^4, \\
&: \ [\, B_{p,t}^s(\partial\Omega)\,]^4 \to [\, B_{p,t}^s(\partial\Omega)\,]^4, \\
\mathcal{L} \ &: \ [\, H_p^{s+1}(\partial\Omega)\,]^4 \to [\, H_p^s(\partial\Omega)\,]^4, \\
&: \ [\, B_{p,t}^{s+1}(\partial\Omega)\,]^4 \to [\, B_{p,t}^s(\partial\Omega)\,]^4,
\end{aligned}$$

are continuous.
Moreover, the following operator equalities hold in appropriate function spaces:

$$\mathcal{K}^{(1)*}\,\mathcal{H}^{(1)} = \mathcal{H}^{(1)}\,\mathcal{K}^{(1)}, \quad \mathcal{L}^{(1)}\,\mathcal{H}^{(1)} = -4^{-1}I_3 + [\,\mathcal{K}^{(1)}\,]^2, \quad \mathcal{H}^{(1)}\,\mathcal{L}^{(1)} = -4^{-1}I_3 + [\,\mathcal{K}^{(1)*}\,]^2,$$

$$\widetilde{\mathcal{K}}^*\,\mathcal{H} = \mathcal{H}\,\mathcal{K}, \qquad \mathcal{L}\,\mathcal{H} = -4^{-1}I_4 + [\,\mathcal{K}\,]^2, \qquad \mathcal{H}\,\mathcal{L} = -4^{-1}I_4 + [\,\widetilde{\mathcal{K}}^*\,]^2.$$

The operators $\mathcal{H}^{(1)}$ and $\mathcal{H}$ possess the following coercivity properties.

THEOREM 3.5 [DL1], [CW1], [DNS1], [DNS2] *There is a positive constant c_1 such that*

$$\big\langle -\mathcal{H}^{(1)}h^{(1)}\,,\, h^{(1)}\big\rangle_{\partial\Omega_1} \ge c_1\,||\,h^{(1)}\,||^2_{[\,H_2^{-1/2}(\partial\Omega_1)\,]^3}$$

for any complex vector-function $h^{(1)} \in [\, H_2^{-1/2}(\partial\Omega_1)\,]^3$.
The operators

$$\begin{aligned}
\mathcal{H}^{(1)} \ &: \ [\, H_p^s(\partial\Omega_1)\,]^3 \to [\, H_p^{s+1}(\partial\Omega_1)\,]^3 \\
&: \ [\, B_{p,t}^s(\partial\Omega_1)\,]^3 \to [\, B_{p,t}^{s+1}(\partial\Omega_1)\,]^3
\end{aligned}$$

are invertible for all $1 < p < \infty$, $1 \le t \le \infty$, and $s \in \mathbb{R}$.
A solution to the equation $A^{(1)}(\partial_x)\,u^{(1)} = 0$ in Ω_1 either from $[\, W_p^1(\Omega_1)\,]^3$ or from $[\, B_{p,t}^1(\Omega_1)\,]^4$ with $1 < p < \infty$ and $1 \le t \le \infty$ can be represented in the form

$$u^{(1)}(x) = V^{(1)}\left([\,\mathcal{H}^{(1)}\,]^{-1}[\,u^{(1)}\,]^+\right)(x), \quad x \in \Omega_1. \tag{3.9}$$

THEOREM 3.6 *There is a positive constant c_2 such that*

$$\Re \, \langle -\mathcal{H}h \, , \, h \rangle_{\partial\Omega} \geq c_2 \, || \, h \, ||^2_{\left[H_2^{-1/2}(\partial\Omega) \right]^4}$$

for any complex vector-function $h \in [\, H_2^{-1/2}(\partial\Omega) \,]^4$.
The operators

$$\begin{aligned} \mathcal{H} \ &: \ [\, H_p^s(\partial\Omega) \,]^4 \to [\, H_p^{s+1}(\partial\Omega) \,]^4 \\ &: \ [\, B_{p,t}^s(\partial\Omega) \,]^4 \to [\, B_{p,t}^{s+1}(\partial\Omega) \,]^4 \end{aligned}$$

are invertible for all $1 < p < \infty$, $1 \leq t \leq \infty$, and $s \in \mathbb{R}$.
A solution to the equation $A(\partial_x)\, U = 0$ in Ω either from $[\, W_p^1(\Omega) \,]^4$ or from $[\, B_{p,t}^1(\Omega) \,]^4$ with $1 < p < \infty$ and $1 \leq t \leq \infty$ can be represented in the form

$$U(x) = V\left([\, \mathcal{H} \,]^{-1} [\, U \,]^+ \right)(x), \quad x \in \Omega. \tag{3.10}$$

Proof. Let $U = (u, \varphi)^\top = V(h)$ be a single layer potential with $h \in [\, H_2^{-1/2}(\partial\Omega) \,]^4$. Clearly, then

$$U = V(h) \in [\, H^1_{2,loc}(\mathbb{R}^3) \,]^4 \cap [C^\infty(\Omega)]^4 \cap [C^\infty(\Omega^-)]^4$$

with $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$. Moreover, U belongs to the Beppo-Levi space in $\mathbb{R}^3$ (for details see, e.g., [DL1], Ch. 11, Part B),

$$U \in [BL(\mathbb{R}^3)]^4 := \left\{ W \ : \ \frac{1}{(1+|x|^2)^{1/2}} \, W \in [L_2(\mathbb{R}^3)]^4, \ \ \partial_j W \in [L_2(\mathbb{R}^3)]^4, \ \ j = 1, 2, 3 \right\},$$

where the norm is defined as

$$|| \, W \, ||^2_{[BL(\mathbb{R}^3)]^4} := || \, (1+|x|^2)^{-1/2} W \, ||^2_{[L_2(\mathbb{R}^3)]^4} + \sum_{k=1}^{3} || \, \partial_k W \, ||^2_{[L_2(\mathbb{R}^3)]^4}.$$

The Beppo-Levi space for an arbitrary unbounded domain of type Ω^- is defined similarly. Note that the semi-norm given by the second summand in the right-hand side of the last equality is indeed an equivalent norm in the space $[BL(\mathbb{R}^3)]^4$ (see [DL1], Ch. 11, Part B, §1, Theorem 1).
With the help of relations (2.9), (2.28) and the Lax-Milgram theorem we can show that the interior and exterior Dirichlet problems in $\Omega^+ := \Omega$ and $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega}$ are uniquely solvable in the spaces $[H_2^1(\Omega^+)]^4$ and $[BL(\Omega^-)]^4$. Furthermore, the linear mapping $\{U\}^\pm \mapsto \{\mathcal{T}U\}^\pm$ is continuous from $[H_2^{\frac{1}{2}}(\partial\Omega)]^4$ into $[H_2^{-\frac{1}{2}}(\partial\Omega)]^4$, i.e., there exists a positive number c such that

$$\|\{\mathcal{T}U\}^\pm\|_{[H_2^{-1/2}(\partial\Omega)]^4} \leq c \, \|\{U\}^\pm\|_{[H_2^{1/2}(\partial\Omega)]^4}.$$

Taking into account the decay property at infinity of the single layer potential $U = V(h)$ we can write Green's formulae for the domains $\Omega^+ := \Omega$ and Ω^-

$$\int\limits_{\Omega^\pm} E(U,U)\,dx = \mp \int\limits_{\partial\Omega} \{\mathcal{T}(\partial,n)U\}^\pm \cdot \{U\}^\pm\,dS, \tag{3.11}$$

where $\partial\Omega^\pm = \partial\Omega$ and due to our agreement the normal vector n on $\partial\Omega$ is directed inward. In view of the jump relations of the single layer potential $U := V(h)$ we easily get from Green's formulae,

$$\begin{aligned}
\Re\,\langle -\mathcal{H}\,h\,,\,h\rangle_{\partial\Omega} &= \Re \int\limits_{\partial\Omega} \big[\,\{U\}^+\cdot\{\mathcal{T}(\partial,n)U\}^+ - \{U\}^-\cdot\{\mathcal{T}(\partial,n)U\}^-\,\big]\;dS \\
&= \Re \int\limits_{\mathbb{R}^3} E(U,U)\,dx \\
&= \Re \int\limits_{\mathbb{R}^3} \big[\,c_{ijlk}\,\partial_i u_j\,\overline{\partial_l u_k} + e_{pjq}\,\partial_p\varphi\,\overline{\partial_q u_j} - e_{pjq}\,\partial_q u_j\,\overline{\partial_p\varphi} + \varepsilon_{pq}\,\partial_p\varphi\,\overline{\partial_q\varphi}\,\big]\;dx \\
&\qquad \{\text{due to relations (2.9) and (2.28)}\} \\
&\geq \delta_3 \int\limits_{\mathbb{R}^3} \sum_{l,k=1}^{3} |s_{lk}(u)|^2\,dx + \delta_4 \int\limits_{\mathbb{R}^3} \sum_{l=1}^{3} |\partial_l\varphi|^2\,dx \\
&\geq \delta_6 \int\limits_{\mathbb{R}^3} \Big[\,\sum_{l,k=1}^{3} |\partial_l u_k|^2 + \sum_{l=1}^{3} |\partial_l\varphi|^2\Big]\,dx \\
&\qquad \{\text{due to Korn's inequality ([KO1], §2, Theorem 3)}\} \\
&\geq \delta_7 \Big[\sum_{k=1}^{3} \big\|(1+|x|^2)^{-1/2}u_k\big\|^2_{L_2(\mathbb{R}^3)} + \sum_{l,k=1}^{3} \|\partial_l u_k\|^2_{L_2(\mathbb{R}^3)} \\
&\qquad + \big\|(1+|x|^2)^{-1/2}\varphi\big\|^2_{L_2(\mathbb{R}^3)} + \sum_{l=1}^{3} \|\partial_l\varphi\|^2_{L_2(\mathbb{R}^3)}\;dx\Big] \\
&\qquad \{\text{due to the equivalence of the norms in Beppo-Levi spaces }\} \\
&\geq \delta_8 \,\big\|\{U\}^\pm\big\|^2_{[H_2^{1/2}(\partial\Omega)]^4} \\
&\qquad \{\text{due to the trace theorem}\} \\
&\geq \delta_9\;\Big[\big\|\{\mathcal{T}U\}^+\big\|^2_{[H_2^{-1/2}(\partial\Omega)]^4} + \big\|\{\mathcal{T}U\}^-\big\|^2_{[H_2^{-1/2}(\partial\Omega)]^4}\Big] \\
&\qquad \{\text{due to the continuity of the mapping } \{U\}^\pm \mapsto \{\mathcal{T}U\}^\pm\} \\
&\geq \delta_{10}\,\big\|\{\mathcal{T}U\}^+ - \{\mathcal{T}U\}^-\big\|^2_{[H_2^{-1/2}(\partial\Omega)]^4} \\
&\geq \delta_{11}\,\|h\|^2_{[H_2^{-1/2}(\partial\Omega)]^4} \\
&\qquad \{\text{due to Theorem 3.2}\}.
\end{aligned}$$

This implies that the operator

$$\mathcal{H} \;:\; [\, H_2^{-1/2}(\partial\Omega)\,]^4 \to [\, H_2^{1/2}(\partial\Omega)\,]^4$$

is invertible. Then by standard well known arguments from the general theory of pseudodifferential operators on manifolds without boundary we conclude that the mappings

$$\begin{aligned}\mathcal{H} \;&:\; \left[H_p^s(\partial\Omega)\right]^4 \to \left[H_p^{s+1}(\partial\Omega)\right]^4 \\ &:\; \left[B_{p,t}^s(\partial\Omega)\right]^4 \to \left[B_{p,t}^{s+1}(\partial\Omega)\right]^4\end{aligned}$$

are invertible for all $1 < p < \infty$, $1 \le t \le \infty$, and $s \in \mathbb{R}$.
The representability of a solution $U \in [\, W_p^1(\Omega)\,]^4 \cap [\, C^\infty(\Omega)\,]^4$ to the homogeneous equation in the form of a single-layer potential can be shown as follows.
Theorem 3.1 and the invertibility of the operator

$$\mathcal{H} \;:\; [\, B_{p,p}^{-1+\frac{1}{p}}(\partial\Omega)\,]^4 \to [\, B_{p,p}^{\frac{1}{p}}(\partial\Omega)\,]^4$$

yield that the vector $U^*(x) = V\,([\,\mathcal{H}\,]^{-1}\{\,U\,\}^+)\,(x) \in [\, W_p^1(\Omega)\,]^4$ solves the Dirichlet problem: $A(\partial_x)\,U^* = 0$ in Ω and $\{\,U^*\,\}^+ = \{\,U\,\}^+$ on $\partial\Omega$. Therefore the difference $\widetilde{U} := U - U^* \in [\, W_p^1(\Omega)\,]^4$ is a solution to the homogeneous Dirichlet problem in Ω and due to the general integral representation formula we have

$$\widetilde{U}(x) = V(\{\mathcal{T}\widetilde{U}\}^+)(x), \quad x \in \Omega.$$

Since $\{\widetilde{U}\}_{\partial\Omega}^+ = 0$, we get $\mathcal{H}\{\mathcal{T}\widetilde{U}\}^+ = 0$ on $\partial\Omega$ by Theorem 3.2. In accordance with the above shown invertibility property of the operator $\mathcal{H}$ the equality $\{\mathcal{T}\widetilde{U}\}^+ = 0$ on $\partial\Omega$ follows, and consequently $U = U^*$ in Ω. This completes the proof for the space $[\, W_p^1(\Omega)\,]^4$. The case of the space $[\, B_{p,t}^1(\Omega)\,]^4$ can be considered quite similarly. □

From Theorems 3.5 and 3.6 follows

THEOREM 3.7 *Let $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbb{R}$, and let*

$$\begin{aligned}\mathcal{B}^{(1)} \;&:=\; I_3 + [\, -\, 2^{-1}I_3 + \mathcal{K}^{(1)}\,]\,[\,\mathcal{H}^{(1)}\,]^{-1}, \\ \mathcal{B} \;&:=\; I_4 - [\, 2^{-1}I_4 + \mathcal{K}\,]\,[\,\mathcal{H}\,]^{-1}.\end{aligned} \tag{3.12}$$

The operators

$$\begin{aligned}\mathcal{B}^{(1)} \;&:\; [\, H_p^s(\partial\Omega_1)\,]^3 \to [\, H_p^{s-1}(\partial\Omega_1)\,]^3 \\ &:\; [\, B_{p,t}^s(\partial\Omega_1)\,]^3 \to [\, B_{p,t}^{s-1}(\partial\Omega_1)\,]^3 \\ \mathcal{B} \;&:\; [\, H_p^s(\partial\Omega)\,]^4 \to [\, H_p^{s-1}(\partial\Omega)\,]^4 \\ &:\; [\, B_{p,t}^s(\partial\Omega)\,]^4 \to [\, B_{p,t}^{s-1}(\partial\Omega_1)\,]^3\end{aligned} \tag{3.13}$$

are invertible.

Moreover, there are positive constants c_3 and c_4 such that for any complex vector-functions $h^{(1)} \in [\, H_2^{1/2}(\partial\Omega_1)\,]^3$ and $h \in [\, H_2^{1/2}(\partial\Omega)\,]^4$ the inequalities

$$\begin{aligned} &\big\langle\, \mathcal{B}^{(1)} h^{(1)} \,,\, h^{(1)} \big\rangle_{\partial\Omega_1} \geq c_3 \,||\, h^{(1)} \,||^2_{[\, H_2^{1/2}(\partial\Omega_1)\,]^3}\,, \\ &\Re\, \langle\, \mathcal{B} h \,,\, h \rangle_{\partial\Omega} \geq c_4 \,||\, h \,||^2_{[\, H_2^{1/2}(\partial\Omega)\,]^4} \end{aligned} \tag{3.14}$$

hold.

Proof. We will prove the theorem for the operator $\mathcal{B}$.
Applying Green's formula (3.11) to the vector $U := V([\mathcal{H}]^{-1}(h))$ in the domain Ω by standard arguments applied in the proof of Theorem 3.5 we can easily show that for the Steklov-Poincaré operator

$$\mathcal{A} := -\,\big[\, 2^{-1} I_4 + \mathcal{K} \,\big]\; [\, \mathcal{H}\,]^{-1}$$

the inequalities

$$\begin{aligned} &\Re\, \langle\, \mathcal{A} h \,,\, h \rangle_{\partial\Omega} \geq 0, \\ &\Re\, \langle\, \mathcal{A} h \,,\, h \rangle_{\partial\Omega} \geq c' \,||\, h \,||^2_{[\, H_2^{1/2}(\partial\Omega)\,]^4} - c'' \,||\, h \,||^2_{[\, H_2^{0}(\partial\Omega)\,]^4} \end{aligned}$$

hold for all $h \in [\, H_2^{1/2}(\partial\Omega)\,]^4$. The equality in the first relation holds only for $h = (a \times x + b, b_4)^\top$, where $a = (a_1, a_2, a_3)^\top$ and $b = (b_1, b_2, b_3)^\top$ are arbitrary constant vectors and b_4 is an arbitrary scalar constant. Here c' and c'' are some positive constants.
Therefore, by (3.12) we derive

$$\begin{aligned} &\Re\, \langle\, \mathcal{B} h \,,\, h \rangle_{\partial\Omega} \geq ||\, h \,||^2_{[\, H_2^{0}(\partial\Omega)\,]^4} > 0 \qquad \text{for} \quad h \neq 0, \\ &\Re\, \langle\, \mathcal{B} h \,,\, h \rangle_{\partial\Omega} \geq c' \,||\, h \,||^2_{[\, H_2^{1/2}(\partial\Omega)\,]^4} - (c'' - 1) \,||\, h \,||^2_{[\, H_2^{0}(\partial\Omega)\,]^4} \qquad \forall\, h \in [\, H_2^{1/2}(\partial\Omega)\,]^4. \end{aligned}$$

The second inequality (coercivity property) implies that the operator

$$\mathcal{B} \;:\; [\, H_2^{1/2}(\partial\Omega)\,]^4 \to [\, H_2^{-1/2}(\partial\Omega)\,]^4 \tag{3.15}$$

is Fredholm with zero index (see, e.g., [Mc1], Theorem 2.34), while the first one shows that the null space of the operator (3.15) is trivial. Therefore (3.15) (i.e. the operators (3.13) with $s = 1/2$, $p = 2$, and $q = 2$) is invertible. Moreover, it is evident that $\mathcal{B}$ is a strongly elliptic pseudo-differential operator of order $+1$.
From the theory of elliptic pseudo-differential operators on compact manifolds without boundary it follows that the operators (3.13) are invertible for all $p \in (1, \infty)$, $t \in [1, \infty]$, and $s \in \mathbb{R}$ (see, e.g., [Hor1], Ch. 19, [DNS3], Ch. 5).
To show the inequality (3.14) we proceed as follows. We introduce the operator

$$2\, B := \mathcal{B} + \mathcal{B}^*,$$

where $\mathcal{B}^*$ is adjoint to (3.15).

Clearly, B has the mapping property

$$B \;:\; [\, H_2^{1/2}(\partial\Omega)\,]^4 \to [\, H_2^{-1/2}(\partial\Omega)\,]^4 \tag{3.16}$$

and, moreover, it is self-adjoint, positive $\langle\, B\, h\,,\, h\rangle_{\partial\Omega} > 0$ for $h \neq 0$, and coercive

$$\langle\, B\, h\,,\, h\rangle_{\partial\Omega} \geq c'\, ||\, h\, ||^2_{\left[\, H_2^{1/2}(\partial\Omega)\,\right]^4} - (c''-1)\, ||\, h\, ||^2_{\left[\, H_2^{0}(\partial\Omega)\,\right]^4} \qquad \forall\, h \in [\, H_2^{1/2}(\partial\Omega)\,]^4$$

in accordance with the equality $\Re\, \langle\, \mathcal{B} h\,,\, h\rangle_{\partial\Omega} = \langle\, B\, h\,,\, h\rangle_{\partial\Omega}$.
Therefore, the operator (3.16) is invertible and the inverse operator

$$B^{-1} \;:\; [\, H_2^{-1/2}(\partial\Omega)\,]^4 \to [\, H_2^{1/2}(\partial\Omega)\,]^4 \tag{3.17}$$

is positive as well.
For arbitrary complex number $\lambda \in \mathbb{C}$ and non-zero vectors $h \in [\, H_2^{1/2}(\partial\Omega)\,]^4$ and $f \in [\, H_2^{-1/2}(\partial\Omega)\,]^4$ there holds the inequality

$$\begin{aligned} 0 \;&\leq\; \langle\, B^{-1}\, (f - \lambda\, B\, h)\,,\, f - \lambda\, B\, h\,\rangle_{\partial\Omega} \\ &=\; \langle\, B^{-1}\, f - \lambda\, h\,,\, f - \lambda\, B\, h\,\rangle_{\partial\Omega} \\ &=\; \langle\, B^{-1}\, f\,,\, f\,\rangle_{\partial\Omega} - \lambda \langle\, h\,,\, f\,\rangle_{\partial\Omega} - \overline{\lambda}\, \langle\, f\,,\, h\,\rangle_{\partial\Omega} + |\lambda|^2\, \langle\, h\,,\, B\, h\rangle_{\partial\Omega}. \end{aligned}$$

Substituting here

$$\lambda = \frac{\langle\, f\,,\, h\,\rangle_{\partial\Omega}}{\langle\, h\,,\, B\, h\rangle_{\partial\Omega}}$$

we arrive at the inequality

$$|\, \langle\, h\,,\, f\,\rangle_{\partial\Omega}\, |^2 \leq \langle\, B^{-1}\, f\,,\, f\,\rangle_{\partial\Omega}\, \langle\, B\, h\,,\, h\rangle_{\partial\Omega}$$

for arbitrary $h \in [\, H_2^{1/2}(\partial\Omega)\,]^4$ and $f \in [\, H_2^{-1/2}(\partial\Omega)\,]^4$.
This inequality yields

$$|\, \langle\, h\,,\, f\,\rangle_{\partial\Omega}\, |^2 \leq \begin{cases} c^*\, \langle\, B\, h\,,\, h\rangle_{\partial\Omega}\, \|\, f\, \|^2_{\left[\, H_2^{-1/2}(\partial\Omega)\,\right]^4}, \\ c^{**}\, \langle\, B^{-1}\, f\,,\, f\rangle_{\partial\Omega}\, \|\, h\, \|^2_{\left[\, H_2^{1/2}(\partial\Omega)\,\right]^4}, \end{cases}$$

with some positive numbers c^* and c^{**} due to the boundedness of the operators (3.16) and (3.17).
From the above inequalities we get

$$\|\, h\, \|^2_{\left[\, H_2^{1/2}(\partial\Omega)\,\right]^4} \leq c^*\, \langle\, B\, h\,,\, h\rangle_{\partial\Omega}, \qquad \|\, f\, \|^2_{\left[\, H_2^{-1/2}(\partial\Omega)\,\right]^4} \leq c^{**}\, \langle\, B^{-1}\, f\,,\, f\rangle_{\partial\Omega},$$

in view of the definition of norms in the corresponding function spaces under consideration.
This completes the proof. □

3.3 Some results for pseudodifferential equations on manifolds with boundary

In this subsection we shall present some principal results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for proving existence theorems for mixed boundary-transmission problems by the potential method. In particular, in our investigation we need some results describing the Fredholm properties of pseudo-differential operators on a compact manifold with boundary. They can be found in [Esk1], [Grb1], [Sh1].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, nonselfintersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\sigma_{\mathcal{A}}(x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}},\ \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \cdots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\sigma_{\mathcal{A}}(x,0,\cdots,0,+1)\,]^{-1}[\,\sigma_{\mathcal{A}}(x,0,\cdots,0,-1)\,], \quad x \in \partial\overline{\mathcal{M}},$$

and introduce the notation

$$\delta_j(x) = \Re\,[\,(2\pi\, i)^{-1} \ln \lambda_j(x)\,],\ \ j = 1, \cdots, N.$$

Here the branch in the logarithmic function $\ln\tau$ is chosen with regard to the inequality $-\pi < \arg\tau \le \pi,\ j = 1, \cdots, N$. Due to the strong ellipticity of $\mathcal{A}$ we have the strong inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$.

Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system.

Remark that in the particular case when $\sigma_{\mathcal{A}}(x,\xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ we have

$$\delta_j(x) = 0 \quad \text{for} \quad j = 1, \cdots, N,$$

since all the eigenvalues $\lambda_j(x)$ ($j = \overline{1,N}$) are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators are characterized by the following theorem.

THEOREM 3.8 *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, i.e., there is a positive constant c_0 such that*

$$\Re\ \sigma_{\mathcal{A}}(x,\xi)\zeta \cdot \zeta \ge c_0\,|\zeta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\zeta \in \mathbb{C}^m$. Then

$$\mathcal{A} \ :\ \widetilde{H}^s_p(\mathcal{M}) \to H^{s-\nu}_p(\mathcal{M}), \tag{3.18}$$

$$:\ \widetilde{B}^s_{p,q}(\mathcal{M}) \to B^{s-\nu}_{p,q}(\mathcal{M}), \tag{3.19}$$

are Fredholm operators with index zero if

$$\frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\,1\le j\le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\,1\le j\le N} \delta_j(x). \tag{3.20}$$

Moreover, the null-spaces and indices of the operators (3.18) *and* (3.19) *are the same (for all values of the parameter $q \in [1, +\infty]$) provided p and s satisfy the inequality* (3.20).

We will essentially use this theorem in the next subsection to prove the existence and regularity results for our boundary-transmission problem (2.29)-(2.36) .

3.4 Some remarks concerning Lipschitz domains

If the domains Ω_1 and Ω are Lipschitz (see, e.g., [Ne1], [Mc1]), then some of the above formulated mapping properties of the layer potentials and corresponding boundary integral operators still hold (cf. [Co1], [Mc1], [Ag1]).
In particular, Theorem 3.1 is true with $p = t = 2$, and $s = -1/2$ for the single layer potential and $s = 1/2$ for the double layer potential; Theorem 3.2 and Lemma 3.3 hold with $p = t = 2$.
The boundary operators $\mathcal{H}^{(1)}$, $\mathcal{K}^{(1)}$, $\mathcal{K}^{(1)*}$, $\mathcal{L}^{(1)}$, $\mathcal{H}$, $\mathcal{K}$, $\widetilde{\mathcal{K}}$, $\mathcal{K}^*$, $\widetilde{\mathcal{K}}^*$, and $\mathcal{L}$ have the following mapping properties (cf. Theorem 3.4)

$$\mathcal{H}^{(1)} \quad : \quad [\,H_2^{-1/2}(\partial\Omega_1)\,]^3 \to [\,H_2^{1/2}(\partial\Omega_1)\,]^3, \tag{3.21}$$

$$\mathcal{K}^{(1)*} \quad : \quad [\,H_2^{1/2}(\partial\Omega_1)\,]^3 \to [\,H_2^{1/2}(\partial\Omega_1)\,]^3,$$

$$\mathcal{K}^{(1)} \quad : \quad [\,H_2^{-1/2}(\partial\Omega_1)\,]^3 \to [\,H_2^{-1/2}(\partial\Omega_1)\,]^3,$$

$$\mathcal{L}^{(1)} \quad : \quad [\,H_2^{1/2}(\partial\Omega_1)\,]^3 \to [\,H_2^{-1/2}(\partial\Omega_1)\,]^3,$$

$$\mathcal{H} \quad : \quad [\,H_2^{-1/2}(\partial\Omega)\,]^4 \to [\,H_2^{1/2}(\partial\Omega)\,]^4, \tag{3.22}$$

$$\mathcal{K}^*,\ \widetilde{\mathcal{K}}^* \quad : \quad [\,H_2^{1/2}(\partial\Omega)\,]^4 \to [\,H_2^{1/2}(\partial\Omega)\,]^4,$$

$$\mathcal{K},\ \widetilde{\mathcal{K}} \quad : \quad [\,H_2^{-1/2}(\partial\Omega)\,]^4 \to [\,H_2^{-1/2}(\partial\Omega)\,]^4,$$

$$\mathcal{L} \quad : \quad [\,H_2^{1/2}(\partial\Omega)\,]^4 \to [\,H_2^{-1/2}(\partial\Omega)\,]^4.$$

The operator equalities in Theorem 3.4 still hold in appropriate function spaces.
The coercivity properties of the operators $\mathcal{H}^{(1)}$ and $\mathcal{H}$ given in Theorems 3.5 and 3.6 remain valid (in fact, the above proof of Theorem 3.6 is word for word for Lipschitz domains). Moreover, the operators (3.21) and (3.22) are invertible. For solutions $u^{(1)} \in [\,W_2^1(\Omega_1)\,]^3$ and $U \in [\,W_2^1(\Omega)\,]^4$ to the corresponding homogeneous differential equations the integral representation formulas (3.9) and (3.10) hold as well.
For the Steklov-Poincaré type operators $\mathcal{B}^{(1)}$ and $\mathcal{B}$ given by (3.12) we still have the mapping properties (3.13) stated in Theorem 3.7 with $s = 1/2$, $p = t = 2$, and coercivity property (3.14), implying invertibility of the corresponding operators (the proof is again word for word).

4 Existence and regularity results

4.1 Reduction to boundary equations

Let us return to the formulation of the boundary-transmission problem (2.29)-(2.36) and derive an equivalent boundary integral formulation of this problem.

Let

$$\widehat{F}_0^{(1)} = (\widehat{F}_{01}^{(1)}, \widehat{F}_{02}^{(1)}, \widehat{F}_{03}^{(1)})^\top \in [\, B_{p,p}^{-1/p}(\partial\Omega_1)\,]^3 \tag{4.1}$$

be some fixed extension of the vector-function $F^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})^\top \in [\, B_{p,p}^{-1/p}(S_1)\,]^3$ onto $\partial\Omega_1$. Note that $\partial\Omega_1 = S_1 \cup \overline{\Gamma}_1$. It is evident that an arbitrary extension of $F^{(1)}$ onto $\partial\Omega_1$ has the form

$$\widehat{F}^{(1)} = \widehat{F}_0^{(1)} + \Phi^{(1)}, \tag{4.2}$$

where

$$\Phi^{(1)} = (\Phi_1^{(1)}, \Phi_2^{(1)}, \Phi_3^{(1)})^\top \in [\, \widetilde{B}_{p,p}^{-1/p}(\Gamma_1)\,]^3 \tag{4.3}$$

is introduced as an unknown vector-function.

Analogously, let

$$\widehat{F}_0 = (\widehat{F}_{01}, \widehat{F}_{02}, \widehat{F}_{03}, \widehat{F}_{04})^\top \in [\, B_{p,p}^{-1/p}(\partial\Omega)\,]^4 \tag{4.4}$$

be some fixed extension of the vector-function $F = (F_1, F_2, F_3, F_4)^\top \in [\, B_{p,p}^{-1/p}(S)\,]^4$ onto $\partial\Omega$. Note that $\partial\Omega = \overline{S} \cup \overline{\Gamma}_1 \cup \overline{\Gamma}$. It is evident that every extension of F onto $\partial\Omega$ can be represented then as

$$\widehat{F} = \widehat{F}_0 + \Phi + \Psi, \tag{4.5}$$

where

$$\begin{aligned} &\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^\top \in [\, \widetilde{B}_{p,p}^{-1/p}(\Gamma_1)\,]^4, \\ &\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)^\top \in [\, \widetilde{B}_{p,p}^{-1/p}(\Gamma)\,]^3 \end{aligned} \tag{4.6}$$

are introduced as unknown vector-functions.

We develop here the so-called indirect boundary integral equation method. We look for a solution pair $(u^{(1)}, U)$ of the boundary-transmission problem (2.29)-(2.36) in the form of the corresponding single layer potentials

$$u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top = V^{(1)}\Big(\big[\, \mathcal{H}^{(1)}\,\big]^{-1}\, h^{(1)} \Big) \quad \text{in} \quad \Omega_1, \tag{4.7}$$

$$U = (u, \varphi)^\top := (u_1, u_2, u_3, u_4)^\top = V\big([\, \mathcal{H}\,]^{-1}\, h \big) \quad \text{in} \quad \Omega, \tag{4.8}$$

where $h^{(1)} = (h_1^{(1)}, h_2^{(1)}, h_3^{(1)})^\top \in [\, B_{p,p}^{1-\frac{1}{p}}(\partial\Omega_1)\,]^3$ and $h = (h_1, h_2, h_3, h_4)^\top \in [\, B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)\,]^4$ are unknown densities.

We have to find the unknown vector-functions

$$\begin{aligned} &h^{(1)} \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega_1)]^3, \quad h \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)]^4, \quad \Phi^{(1)} \in [\widetilde{B}_{p,p}^{-1/p}(\Gamma_1)]^3, \\ &\Phi \in [\widetilde{B}_{p,p}^{-1/p}(\Gamma_1)]^4, \quad \Psi \in [\widetilde{B}_{p,p}^{-1/p}(\Gamma)]^4, \end{aligned} \tag{4.9}$$

such that the following Neumann, Dirichlet, and transmission conditions are satisfied:

$$[\, -\, 2^{-1}\, I_3 + \mathcal{K}^{(1)}\,]\,[\,\mathcal{H}^{(1)}\,]^{-1}\, h^{(1)} - \Phi^{(1)} = \widehat{F}_0^{(1)} \quad \text{on} \quad \partial\Omega_1, \tag{4.10}$$

$$[\, 2^{-1}\, I_4 + \mathcal{K}\,]\,[\,\mathcal{H}\,]^{-1}\, h - \Phi - \Psi = \widehat{F}_0 \quad \text{on} \quad \partial\Omega, \tag{4.11}$$

$$r_{\Gamma}\, h_k = f_k \quad \text{on} \quad \Gamma, \quad k = 1,2,3,4, \tag{4.12}$$

$$r_{\Gamma_1}\, h_j^{(1)} - r_{\Gamma_1}\, h_j = g_j \quad \text{on} \quad \Gamma_1, \quad j = 1,2,3, \tag{4.13}$$

$$\Phi_j^{(1)} - \Phi_j = G_j - r_{\Gamma_1}\, \widehat{F}_{0j}^{(1)} + r_{\Gamma_1}\, \widehat{F}_{0j} \quad \text{on} \quad \Gamma_1, \;\; j = 1,2,3, \tag{4.14}$$

$$r_{\Gamma_1}\, h_4 = g_4 \quad \text{on} \quad \Gamma_1. \tag{4.15}$$

We assume that the following compatibility conditions hold (cf. (2.38)):

$$G_j^* := G_j - r_{\Gamma_1}\, \widehat{F}_{0j}^{(1)} + r_{\Gamma_1}\, \widehat{F}_{0j} \in \widetilde{B}_{p,p}^{-1/p}(\Gamma_1), \;\; j = 1,2,3, \quad G^* := (G_1^*, G_2^*, G_3^*)^\top. \tag{4.16}$$

REMARK 4.1 *Equations* (4.10) *and* (4.11) *guarantee that the vectors* $\{T^{(1)}(\partial,n)u^{(1)}\}^+$ *and* $\{\mathcal{T}(\partial,n)U\}^+$ *coincide with the vector-functions* $\widehat{F}_0^{(1)} + \Phi^{(1)}$ *and* $\widehat{F}_0 + \Phi + \Psi$ *on* $\partial\Omega_1$ *and* $\partial\Omega$*, respectively, due to* (4.7) *and* (4.8)*, and Theorem* 3.2*. Therefore, the vector-functions* $\widehat{F}_0^{(1)} + \Phi^{(1)}$ *and* $\widehat{F}_0 + \Phi + \Psi$ *on* $\partial\Omega_1$ *and* $\partial\Omega$ *must satisfy the following solvability conditions for the Neumann-type data:*

$$\Big\langle \{T^{(1)}(\partial,n)u^{(1)}\}^+ \,,\, \chi^{(1)} \Big\rangle_{\partial\Omega_1} = \Big\langle [\, \widehat{F}_0^{(1)} + \Phi^{(1)}\,] \,,\, \chi^{(1)} \Big\rangle_{\partial\Omega_1} = 0, \tag{4.17}$$

$$\Big\langle \{\mathcal{T}(\partial,n)U\}^+ \,,\, \chi \Big\rangle_{\partial\Omega} = \Big\langle [\, \widehat{F}_0 + \Phi + \Psi\,] \,,\, \chi \Big\rangle_{\partial\Omega} = 0, \tag{4.18}$$

where

$$\chi^{(1)} = a \times x + b, \qquad \chi = (a \times x + b, c)$$

whith a *and* b *as arbitrary three-dimensional constant vectors, and* c *as an arbitrary scalar constant.*

Below we will show that the system (4.10)-(4.15) *with respect to the unknowns* (4.9) *is uniquely solvable and consequently the conditions* (4.17) *and* (4.18) *are satisfied automatically since the vectors*

$$\{T^{(1)}(\partial,n)u^{(1)}\}^+ = [\, -\, 2^{-1}\, I_3 + \mathcal{K}^{(1)}\,]\,[\mathcal{H}^{(1)}\,]^{-1}\, h^{(1)}$$

and

$$\{\,\mathcal{T}(\partial,n)U\,\}^+ = [\,2^{-1}\, I_4 + \mathcal{K}\,]\,[\,\mathcal{H}\,]^{-1}\, h$$

with arbitrary $h^{(1)}$ *and* h *are orthogonal to the vectors* $\chi^{(1)}$ *and* χ*. This can be shown with the help of the corresponding Green's formulas.*

The matrix operator generated by the left-hand side expressions in the system (4.10)-(4.15) we denote by $\mathcal{N}$. So this system can be rewritten formally as

$$\mathcal{N}\, X = Y, \tag{4.19}$$

where

$$X := [\, h^{(1)}, h, \Phi^{(1)}, \Phi, \Psi \,]^\top$$

is the unknown vector function,

$$Y := [\, \widehat{F}_0^{(1)}, \widehat{F}_0, f, g^*, G^*, g_4 \,]^\top$$

is a known vector constructed by the right-hand side functions in equations (4.10)-(4.15) with $g^* = (g_1, g_2, g_3)^\top$, and

$$\mathcal{N} := \begin{bmatrix} r_{\partial\Omega_1}\,\mathcal{A}^{(1)} & 0 & -r_{\partial\Omega_1}\, I_3 & 0 & 0 \\ 0 & -r_{\partial\Omega}\,\mathcal{A} & 0 & -r_{\partial\Omega}\, I_4 & -r_{\partial\Omega}\, I_4 \\ 0 & r_{\Gamma}\, I_4 & 0 & 0 & 0 \\ r_{\Gamma_1}\, I_3 & -r_{\Gamma_1}\, I^*_{3\times 4} & 0 & 0 & 0 \\ 0 & 0 & r_{\Gamma_1}\, I_3 & -r_{\Gamma_1}\, I^*_{3\times 4} & 0 \\ 0 & r_{\Gamma_1}\, I^*_{1\times 4} & 0 & 0 & 0 \end{bmatrix} \tag{4.20}$$

with

$$I^*_{3\times 4} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad I^*_{1\times 4} := [\,0\;0\;0\;1\,]. \tag{4.21}$$

Here $\mathcal{A}^{(1)}$ and $\mathcal{A}$ are Steklov-Poincaré type operators

$$\mathcal{A}^{(1)} := [\; -2^{-1} I_3 + \mathcal{K}^{(1)} \;]\,[\; \mathcal{H}^{(1)} \;]^{-1}, \qquad \mathcal{A} := -[\; 2^{-1} I_4 + \mathcal{K} \;]\,[\; \mathcal{H} \;]^{-1}. \tag{4.22}$$

It can be easily verified that the operator $\mathcal{N}$ has the following mapping properties

$$\begin{aligned} \mathcal{N} \;&:\; \mathbf{X}^{(1)}_{s,p} \;\to\; \mathbf{Y}^{(1)}_{s,p}, \\ &:\; \mathbf{X}^{(2)}_{s,p,t} \;\to\; \mathbf{Y}^{(2)}_{s,p,t}, \end{aligned} \tag{4.23}$$

where $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$,

$$
\begin{aligned}
\mathbf{X}^{(1)}_{s,p} &:= [H^s_p(\partial\Omega_1)]^3 \times [H^s_p(\partial\Omega]^4 \times [\widetilde{H}^{s-1}_p(\Gamma_1)]^3 \times [\widetilde{H}^{s-1}_p(\Gamma_1)]^4 \times [\widetilde{H}^{s-1}_p(\Gamma)]^4,\\
\mathbf{Y}^{(1)}_{s,p} &:= [H^{s-1}_p(\partial\Omega_1)]^3 \times [H^{s-1}_p(\partial\Omega]^4 \times [H^s_p(\Gamma)]^4 \times [H^s_p(\Gamma_1)]^3\\
&\quad \times[\widetilde{H}^{s-1}_p(\Gamma_1)]^3 \times H^s_p(\Gamma_1),\\
\mathbf{X}^{(2)}_{s,p,t} &:= [B^s_{p,t}(\partial\Omega_1)]^3 \times [B^s_{p,t}(\partial\Omega]^4 \times [\widetilde{B}^{s-1}_{p,t}(\Gamma_1)]^3 \times [\widetilde{B}^{s-1}_{p,t}(\Gamma_1)]^4 \times [\widetilde{B}^{s-1}_{p,t}(\Gamma)]^4,\\
\mathbf{Y}^{(2)}_{s,p,t} &:= [B^{s-1}_{p,t}(\partial\Omega_1)]^3 \times [B^{s-1}_{p,t}(\partial\Omega]^4 \times [B^s_{p,t}(\Gamma)]^4 \times [B^s_{p,t}(\Gamma_1)]^3\\
&\quad \times[\widetilde{B}^{s-1}_{p,t}(\Gamma_1)]^3 \times B^s_{p,t}(\Gamma_1);
\end{aligned}
\tag{4.24}
$$

here the symbol $\times$ denotes the direct (Cartesian) product of spaces.
Note that $\mathbf{X}^{(1)}_{\frac12,2} = \mathbf{X}^{(2)}_{\frac12,2,2}$ and $\mathbf{Y}^{(1)}_{\frac12,2} = \mathbf{Y}^{(2)}_{\frac12,2,2}$ (see Subsection 2.4).
From the embedding relations for the components of the vector X (see (2.37), (4.1), (4.4), and (4.16)) it follows that

$$
X \in \mathbf{X}^{(2)}_{1-\frac1p,p,p} \quad \text{and} \quad Y \in \mathbf{Y}^{(2)}_{1-\frac1p,p,p}\,. \tag{4.25}
$$

Now, our goal is to show that the system of pseudodifferential equations (4.10)-(4.15) is uniquely solvable in appropriate function spaces. Clearly, this is equivalent to the invertibility of the operators (4.23). Finally, this invertibility property of the operator $\mathcal{N}$ leads to the existence results for the original boundary transmission problem.

4.2 Existence theorems and regularity of solutions

As we will see below the operator (4.23) is not invertible for all $s \in \mathbb{R}$. The interval $a < s < b$ of invertibility depends on p and on some parameters γ' and γ'' which are determined by the eigenvalues of special matrices constructed by means of the principal homogeneous symbols of the operators $\mathcal{A}^{(1)}$ and $\mathcal{A}$ given by (4.22). Note that the numbers γ' and γ'' define also the smoothness exponents for the solutions to the original boundary transmission problem in a neighbourhood of curves $\partial\Gamma_1$ and $\partial\Gamma$ (see Remark 4.6).
We start with the following theorem.

THEOREM 4.2 *Let the conditions*

$$
1 < p < \infty, \quad 1 \le t \le \infty, \quad \frac1p - \frac12 + \gamma'' < s < \frac1p + \frac12 + \gamma' \tag{4.26}
$$

be satisfied with γ' and γ'' given by (4.81), (4.82), *and* (4.85). *Then the operators*

$$
\begin{aligned}
\mathcal{N} \;:\; \mathbf{X}^{(1)}_{s,p} &\to \mathbf{Y}^{(1)}_{s,p}\\
:\; \mathbf{X}^{(2)}_{s,p,t} &\to \mathbf{Y}^{(2)}_{s,p,t}
\end{aligned}
\tag{4.27}
$$

are invertible.

Proof. We prove the theorem in several steps. First we show that the operators (4.27) are Fredholm with zero index and afterwards we establish that the corresponding null-spaces are trivial.
Step 1. To this end let us consider the following auxiliary system which is closely related to the system (4.10)-(4.15) (see also (3.12)):

$$\mathcal{B}^{(1)}\,\widetilde{h}^{(1)} - \widetilde{\Phi}^{(1)} = \widetilde{F}^{(1)} \quad \text{on} \quad \partial\Omega_1, \tag{4.28}$$

$$-\mathcal{B}\,\widetilde{h} - \widetilde{\Phi} - \widetilde{\Psi} = \widetilde{F} \quad \text{on} \quad \partial\Omega, \tag{4.29}$$

$$r_{\Gamma}\,\widetilde{h} = \widetilde{f} \quad \text{on} \quad \Gamma, \tag{4.30}$$

$$r_{\Gamma_1}\,\widetilde{h}^{(1)} - r_{\Gamma_1}\,\widetilde{h}^{*} = \widetilde{g}^{*} \quad \text{on} \quad \Gamma_1, \tag{4.31}$$

$$\widetilde{\Phi}^{(1)} - \widetilde{\Phi}^{*} = \widetilde{G}^{*} \quad \text{on} \quad \Gamma_1, \tag{4.32}$$

$$r_{\Gamma_1}\,\widetilde{h}_4 = \widetilde{g}_4 \quad \text{on} \quad \Gamma_1. \tag{4.33}$$

Here the unknown vectors are

$$\begin{aligned}
&\widetilde{h}^{(1)} = (\widetilde{h}_1^{(1)}, \widetilde{h}_2^{(1)}, \widetilde{h}_3^{(1)})^\top,\\
&\widetilde{h} = (\widetilde{h}^{*}, \widetilde{h}_4)^\top, \quad \widetilde{h}^{*} = (\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3)^\top\\
&\widetilde{\Phi}^{(1)} = (\widetilde{\Phi}_1^{(1)}, \widetilde{\Phi}_2^{(1)}, \widetilde{\Phi}_3^{(1)})^\top,\\
&\widetilde{\Phi} = (\widetilde{\Phi}^{*}, \widetilde{\Phi}_4)^\top, \quad \widetilde{\Phi}^{*} = (\widetilde{\Phi}_1, \widetilde{\Phi}_2, \widetilde{\Phi}_3)^\top,\\
&\widetilde{\Psi} = (\widetilde{\Psi}^{*}, \widetilde{\Psi}_4)^\top, \quad \widetilde{\Psi}^{*} = (\widetilde{\Psi}_1, \widetilde{\Psi}_2, \widetilde{\Psi}_3)^\top.
\end{aligned} \tag{4.34}$$

The given right-hand side vector-functions in (4.28)-(4.33) have the structure:

$$\begin{aligned}
&\widetilde{F}^{(1)} = (\widetilde{F}_1^{(1)}, \widetilde{F}_2^{(1)}, \widetilde{F}_3^{(1)})^\top, \quad \widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_4)^\top,\\
&\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3, \widetilde{f}_4)^\top, \quad \widetilde{g}^{*} = (\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3)^\top,\\
&\widetilde{G}^{*} = (\widetilde{G}_1^{*}, \widetilde{G}_2^{*}, \widetilde{G}_3^{*})^\top, \quad \text{and} \quad \widetilde{g}_4 \quad \text{is a scalar function.}
\end{aligned}$$

Denote

$$\widetilde{X} := (\widetilde{h}^{(1)}, \widetilde{h}, \widetilde{\Phi}^{(1)}, \widetilde{\Phi}, \widetilde{\Psi})^\top, \quad \widetilde{Y} := (\widetilde{F}^{(1)}, \widetilde{F}, \widetilde{f}, \widetilde{g}^{*}, \widetilde{G^{*}}, \widetilde{g}_4)^\top.$$

We assume that either

$$\widetilde{X} \in \mathbf{X}_{s,p}^{(1)}, \qquad \widetilde{Y} \in \mathbf{Y}_{s,p}^{(1)}, \tag{4.35}$$

or

$$\widetilde{X} \in \mathbf{X}_{s,p,t}^{(2)}, \qquad \widetilde{Y} \in \mathbf{Y}_{s,p,t}^{(2)}. \tag{4.36}$$

Denote by $\widetilde{\mathcal{N}}$ the operator generated by the left-hand side expressions of the system (4.28)-(4.33). Then system (4.28)-(4.33) takes the form

$$\widetilde{\mathcal{N}}\,\widetilde{X} = \widetilde{Y}, \tag{4.37}$$

where

$$\widetilde{\mathcal{N}} := \begin{bmatrix} r_{\partial\Omega_1}\,\mathcal{B}^{(1)} & 0 & -r_{\partial\Omega_1}\,I_3 & 0 & 0 \\ 0 & -r_{\partial\Omega}\,\mathcal{B} & 0 & -r_{\partial\Omega}\,I_4 & -r_{\partial\Omega}\,I_4 \\ 0 & r_{\Gamma}\,I_4 & 0 & 0 & 0 \\ r_{\Gamma_1}\,I_3 & -r_{\Gamma_1}\,I^*_{3\times 4} & 0 & 0 & 0 \\ 0 & 0 & r_{\Gamma_1}\,I_3 & -r_{\Gamma_1}\,I^*_{3\times 4} & 0 \\ 0 & r_{\Gamma_1}\,I^*_{1\times 4} & 0 & 0 & 0 \end{bmatrix} \tag{4.38}$$

with $\mathcal{B}^{(1)}$ and $\mathcal{B}$ given by (3.12).
It is evident that the operator $\widetilde{\mathcal{N}}$ has the same mapping properties as $\mathcal{N}$ (see (4.23))

$$\begin{aligned} \widetilde{\mathcal{N}} \; &: \; \mathbf{X}^{(1)}_{s,p} \;\to\; \mathbf{Y}^{(1)}_{s,p} \\ &: \; \mathbf{X}^{(2)}_{s,p,t} \;\to\; \mathbf{Y}^{(2)}_{s,p,t} \end{aligned} \tag{4.39}$$

with $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$. Moreover, the difference

$$\widetilde{\mathcal{N}} - \mathcal{N} = \begin{bmatrix} r_{\partial\Omega_1}\,I_3 & 0 & 0 & 0 & 0 \\ 0 & -r_{\partial\Omega}\,I_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.40}$$

is a compact operator from $\mathbf{X}^{(1)}_{s,p}$ into $\mathbf{Y}^{(1)}_{s,p}$ (respectively from $\mathbf{X}^{(2)}_{s,p,t}$ into $\mathbf{Y}^{(2)}_{s,p,t}$) due to the compactness of the embedding $H^s_p \subset H^{s-1}_p$ (respectively $B^s_{p,t} \subset B^{s-1}_{p,t}$).
In what follows we show that the operator (4.39) is Fredholm with zero index. This implies the same properties for the operator $\mathcal{N}$ (see (4.27)) due to the compactness of the difference (4.40).
In view of Theorem 3.7 the operators

$$\begin{aligned} \mathcal{B}^{(1)} \; &: \; [\,H^s_p(\partial\Omega_1)\,]^3 \to [\,H^{s-1}_p(\partial\Omega_1)\,]^3 \\ &: \; [\,B^s_{p,t}(\partial\Omega_1)\,]^3 \to [\,B^{s-1}_{p,t}(\partial\Omega_1)\,]^3 \\ \mathcal{B} \; &: \; [\,H^s_p(\partial\Omega)\,]^4 \to [\,H^{s-1}_p(\partial\Omega)\,]^4 \\ &: \; [\,B^s_{p,t}(\partial\Omega)\,]^4 \to [\,B^{s-1}_{p,t}(\partial\Omega)\,]^4 \end{aligned} \tag{4.41}$$

are invertible. We denote the corresponding inverse operators by $[\mathcal{B}^{(1)}]^{-1} := [\ [\mathcal{B}^{(1)}]^{-1}_{kj}\ ]_{3\times 3}$ and $\mathcal{B}^{-1} := [\,\mathcal{B}^{-1}_{kj}\,]_{4\times 4}$, respectively. Further, let

$$\mathcal{C} := \begin{bmatrix} \mathcal{B}^{-1}_{11} & \mathcal{B}^{-1}_{12} & \mathcal{B}^{-1}_{13} & \mathcal{B}^{-1}_{14} \\ \mathcal{B}^{-1}_{21} & \mathcal{B}^{-1}_{22} & \mathcal{B}^{-1}_{23} & \mathcal{B}^{-1}_{24} \\ \mathcal{B}^{-1}_{31} & \mathcal{B}^{-1}_{32} & \mathcal{B}^{-1}_{33} & \mathcal{B}^{-1}_{34} \end{bmatrix}_{3\times 4}. \tag{4.42}$$

From the first two equations of the system (4.28)-(4.33) we can find

$$\widetilde{h}^{(1)} = [\,\mathcal{B}^{(1)}\,]^{-1}\,\widetilde{\Phi}^{(1)} + [\,\mathcal{B}^{(1)}\,]^{-1}\,\widetilde{F}^{(1)} \quad \text{on} \quad \partial\Omega_1, \tag{4.43}$$

$$\widetilde{h} = -\mathcal{B}^{-1}\,\widetilde{\Phi} - \mathcal{B}^{-1}\,\widetilde{\Psi} - \mathcal{B}^{-1}\,\widetilde{F} \quad \text{on} \quad \partial\Omega. \tag{4.44}$$

In particular, we get from (4.44)

$$\widetilde{h}^{*} = (\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3)^{\top} = -\mathcal{C}\,\widetilde{\Phi} - \mathcal{C}\,\widetilde{\Psi} - \mathcal{C}\,\widetilde{F} \quad \text{on} \quad \partial\Omega, \tag{4.45}$$

$$\widetilde{h}_4 = -\sum_{k=1}^{4} \mathcal{B}^{-1}_{4k}\,\Big(\widetilde{\Phi}_k + \widetilde{\Psi}_k + \widetilde{F}_k\Big) \quad \text{on} \quad \partial\Omega. \tag{4.46}$$

If we substitute the expressions (4.43)-(4.46) into the equations (4.30)-(4.33) we arrive at the system which is equivalent to the coupled equations (4.28)-(4.33)

$$\widetilde{h}^{(1)} = [\,\mathcal{B}^{(1)}\,]^{-1}\,\widetilde{\Phi}^{(1)} + [\,\mathcal{B}^{(1)}\,]^{-1}\,\widetilde{F}^{(1)} \quad \text{on} \quad \partial\Omega_1, \tag{4.47}$$

$$\widetilde{h} = -\mathcal{B}^{-1}\,\widetilde{\Phi} - \mathcal{B}^{-1}\,\widetilde{\Psi} - \mathcal{B}^{-1}\,\widetilde{F} \quad \text{on} \quad \partial\Omega, \tag{4.48}$$

$$-r_{_\Gamma}\,\mathcal{B}^{-1}\,\widetilde{\Phi} - r_{_\Gamma}\,\mathcal{B}^{-1}\,\widetilde{\Psi} = Q_1 \quad \text{on} \quad \Gamma, \tag{4.49}$$

$$r_{_{\Gamma_1}}[\mathcal{B}^{(1)}]^{-1}\,\widetilde{\Phi}^{(1)} + r_{_{\Gamma_1}}\mathcal{C}\,\widetilde{\Phi} + r_{_{\Gamma_1}}\mathcal{C}\,\widetilde{\Psi} = Q_2 \quad \text{on} \quad \Gamma_1, \tag{4.50}$$

$$\widetilde{\Phi}^{(1)} - \widetilde{\Phi}^{*} = Q_3 \quad \text{on} \quad \Gamma_1, \tag{4.51}$$

$$-\sum_{k=1}^{4} r_{_{\Gamma_1}}\,\mathcal{B}^{-1}_{4k}\,\widetilde{\Phi}_k - \sum_{k=1}^{4} r_{_{\Gamma_1}}\,\mathcal{B}^{-1}_{4k}\,\widetilde{\Psi}_k = Q_4 \quad \text{on} \quad \Gamma_1, \tag{4.52}$$

where

$$Q_1 := \widetilde{f} + r_{_\Gamma}\,\mathcal{B}^{-1}\,\widetilde{F}, \quad Q_2 := \widetilde{g}^{*} - r_{_{\Gamma_1}}[\,\mathcal{B}^{(1)}\,]^{-1}\,\widetilde{F}^{(1)} - r_{_{\Gamma_1}}\mathcal{C}\,\widetilde{F}, \tag{4.53}$$

$$Q_3 := \widetilde{G}^{*}, \quad Q_4 := \widetilde{g}_4 + \sum_{k=1}^{4} r_{_{\Gamma_1}}\,\mathcal{B}^{-1}_{4k}\,\widetilde{F}_k. \tag{4.54}$$

We express the vector $\widetilde{\Phi}^{(1)}$ by equation (4.51) and insert it into (4.50):

$$\widetilde{h}^{(1)} = [\,\mathcal{B}^{(1)}\,]^{-1}\,\widetilde{\Phi}^{(1)} + [\,\mathcal{B}^{(1)}\,]^{-1}\,\widetilde{F}^{(1)} \quad \text{on} \quad \partial\Omega_1, \tag{4.55}$$

$$\widetilde{h} = -\mathcal{B}^{-1}\,\widetilde{\Phi} - \mathcal{B}^{-1}\,\widetilde{\Psi} - \mathcal{B}^{-1}\,\widetilde{F} \quad \text{on} \quad \partial\Omega, \tag{4.56}$$

$$\widetilde{\Phi}^{(1)} = \widetilde{\Phi}^{*} + Q_3 \quad \text{on} \quad \Gamma_1, \tag{4.57}$$

$$-r_{\Gamma}\,\mathcal{B}^{-1}\,\widetilde{\Phi} - r_{\Gamma}\,\mathcal{B}^{-1}\,\widetilde{\Psi} = Q_1 \quad \text{on} \quad \Gamma, \tag{4.58}$$

$$r_{\Gamma_1}[\mathcal{B}^{(1)}]^{-1}\,\widetilde{\Phi}^{*} + r_{\Gamma_1}\mathcal{C}\,\widetilde{\Phi} + r_{\Gamma_1}\mathcal{C}\,\widetilde{\Psi} = Q_2 - r_{\Gamma_1}[\mathcal{B}^{(1)}]^{-1}\,Q_3 \quad \text{on} \quad \Gamma_1, \tag{4.59}$$

$$-\sum_{k=1}^{4} r_{\Gamma_1}\,\mathcal{B}^{-1}_{4k}\,\widetilde{\Phi}_k - \sum_{k=1}^{4} r_{\Gamma_1}\,\mathcal{B}^{-1}_{4k}\,\widetilde{\Psi}_k = Q_4 \quad \text{on} \quad \Gamma_1. \tag{4.60}$$

Multiplying the fourth and sixth equations by -1 and combining then the last two equations after an obvious rearrangement we can rewrite the system as

$$\widetilde{h}^{(1)} = [\,\mathcal{B}^{(1)}\,]^{-1}\,\widetilde{\Phi}^{(1)} + [\,\mathcal{B}^{(1)}\,]^{-1}\,\widetilde{F}^{(1)} \quad \text{on} \quad \partial\Omega_1, \tag{4.61}$$

$$\widetilde{h} = -\mathcal{B}^{-1}\,\widetilde{\Phi} - \mathcal{B}^{-1}\,\widetilde{\Psi} - \mathcal{B}^{-1}\,\widetilde{F} \quad \text{on} \quad \partial\Omega, \tag{4.62}$$

$$\widetilde{\Phi}^{(1)} = \widetilde{\Phi}^{*} + Q_3 \quad \text{on} \quad \Gamma_1, \tag{4.63}$$

$$\mathcal{P}^{(1)}\,\widetilde{\Phi} + \mathcal{P}^{(3)}\,\widetilde{\Psi} = Q_1^{*} \quad \text{on} \quad \Gamma_1, \tag{4.64}$$

$$\mathcal{P}^{(4)}\,\widetilde{\Phi} + \mathcal{P}^{(2)}\,\widetilde{\Psi} = Q_2^{*} \quad \text{on} \quad \Gamma, \tag{4.65}$$

where

$$Q_1^{*} := (Q_2 - r_{\Gamma_1}[\mathcal{B}^{(1)}]^{-1}\,Q_3,\ -Q_4)^{\top}, \quad Q_2^{*} := -Q_1, \tag{4.66}$$

with either $(Q_1^{*}, Q_2^{*})^{\top} \in \big[H^s_p(\Gamma_1)\big]^4 \times \big[H^s_p(\Gamma)\big]^4$ or $(Q_1^{*}, Q_2^{*})^{\top} \in [\,B^s_{p,t}(\Gamma_1)\,]^4 \times [\,B^s_{p,t}(\Gamma)\,]^4$, and

$$\begin{aligned}
&\mathcal{P}^{(1)} := r_{\Gamma_1}\,\mathcal{B}^{-1} + r_{\Gamma_1}\,\mathcal{D}, && \mathcal{P}^{(3)} := r_{\Gamma_1}\,\mathcal{B}^{-1}, \\
&\mathcal{P}^{(2)} := r_{\Gamma}\,\mathcal{B}^{-1}, && \mathcal{P}^{(4)} := r_{\Gamma}\,\mathcal{B}^{-1}, \\
&\mathcal{D} = [\ \mathcal{D}_{kj}\ ]_{4\times 4}, && \mathcal{D}_{kj} := [\,\mathcal{B}^{(1)}\,]^{-1}_{kj}, \quad k, j = 1, 2, 3, \\
&\mathcal{D}_{i4} := 0, && \mathcal{D}_{4i} := 0, \qquad\quad i = 1, 2, 3, 4.
\end{aligned} \tag{4.67}$$

In matrix form the system (4.61)-(4.65) reads as follows

$$\widetilde{\mathcal{N}}^{(1)}\,\widetilde{X} = \widetilde{Y}_0 \tag{4.68}$$

with

$$\widetilde{\mathcal{N}}^{(1)} := \begin{bmatrix} r_{_{\partial\Omega_1}} I_3 & 0 & -r_{_{\partial\Omega_1}} [\mathcal{B}^{(1)}]^{-1} & 0 & 0 \\ 0 & r_{_{\partial\Omega}} I_4 & 0 & r_{_{\partial\Omega}} \mathcal{B}^{-1} & r_{_{\partial\Omega}} \mathcal{B}^{-1} \\ 0 & 0 & r_{_{\Gamma_1}} I_3 & -r_{_{\Gamma_1}} I^*_{3\times 4} & 0 \\ 0 & 0 & 0 & r_{_{\Gamma_1}} \mathcal{P}^{(1)} & r_{_{\Gamma_1}} \mathcal{P}^{(3)} \\ 0 & 0 & 0 & r_{_\Gamma} \mathcal{P}^{(4)} & r_{_\Gamma} \mathcal{P}^{(2)} \end{bmatrix}. \tag{4.69}$$

Here

$$\widetilde{X} = [\ \widetilde{h}^{(1)}, \widetilde{h}, \widetilde{\Phi}^{(1)}, \widetilde{\Phi}, \widetilde{\Psi}\ ]^\top$$

is the sought unknown vector. For the known right-hand side vector

$$\widetilde{Y}_0 := [\ [\mathcal{B}^{(1)}]^{-1}\, \widetilde{F}^{(1)},\ -\mathcal{B}^{-1}\, \widetilde{F},\ Q_3,\ Q_1^*,\ Q_2^*\ ]^\top$$

we assume that it belongs to the space

$$\mathbf{Z}^{(1)}_{s,p} := [H^s_p(\partial\Omega_1)]^3 \times [H^s_p(\partial\Omega]^4 \times [\widetilde{H}^{s-1}_p(\Gamma_1)]^3 \times [H^s_p(\Gamma_1)]^4 \times [H^s_p(\Gamma)]^4$$

or (4.70)

$$\mathbf{Z}^{(2)}_{s,p,t} := [B^s_{p,t}(\partial\Omega_1)]^3 \times [B^s_{p,t}(\partial\Omega)]^4 \times [\widetilde{B}^{s-1}_{p,t}(\Gamma_1)]^3 \times [B^s_{p,t}(\Gamma_1)]^4 \times [B^s_{p,t}(\Gamma)]^4.$$

It can easily be seen that the system (4.28)-(4.33) is equivalently reduced to the system (4.61)-(4.65) (i.e. (4.37) and (4.68) are equivalent equations).
Note that the operator $\widetilde{\mathcal{N}}^{(1)}$ has the mapping property

$$\begin{aligned} \widetilde{\mathcal{N}}^{(1)} \ &: \ \mathbf{X}^{(1)}_{s,p} \ \to \ \mathbf{Z}^{(1)}_{s,p} \\ &: \ \mathbf{X}^{(2)}_{s,p,t} \ \to \ \mathbf{Z}^{(2)}_{s,p,t}. \end{aligned} \tag{4.71}$$

Moreover, it is evident that the operators defined on the interface Γ_1 and on the Dirichlet boundary part Γ respectively

$$\begin{aligned} \mathcal{P}^{(1)} := r_{_{\Gamma_1}} \mathcal{B}^{-1} + r_{_{\Gamma_1}} \mathcal{D} \ &: \ [\widetilde{H}^{s-1}_p(\Gamma_1)]^4 \to [H^s_p(\Gamma_1)]^4, \\ &: \ [\widetilde{B}^{s-1}_{p,t}(\Gamma_1)]^4 \to [B^s_{p,t}(\Gamma_1)]^4, \end{aligned} \tag{4.72}$$

$$\begin{aligned} \mathcal{P}^{(2)} := r_{_\Gamma} \mathcal{B}^{-1} \ &: \ [\widetilde{H}^{s-1}_p(\Gamma)]^4 \to [H^s_p(\Gamma)]^4, \\ &: \ [\widetilde{B}^{s-1}_{p,t}(\Gamma)]^4 \to [B^s_{p,t}(\Gamma)]^4, \end{aligned} \tag{4.73}$$

are strongly elliptic pseudodifferential operators due to the strong ellipticity and invertibility of the operators $\mathcal{B}$ and $\mathcal{B}^{(1)}$ (see Theorem 3.7), while the operators

$$\mathcal{P}^{(3)} := r_{_{\Gamma_1}} \mathcal{B}^{-1} \;\; : \;\; [\widetilde{H}^{s-1}_p(\Gamma)]^4 \to [H^s_p(\Gamma_1)]^4, \\ \;\; : \;\; [\widetilde{B}^{s-1}_{p,t}(\Gamma)]^4 \to [B^s_{p,t}(\Gamma_1)]^4 \tag{4.74}$$

$$\mathcal{P}^{(4)} := r_{_\Gamma} \mathcal{B}^{-1} \;\; : \;\; [\widetilde{H}^{s-1}_p(\Gamma_1)]^4 \to [H^s_p(\Gamma)]^4, \\ \;\; : \;\; [\widetilde{B}^{s-1}_{p,t}(\Gamma_1)]^4 \to [B^s_{p,t}(\Gamma)]^4 \tag{4.75}$$

are compact (in fact, C^∞-smoothing) operators since $\overline{\Gamma} \cap \overline{\Gamma}_1 = \varnothing$.
Let us consider the matrix operator

$$\widetilde{\mathcal{N}}^{(2)} := \begin{bmatrix} r_{_{\partial\Omega_1}} I_3 & 0 & -r_{_{\partial\Omega_1}} [\,\mathcal{B}^{(1)}\,]^{-1} & 0 & 0 \\ 0 & r_{_{\partial\Omega}} I_4 & 0 & r_{_{\partial\Omega}} \mathcal{B}^{-1} & r_{_{\partial\Omega}} \mathcal{B}^{-1} \\ 0 & 0 & r_{_{\Gamma_1}} I_3 & -r_{_{\Gamma_1}} I^*_{3\times 4} & 0 \\ 0 & 0 & 0 & r_{_{\Gamma_1}} \mathcal{P}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & r_{_\Gamma} \mathcal{P}^{(2)} \end{bmatrix}, \tag{4.76}$$

which, clearly, has the same mapping properties as the operator $\widetilde{\mathcal{N}}^{(1)}$ (see (4.71)).
Then

$$\widetilde{\mathcal{N}}^{(1)} - \widetilde{\mathcal{N}}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{_{\Gamma_1}} \mathcal{P}^{(3)} \\ 0 & 0 & 0 & r_{_\Gamma} \mathcal{P}^{(4)} & 0 \end{bmatrix}, \tag{4.77}$$

and it is evident that the operators

$$\widetilde{\mathcal{N}}^{(1)} - \widetilde{\mathcal{N}}^{(2)} \;\; : \;\; \mathbf{X}^{(1)}_{s,p} \to \mathbf{Z}^{(1)}_{s,p} \\ \;\; : \;\; \mathbf{X}^{(2)}_{s,p,t} \to \mathbf{Z}^{(2)}_{s,p,t} \tag{4.78}$$

are compact.
Therefore, the Fredholm properties and indices of the operators $\widetilde{\mathcal{N}}^{(2)}$ and $\widetilde{\mathcal{N}}^{(1)}$ are the same.
Step 2. Here we establish the invertibility of the upper triangular operator

$$\widetilde{\mathcal{N}}^{(2)} \;\; : \;\; \mathbf{X}^{(1)}_{s,p} \to \mathbf{Z}^{(1)}_{s,p} \\ \;\; : \;\; \mathbf{X}^{(2)}_{s,p,t} \to \mathbf{Z}^{(2)}_{s,p,t}\,. \tag{4.79}$$

Due to the structure of the matrix (4.76) it suffices to study the invertibility of the operators (4.72)-(4.73).

We denote by $\sigma_1(x,\xi_1,\xi_2)$ and $\sigma_2(x,\xi_1,\xi_2)$ the principal homogeneous symbol matrices of the operators $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$, where $\xi' = (\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{0\}$, and $x \in \partial\Gamma_1$ or $x \in \partial\Gamma$, respectively. The entries of these matrices are homogeneous function in $\xi' = (\xi_1,\xi_2)$ of order -1 and due to the strong ellipticity of the operators $\mathcal{B}^{-1}$ and $[\,\mathcal{B}^{(1)}\,]^{-1}$ there is a positive constant c such that for all $\xi' \in \mathbb{R}^2 \setminus \{0\}$ and for all $\eta \in \mathbb{C}^4$

$$\Re\,\sigma_j(x,\xi_1,\xi_2)\,\eta\cdot\eta \geq c\,|\xi'|^{-1}\,|\eta|^2 \tag{4.80}$$

with $x \in \overline{\Gamma}_1$ for $j=1$ and $x \in \overline{\Gamma}$ for $j=2$.

Further, we set

$$\gamma_1{}' := \inf_{x\in\partial\Gamma_1,\,1\leq j\leq 4} \frac{1}{2\pi}\,\arg\lambda_j^{(1)}(x), \quad \gamma_1{}'' := \sup_{x\in\partial\Gamma_1,\,1\leq j\leq 4} \frac{1}{2\pi}\,\arg\lambda_j^{(1)}(x), \tag{4.81}$$

$$\gamma_2{}' := \inf_{x\in\partial\Gamma,\,1\leq j\leq 4} \frac{1}{2\pi}\,\arg\lambda_j^{(2)}(x), \quad \gamma_2{}'' := \sup_{x\in\partial\Gamma,\,1\leq j\leq 4} \frac{1}{2\pi}\,\arg\lambda_j^{(2)}(x), \tag{4.82}$$

where $\lambda_j^{(1)}(x)$ ($j=\overline{1,4}$, $x\in\partial\Gamma_1$) are the eigenvalues of the matrix

$$[\,\sigma_1(x,0,+1)\,]^{-1}\,\sigma_1(x,0,-1) \tag{4.83}$$

and $\lambda_j^{(2)}(x)$ ($j=\overline{1,4}$, $x\in\partial\Gamma$) are the eigenvalues of the matrix

$$[\,\sigma_2(x,0,+1)\,]^{-1}\,\sigma_2(x,0,-1). \tag{4.84}$$

Note that in general $\gamma_j{}'$ and $\gamma_j{}''$ ($j=1,2$) depend on the elastic and electric constants, the geometry of the curves $\partial\Gamma_1$ and $\partial\Gamma$, and belong to the interval $(-\frac{1}{2},\,\frac{1}{2})$ due to the inequalities (4.80). We put

$$\gamma' := \min\{\gamma_1{}',\,\gamma_2'\}, \qquad \gamma'' := \max\{\gamma_1{}'',\,\gamma_2''\} \tag{4.85}$$

in order to acquire the worst case and to get global regularity results. From Theorems 3.6 and 3.7 we conclude that if the parameters $s,r\in\mathbb{R}$, $1<p<\infty$, $1\leq t\leq\infty$, satisfy the conditions

$$\frac{1}{p}-\frac{1}{2}+\gamma_1{}'' < r < \frac{1}{p}+\frac{1}{2}+\gamma_1{}', \qquad \frac{1}{p}-\frac{1}{2}+\gamma_2{}'' < s < \frac{1}{p}+\frac{1}{2}+\gamma_2{}', \tag{4.86}$$

then the operators

$$\begin{aligned}
\mathcal{P}^{(1)} = r_{\Gamma_1}\mathcal{D} + r_{\Gamma_1}\mathcal{B}^{-1} \ &: \ [\widetilde{H}_p^{r-1}(\Gamma_1)]^4 \to [H_p^r(\Gamma_1)]^4 \\
&: \ [\widetilde{B}_{p,t}^{r-1}(\Gamma_1)]^4 \to [B_{p,t}^r(\Gamma_1)]^4, \\
\mathcal{P}^{(2)} = r_{\Gamma}\,\mathcal{B}^{-1} \ &: \ [\widetilde{H}_p^{s-1}(\Gamma)]^4 \to [H_p^s(\Gamma)]^4 \\
&: \ [\widetilde{B}_{p,t}^{s-1}(\Gamma)]^4 \to [B_{p,t}^s(\Gamma)]^4
\end{aligned} \tag{4.87}$$

are invertible.
Therefore the operator (4.79) is invertible if s, p, and t satisfy conditions (4.26).
Step 3. From the invertibility of the operator $\widetilde{\mathcal{N}}^{(2)}$ (see (4.79)) it follows that the operator $\widetilde{\mathcal{N}}^{(1)}$ (see (4.71)) is Fredholm with zero index due to the compactness of the difference $\widetilde{\mathcal{N}}^{(2)} - \widetilde{\mathcal{N}}^{(1)}$ (see (4.78)). Moreover, the null-spaces of the operators (4.71) are the same for all values of the parameters s and p satisfying the inequality (4.86), and for all $t \in [1, +\infty])$ due to Theorem 3.8.
Since the operators $\widetilde{\mathcal{N}}^{(1)}$ and $\widetilde{\mathcal{N}}$ (see (4.71) and (4.39)) correspond to the equivalent systems (in the sense described in Step 1) we conclude that the operator $\widetilde{\mathcal{N}}$ is Fredholm with zero index for all values of the parameters s and p satisfying the inequality (4.86), and for all $t \in [1, +\infty]$. This yields that the operator $\mathcal{N}$ (see (4.27)) is Fredholm with zero index for s, p, and t satisfying the conditions (4.26) due to the compactness of the perturbation (4.40). Moreover, with the help of Theorem 3.7 and the embedding theorems for the Bessel potential and Besov spaces it can be shown that the corresponding null-spaces do not depend on the parameters s, p, and t if (4.26) is satisfied (which automatically yield the restrictions (4.86)).
Step 4. To show the invertibility of the operator (4.27) it suffices to verify that the corresponding null-space is trivial for at least one triple s, p, t satisfying the conditions (4.26).
Let us consider the homogeneous version of the system (4.10)-(4.15) and show that it has only the trivial solution in the space $\mathbf{X}^{(2)}_{s,p,t}$ with $s = 1/2$, $p = 2$, $t = 2$, i.e., the operator $\mathcal{N}$ has the trivial kernel in the space $\mathbf{X}^{(2)}_{\frac{1}{2},2,2}$. We recall that $\mathbf{X}^{(1)}_{\frac{1}{2},2} = \mathbf{X}^{(2)}_{\frac{1}{2},2,2}$ and $\mathbf{Y}^{(1)}_{\frac{1}{2},2} = \mathbf{Y}^{(2)}_{\frac{1}{2},2,2}$.
Further, let the vector $X = [\, h^{(1)}, h, \Phi^{(1)}, \Phi, \Psi \,]^\top \in \mathbf{X}^{(1)}_{\frac{1}{2},2}$ be a solution to the homogeneous version of the system (4.10)-(4.15) and construct the single layer potentials (cf. (4.7)-(4.8))

$$\begin{array}{lcll} u^{(1)} & = & V^{(1)}\big([\, \mathcal{H}^{(1)} \,]^{-1} h^{(1)} \big) & \text{in} \;\; \Omega_1, \\ U & = & V\big([\, \mathcal{H} \,]^{-1} h \big) & \text{in} \;\; \Omega. \end{array}$$

We can easily show that the system (4.10)-(4.15) corresponds to the homogeneous boundary-transmission problem (2.29)-(2.36) for the pair $(\, u^{(1)} \,,\, U \,)$ in the space $[W^1_2(\Omega_1)]^3 \times [W^1_2(\Omega)]^4$. Moreover, we can assume that in this case we can take the fixed extensions $\widehat{F}^{(1)}_0$ and $\widehat{F}_0$ as zero vector-functions (otherwise we can combine them with the unknown vectors $\Phi^{(1)}$, Φ, and Ψ due to the inclusions $\widehat{F}^{(1)}_0 \in [\widetilde{H}^{-1/2}_2(\Gamma_1)]^3$ and $\widehat{F}_0 \in [\widetilde{H}^{-1/2}_2(\Gamma_1 \cup \Gamma)]^4$).
With the help of the uniqueness Theorem 2.1 we conclude that $u^{(1)} = 0$ in Ω_1 and $U = 0$ in Ω. Whence $\{u^{(1)}\}^+ = h^{(1)} = 0$ on $\partial\Omega_1$ and $\{U\}^+ = h = 0$ on $\partial\Omega$ follow. From equations (4.10)-(4.11) we get then $\Phi^{(1)} = 0$ on $\partial\Omega_1$ and $\Phi = \Psi = 0$ on $\partial\Omega$. Thus the null space of the operator $\mathcal{N}$ is trivial in $\mathbf{X}^{(1)}_{\frac{1}{2},2}$, and consequently in $\mathbf{X}^{(1)}_{s,p}$ and $\mathbf{X}^{(2)}_{s,p,t}$ for arbitrary s, p, t satisfying the conditions (4.26). The proof is completed. □

REMARK 4.3 *Theorem* 4.2 *implies that the equation* (4.19) *is uniquely solvable for an arbitrary right-hand side vector from the space either* $\mathbf{Y}^{(1)}_{s,p}$ *or* $\mathbf{Y}^{(2)}_{s,p,t}$ *and the solution belongs* *to the space $\mathbf{X}^{(1)}_{s,p}$ or $\mathbf{X}^{(2)}_{s,p,t}$, respectively. Moreover, the solution vectors $h^{(1)}$ and h which actually coincide with the traces $[\,u^{(1)}\,]^+_{\partial\Omega_1}$ and $[\,U\,]^+_{\partial\Omega}$ are defined uniquely and they do not depend on the forms of extensions of the vectors $F^{(1)}$ and F involved in the conditions* (2.31), (2.32). *Analogously, the solution vectors $\Phi^{(1)}, \Phi$, and Ψ are defined uniquely for fixed extensions of the vectors $F^{(1)}$ and F, but they depend on the forms of these extensions. However, the vectors $\widehat{F}^{(1)}_0 + \Phi^{(1)}$ and $\widehat{F}_0 + \Phi + \Psi$ are defined uniquely and they do not depend on the form of the extensions.*

Now we are in the position to formulate the basic existence and uniqueness results for the boundary-transmission problem under consideration.

THEOREM 4.4 *Let*

$$\frac{4}{3-2\gamma''} < p < \frac{4}{1-2\gamma'} \tag{4.88}$$

and the compatibility condition (4.16) *hold.*
Then the boundary-transmission problem (2.29)-(2.36) *has a unique solution which can be represented by formulas*

$$u^{(1)} = V^{(1)}\big(\,[\,\mathcal{H}^{(1)}\,]^{-1}\,h^{(1)}\big) \quad \text{in} \quad \Omega_1, \qquad U = (u,\varphi)^\top = V\big(\,[\,\mathcal{H}\,]^{-1}\,h\big) \quad \text{in} \quad \Omega,$$

where the densities $h^{(1)}$ and h are defined by the system (4.10)-(4.15).

Proof. The existence of a solution pair $(u^{(1)}, U)$ in the class $[W^1_p(\Omega_1)]^3 \times [W^1_p(\Omega)]^4$ with p satisfying (4.88) follows from Theorem 4.2 with $s = 1 - \frac{1}{p}$. Due to the inequalities

$$-\frac{1}{2} < \gamma' \le \gamma'' < \frac{1}{2}$$

we have

$$p = 2 \in \left(\frac{4}{3-2\gamma''}\,,\;\frac{4}{1-2\gamma'}\right).$$

Therefore the unique solvability for $p=2$ is a consequence of Theorem 2.1.
To show the uniqueness result for all other values of p from the interval (4.88) we proceed as follows. Let a pair

$$(u^{(1)}, U) \in [W^1_p(\Omega_1)]^3 \times [W^1_p(\Omega)]^4 \tag{4.89}$$

with p satisfying (4.88) be a solution to the homogeneous boundary-transmission problem. Then, it is evident that there exist the traces

$$[\,u^{(1)}\,]^+ \in [B^{1-\frac{1}{p}}_{p,p}(\partial\Omega_1)]^3, \quad [\,U\,]^+ \in [B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)]^4 \tag{4.90}$$

and the vectors $u^{(1)}$ and U in Ω_1 and Ω respectively are represented in the form (cf. (4.7)-(4.8))

$$u^{(1)} = V^{(1)}\big(\,[\,\mathcal{H}^{(1)}\,]^{-1}\,[\,u^{(1)}\,]^+\big) \quad \text{in} \quad \Omega_1, \tag{4.91}$$

$$U = V\big(\,[\,\mathcal{H}\,]^{-1}\,[\,U\,]^+\big) \quad \text{in} \quad \Omega, \tag{4.92}$$

due to Theorems 3.4 and 3.5.
By the same arguments as in Subsection 4.1 we arrive at the homogeneous system

$$\mathcal{N}\, X = 0,$$

where

$$X := \big[\, [\, u^{(1)}\,]^+, [\, U\,]^+, \Phi^{(1)}, \Phi, \Psi \,\big]^\top \in \mathbf{X}^{(1)}_{1-\frac{1}{p},p}\,.$$

Due to Theorem 4.2 and Remark 4.3 we conclude that $[\, u^{(1)}\,]^+_{\partial\Omega_1} = 0$ and $[\, U\,]^+_{\partial\Omega} = 0$, which imply that $u^{(1)} = 0$ in Ω_1 and $U = 0$ in Ω. □

Finally, we can prove the following regularity result for the solution of the boundary-transmission problem.

THEOREM 4.5 *Let*

$$\frac{4}{3-2\gamma''} < p < \frac{4}{1-2\gamma'}\ ,\ 1 < r < \infty,\ 1 \leq t \leq \infty,\ \frac{1}{r} - \frac{1}{2} + \gamma'' < s < \frac{1}{r} + \frac{1}{2} + \gamma', \quad (4.93)$$

and the compatibility condition (*cf.* (4.16))

$$G_j^* := G_j - r_{\Gamma_1} \widehat{F}^{(1)}_{0j} + r_{\Gamma_1} \widehat{F}_{0j} \in \widetilde{B}^{s-1}_{r,t}(\Gamma_1), \quad j = 1,2,3,$$

be satisfied, $u^{(1)} \in [W^1_p(\Omega_1)]^3$ *and* $U \in [W^1_p(\Omega)]^4$ *be a unique solution pair of the boundary-transmission problem* (2.29)-(2.37).
Then the following hold:
i) *if*

$$F^{(1)} \in [B^{s-1}_{r,r}(S_1)]^3, \quad F \in [B^{s-1}_{r,r}(S)]^4, \quad f \in [B^{s}_{r,r}(\Gamma)]^4,$$
$$g \in [B^{s}_{r,r}(\Gamma_1)]^4, \qquad G \in [B^{s-1}_{r,r}(\Gamma_1)]^3,$$

then $u^{(1)} \in [\, H_r^{s+\frac{1}{r}}(\Omega_1)\,]^3$ *and* $U \in [\, H_r^{s+\frac{1}{r}}(\Omega)\,]^4$*;*
ii) *if*

$$F^{(1)} \in [B^{s-1}_{r,t}(S_1)]^3, \quad F \in [B^{s-1}_{r,t}(S)]^4, \quad f \in [B^{s}_{r,t}(\Gamma)]^4,$$
$$g \in [B^{s}_{r,t}(\Gamma_1)]^4, \qquad G \in [B^{s-1}_{r,t}(\Gamma_1)]^3,$$

then

$$u^{(1)} \in [\, B_{r,t}^{s+\frac{1}{r}}(\Omega_1)\,]^3, \quad U \in [\, B_{r,t}^{s+\frac{1}{r}}(\Omega)\,]^4; \quad (4.94)$$

iii) *if* $\alpha > 0$ *is not integer and*

$$\begin{aligned} &F^{(1)} \in [B^{\alpha-1}_{\infty,\infty}(S_1)]^3, \quad F \in [B^{\alpha-1}_{\infty,\infty}(S)]^4, \quad f \in [C^\alpha(\overline{\Gamma})]^4, \\ &g \in [C^\alpha(\Gamma_1)]^4, \qquad G \in [B^{\alpha-1}_{\infty,\infty}(\Gamma_1)]^3, \end{aligned} \quad (4.95)$$

then

$$u^{(1)} \in \bigcap_{\alpha'<\mu} [\, C^{\alpha'}(\overline{\Omega}_1)\,]^3, \quad U \in \bigcap_{\alpha'<\mu} [\, C^{\alpha'}(\,\overline{\Omega}\,)\,]^4, \quad (4.96)$$

where $\mu = \min\{\alpha,\ \frac{1}{2} + \gamma'\}$.

Proof. The proof of items i) and ii) easily follows from Theorems 4.2 , 4.4, and Remark 4.3.
To prove (iii) we use the following embedding relations (see, e.g., [Tr1], [Tr2])

$$C^{\alpha}(\mathcal{M}) = B^{\alpha}_{\infty,\infty}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{\infty,1}(\mathcal{M}) \subset B^{\alpha-\varepsilon}_{\infty,t}(\mathcal{M})$$
$$\subset B^{\alpha-\varepsilon}_{r,t}(\mathcal{M}) \subset C^{\alpha-\varepsilon-k/r}(\mathcal{M}), \tag{4.97}$$

where ε is an arbitrary small positive number, $\mathcal{M} \subset \mathbb{R}^3$ is a compact k-dimensional ($k = 2, 3$) smooth manifold with smooth boundary, $1 \le t \le \infty$, $1 < r < \infty$, $\alpha - \varepsilon - k/r > 0$, α and $\alpha - \varepsilon - k/r$ are not integers.
From (4.95) and the embedding (4.97) the condition (4.94) follows with any $s \le \alpha - \varepsilon$.
Bearing in mind (4.93) and taking r sufficiently large and ε sufficiently small, we can put

$$s = \alpha - \varepsilon \quad \text{if} \quad \frac{1}{r} - \frac{1}{2} + \gamma'' < \alpha - \varepsilon < \frac{1}{r} + \frac{1}{2} + \gamma', \tag{4.98}$$

and

$$s \in \left(\frac{1}{r} - \frac{1}{2} + \gamma'', \frac{1}{r} + \frac{1}{2} + \gamma'\right) \quad \text{if} \quad \frac{1}{r} + \frac{1}{2} + \gamma' < \alpha - \varepsilon. \tag{4.99}$$

By (4.94) for the solution vectors we have $u^{(1)} \in [\, B^{s+\frac{1}{r}}_{r,t}(\Omega_1)\,]^3$ and $U \in [\, B^{s+\frac{1}{r}}_{r,t}(\Omega)\,]^4$ with

$$s + \frac{1}{r} = \alpha - \varepsilon + \frac{1}{r}$$

if (4.98) holds, and with

$$s + \frac{1}{r} \in \left(\frac{2}{r} - \frac{1}{2} + \gamma'', \frac{2}{r} + \frac{1}{2} + \gamma'\right)$$

if (4.99) holds. In the last case we can take

$$s + \frac{1}{r} = \frac{2}{r} + \frac{1}{2} + \gamma' - \varepsilon\,.$$

Therefore, we have either

$$u^{(1)} \in [\, B^{\alpha-\varepsilon+\frac{1}{r}}_{r,t}(\Omega_1)\,]^3, \quad U \in [\, B^{\alpha-\varepsilon+\frac{1}{r}}_{r,t}(\Omega)\,]^4,$$

or

$$u^{(1)} \in [\, B^{\frac{1}{2}+\frac{2}{r}+\gamma'-\varepsilon}_{r,t}(\Omega_1)\,]^3, \quad U \in [\, B^{\frac{1}{2}+\frac{2}{r}+\gamma'-\varepsilon}_{r,t}(\Omega)\,]^4,$$

in accordance with the inequalities (4.98) and (4.99). The last embedding in (4.97) (with $k = 3$) yields then that either

$$u^{(1)} \in [\, C^{\alpha-\varepsilon-\frac{2}{r}}(\overline{\Omega}_1)\,]^3, \quad U \in [\, C^{\alpha-\varepsilon-\frac{2}{r}}(\overline{\Omega})\,]^4,$$

or

$$u^{(1)} \in [\, C^{\frac{1}{2}-\varepsilon+\gamma'-\frac{1}{r}}(\overline{\Omega}_1)\,]^3, \quad U \in [\, C^{\frac{1}{2}-\varepsilon+\gamma'-\frac{1}{r}}(\overline{\Omega})\,]^4,$$

which lead to the inclusions

$$u^{(1)} \in [\, C^{\mu-\varepsilon-\frac{2}{r}}(\overline{\Omega}_1)\,]^3, \quad U \in [\, C^{\mu-\varepsilon-\frac{2}{r}}(\overline{\Omega})\,]^4, \tag{4.100}$$

where $\mu = \min\{\alpha,\ \dfrac{1}{2}+\gamma'\}$. Since r is sufficiently large and ε is sufficiently small, the embedding (4.100) completes the proof. □

REMARK 4.6 *More detailed analysis based on the asymptotic expansions of solutions* (*see* [CD1], [CD2], [CDD1]) *shows that for sufficiently smooth boundary data* (*e.g.,* C^∞-*smooth data say*) *the principal singular terms of the solution vectors* $u^{(1)}$ *and* U *near the curves* $\partial\Gamma_1$ *and* $\partial\Gamma$ *can be represented as a product of a "good" vector-function and the factor* $[\,\varrho(x)\,]^{\alpha_j+i\,\beta_j}$ *with* $\beta_j \neq 0$, *in general. Here* $\varrho(x)$ *is the distance from a reference point* x *to the curves* $\partial\Gamma_1$ *or* $\partial\Gamma$. *Therefore, near these curves the principal singular terms of the corresponding stress vectors* $T^{(1)}\,u^{(1)}$ *and* $\mathcal{T}U$ *are represented as a product of a "good" vector-function and the factor* $[\,\varrho(x)\,]^{-1+\alpha_j+i\,\beta_j}$. *The numbers* $\beta_j \neq 0$ *describe the oscillating character of the stress singularities.*
The exponents $\alpha_j + i\,\beta_j$ *are related to the corresponding eigenvalues of the matrices* (4.83) *and* (4.84) *by the equalities*

$$\alpha_j = \frac{1}{2} + \frac{\arg\,\lambda_j}{2\pi}, \quad \beta_j = -\frac{\ln|\lambda_j|}{2\pi}. \tag{4.101}$$

Due to (4.85) *and Theorem* 4.5 *it follows that the stress singularities at the curves* $\partial\Gamma_1$ *and* $\partial\Gamma$ *behave like* $\mathcal{O}\,(\,[\varrho(x)]^{-\frac{1}{2}+\gamma'}\,)$. *In contrast to the classical pure elasticity case* (*where* $\gamma' = \gamma'' = 0$), *here* γ' *depends on the material parameters and is different of zero, in general* (*see the example below*). *This is related to the fact that our transmission problem and, consequently, the corresponding strongly elliptic boundary integral equations system are not self-adjoint. This implies that the eigenvalues* λ_j *are complex numbers, in general.*

REMARK 4.7 *The above approach can be extended to the case when an interface crack* Γ_1^* *occurs on the surface* Γ_1, $\overline{\Gamma_1^*}$ *is a proper part of* Γ_1. *In this case, the corresponding analysis shows that for sufficiently smooth boundary data the stress vectors* $T^{(1)}\,u^{(1)}$ *and* $\mathcal{T}U$ *have singularity of type* $[\,\varrho_*(x)\,]^{-\frac{1}{2}+i\,\beta_j}$ *in a neighbourhood of the curve* $\partial\Gamma_1^*$, *where* $\varrho_*(x)$ *is the distance from a reference point* x *to the curve* $\partial\Gamma_1^*$. *The reason of such behaviour near the curve* $\partial\Gamma_1^*$ *is that the Dirichlet boundary condition for the electric potential function* φ *is given on the whole surface* Γ_1. *This gives us the possibility to reduce the corresponding transmission problem to a system of pseudodifferential equations with a positive definite principal homogeneous symbol matrix* $\sigma_*(x,\xi_1,\xi_2)$ *for all* $x \in \partial\Gamma_1^*$. *Therefore, the eigenvalues* λ_j^* *of the matrix* $[\,\sigma_*(x,0,+1)\,]^{-1}\,\sigma_*(x,0,-1)$ *are positive. This implies* $\gamma' = \gamma'' = 0$, *while* $\beta_j \neq 0$, *in general.*

REMARK 4.8 *It can be shown that if the boundaries $\partial\Omega$ and $\partial\Omega_1$, and the submanifolds S, S_1, Γ, and Γ_1 are piecewise smooth Lipschitz surfaces then Theorems 4.2 and 4.4 still remain valid with $p = 2$, $s = 1/2$, and $t = 2$. Actually, it follows from the results collected in Subsection 3.4 and the proofs of Theorems 4.2 and 4.4.*

Example

Here we apply our approach to practical examples to show the dependence of the characteristics $\gamma_k^{'}$ and $\gamma_k^{''}$ $(k = 1, 2)$ on the material parameters.
We assume that the domain Ω_1 is occupied by the isotropic metallic material *silver-palladium alloy* with Lamé constants $\lambda = 1.0 \cdot 10^{11}$ Pa and $\mu = 3.17 \cdot 10^{10}$ Pa, whereas the domain Ω is occupied by different piezoelectric media. We consider the piezoelectric materials $BaTiO_3$ (with the crystal symmetry of the class **4mm**), PZT-4 and PZT-5A (with the crystal symmetry of the class **6mm**). Their material constants are given in the tables below:

	c_{11} (Pa)	c_{12} (Pa)	c_{13} (Pa)	c_{33} (Pa)	c_{44} (Pa)	c_{66} (Pa)
$BaTiO_3$	$2.75 \cdot 10^{11}$	$1.79 \cdot 10^{11}$	$1.52 \cdot 10^{11}$	$1.69 \cdot 10^{11}$	$5.43 \cdot 10^{10}$	$1.13 \cdot 10^{11}$
PZT-4	$1.39 \cdot 10^{11}$	$7.80 \cdot 10^{10}$	$7.40 \cdot 10^{10}$	$1.15 \cdot 10^{11}$	$2.56 \cdot 10^{10}$	$3.05 \cdot 10^{10}$
PZT-5A	$1.20 \cdot 10^{11}$	$7.52 \cdot 10^{10}$	$7.51 \cdot 10^{10}$	$1.11 \cdot 10^{11}$	$2.11 \cdot 10^{10}$	$2.26 \cdot 10^{10}$

	e_{15} (C/m^2)	e_{31} (C/m^2)	e_{33} (C/m^2)	ε_{11} (F/m)	ε_{33} (F/m)
$BaTiO_3$	21.30	-2.69	3.65	$1.75 \cdot 10^{-8}$	$9.89 \cdot 10^{-10}$
PZT-4	12.70	-5.20	15.10	$6.50 \cdot 10^{-9}$	$5.60 \cdot 10^{-9}$
PZT-5A	12.29	-5.35	15.78	$8.14 \cdot 10^{-9}$	$7.32 \cdot 10^{-9}$

We remark that the constants c_{ijkl}, e_{ikl}, and c_{pq}, e_{pq} are related by the following rule:

$$c_{f(ij)f(kl)} = c_{ijkl}, \quad e_{i\,f(kl)} = e_{ikl},$$

where

$$f(11) = 1, \qquad f(22) = 2, \qquad f(33) = 3,$$
$$f(23) = f(32) = 4, \quad f(13) = f(31) = 5, \quad f(12) = f(21) = 6.$$

Moreover, for the above piezoelectric materials there hold:

$$c_{kj} = c_{jk}, \quad c_{11} = c_{22}, \quad c_{13} = c_{23}, \quad c_{44} = c_{55},$$
$$c_{ij} = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad i, j = 4, 5, 6;$$
$$e_{24} = e_{15}, \quad e_{31} = e_{32}, \quad e_{1i} = e_{2j} = e_{3k} = 0 \text{for} i \neq 5,\ j \neq 4,\ k > 3;$$
$$\varepsilon_{11} = \varepsilon_{22}, \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0.$$

To compute the smoothness, that means the singularity exponents mentioned in Theorem 4.5 and Remark 4.6 for the corresponding solutions of the boundary transmission problem near the curves $\partial\Gamma_1$ and $\partial\Gamma$, we have to find the eigenvalues of the matrices (4.83) and (4.84).
To this end we have to go over to an appropriate local co-ordinate system in a neighbouhood of $x \in \partial\Gamma_1 \cup \partial\Gamma$ and to calculate the principal homogeneous symbol matrices of the pseudodifferential operators $\mathcal{H}^{(1)}$, $-2^{-1}I_3+\mathcal{K}^{(1)}$, $\mathcal{H}$, and $2^{-1}I_4+\mathcal{K}$. Denoting these symbols by

$$M^{(1)}(x,\xi_1,\xi_2) := \sigma(\mathcal{H}^{(1)})(x,\xi_1,\xi_2),\ N^{(1)}(x,\xi_1,\xi_2) := \sigma(\,-\,2^{-1}I_3+\mathcal{K}^{(1)})(x,\xi_1,\xi_2),\ x \in \partial\Gamma_1,$$
$$M(x,\xi_1,\xi_2) := \sigma(\mathcal{H})(x,\xi_1,\xi_2), \qquad N(x,\xi_1,\xi_2) := \sigma(2^{-1}I_4+\mathcal{K})(x,\xi_1,\xi_2), \qquad x \in \partial\Gamma,$$

we get for the principal homogeneous symbol matrices of the operators $\mathcal{B}^{(1)}$ and $\mathcal{B}$ (see (3.12))

$$\sigma(\mathcal{B}^{(1)}) = N^{(1)}\,[\,M^{(1)}\,]^{-1}, \quad \sigma(\mathcal{B}) = -N\ M^{-1}.$$

Due to equalities (4.67) we have then

$$\begin{aligned} &\sigma_1(x,\xi_1,\xi_2) := \sigma(\mathcal{P}^{(1)}) = -M\ N^{-1} + K \ \text{ for } \ x \in \partial\Gamma_1, \\ &\sigma_2(x,\xi_1,\xi_2) := \sigma(\mathcal{P}^{(2)}) = -M\ N^{-1} \ \text{ for } \ x \in \partial\Gamma, \end{aligned} \tag{4.102}$$

where

$$K := \begin{bmatrix} M^{(1)}\,[\,N^{(1)}\,]^{-1} & [\,0\,]_{3\times1} \\ [\,0\,]_{1\times3} & 0 \end{bmatrix}_{4\times4}.$$

The eigenvalues of the matrices $[\,\sigma_k(x,0,+1)\,]^{-1}\,\sigma_k(x,0,-1)$, $k=1,2$, are computed with the help of the package "Mathematica" (version 5).

Global singularity effects. We show numerical results concerning the global dominant stress singularities in the composed structure. They characterise possible maximal stress singularities in the closed domain $\overline{\Omega} \cup \overline{\Omega_1}$ and are related to the computation of the paratmeters $\gamma_k^{'}$ and $\gamma_k^{''}$.
The calculations have shown that arg $\lambda_j^{(1)}(x)$ and arg $\lambda_j^{(2)}(x)$ $(j=1,2,3,4)$ do not depend on the reference point x. Moreover, $\gamma_k^{'} \le 0$, $\gamma_k^{''} \ge 0$, and $\gamma_k^{'} = -\gamma_k^{''}$, $k=1,2$ (see (4.82)-(4.81)). The computed values of $\gamma_1^{'}$ and $\gamma_2^{'}$ corresponding to the considered three cases are as follows

	$BaTiO_3$	PZT-4	PZT-5A
$\gamma_1^{'}$	-0.06	-0.08	-0.09
$\gamma_2^{'}$	-0.12	-0.12	-0.13.

We recall that the real part of the stress singularity exponent near the curve where the type of boundary conditions change (the curve Γ) is $-\frac{1}{2}+\gamma_2^{'}$, while near the curve Γ_1 (where the interface intersects the exterior boundary) it equals to $-\frac{1}{2}+\gamma_1^{'}$. From the above numerical results it is evident that the stress singularities near the curve Γ are higher than near the curve Γ_1:

$$\begin{array}{lcccr} & \mathrm{BaTiO_3} & \text{PZT-4} & \text{PZT-5A} & \\ \text{Stress singularity exponent at } \Gamma_1 & -0.56 & -0.58 & -0.59 & (4.103) \\ \text{Stress singularity exponent at } \Gamma & -0.62 & -0.62 & -0.63\,. & \end{array}$$

We have done also computations showing the influence of the electric constants $e_{15},\ e_{31},\ e_{33}$ on the parameters $\gamma_1^{'}$ and $\gamma_2^{'}$. To this end we have performed the above calculations (for $BaTiO_3$ and PZT-5A) with the constants $t\,e_{kj}$ for e_{kj}. Here t ranges in the interval (0.0, 100.0). The corresponding graphs are presented in Figure 2 (for $BaTiO_3$) and Figure 3 (for PZT-5A). We see that for small values of the electric constants (i.e., for small values of t) the parameters $\gamma_1^{'}$ and $\gamma_2^{'}$ vanish, and consequently the stress singularity exponents are equal to $-\frac{1}{2}$ (as in the classical pure elasticity case). As the numerical experiment shows the growth of t implies the bounded monotonic growth of the values $|\gamma_1^{'}|$ and $|\gamma_2^{'}|$ with a certain stabilization.

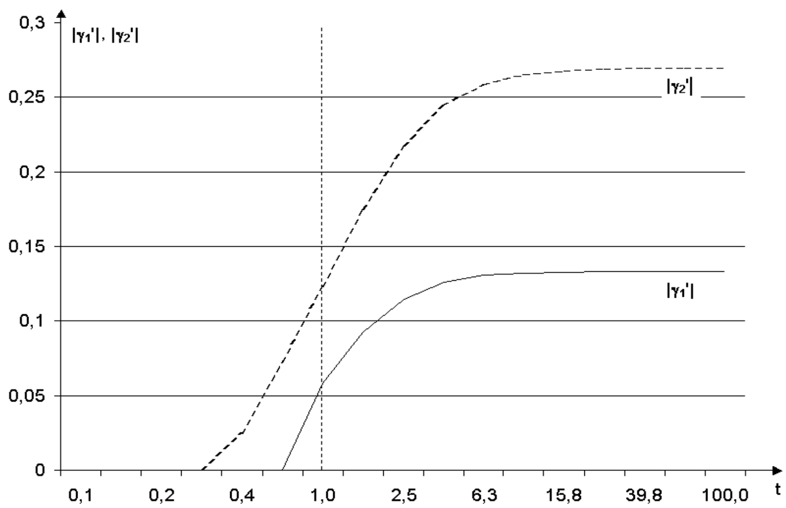

Figure 2: Dependence of $|\gamma'_1|$ and $|\gamma'_2|$ on t for $BaTiO_3$

Local singularity effects at different edges. Here we compare the dominant stress singularity exponents calculated for the curves $\partial\Gamma$ and $\partial\Gamma_1$.

Note that factors of type $[\,\varrho(x)\,]^{a_j+i\,b_j}$ appear in the singular edge terms of the stress fields, where the exponents are complex valued, in general. Here $\varrho(x)$ is the distance from a reference point x to the curves $\partial\Gamma_1$ or $\partial\Gamma$. The exponents $a_j + i\,b_j$ are related to the eigenvalues of the matrices (4.102) by the equalities (see Remark 4.6)

$$a_j = -\frac{1}{2} + \frac{\arg\,\lambda_j}{2\pi}, \quad b_j = -\frac{\ln|\lambda_j|}{2\pi}.$$

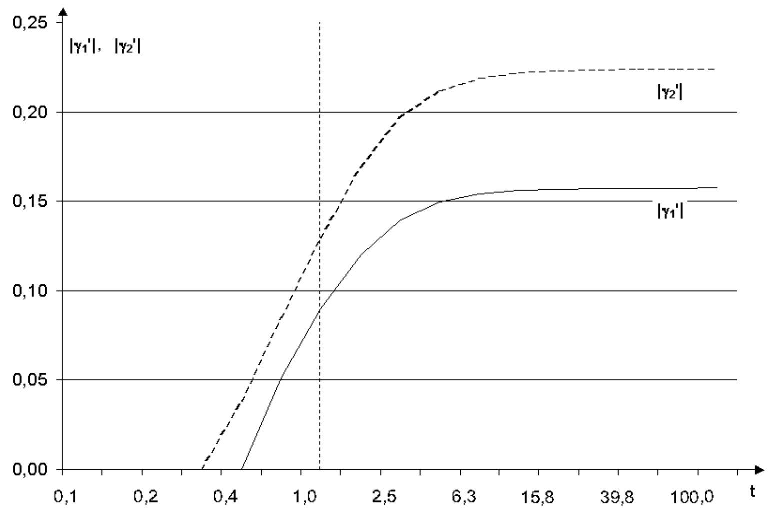

Figure 3: Dependence of $|\gamma'_1|$ and $|\gamma'_2|$ on t for PZT-5A

The computations have shown that for the above mentioned piezoelectric materials $BaTiO_3$, PZT-4, and PZT-5A the parameters b_j (characterizing the oscillating singularity effect) vanish, which means that the moduli of all four eigenvalues equal to 1. Moreover, two of them (λ_1 and λ_2 say) are mutally inverse complex numbers:

$$\lambda_1 = \exp\{-i\,\theta\}, \quad \lambda_2 = \exp\{i\,\theta\}, \quad \theta > 0;$$

another two eigenvalues are equal to 1: $\lambda_3 = \lambda_4 = 1$. Therefore, the eigenvalue λ_1 with the negative argument $-\theta$ corresponds to the dominant stress singularity term whose singularity exponents $(-\frac{1}{2} - \frac{\theta}{2\pi})$ are given in the table (4.103).
However, the calculations performed for the disturbed electric constants te_{15}, te_{31}, te_{33} have shown that for small t there appear oscillating singularities. For all values of the parameter t two of the eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ are again mutually inverse complex numbers and $\lambda_3(t) = \lambda_4(t) = 1$.
The corresponding tables for the principal singularity exponents $a_1 + i\,b_1$ corresponding to the curves $\partial\Gamma$ and $\partial\Gamma_1$ are as follows:

(i) for $BaTi_3$

t	a_1 $(\partial\Gamma)$	b_1 $(\partial\Gamma)$	a_1 $(\partial\Gamma_1)$	b_1 $(\partial\Gamma_1)$
0.000	−0.500	0.059	−0.500	0.061
0.158	−0.500	0.053	−0.500	0.059
0.251	−0.500	0.044	−0.500	0.055
0.398	−0.518	0.000	−0.500	0.047
0.631	−0.573	0.000	−0.500	0.016
1.000	−0.623	0.000	−0.558	0.000
1.585	−0.675	0.000	−0.592	0.000 ;

(ii) for PZT-5A

t	$a_1\ (\partial\Gamma)$	$b_1\ (\partial\Gamma)$	$a_1\ (\partial\Gamma_1)$	$b_1\ (\partial\Gamma_1)$
0.000	−0.500	0.057	−0.500	0.059
0.158	−0.500	0.050	−0.500	0.055
0.251	−0.500	0.036	−0.500	0.048
0.398	−0.538	0.000	−0.500	0.027
0.631	−0.585	0.000	−0.551	0.000
1.000	−0.630	0.000	−0.590	0.000
1.585	−0.669	0.000	−0.620	0.000 .

The results obtained for the principal singularity exponent $a_1(t) + i\,b_1(t)$ for $BaTiO_3$ are given in Figure 4 below.

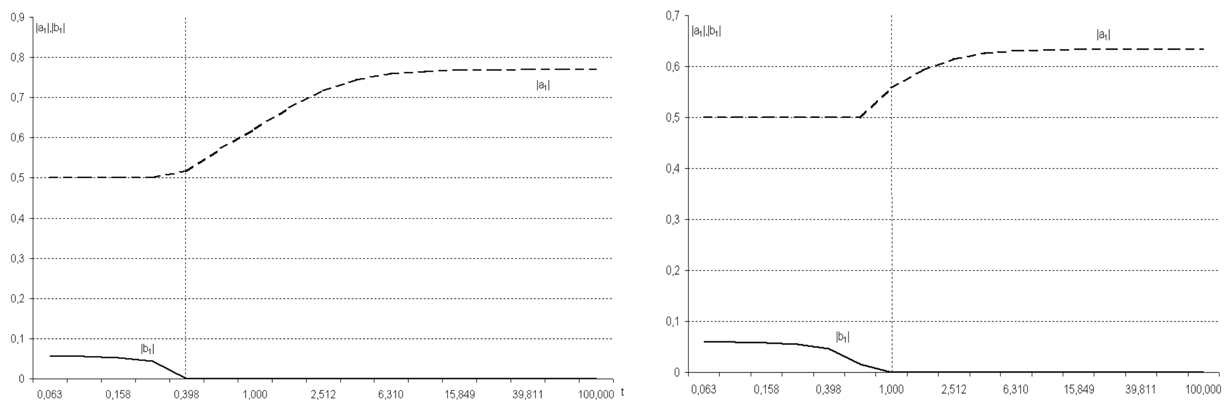

Figure 4: Dependence of $|a_1|$ and $|b_1|$ on t for $BaTiO_3$ at the curve $\partial\Gamma$ and $\partial\Gamma_1$

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