Universität Stuttgart

Regularity results for linear elliptic boundary value problems in polygons

Lectures at the Charles University Prague, Oct.05

Prof. Dr. Anna-Margarete Sändig

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Vorlesungsskript 2006/010

Universität Stuttgart

Regularity results for linear elliptic
boundary value problems in polygons

Lectures at the Charles University Prague, Oct.05

Prof. Dr. Anna-Margarete Sändig

**Berichte aus dem Institut für
Angewandte Analysis und Numerische
Simulation**

Vorlesungsskript 2006/010

Institut für Angewandte Analysis und Numerische Simulation (IANS)
Fakultät Mathematik und Physik
Fachbereich Mathematik
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: ians-preprints@mathematik.uni-stuttgart.de
WWW: http://preprints.ians.uni-stuttgart.de

ISSN **1611-4176**

Contents

Chapter 1

Introduction

The general theory of linear elliptic boundary value problems in **smooth domains** was developed in the second half of the 20th century. Fundamental results are:
a priori estimates for the solutions in different function spaces, the Fredholm property of the operator corresponding to the boundary value problem, regularity theorems.

The treatment of linear elliptic boundary value problems in **domains with corners** required a new theory; in particular, regularity results for smooth domains do not hold in non-smooth domains. The pioneering work for a general theory in domains with angular and conical points was done by G.I.Eskin(1963/70), Ya.B. Lopatinsky(1963) and V.A.Kondrat'ev(1963/67). The Mellin-technique was developed in order to prove a priori estimates and Fredholm properties in weighted Sobolev spaces, so called Kondrat'ev spaces. Asymptotic expansions of the solutions near conical points play a central role and describe the regularity of the solution accurately.

The 6 hours-course (10.-14.10.2005) accompanied by discussions and consultations will be a short and compact introduction into the theory of elliptic boundary value problems in polygons.

- We will start with an introducing example, the Dirichlet problem for the Laplacian, and demonstrate its treatment.
- We explain the Mellin transform and the Kondrat'ev spaces.
- We derive a decomposition of the solution in a singular and a regular part.
- Finally, we consider general elliptic boundary value problems and illustrate the general theory by examples.

The goal is to enable the participants of the course to calculate the singular terms of solutions for different examples and thus to get regularity results.

Chapter 2

An example

We start with an introducing example, the Dirichlet problem for the Poisson equation in a bounded domain $\Omega \in I\!R^2$. The weak formulation reads: Find $u \in \overset{\circ}{W}{}^{1,2}(\Omega) = V$ such that for $f \in V'$

$$a(u,v) = \int\limits_{\Omega} \nabla u \cdot \nabla v = \langle f, v\rangle \quad \forall v \in V\,. \tag{2.1}$$

If f is smooth enough, then the question appears, whether the solution u is smooth as well. For instance, if f belongs to $L_2(\Omega)$, is then $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap W^{2,2}(\Omega)$? If the answer is yes, then u satisfies the strong formulated problem

$$-\Delta u = f \quad \text{in}\,\Omega \tag{2.2}$$
$$u = 0 \quad \text{on}\,\partial\Omega \tag{2.3}$$

where $\Delta : W^{2,2}(\Omega) \to L_2(\Omega)$ and the trace of u on $\partial\Omega$ is well defined. Unfortunately, the answer is negative if Ω is a polygon with reentrant corners, see Figure 2.1.

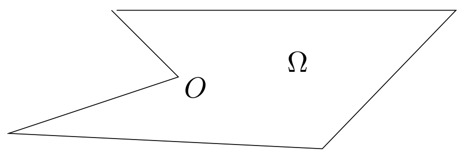

Figure 2.1: polygon with a reentrant corner $O = (0,0)$

We describe how to proceed in order to get regularity results for the solution u in polygons. For simplicity we assume the domain has only one corner point O, see Figure 2.2.

1. Step: Localisation

Let be $B_\varepsilon(O) = \{x : |x| < \varepsilon\}$ a circle with small radius ε and $\eta \in C_0^\infty(B_\varepsilon(O))$ a cutoff function with $0 \le \eta \le 1, \eta \equiv 1$ on $B_{\frac{\varepsilon}{2}}(O)$. We consider a weak solution $u \in V$ of problem

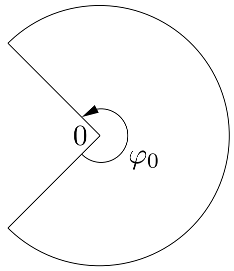

Figure 2.2: Domain with one corner point

(2.1) and assume that $f \in L_2(\Omega)$. The function ηu is well-defined for $u \in W^{1,2}(\Omega)$ and it holds in the distribution sense

$$\begin{aligned} -\Delta \eta u = -\eta \Delta u - u \Delta \eta - 2\nabla u \nabla \eta &= F \quad \text{in}\, \Omega \\ \eta u &= 0 \quad \text{on}\, \partial\Omega\,, \end{aligned}$$

where $F \in L_2(\Omega)$.
We investigate the behaviour of $\eta u = v$ in the infinite cone

$$K = K_{\varphi_0} = \{(r, \varphi), 0 < r < \infty, 0 < \varphi < \varphi_0\}.$$

The function v satisfies the boundary value problem

$$\begin{aligned} -\Delta v &= F \quad \text{in}\, K, \\ v &= 0 \quad \text{on}\, \partial K\,. \end{aligned} \tag{2.4}$$

2. Step: Solutions in K

A solution of (2.4) can be decomposed additively into the general solution of the homogeneous problem and a particular solution of the inhomogeneous problem. We start with the homogeneous problem:

$$\begin{aligned} -\Delta v &= 0 \quad \text{in}\, K, \\ v &= 0 \quad \text{on}\, \partial K \end{aligned} \tag{2.5}$$

We introduce polar coordinates (r, φ) in (2.5) and get

$$\begin{aligned} \Delta v = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right] v(r,\varphi) &= 0 \quad \text{in}\, K, \\ v(r,0) &= 0, \\ v(r,\varphi_0) &= 0. \end{aligned} \tag{2.6}$$

Similar to the derivation of the solution to the Dirichlet problem in the circle by the Fourier method we use the separation-ansatz

$$v(r, \varphi) = r^{\alpha} \varphi_{\alpha}(\varphi)\,.$$

Then, problem (2.6) reads

$$\begin{aligned} \alpha(\alpha-1)r^{\alpha-2}\Phi_\alpha(\varphi)+\alpha r^{\alpha-2}\Phi_\alpha(\varphi)+r^{\alpha-2}\Phi''_\alpha(\varphi)=0 \quad &\text{for } 0<\varphi<\varphi_0, 0<r<\infty \\ r^\alpha\Phi_\alpha(\varphi)=0 \quad &\text{for } \varphi=0, \varphi=\varphi_0, 0<r<\infty. \end{aligned} \tag{2.7}$$

The resulting quadratic boundary eigenvalue problem

$$\begin{aligned} \alpha^2\Phi_\alpha(\varphi)+\Phi''_\alpha(\varphi)=0 \quad &\text{for } 0<\varphi<\varphi_0, \\ \Phi_\alpha(\varphi)=0 \quad &\text{for } \varphi=0, \varphi=\varphi_0 \end{aligned} \tag{2.8}$$

has the eigenvalues $\alpha_k = \frac{k\pi}{\varphi_0}$, $k=\pm1,\pm2,\cdots$ with the corresponding eigensolutions

$$\Phi_{\alpha_k}(\varphi)=\sin\alpha_k\varphi=\sin\frac{k\pi}{\varphi_0}\varphi. \tag{2.9}$$

Thus, we have constructed solutions of the homogeneous boundary value problem (2.6):

$$v_k(r,\varphi)=r^{\alpha_k}\sin\alpha_k\varphi=r^{\frac{k\pi}{\varphi_0}}\sin\frac{k\pi}{\varphi_0}\varphi \quad \text{for } k=\pm1,\pm2,\cdots \tag{2.10}$$

Since we consider solutions in $W^{1,2}(K)$, we obtain the restriction $\alpha_k \geq 0$ and therefore $k=1,2,\cdots$. If we want to ensure, that $v\in W^{2,2}(\Omega)$, then we have to demand that $\alpha_k\geq 1$. This is not satisfied for reentrant corners where $\alpha_1=\frac{\pi}{\varphi_0}<1$.

We decompose the solution $v_{\text{hom}}\in W^{1,2}(K)$ of the homogeneous problem in the cone K into the sum

$$v_{\text{hom}}=(\eta u)_{\text{hom}}=cr^{\frac{\pi}{\varphi_0}}\sin\frac{\pi}{\varphi_0}\varphi+v_{\text{hom,reg}}$$

with $v_{\text{hom,reg}}=\sum_{k=2}^\infty c_k r^{\frac{k\pi}{\varphi_0}}\sin\frac{k\pi}{\varphi_0}\varphi\in W^{2,2}(K)$. Adding a sufficiently smooth particular solution v_p of the inhomogeneous problem

$$\begin{aligned} -\Delta v = F \quad &\text{in } K, \\ v=0 \quad &\text{on } \partial K\,, \end{aligned}$$

it follows

$$\eta u = v = v_{\text{hom}}+v_p = cr^{\frac{\pi}{\varphi_0}}\sin\frac{\pi}{\varphi_0}\varphi+v_{\text{reg}}. \tag{2.11}$$

Later in theorem 2, we will prove that $v_{\text{reg}}=O(r^{\frac{2\pi}{\varphi_0}\varphi})$ and therefore it follows that $v_{\text{reg}}\in W^{2,2}(\Omega)$. If we multiply ηu with a cutoff function $\tilde\eta$ such that $\tilde\eta\equiv 1$ on $\operatorname{supp}\eta$, then we get by substituting $\eta\tilde\eta=\kappa$

$$\kappa u=\tilde\eta c r^{\frac{\pi}{\varphi_0}}\sin\frac{\pi}{\varphi_0}+\tilde\eta v_{\text{reg}}. \tag{2.12}$$

3. Step: Solution in the bounded domain Ω

We decompose the solution of problem (2.2) in the domain Ω as follows:

$$\begin{aligned} u &= \kappa u+(1-\kappa)u \\ &= \tilde\eta c r^{\frac{\pi}{\varphi_0}}\sin\frac{\pi}{\varphi_0}+u_{\text{reg}}. \end{aligned} \tag{2.13}$$

Here $(1-\kappa)u$ is regular. If we start with (2.11), then we get an equivalent decomposition of the solution

$$\begin{aligned} u &= \eta u - (1-\eta)u \\ &= c r^{\frac{\pi}{\varphi_0}} \sin \frac{\pi}{\varphi_0}\varphi + \hat{u}_{\text{reg}}, \end{aligned} \tag{2.14}$$

where $\hat{u}_{\text{reg}} \in W^{2,2}(\Omega)$. The coefficient c characterises the intensity of the singular function and is called "singular coefficient". In the linear elasticity this coefficient is called generalized stress intensity factor, in fracture mechanics ($\varphi_0 = 2\pi$) it is called K–factor or stress intensity factor. It plays a role in fracture criteria in linear elasticity. The coefficient c can be computed in several ways. We prove a formula which goes back to Mazya Plamenevskii,[7].

Lemma 1 (coefficient formula) *If* $\frac{\pi}{\varphi_0} < 1$ *the coefficient* c *in* (2.14) *can be calculated by the formula*

$$c = -\frac{1}{\pi}\int\limits_{\Omega} (f s_- + u\Delta s_-)\, \mathrm{d}x\,, \tag{2.15}$$

where $f \in L_2(\Omega), s_- = \eta r^{-\frac{\pi}{\varphi_0}} \sin(-\frac{\pi}{\varphi_0})\varphi$. *That means*

$$c = \frac{1}{\pi}\int\limits_{\Omega} f\eta r^{-\frac{\pi}{\varphi_0}} \sin(\frac{\pi}{\varphi_0}\varphi) + u\Delta(\eta r^{-\frac{\pi}{\varphi_0}} \sin(\frac{\pi}{\varphi_0}\varphi))\, \mathrm{d}x\,. \tag{2.16}$$

Proof

We consider the domain $\Omega\backslash\overline{B}_\delta = \Omega_\delta$, where B_δ is a circle with radius δ and centre O.

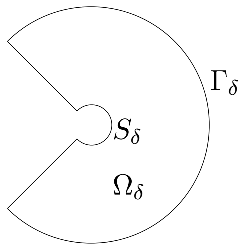

Figure 2.3: The domain Ω_δ

Here, δ is so small that $\eta \equiv 1$ on $\overline{B}_\delta$. In the domain Ω_δ we apply the second Green's formula to the functions u (given by (2.14)), and s_-:

$$\int\limits_{\Omega_\delta} (\Delta u s_- - u\Delta s_-)\, \mathrm{d}x = \int\limits_{\partial\Omega_\delta} \left(\frac{\partial u}{\partial n}s_- - u\frac{\partial s_-}{\partial n}\right) \mathrm{d}\sigma\,. \tag{2.17}$$

It is $\partial\Omega_\delta = \Gamma_\delta \cup S_\delta$, Γ_δ is a part of the boundary $\partial\Omega$ and S_δ is a part of the boundary ∂B_δ (see Figure 2.3). Since s_- and u vanish on Γ_δ, the right hand side of (2.17) can be written as

$$\begin{aligned}
\int\limits_{S_\delta} & \left(\frac{\partial u}{\partial n} s_- - u\frac{\partial s_-}{\partial n}\right) \mathrm{d}\sigma = \\
= & -\delta \int\limits_0^{\varphi_0} c\left[\left(\frac{\pi}{\varphi_0}\right)\delta^{\frac{\pi}{\varphi_0}-1}\sin\left(\frac{\pi}{\varphi_0}\right)\varphi + \ldots\right]\left[\delta^{-\frac{\pi}{\varphi_0}}\sin\left(\frac{-\pi}{\varphi_0}\varphi\right)\right] \mathrm{d}\varphi \\
& -\delta \int\limits_0^{\varphi_0} c\left[\delta^{\frac{\pi}{\varphi_0}}\sin\left(\frac{\pi}{\varphi_0}\varphi\right) + \cdots\right]\left[\frac{\pi}{\varphi_0}\delta^{-\frac{\pi}{\varphi_0}-1}\sin\left(\frac{-\pi}{\varphi_0}\varphi\right)\right] \mathrm{d}\varphi \\
= & \; 2c\frac{\pi}{\varphi_0}\int\limits_0^{\varphi_0}\left(\sin\left(\frac{\pi}{\varphi_0}\varphi\right)\right)^2 \mathrm{d}\varphi \\
& + \text{expressions with positive exponents of } \delta.
\end{aligned}$$

For $\delta \to 0$, we get for the right hand side of (2.17)

$$\lim_{\delta\to 0}\int\limits_{\partial\Omega_\delta}\left(\frac{\partial u}{\partial n}s_- - u\frac{\partial s_-}{\partial n}\right)\mathrm{d}\sigma = 2c\frac{\pi}{\varphi_0}\frac{\varphi_0}{2} = c\pi.$$

Since $s_- \in L_2(\Omega)$ and $u\Delta s_- = u\Delta\left(\eta r^{-\frac{\pi}{\varphi_0}}\sin(\frac{-\pi}{\varphi}\varphi)\right) \in L_2(\Omega)$, we obtain formula (2.16). ∎

There are some open problems:

- Does this special ansatz lead to all possible singular terms?
- Can every weak solution decomposed as in (2.13) or (2.14) ?
- Which assumption has the right hand side f to satisfy ?
- Is it possible to apply this method to general elliptic boundary value problems too?

These questions are answered by articels of [3], [4], [5], [6], [7] and [8]. Here, we follow the ideas of Kondrat'ev, who has developed a Mellin-technique for the treatment of linear elliptic boundary value problems in domains with conical points. The appropriate function spaces for the application of this technique are weighted Sobolev spaces.

Exercises

1. Let be $\Omega_0 = \{(r,\varphi) : 0 < r < r_0, 0 < \varphi < \varphi_0\}$.
   For which α is $v(r,\varphi) = r^\alpha \sin(\alpha\varphi)$ from $L_2(\Omega_0), W^{1,2}(\Omega_0)$ or $W^{2,2}(\Omega_0)$ respectively ?
2. Construct solutions of the following homogeneous boundary value problems in K
   a) Neumann problem:

$$\begin{aligned}
-\Delta v &= 0 \quad \text{in}\, K \\
\frac{\partial v}{\partial n} &= 0 \quad \text{on}\, \partial K\,.
\end{aligned}$$

b) Mixed boundary value problem:

$$\begin{aligned} \Delta v &= 0 \quad \text{in}\, K \\ \frac{\partial v}{\partial n} &= 0 \quad \text{for}\, \varphi = \varphi_0 \\ v &= 0 \quad \text{for}\, \varphi = 0\,. \end{aligned}$$

3. Derive a coefficient formula analogously to Lemma 1, where instead s_- the function $s_-^* = r^{-\frac{\pi}{\varphi_0}} \sin(-\frac{\pi}{\varphi_0}\varphi)$ appears.

Chapter 3

Mellin transformation and weighted Sobolev spaces

In this chapter we present tools which allow to transfer the ideas of chapter 2 to general elliptic boundary value problems. We start with the Mellin transformation.

3.1 Mellin transformation

The Fourier transformation for functions $u \in L^1(\mathbb{R}^1)$ is defined by

$$\mathcal{F}[u](\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u(x) e^{-i\xi x} \, \mathrm{d}x, \quad \xi \in \mathbb{R}^1.$$

Now, we consider the so called complex Fourier transformation

$$\mathcal{F}[f](\lambda) = \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{-i\lambda x} \, \mathrm{d}x, \quad \lambda \in \mathbb{C}. \tag{3.1}$$

The Mellin transformation is defined as

$$\mathcal{M}[f](\alpha) = \tilde{f}(\alpha) := \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{\infty} r^{-\alpha-1} f(r) \, \mathrm{d}r, \quad \alpha \in \mathbb{C}. \tag{3.2}$$

If we set $x = \ln r$ in (3.1), then it follows

$$\begin{aligned}
\mathcal{F}[f](\lambda) \quad &= \quad \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{-ix\lambda} \, \mathrm{d}x \\
&\overset{e^x = r}{=} \quad \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{\infty} f(\ln r) r^{-i\lambda-1} \, \mathrm{d}r \\
&= \quad \mathcal{M}[f(\ln r)](i\lambda) = \mathcal{M} \circ \mathcal{E}(f)(i\lambda),
\end{aligned}$$

where $\mathcal{E}(f) = f(\ln r) = F(r)$ is called Euler transformation. With it, we can express the relation between the complex Fourier transformation and the Mellin transformation

$$\mathcal{F} = \mathcal{M} \circ \mathcal{E},\ \mathcal{M}[F] = \mathcal{F}[f]\,. \tag{3.3}$$

Taking into account the properties of the complex Fourier transformation we obtain the following properties of the Mellin transformation:

(M1) Let be $h = -\operatorname{Re}\alpha$ (real part of α) and

$$L^2_{\frac{2h-1}{2}}(I\!R^+) = \{f : \int\limits_0^\infty |f(r)|^2 r^{2h-1}\,\mathrm{d}r < \infty\}\,.$$

The restriction $\mathcal{M}_h = \mathcal{M}|_{L^2_{\frac{2h-1}{2}}(I\!R^+)}$ is an isomorphism between the spaces $L^2_{\frac{2h-1}{2}}(I\!R^+)$ and $L^2(-h + iI\!R)$ that means

$$\mathcal{M}_h : L^2_{\frac{2h-1}{2}}(I\!R^+) \leftrightarrow L^2(-h + iI\!R)\,.$$

(M2) The inverse Mellin transformation is given by

$$\mathcal{M}_h^{-1}[\tilde{f}](r) = \frac{1}{i\sqrt{2\pi}} \int\limits_{-h-i\infty}^{-h+i\infty} r^\alpha \tilde{f}(\alpha)\,\mathrm{d}\alpha\,. \tag{3.4}$$

For a fixed $h = -\operatorname{Re}\alpha$ it holds the Parseval identity:

$$\int\limits_0^\infty |f(r)|^2 r^{2h-1}\,\mathrm{d}r = \frac{1}{i} \int\limits_{-h-i\infty}^{-h+i\infty} |\tilde{f}(\alpha)|^2\,\mathrm{d}\alpha.$$

(M3) It holds for the derivatives

$$\mathcal{M}[(r\frac{\mathrm{d}}{\mathrm{d}r})^k f](\alpha) = \alpha^k \mathcal{M}[f](\alpha) = \alpha^k \tilde{f}(\alpha). \tag{3.5}$$

(M4) The Parseval identity for derivatives reads

$$\int\limits_0^\infty \left| \left(r\frac{\mathrm{d}}{\mathrm{d}r} \right)^k f \right|^2 r^{2h-1}\,\mathrm{d}r = \frac{1}{i} \int\limits_{-h-i\infty}^{-h+i\infty} |\alpha|^{2k} |\tilde{f}(\alpha)|^2\,\mathrm{d}\alpha.$$

We take a look at the introducing example (2.2) to understand, why the Mellin transformation is important for boundary value problems in an infinite cone.
The 2D Laplace operator in polar coordinates reads

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = r^{-2}\left[r^2 \frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \varphi^2} \right] \\ &= r^{-2}\left[\left(r\frac{\partial}{\partial r} \right)^2 + \frac{\partial^2}{\partial \varphi^2} \right]. \end{aligned}$$

The application of the Mellin transformation yields to

$$\begin{aligned}\mathcal{M}(r^2\Delta u) &= \mathcal{M}\left[\left(r\frac{\partial}{\partial r}\right)^2 u(r,\varphi) + \frac{\partial^2}{\partial\varphi^2}u(r,\varphi)\right] \\ &\overset{(M3)}{=} \alpha^2\tilde{u}(\alpha,\varphi) + \frac{\partial^2}{\partial\varphi^2}\tilde{u}(\alpha,\varphi). \qquad (3.6)\end{aligned}$$

This Mellin transformed equation coincides with the differential equation (2.8), which we have got from the ansatz $r^\alpha\Phi_\alpha(\varphi) = v(r,\varphi)$.
If we consider the inhomogeneous Laplace equation

$$\Delta u = F \quad \text{in } K,$$

then the Mellin transformation

$$\mathcal{M}(r^2\Delta u) = \mathcal{M}(r^2F)$$

should be defined. Looking at the properties $(M1)$ and $(M2)$ this demands the existence of the integrals

$$\int_0^\infty r^{2h-1}\left|r^2F(r)\right|^2 \,\mathrm{d}r, \qquad (3.7)$$

i.e. the right hand side F should be in a Sobolev space with the weight r^β. These weighted spaces were established by Kondrat'ev in 1967 [5] and are called Kondrat'ev spaces nowadays.

3.2 Weighted Sobolev spaces

Let us define the following weighted Sobolev spaces.

Definition 1 *Let K be an infinite cone $K = \{(r,\varphi) : 0 < r < \infty,\, 0 < \varphi < \varphi_0\}$ and*

$$C^\infty_{\{0\}}(K) = \{u \in C^\infty(\overline{K}) : \operatorname{supp} u \,\text{is bounded}\,, \operatorname{supp} u \cap \{0\} = \emptyset\}.$$

For a given $\beta \in I\!R$ the space $V^{k,p}_\beta(K)$ is the closure of $C^\infty_{\{0\}}(K)$ with respect to the norm

$$\|u\|_{V^{k,p}_\beta(K)} = \left(\sum_{|\alpha|\le k}\int_K |D^\alpha u(x)|^p\, |x|^{p(\beta-k+|\alpha|)}\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

In particular, it is $V^{0,p}_\beta(K) =: L^p_\beta(K)$.

Lemma 2 *Let F be in $L^2_\beta(K)$ with $\beta = h+1$. Then $r^2F \in L^2_{\frac{2h-1}{2}}(I\!R^+)$ and therefore r^2F is Mellin-transformable.*

Proof

$F \in L^2_\beta(K)$ means, that

$$\int_K |x|^{2\beta} |F(x)|^2 \, \mathrm{d}x < \infty \, .$$

Now, in polar coordinates,

$$\int_K |x|^{2\beta}|F(x)|^2 \, \mathrm{d}x = \int_0^{\varphi_0} \int_0^\infty r^{2\beta} |F(r,\varphi)|^2 r \, \mathrm{d}r \, \mathrm{d}\varphi = \int_0^{\varphi_0} \left(\int_0^\infty r^{2h-1} |r^2 F(r,\varphi)|^2 \, \mathrm{d}r \right) \mathrm{d}\varphi < \infty$$

and so

$$\int_0^\infty r^{2h-1} |r^2 F(r,\varphi)|^2 \, \mathrm{d}r < \infty \, , a.e. \, .$$

■

It holds the following theorem about the solvability of the model problem

$$\begin{aligned} \Delta_{r,\varphi} u(r,\varphi) &= F(r,\varphi) \quad && \text{in } K, \\ u(r,\varphi) &= 0 && \text{on } \partial K. \end{aligned} \tag{3.8}$$

Theorem 1 *Let be $F \in L^2_\beta(K)$ with $\beta = h+1$. A solution $u_h \in V^{2,2}_\beta(K)$ of the boundary value problem* (3.8) *exists if and only if there are no eigenvalues α of the Mellin transformed boundary value problem*

$$\begin{aligned} \alpha^2 \tilde{u}(\alpha,\varphi) + \tfrac{\partial^2 \tilde{u}(\alpha,\varphi)}{\partial \varphi^2} &= 0, \quad 0 < \varphi < \varphi_0 \, , \\ \tilde{u}(\alpha,0) = \tilde{u}(\alpha,\varphi_0) &= 0 \end{aligned} \tag{3.9}$$

on the line $\operatorname{Re}\alpha = -h = -\beta + 1$. *Furthermore, the estimate*

$$\|u_h\|_{V^{2,2}_\beta(K)} \le c \|F\|_{L^2_\beta(K)} \tag{3.10}$$

holds with a constant c, independent of u_h.

Sketch of the proof

According to lemma 2 the following procedure is justified: multiplication of the differential equation in (3.8) with r^2, application of the Mellin transformation. This yields to

$$\begin{aligned} \alpha^2 \tilde{u}(\alpha,\varphi) + \tfrac{\partial^2 \tilde{u}(\alpha,\varphi)}{\partial \varphi^2} &= \mathcal{M}(r^2 F)(\alpha,\varphi) \quad && 0 < \varphi < \varphi_0 \\ \tilde{u}(\alpha,\varphi) &= 0 && \text{for } \varphi = 0, \varphi = \varphi_0. \end{aligned} \tag{3.11}$$

We introduce the operator belonging to the boundary value problem (3.11).
Let be $I_{\varphi_0} = (0,\varphi_0)$, $r^2 \Delta_{r,\varphi} = L(r\partial r, \varphi, \partial\varphi)$, and

$$\begin{aligned} L(\alpha,\varphi,\partial\varphi) \;&:\; W^{2,2}(I_{\varphi_0}) \to L^2(I_{\varphi_0}), \\ &:\; \tilde{u} \to \alpha^2\tilde{u} + \frac{\partial^2\tilde{u}(\alpha,\varphi)}{\partial\varphi^2}. \end{aligned}$$

Furthermore, let be T the trace operator

$$T : W^{2,2}(I_{\varphi_0}) \to W^{\frac{3}{2},2}(\partial I_{\varphi_0}),$$

which has the values $u(0)$ and $u(\varphi_0)$ in our particular case. Then

$$A(\alpha) = (L(\alpha,\varphi,\partial\varphi), T) : W^{2,2}(I_{\varphi_0}) \to L^2(I_{\varphi_0}) \times W^{\frac{3}{2},2}(\partial I_{\varphi_0})$$

is well-defined.

Furthermore

- $\{A(\alpha)\}_\alpha$ is a family of linear continous operators.
- The spectrum of the bundle $\{A(\alpha)\}_\alpha$ consists of countable many eigenvalues.
- The operator $A(\alpha)$ is invertible, if and only if α is no eigenvalue. An a-priori estimate holds in this case.
- If the line $\mathrm{Re}\alpha = -h$ is free of eigenvalues, then $A^{-1}(\alpha)$ exists there.

The inverse Mellin transformation M_h^{-1}, which is described by formula (3.4) yields the solution. Now, we take a look at the estimate (3.10). It is

$$\begin{aligned} \|u_h\|^2_{V^{2,2}_\beta(K)} \;&=\; \int\limits_K r^{2(\beta-2)}|u_h|^2\,\mathrm{d}x + \sum_{|\gamma|=1}\int\limits_K r^{2(\beta-1)}|D^\gamma u_h|^2\,\mathrm{d}x \\ &\quad + \sum_{|\gamma|=2}\int\limits_K r^{2\beta}|D^\gamma u_h|^2\,\mathrm{d}x \\ &\leq\; const\left(\int\limits_0^{\varphi_0}\int\limits_0^{\infty} r^{2\beta-3}|u_h(r,\varphi)|^2\,\mathrm{d}r\,\mathrm{d}\varphi\right) \\ &\quad + \int\limits_0^{\varphi_0}\int\limits_0^{\infty} r^{2\beta-3}\left(\left|r\frac{\partial}{\partial r}u_h\right|^2 + \left|\frac{\partial}{\partial\varphi}u_h\right|^2\right)\mathrm{d}r\,\mathrm{d}\varphi \\ &\quad + \int\limits_0^{\varphi_0}\int\limits_0^{\infty} r^{2\beta-3}\left(\left|\left(r\frac{\partial}{\partial r}\right)^2 u_h\right|^2 + \left|r\frac{\partial}{\partial r}\frac{\partial}{\partial\varphi}u_h\right|^2 + \left|\frac{\partial^2 u_h}{\partial\varphi^2}\right|^2\right)\mathrm{d}r\,\mathrm{d}\varphi \end{aligned}$$

$$
\begin{aligned}
&\overset{\substack{2\beta-3=2h-1\\(M4)}}{\leq} \quad c \int\limits_{-h-i\infty}^{-h+i\infty} \left(|\alpha|^4 \, \|\tilde{u}\|^2_{L_2(I_{\varphi_0})} + |\alpha|^2 \left\| \frac{\partial \tilde{u}}{\partial \varphi} \right\|^2_{L_2(I_{\varphi_0})} + |\alpha|^0 \| \frac{\partial^2 \tilde{u}}{\partial \varphi^2} \|^2_{L_2(I_{\varphi_0})} \right) &(3.12)\\
&\overset{[5]}{\leq} \quad const \int\limits_{-h-i\infty}^{-h+i\infty} \|\mathcal{M}(r^2 F)\|^2_{L_2(I_{\varphi_0})} \, \mathrm{d}\alpha &(3.13)\\
&\overset{(M4)}{=} \quad const \int\limits_{0}^{\infty} \|r^2 F\|_{L_2(I_{\varphi_0})} r^{2h-1} \, \mathrm{d}r = const \int\limits_{K} |F|^2 |x|^{2\beta} \, \mathrm{d}x. \\
&= \quad c\|F\|^2_{L^2_\beta(K)}.
\end{aligned}
$$

The decisive step to get from (3.12) to (3.13) is based on results of M.S. Agranowich/ M.I. Vishik (see [1],[2]) related to the unique solvability of parameter depending elliptic boundary value problems. This statement holds for the family $A(\alpha)$ above (see also the remark to the a-priori estimates).

Exercises

4. Consider the biharmonic operator $\Delta^2 = \Delta\Delta$. Calculate $\mathcal{M}(r^4 \Delta^2 u)$.

Chapter 4

Expansion based on the Mellin technique

Now, we apply the Mellin technique in order to get an expansion of the solution v of the model problem

$$\begin{aligned} \Delta v = F \quad & \text{in}\, K \\ v = 0 \quad & \text{on}\, \partial K \end{aligned} \tag{4.1}$$

into singular and regular terms

$$v = v_{\text{sing}} + v_{\text{reg}} \tag{4.2}$$

We assume that $F \in L^2_{\beta_1}(K) \cap L^2_{\beta}(K)$ with $\beta_1 \leq \beta$, i.e. the right hand side behaves better as $L^2_{\beta}(K)$ on the top of the cone. Assuming additionally that there are no eigenvalues of $A(\alpha)$ on the line $\mathrm{Re}\,\alpha = -h = -\beta_1 + 1$, then according to theorem 1, a solution $u_{h_1} \in V^{2,2}_{\beta_1}(K)$ exists. The connection between the solution u_h and u_{h_1} yields the following theorem

Theorem 2 *Let be $F \in L^2_{\beta}(K) \cap L^2_{\beta_1}(K)$, where $\beta_1 = h_1 + 1 \leq \beta = h + 1$. We assume that $A(\alpha)$ has no eigenvalues on the lines*

$$\mathrm{Re}\ \alpha = -h = -\beta + 1 \quad \textit{and} \quad \mathrm{Re}\ \alpha = -h_1 = -\beta_1 + 1\,.$$

Then, it holds for a solution $u_h \in V^{2,2}_{\beta}(K)$ of (4.1)

$$u_h = \sum_{-h < \mathrm{Re}\,\alpha_i < -h_1} c_i r^{\alpha_i} \Phi_i(\alpha_i, \varphi) + u_{h_1}, \tag{4.3}$$

$u_{h_1} \in V^{2,2}_{\beta_1}(K)$ and $\Phi_i(\alpha_i, \varphi) = \Phi_{\alpha_i}(\varphi)$ is defined by (2.6).

Sketch of the proof

The solution u_h can be written as

$$u_h(r, \varphi) = \frac{1}{i\sqrt{2\pi}} \int\limits_{-h-i\infty}^{-h+i\infty} r^{\alpha} A^{-1}(\alpha)(\mathcal{M}(r^2 F), 0)\, \mathrm{d}\alpha\,. \tag{4.4}$$

We remark that the eigenvalues of $A(\alpha)$ are poles of $A^{-1}(\alpha)$. We calculate the complex integral (4.4) on the line $(-h - i\infty, -h + i\infty)$ as a limit (figure 4.1)

$$
\begin{aligned}
u_h \quad = \quad & \lim_{N\to\infty} \left(\frac{1}{i\sqrt{2\pi}} \int\limits_{I_1(N)} r^\alpha A^{-1}(\alpha)[\mathcal{M}(r^2F), 0]\,\mathrm{d}\alpha \right. \\
& + \frac{1}{i\sqrt{2\pi}} \int\limits_{I_2(N)} r^\alpha A^{-1}(\alpha)[\mathcal{M}(r^2F), 0]\,\mathrm{d}\alpha \\
& \left. + \frac{1}{i\sqrt{2\pi}} \int\limits_{I_3(N)} r^\alpha A^{-1}(\alpha)[\mathcal{M}(r^2F), 0]\,\mathrm{d}\alpha \right) \\
& -\sqrt{2\pi} \sum_j \text{residuals}(\alpha_j) \text{ of } r^\alpha A^{-1}(\alpha)[\mathcal{M}(r^2F, 0)].
\end{aligned}
$$

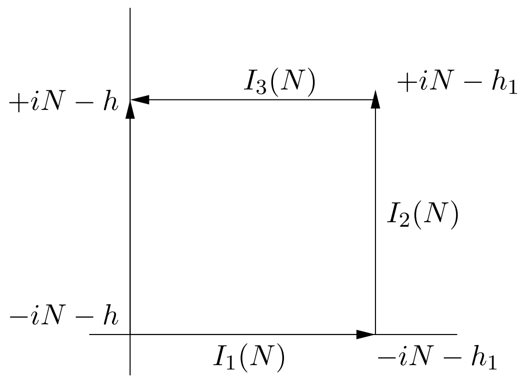

Figure 4.1: way of integration

One can prove [5] that $\lim_{N\to\infty} \int\limits_{I_1(N)} r^\alpha A^{-1}(\alpha)[\mathcal{M}(r^2F), 0]\,\mathrm{d}\alpha = 0$ and $\lim_{N\to\infty} \int\limits_{I_3(N)} r^\alpha A^{-1}(\alpha)[\mathcal{M}(r^2F), 0]\,\mathrm{d}\alpha = 0$. Therefore,

$$
u_h = -\sqrt{2\pi} \sum_j \text{residuals}(\alpha_j) + u_{h_1}.
$$

We calculate the residuals (α_j). Let α_j be an eigenvalue of $A(\alpha)$. In our model problem, $\alpha_j = \frac{j\pi}{\varphi_0}$ is a simple eigenvalue. Thus, α_j is a simple pole and

$$
A^{-1}(\alpha) = \frac{1}{(\alpha - \alpha_j)} A_{-1} + A_0 + (\alpha - \alpha_j)A_1 + (\alpha - \alpha_j)^2 A_2 + \cdots,
$$

where A_{-1} is mapping into the eigenspace of α_j. Furthermore, one can prove that $\mathcal{M}(r^2F)$ is a holomorphic function in the finite strip $-h < \operatorname{Re}\alpha < -h_1$ and has the form

$$
\mathcal{M}(r^2F)(\alpha, \varphi) = \sum_{l=0}^{\infty} a_l(\varphi)(\alpha - \alpha_j)^l .
$$

Therefore, it is in a neighbourhood of α_j

$$r^{\alpha}A^{-1}(\alpha)[\mathcal{M}(r^2F),0] =$$

$$r^{\alpha_j}r^{\alpha-\alpha_j}\left[\tfrac{1}{\alpha-\alpha_j}A_{-1}+A_0+\cdots\right]\left[a_0(\varphi)+a_1(\varphi)(\alpha-\alpha_j)^1+a_2(\varphi)(\alpha-\alpha_j)^2+\cdots,0\right]. \tag{4.5}$$

Since

$$r^{\alpha-\alpha_j}=e^{(\alpha-\alpha_j)\ln r}=1+\frac{(\alpha-\alpha_j)\ln r}{1!}+\frac{(\alpha-\alpha_j)^2\ln r^2}{2!}+\cdots$$

we get for the coefficient in front of $\frac{1}{\alpha-\alpha_j}$ the expression

$$r^{\alpha_j}A_{-1}(\mathcal{M}(r^2F),0)=c_jr^{\alpha_j}\sin\frac{j\pi}{\varphi_0}\varphi.$$

It follows (4.3). ■

Remark

Setting

$$u_h=\eta u=\sum_{-h<\mathrm{Re}\;\alpha_i<-h_1}c_ir^{\alpha_i}\Phi_i(\alpha_i,\varphi)+u_{h_1}$$

in the cone, then $u_{h_1}=\mathcal{O}(r^{\gamma}), \mathrm{Re}\,\gamma\geq -h_1=-\beta_1+1$. We get in the domain Ω

$$u=\eta u+(1-\eta)u=\sum_{-h<\mathrm{Re}\;\alpha_i<-h_1}c_ir^{\alpha_i}\Phi_i(\alpha_i,\varphi)+u_{h_1}+(1-\eta)u.$$

Since $(1-\eta)u$ vanishes in the neighbourhood of the corner, it is

$$u_{\text{reg}}=u_{h_1}+(1-\eta)u\in V^{2,2}_{\beta_1}(\Omega)\text{ and }u_{\text{reg}}=\mathcal{O}(r^{\gamma}).$$

Corollary

Let be $u\in\overset{\circ}{W}{}^{1,2}(\Omega)$ a weak solution of the Dirichlet problem

$$\begin{aligned}\Delta u&=f\quad\text{in}\,\Omega\\ u&=0\quad\text{on}\,\partial\Omega,\end{aligned}\tag{4.6}$$

where $f\in L^2(\Omega)$. f belongs also to $L^2_1(\Omega)$, since

$$\|f\|^2_{L^2_1(\Omega)}=\int_\Omega r^2|f|^2\,\mathrm{d}x\leq c\int_\Omega|f|^2\,\mathrm{d}x\,.$$

No eigenvalues of $A(\alpha)$ are situated on the line $\mathrm{Re}\;\alpha=-h=-\beta+1=0$, i.e. for $\beta=1$. Therefore, a solution $u\in V^{2,2}_1(\Omega)$ of (4.6) exists. This solution is in $W^{1,2}(\Omega)$, since

$$\begin{aligned}\|u\|^2_{W^{1,2}(\Omega)}&=\sum_{|\gamma|=1}\int_\Omega|D^\gamma u|^2\,\mathrm{d}x+\int_\Omega|u|^2\,\mathrm{d}x &(4.7)\\ &\leq const\int_\Omega\left(r^{-2}|u|^2+r^0\sum_{|\gamma|=1}|D^\gamma u|^2+r^2\sum_{|\gamma|=2}|D^\gamma u|^2\right)\mathrm{d}x &(4.8)\\ &=const\|u\|^2_{V^{2,2}_1(\Omega)}. &(4.9)\end{aligned}$$

Here, we have used that $1 \leq cr^{-2}$, thus $r^2 \leq c$. Furthermore, it is $u = 0$ on $\partial\Omega$. The trace theorem yields that $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$. Since the solution $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$ is unique, it must be that $u \in V_1^{2,2}(\Omega)$.
Now, we consider the line $\mathrm{Re}\ \alpha = -h_1 = 0 + 1 = 1$, i.e. we choose $\beta = 0$. This line contains no eigenvalues for $\varphi_0 \neq \pi$. Therefore , it holds, according to theorem 2: the weak solution $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap V_1^{2,2}(\Omega)$ has the form

$$u = c_1 r^{\frac{\pi}{\varphi_0}} \sin \frac{\pi}{\varphi_0}\varphi + u_{\text{reg}}, \tag{4.10}$$

where $u_{\text{reg}} \in V_0^{2,2}(\Omega)$. It follows from (4.10)

$$u \in W^{1,p}(\Omega), p < \frac{2}{1 - \frac{\pi}{\varphi_0}}\,.$$

The worst case is $p = 4\,(\varphi_0 = 2\pi - \varepsilon)$. Furthermore, one can prove that $u \in V^{2,2}_{1-\frac{\pi}{\varphi_0}+\varepsilon}(\Omega)$ and $u \in W^{1+\frac{\pi}{\varphi_0}-\varepsilon,2}(\Omega)$.

Chapter 5

General elliptic boundary value problems

Up to now, we confined to the Dirichlet problem for the Laplace operator. Now, we characterise more general boundary value problems, which can be treated with the Mellin technique. We start with elliptic boundary value problems in smooth domains and turn then to domains with corners.

Elliptic boundary value problems in smooth domains

First, we introduce linear elliptic operators, consisting of the principal part. Let

$$L(D_x) = \sum_{|\alpha|=l} a_\alpha D_x^\alpha \tag{5.1}$$

be a homogeneous linear differential operator in $\mathbb{R}^n$, $\alpha = (\alpha_1, \dots \alpha_n)$ is a multi-index, l is the order of L, $D_x = -i\partial_x$, $\partial_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$, $D_x^\alpha = (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}$.

Definition 2 *L is elliptic, if $L(\xi) = \sum_{|\alpha|=l} a_\alpha \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \neq 0$ for $\xi \in \mathbb{R}^n \backslash \{0\}$.*

Example

$$\begin{aligned} L(D_x) &= -\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\,, \\ L(\xi) &= |\xi|^2\,. \end{aligned}$$

Remark

For $n \geq 3$, it follows from the ellipticity that $l = 2m$, i.e. the order is even.
Now we define the ellipticity for a more general class of differential operators.

Definition 3 *The operator* $\mathcal{L}(x, D_x) = \sum_{|\alpha|\leq 2m} a_\alpha(x) D_x^\alpha$ *with coefficients* $a_\alpha \in C^\infty(\overline{\Omega})$ *is elliptic in* $\overline{\Omega}$*, if the principal parts with frozen coefficients*

$$L(x_0, D_x) = \sum_{|\alpha|=2m} a_\alpha(x_0) D_x^\alpha$$

are elliptic for all points $x_0 \in \overline{\Omega}$*.*

To describe boundary value problems, we have to introduce differential boundary operators. Its properties are described in local coordinates. We consider smooth bounded domains in $I\!R^2$, i.e. a normal and a tangent vector in each boundary point exists; x_τ and respectively x_ν denote the corresponding coordinates.

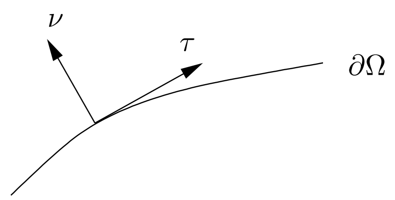

Figure 5.1: normal vector and tangent vector

We consider tubular coordinates $x = (x_\tau, x_\nu) \in \partial\Omega \times (-1, 1)$. A boundary point $y \in \partial\Omega$ is described by $y = (x_\tau, 0)$. The boundary operators $\mathcal{B}_1, \ldots, \mathcal{B}_m$ have the form

$$\mathcal{B}_i(y, D) = \sum_{|\beta|\leq m_i} b_\beta(y, D_\tau) D_\nu, \quad m_i \leq 2m - 1,$$

their principal parts with frozen coefficients $y_0 \in \partial\Omega$ are:

$$B_i(y_0, D) = \sum_{|\beta|=m_i} b_\beta(y_0, D_\tau) D_\nu \, .$$

Definition 4 *The tuple of operators*

$$\{\mathcal{L}(y, D), \mathcal{B}_1(y, D), \ldots, \mathcal{B}_m(y, D)\}, \quad y \in \partial\Omega,$$

satisfies a Shapiro-Lopatinski condition in $y_0 \in \partial\Omega$*, if the following initial value problem has only the trivial solution for all* $\xi_\tau \in I\!R\backslash\{0\}$

$$\begin{aligned} L\left(y_0, \xi_\tau; -i\tfrac{\partial}{\partial z}\right) w(z) &= 0 \quad \textit{for } z > 0\,, \\ B_i\left(y_0, \xi_\tau; -i\tfrac{\partial}{\partial z}\right) w(z) &= 0 \quad \textit{for } z = 0,\ i = 1, \ldots m\,, \\ |w(z)| &\to 0 \quad \textit{for } z \to \infty\,. \end{aligned}$$

Here we have set $x_\nu = z$, $\frac{\partial}{\partial \nu} = -i\frac{\partial}{\partial z}$, L and B_i denote the principal parts of $\mathcal{L}$ and $\mathcal{B}_i$ in the local coordinates (x_τ, x_ν).

Example

We consider again the Dirichlet problem for the Laplace operator in $I\!R^2$:

$$L\left(y_0, \xi_\tau, -i\frac{\partial}{\partial z}\right) w(z) = (\xi_\tau)^2 w(z) - \frac{\partial^2 w(z)}{\partial z^2} = 0 \quad \text{for } z > 0 \tag{5.2}$$

$$w(0) = 0 \tag{5.3}$$

$$\lim_{z\to\infty} |w(z)| = 0. \tag{5.4}$$

The solution of the ordinary differential equation (5.2) is

$$w(z) = c_1 e^{\xi_\tau z} + c_2 e^{-\xi_\tau z}.$$

The boundary condition $w(0) = c_1 + c_2 = 0$ and the decay condition (5.4) yield $w(z) \equiv 0$. Now we are able to define elliptic boundary value problems.

Definition 5 *The boundary value problem with boundary operators of order* $m_i \le 2m - 1$

$$\begin{array}{ll} \mathcal{L}(x, D)u(x) = f(x) & \text{for } x \in \Omega \\ \mathcal{B}_i(y, D)u(y) = g_i(y) & \text{for } y \in \partial\Omega, \ i = 1, \dots, m \end{array} \tag{5.5}$$

is called elliptic in $\overline{\Omega}$, *if*

1.) *the operator* $\mathcal{L}(x, D)$ *is elliptic for all* $x \in \overline{\Omega}$.

2.) *the tuple* $(\mathcal{L}(x, D), \mathcal{B}_i(y, D), \dots, \mathcal{B}_m(y, D))$ *satisfies for all* $y \in \partial\Omega$ *the Shapiro-Lopatinski condition.*

In smooth domains the following basic result holds for elliptic boundary value problems it holds.

Theorem 3 (Main Theorem) [10, S.189, Satz 13.1]
Let Ω *be a bounded* C^∞*- domain in* $I\!R^n$. *The following three statements are equivalent.*

(i) *The boundary value problem* (5.5) *is elliptic.*

(ii) *The operator*

$$(\mathcal{L}(x, D), B(y, D)) : W^{2m,2}(\Omega) \to L^2(\Omega) \times \prod_{i=1}^{m} W^{2m-m_i-\frac{1}{2},2}(\partial\Omega)$$

is a Fredholm operator.

(iii) *For* $u \in W^{2m,2}(\Omega)$ *it holds the a-priori estimate*

$$\begin{aligned} \|u\|_{W^{2m,2}(\Omega)} \le\ & C(\|\mathcal{L}(x, D)u\|_{L^2(\Omega)} + \sum_{i=1}^{m} \|\mathcal{B}_i(y, D)u\|_{W^{2m-m_i-\frac{1}{2},2}(\partial\Omega)} \\ & + \|u\|_{W^{2m-1,2}(\Omega)}). \end{aligned}$$

Elliptic boundary value problems in domains with one corner point

Let Ω be a bounded domain in $\mathbb{R}^2$ with one corner point O , see Figure 2.2.

Definition 6 *The boundary value problem*

$$\begin{array}{ll} \mathcal{L}(x,D)u(x) = f(x) & \text{for } x \in \Omega \\ \mathcal{B}_i(y,D)u(y) = g_i(y) & \text{for } y \in \partial\Omega\backslash\{O\},\ i = 1,\dots,m. \end{array} \tag{5.6}$$

is elliptic, if

(i) *the operator $\mathcal{L}(x,D)$ is elliptic in $\overline{\Omega}$,*

(ii) *the tuple $(\mathcal{L}(y,D),\mathcal{B}_1(y,D),\dots,\mathcal{B}_m(y,D))$ satisfies the Shapiro-Lopatinski condition for all $y \in \partial\Omega\backslash\{O\}$.*

Theorem 4 (Main Theorem) *Let be $\beta \in \mathbb{R}$ and let the boundary value problem* (5.6) *be elliptic. We assume for the Mellin transformed eigenvalue problem, which belongs to the principal parts with frozen coefficients in O, that the line* $\mathrm{Re}\ \alpha = -\beta + 1 = -h$ *is free of eigenvalues. Then the operator*

$$(\mathcal{L}(x,D),\mathcal{B}_1(y,D),\dots,\mathcal{B}_m(y,D)) : V_\beta^{2m,2}(\Omega) \to L^2_\beta(\Omega) \times \prod_{i=1}^m V_\beta^{2m-m_i-\frac{1}{2},2}(\partial\Omega)$$

is a Fredholm operator. It holds the estimate in weighted Sobolev spaces

$$\|u\|_{V_\beta^{2m,2}(\Omega)} \le c\left(\|\mathcal{L}(x,D)u\|_{L^2_\beta(\Omega)} + \sum_{i=1}^m \|\mathcal{B}_i u\|_{W^{2m-m_i-\frac{1}{2},2}(\partial\Omega)} + \|u\|_{V_\beta^{2m-1,2}(\Omega)}\right).$$

Furthermore, the following theorem is valid:

Theorem 5 (Regularity Theorem) *Let β and β_1 be real numbers with $\beta_1 \le \beta$ and $0 \le \beta - \beta_1 \le 1$. Suppose, the boundary value problem* (5.6) *is elliptic. We assume for the Mellin transformed eigenvalue problem, which belongs to the principal parts with frozen coefficients in O, that the lines* $\mathrm{Re}\ \alpha = -\beta + 1 = -h$ *and* $\mathrm{Re}\ \alpha = -\beta_1 + 1 = -h_1$ *are free of eigenvalues. Furthermore, let be $f \in L^2_{\beta_1}(\Omega)$ and $g_i \in V_{\beta_1}^{2m-m_i-\frac{1}{2},2}(\partial\Omega), i = 1,\dots,m$. A solution $u \in V_\beta^{2m}(\Omega)$ can be written as*

$$u = \sum_{\gamma\in\Gamma} c_\gamma u_\gamma(r,\varphi) + u_{reg}\,, \tag{5.7}$$

where $u_{reg} \in V_{\beta_1}^{2m}(\Omega)$. Here $\gamma = (\mu,\sigma,\kappa)$ is a multi-index and

$$\Gamma = \{(\mu,\sigma,\kappa) : \mu = 1,\dots,N, \sigma = 1,\dots I_\mu, \kappa = 0,\dots,\kappa_{\mu_\sigma} - 1\}$$

$$u_\gamma = r^{\alpha_\mu} \sum_{q=0}^k \frac{1}{q}(\ln r)^q u_\mu^{k-q,\sigma}(\varphi).$$

N is the number of the eigenvalues in the strip $-h <$ $\mathrm{Re}\ \alpha$ $< -h_1$, I_μ is the dimension of the eigenspace to the eigenvalue α_μ, κ_{μ_σ} is the length of the corresponding Jordan chain, c_γ are constants.

Chapter 6

Elliptic system of second order

In the previous section we have considered general elliptic boundary value problems to a scalar equation of higher order in a plane domain with a corner point. The theory for elliptic systems in nonsmooth domains is well developed too. Here, we give an example in order to explain how to proceed. This example is to find in [9].

6.1 Lamè equations in a plane domain with a corner point

The field equations of linear elasticity for isotropic materials read in the plane strain case:

$$\begin{aligned} -\mathrm{div}\sigma(u) &= -\mathrm{div}\begin{pmatrix} \lambda\mathrm{div}u + 2\mu\partial_1 u_1 & \mu(\partial_1 u_2 + \partial_2 u_1) \\ \mu(\partial_1 u_2 + \partial_2 u_1) & \lambda\mathrm{div}u + 2\mu\partial_2 u_2 \end{pmatrix} \\ &= -\begin{pmatrix} \mu\Delta u_1 + (\lambda+\mu)\partial_1\mathrm{div}u \\ \mu\Delta_2 + (\lambda+\mu)\partial_2\mathrm{div}u \end{pmatrix} = -Lu = f\,, \end{aligned}$$

where $u = (u_1, u_2)^\top$ is the displacement field, λ and μ are the Lamè parameters, σ denotes the stress tensor, L the Lamè operator.
We consider the mixed boundary value problem

$$\begin{aligned} -Lu &= f \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \Gamma_0\,, \\ \sigma(u)n &= g_1 \quad \text{on } \Gamma_1\,, \end{aligned}$$

where $\Gamma_1 \cap \Gamma_0 = \emptyset, \quad \overline{\Gamma}_1 \cup \overline{\Gamma}_0 = \partial\Omega$. Analogously to the considerations for boundary value problems for the Laplacian we localize the problem in a vicinity of an angular point O, using a partition of unity. Thus we get different boundary value problems (Dirichlet, Neumann and mixed boundary value problems) in an infinite cone K.

Introducing polar coordinates in K and using the ansatz

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = r^\alpha \begin{pmatrix} h_1(\alpha,\varphi) \\ h_2(\alpha,\varphi) \end{pmatrix} \tag{6.1}$$

we get a system of ordinary differential equations for the Lamè system, (3.6)

$$\begin{aligned}
&\mu(\alpha^2 h_1 + h_1'') + (\lambda+\mu)\cdot\{h_1[\left(\tfrac{\alpha^2}{2}-\alpha\right)\cos 2\varphi + \tfrac{\alpha^2}{2}] + h_1'(1-\alpha)\sin 2\varphi + \\
&h_1''\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\varphi\right) + h_2\left(\tfrac{\alpha^2}{2}-\alpha\right)\sin 2\varphi + h_2'(\alpha-1)\cos 2\varphi + h_2''\left(-\tfrac{1}{2}\sin 2\varphi\right)\} = 0 \\
&\mu(\alpha^2 h_2 + h_2'') + (\lambda+\mu)\cdot\{h_2[\left(\tfrac{-\alpha^2}{2}+\alpha\right)\cos 2\varphi + \tfrac{\alpha^2}{2}] + h_2'(\alpha-1)\sin 2\varphi + \\
&h_2''\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\varphi\right) + h_1\left(\tfrac{\alpha^2}{2}-\alpha\right)\sin 2\varphi + h_1'(\alpha-1)\cos 2\varphi - h_1''\left(\tfrac{1}{2}\sin 2\varphi\right)\} = 0.
\end{aligned} \tag{6.2}$$

This system has four linearly independent solutions $\boldsymbol{h}(\alpha,\varphi) = (h_1(\alpha,\varphi), h_2(\alpha,\varphi))^\top$ in the form

for $\alpha \neq 0$

$$\begin{aligned}
\boldsymbol{h}(\alpha,\varphi) &= c_1(\alpha)\begin{pmatrix} \cos\alpha\varphi \\ -\sin\alpha\varphi \end{pmatrix} + c_2(\alpha)\begin{pmatrix} \sin\alpha\varphi \\ \cos\alpha\varphi \end{pmatrix} \\
&+ c_3(\alpha)\begin{pmatrix} \cos(\alpha-2)\varphi \\ -\sin(\alpha-2)\varphi - A(\alpha)\sin\varphi \end{pmatrix} + c_4(\alpha)\begin{pmatrix} \sin(\alpha-2)\varphi \\ \cos(\alpha-2)\varphi + A(\alpha)\cos\alpha\varphi \end{pmatrix}
\end{aligned}$$

where $A(\alpha) = \frac{2(\lambda+3\mu)}{(\lambda+\mu)\alpha}$,

for $\alpha = 0$

$$\boldsymbol{h}(0,\varphi) = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 2c+\sin 2\varphi \\ -\cos 2\varphi \end{pmatrix} + c_4\begin{pmatrix} \cos 2\varphi \\ -2c\varphi + \sin 2\varphi \end{pmatrix}$$

with $c = \frac{\lambda+3\mu}{\lambda+\mu}$.

Now, we consider different boundary conditions on ∂K:
D-D, N-N and N-D conditions for $\varphi = 0$ and $\varphi = \varphi_0$. This leads to boundary eigenvalue problems analogously to (2.8). The corresponding eigenvalues and eigensolutions occur in an asymptotic expansion similar to (2.11).

1. D-D conditions

We have to determine nontrivial vectors $\boldsymbol{h}(\alpha,\varphi)$, such that $\boldsymbol{h}(\alpha,0)=\boldsymbol{h}(\alpha,\varphi_0)=0$. Using the above representation of $\boldsymbol{h}(\alpha,\varphi)$ we get the equation system
for $\alpha \neq 0$

$$\begin{aligned}\boldsymbol{h}(\alpha,0) &= c_1(\alpha)\begin{pmatrix}1\\0\end{pmatrix}+c_2(\alpha)\begin{pmatrix}0\\1\end{pmatrix}+c_3(\alpha)\begin{pmatrix}1\\0\end{pmatrix}\\ &\quad +c_4(\alpha)\begin{pmatrix}0\\1+A(\alpha)\end{pmatrix}=0 \qquad (6.3)\\ \boldsymbol{h}(\alpha,\varphi_0) &= c_1(\alpha)\begin{pmatrix}\cos\alpha\varphi_0\\-\sin\alpha\varphi_0\end{pmatrix}+c_2(\alpha)\begin{pmatrix}\sin\alpha\varphi_0\\\cos\alpha\varphi_0\end{pmatrix}\\ &\quad +c_3(\alpha)\begin{pmatrix}\cos(\alpha-2)\varphi_0\\-\sin(\alpha-2)\varphi_0-A(\alpha)\sin\alpha\varphi_0\end{pmatrix}\\ &\quad +c_4(\alpha)\begin{pmatrix}\sin(\alpha-2)\varphi_0\\\cos(\alpha-2)\varphi_0+A(\alpha)\cos\alpha\varphi_0\end{pmatrix}=0. \qquad (6.4)\end{aligned}$$

There are nontrivial solutions if the determinant

$$D_{D-D}(\alpha,\varphi_0)=\begin{vmatrix}1&0&1&0\\0&1&0&1+A(\alpha)\\\cos\alpha\varphi_0&\sin\alpha\varphi_0&\cos(\alpha-2)\varphi_0&\sin(\alpha-2)\varphi_0\\-\sin\alpha\varphi_0&\cos\alpha\varphi_0&-\sin(\alpha-2)\varphi_0-A(\alpha)\sin\alpha\varphi_0&\cos(\alpha-2)\varphi_0+A(\alpha)\cos\varphi_0\end{vmatrix}=$$

$$4\sin^2\alpha\varphi_0-A(\alpha)^2\sin^2\varphi_0=0 \quad \text{or} \quad (\lambda+\mu)^2\alpha^2\sin^2\alpha\varphi_0-(\lambda+3\mu)^2\sin^2\varphi_0=0.$$

The zeros $\alpha_1,\alpha_2,\cdots$ are in general complex numbers and they can be computed (see Figure 6.1). Figure 6.1 illustrates the distribution of the eigenvalues, the dotted lines describe real eigenvalues, whereas the full lines indicate the real parts of the conjugate pair of complex eigenvalues; $G=-\frac{\lambda+\mu}{\mu}, 0<\omega_0=\varphi_0\leq 2\pi$ is the opening angle.

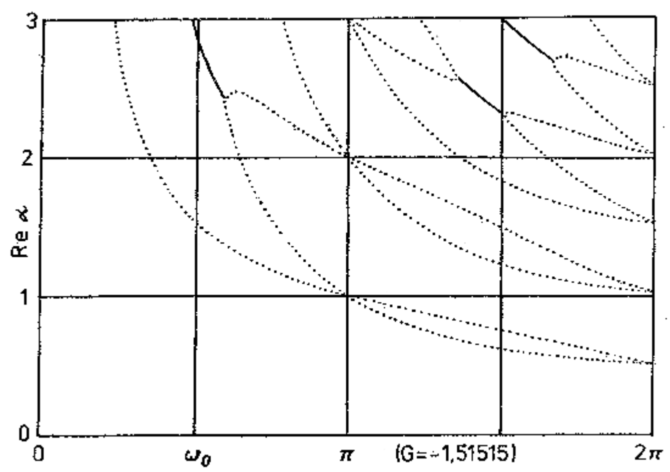

Figure 6.1: eigenvalues for the D-D problem

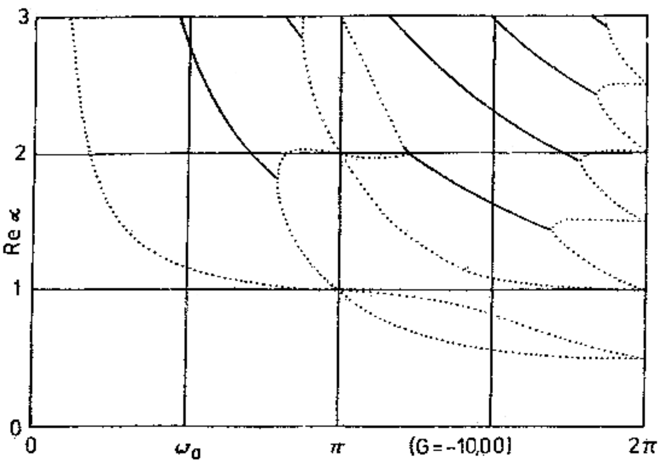

Figure 6.2: eigenvalues for the D-D problem

Note that $\alpha = 0$ is no eigenvalue.
The corresponding eigenfunctions read for $\varphi_0 \neq \pi, \varphi_0 \neq 2\pi$

$$\boldsymbol{h}_{D-D}(\alpha_\nu,\varphi) = (-\cos\alpha_\nu\varphi_0 + \cos(\alpha_\nu - 2)\varphi_0)\begin{pmatrix} -\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi \\ (1-A_\nu)\sin\alpha_\nu\varphi - \sin(\alpha_\nu-2)\varphi \end{pmatrix} +$$
$$(-(1-A_\nu)\sin\alpha_\nu\varphi_0 + \sin(\alpha_\nu-2)\varphi_0)\begin{pmatrix} -(1+A_\nu)\sin\alpha_\nu\varphi + \sin(\alpha_\nu-2)\varphi) \\ -\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi) \end{pmatrix} \qquad (6.5)$$

with $A_\nu = \frac{2(\lambda+3\mu)}{(\lambda+\mu)\alpha_\nu}$.
For $\varphi_0 = \pi$ we have $\alpha_\nu = \nu$, for $\varphi_0 = 2\pi$ we get $\alpha_\nu = \frac{\nu}{2}$, $\nu = 1, 2, \cdots$. In these cases we have two independent eigensolutions

$$\boldsymbol{h}^1_{D-D}(\alpha_\nu,\varphi) = \begin{pmatrix} -\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi \\ (1-A_\nu)\sin\alpha_\nu\varphi - \sin(\alpha_\nu-2)\varphi \end{pmatrix},$$
$$\boldsymbol{h}^2_{D-D}(\alpha_\nu,\varphi) = \begin{pmatrix} -(1+A_\nu)\sin\alpha_\nu\varphi + \sin(\alpha_\nu-2)\varphi \\ -\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi \end{pmatrix}.$$

Theorem 6 *The singular functions of the weak solutions $u \in V$ of the Dirichlet problem have the following form :*

(i) if $\varphi_0 \neq \pi, \varphi_0 \neq 2\pi$ and $\alpha_\nu(\lambda,\mu,\omega_0) \neq 0$ is a simple zero of

$$D_{D-D}(\alpha) = 4\sin^2\alpha\varphi_0 - A^2(\alpha)\sin^2\varphi_0 = 0\,,$$

the singular function occur:

$$\mathbf{u}_\nu(r,\varphi) = r^{\alpha_\nu}\boldsymbol{h}_{D-D}(\alpha_\nu,\varphi) =$$
$$= r^{\alpha_\nu}((-\cos\alpha_\nu\varphi_0 + \cos(\alpha_\nu-2)\varphi_0)\begin{pmatrix} -\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi \\ (1-A_\nu)\sin\alpha_\nu\varphi - \sin(\alpha_\nu-2)\varphi \end{pmatrix} +$$
$$(-(1-A_\nu)\sin\alpha_\nu\varphi_0 + \sin(\alpha_\nu-2)\varphi_0)\begin{pmatrix} -(1+A_\nu)\sin\alpha_\nu\varphi + \sin(\alpha_\nu-2)\varphi) \\ -\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi) \end{pmatrix})$$

The asymptotic expansion of a weak solution $\mathbf{u} \in [\overset{\circ}{H}{}^1(\Omega)]^2$ *reads near a reentrant corner*

$$\mathbf{u} = \sum_{\nu=1}^{2} c_\nu \mathbf{u}_\nu(r,\varphi) + \mathbf{u}_{reg}\,,\ \frac{1}{2} < \mathrm{Re}\alpha_\nu < 1\,.$$

(ii) If $\varphi_0 \neq 0, \varphi_0 \neq 2\pi$ *and* $\alpha_\nu(\lambda,\mu,\varphi_0) \neq 0$ *is a double zero of*

$$D_{D-D}(\alpha) = 4\sin^2\alpha\varphi_0 - A^2(\alpha)\sin^2\varphi_0 = 0\,,$$

that means, additionally α_ν *satisfies the conditions* $\alpha_\nu\varphi_0 = \tan\alpha_\nu\varphi_0$ *and*

$$\left(\frac{\sin\varphi_0}{\varphi_0}\right)^2 = \left(\frac{\lambda+3\mu}{\lambda+\mu}\cos\alpha_\nu\varphi_0\right)^2,\ \cos\alpha_\nu\varphi_0 \neq 0,\ \sin\alpha_\nu\varphi_0 \neq 0\,,$$

then the singular functions

$$\mathbf{u}_\nu^{(1)} = r^{\alpha_\nu}\boldsymbol{h}_{D-D}(\alpha_\nu,\varphi),$$

$$\mathbf{u}_\nu^{(2)} = r^{\alpha_\nu}(\boldsymbol{H}_{D-D}(\alpha_\nu) + \ln r\,\boldsymbol{h}_{D-D}(\alpha_\nu,\varphi)),$$

appear, where

$$\boldsymbol{H}_{D-D}(\alpha_\nu) = \frac{d}{da}\boldsymbol{h}_{D-D}(\alpha,\varphi)|_{\alpha=\alpha_\nu}.$$

The asymptotic expansion of a weak solution $\mathbf{u} \in [\overset{\circ}{H}{}^1(\Omega)]^2$ *reads near a reentrant corner*

$$\mathbf{u} = c_1\mathbf{u}_\nu^{(1)} + c_2 bf u_\nu^{(2)} + \mathbf{u}_{reg}\,,\ \frac{1}{2} < \mathrm{Re}\alpha_\nu < 1\,.$$

(iii) If $\omega_0 = \pi$*, then* $\alpha_\nu(\lambda,\mu,\pi) = \nu$*,* $\nu = 1,2,\ldots$ *and no proper singular function exists.*

(iv) If $\omega = 2\pi$*, then* $\alpha_\nu(\lambda,\mu,2\pi) = \frac{\nu}{2}, \nu = 1,2,\ldots$ *and*

$$\begin{aligned}
\mathbf{u}_\nu^1(r,\varphi) &= r^{\frac{\nu}{2}}\begin{pmatrix} -\cos\frac{\nu}{2}\varphi + \cos(\frac{\nu}{2}-2)\varphi \\ (1-A_\nu)\sin\frac{\nu}{2}\varphi - \sin(\frac{\nu}{2}-2)\varphi \end{pmatrix} \\
\mathbf{u}_\nu^2(r,\varphi) &= r^{\frac{\nu}{2}}\begin{pmatrix} -(1+A_\nu)\sin\frac{\nu}{2}\varphi + \sin(\alpha_\nu-2)\varphi \\ -\cos\frac{\nu}{2}\varphi + \cos(\alpha_\nu-2)\varphi \end{pmatrix}
\end{aligned}$$

are the linearly independent pairs of singular functions. The asymptotic expansion of a weak solution $\mathbf{u} \in [\overset{\circ}{H}{}^1(\Omega)]^2$ *reads near a crack*

$$\mathbf{u} = c_1\mathbf{u}_1^1(r,\varphi) + c_2\mathbf{u}_1^2(r,\varphi) + \mathbf{u}_{reg}\,.$$

2. N-D conditions

The Neumann condition $\sigma(\mathbf{u})\cdot\mathbf{n}=\mathbf{0}$ written in polar coordinates, has the following form for displacement fields (6.2) at $\varphi=\varphi_0$:

$$\begin{pmatrix} ((\lambda+\mu)\sin^2\varphi_0+\mu)+(\lambda+\mu)\cos\varphi_0\sin\varphi_0+ \\ \alpha(\lambda+\mu)\cos\varphi_0\sin\varphi_0+\alpha((\lambda+\mu)\sin^2\varphi_0-\mu) \\ \\ (\lambda+\mu)\sin\varphi_0\cos\varphi_0-(\lambda+\mu)\cos^2\varphi_0+\mu+ \\ \alpha(\mu-(\lambda+\mu)\cos^2\varphi_0)-\alpha((\lambda+\mu)\cos\varphi_0\sin\varphi_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.6}$$

The homogenous N-D conditions (N for $\varphi=0$, D for $\varphi=\varphi_0$) at ∂K read

$$\text{N:} \qquad \begin{pmatrix} h_1'(\alpha,0)+\alpha h_2(\alpha,0) \\ (\lambda+2\mu)h_2'(\alpha,0)+\lambda\alpha h_1(\alpha,0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\text{D:} \qquad \begin{pmatrix} h_1(\alpha,\varphi_0) \\ h_2(\alpha,\varphi_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There are nontrivial solutions $\boldsymbol{h}$ of the mixed boundary value problem in K, if the following determinant vanishes :
for $\alpha \neq 0$:

$$\begin{aligned} D_{N-D}(\alpha) &= \begin{vmatrix} \cos\alpha\varphi_0 & \sin\alpha\varphi_0 & \cos(\alpha-2)\varphi_0 & \sin(\alpha-2)\varphi_0 \\ -\sin\alpha\varphi_0 & \cos\alpha\varphi_0 & -\sin(\alpha-2)\varphi_0-A\sin\alpha\varphi_0 & \cos(\alpha-2)\varphi_0+A\cos\varphi_0 \\ 0 & 2\alpha & 0 & \alpha A(\alpha)+2\alpha-2 \\ -2\mu\alpha & 0 & (\lambda+2\mu)(2-\alpha A(\alpha))-2\mu\alpha & 0 \end{vmatrix} \\ &= 16\mu\left(\alpha^2\sin^2\varphi_0-\frac{(\lambda+2\mu)^2}{(\lambda+\mu)^2}+\sin^2\alpha\varphi_0\frac{\lambda+3\mu}{\lambda+\mu}\right)=0\,. \end{aligned}$$

The eigenvalues α_ν are the zeros of the equation

$$\sin^2\alpha\varphi_0=\frac{-\alpha^2\sin^2\omega_0(\lambda+\mu)^2+(\lambda+2\mu)^2}{(\lambda+\mu)(\lambda+3\mu)}. \tag{6.7}$$

Note, that $\alpha=0$ is no eigenvalue. Figure 6.3 shows the distribution of eigenvalues in this case.

Now, we describe the singular functions for the mixed boundary value problem.

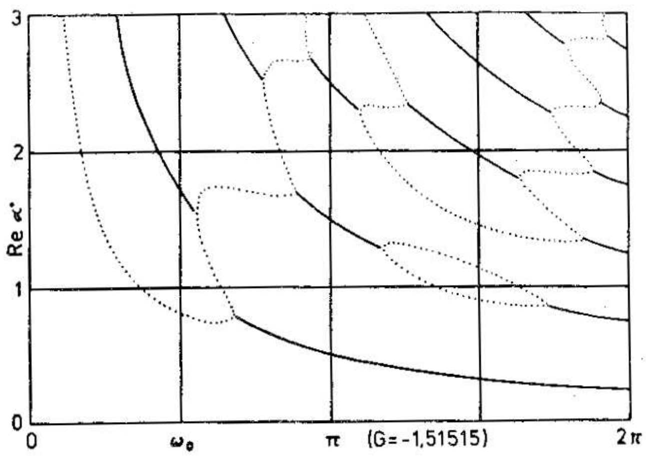

Figure 6.3: eigenvalues for the N-D problem

Theorem 7 *(i) If $\alpha_\nu = \alpha_\nu(\lambda, \mu, \varphi_0)$ is a simple zero of $D_{N-D}(\alpha)$ then*

$$\mathbf{u}_\nu(r,\varphi) = r^{\alpha_\nu}(c_1(\alpha_\nu,\varphi_0)\boldsymbol{h}^1_{N-D}(\alpha_\nu,\varphi) + c_2(\alpha_\nu,\varphi_0)\boldsymbol{h}^2_{N-D}(\alpha_\nu,\varphi))$$

is a singular function for the mixed boundary value problem with

$$\begin{aligned}
c_1(\alpha_\nu,\varphi_0) &= \left(\frac{2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1\right)\cos\alpha_\nu\varphi_0 + \cos(\alpha_\nu-2)\varphi_0,\\
c_2(\alpha_\nu,\varphi_0) &= \left(\frac{2\mu}{(\lambda+\mu)\alpha_\nu} - 1\right)\sin\alpha_\nu\varphi_0 + \sin(\alpha_\nu-2)\varphi_0\,,\\
\boldsymbol{h}^1_{N-D}(\alpha_\nu,\varphi) &= \begin{pmatrix} \left(\frac{-2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1\right)\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi \\ \left(\frac{2\mu}{(\lambda+\mu)\alpha_\nu} - 1)(-\sin\alpha_\nu\varphi) - \sin(\alpha_\nu-2)\varphi\right) \end{pmatrix},\\
\boldsymbol{h}^2_{N-D}(\alpha_\nu,\varphi) &= \begin{pmatrix} \left(\frac{2\mu}{(\lambda+\mu)\alpha_\nu} - 1)\sin\alpha_\nu\varphi + \sin(\alpha_\nu-2)\varphi\right) \\ \left(\frac{-2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1\right)\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi \end{pmatrix},
\end{aligned}$$

(ii) If $\alpha_\nu = \alpha_\nu(\lambda, \mu, \varphi_0)$ is a double zero of $D_{N-D}(\alpha)$ then two singular functions occur :

$$\begin{aligned}
\mathbf{u}^{(1)}_\nu(r,\varphi) &= r^{\alpha_\nu}(c_1(\alpha_\nu,\varphi_0)\boldsymbol{h}^1_{N-D}(\alpha_\nu,\varphi) + c_2(\alpha_\nu,\varphi_0)\boldsymbol{h}^2_{N-D}(\alpha_\nu,\varphi))\\
&= r^{\alpha_\nu}\boldsymbol{h}_{N-D}(\alpha_\nu,\varphi), \qquad (6.8)\\
\mathbf{u}^{(2)}_\nu(r,\varphi) &= r^{\alpha_\nu}(\boldsymbol{H}_{N-D}(\alpha_\nu,\varphi) + \ln r\frac{d}{d\alpha}\boldsymbol{h}_{N-D}(\alpha,\varphi)|_{\alpha=\alpha_\nu})\,, \qquad (6.9)
\end{aligned}$$

where

$$\mathbf{H}_{N-D}(\alpha_\nu,\varphi) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathbf{h}_{N-D}(\alpha,\varphi)|_{\alpha=\alpha_\nu}\,.$$

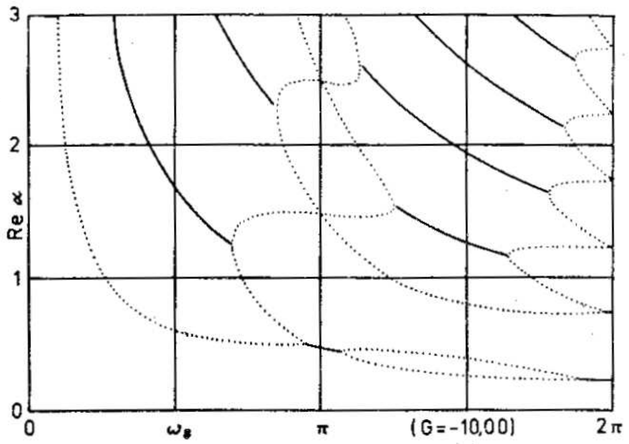

Figure 6.4: eigenvalues for thr N-D problem

The equations

$$\alpha_\nu\varphi_0 \sin\alpha_\nu\varphi_0 \cos\alpha_\nu\varphi_0 = \sin^2\alpha_\nu\varphi_0 - \frac{(\lambda+2\mu)^2}{(\lambda+\nu)(\lambda+3\mu)}\,, \tag{6.10}$$

$$\frac{\sin^2\varphi_0}{\varphi_0^2} = -\frac{\sin\alpha_\nu\varphi_0\cos\alpha_\nu\varphi_0}{\alpha_\nu\varphi_0}\frac{(\lambda+3\mu)}{(\lambda+\mu)} \tag{6.11}$$

are sufficient and necessary, that α_ν is a double eigenvalue.

The asymptotic expansion of a weak solution $\mathbf{u} \in [H^1(\Omega)]^2$ with $\mathbf{u}|_{\Gamma_0} = 0$ reads near a corner point ($\varphi_0 = \pi$ for changing boundary conditions included) for both cases (i) and (ii)

$$\mathbf{u} = \sum_{0<\mathrm{Re}\alpha_\nu<1} c_\nu \mathbf{u}_\nu(r,\varphi) + \mathbf{u}_{\mathrm{reg}}\,.$$

3. N-N conditions

In this case we have to find the singular functions as nontrivial solutions $\mathbf{h} = \mathbf{h}(\alpha,\varphi_0,\varphi)$ of the boundary eigenvalue problem (6.2), (6.6), where the homogenous Neumann condition should be satisfied for $\varphi = 0$ and $\varphi = \varphi_0$. This leads to the condition, that the following determinant vanishes :

for $\alpha \neq 0$

$$D_{N-N}(\alpha) = 32\mu^3\alpha^2(\alpha^2\sin^2\varphi_0 - \sin^2\alpha\varphi_0) = 0. \tag{6.12}$$

Note, that α does not depend on the material parameter λ and μ.
If $\alpha = 0$, then we have two linearly independent solutions

$$\boldsymbol{h}^1_{N-N}(0,\varphi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \ \boldsymbol{h}^2_{N-N}(0,\varphi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.13}$$

Fig. 6.5 shows the distribution of the eigenvalues in the strip $0 \leq \mathrm{Re}\ \alpha \leq 3$. The dotted lines indicates the real eigenvalues, the full lines the real parts of the complex eigenvalues. In the N-N case we get the following theorem :

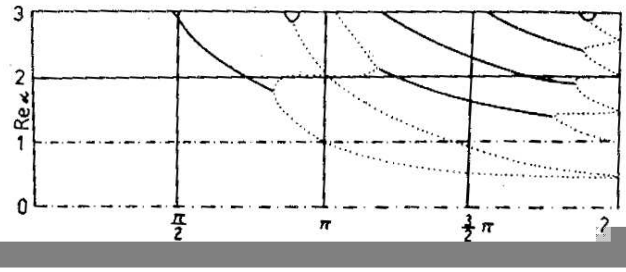

Figure 6.5: eigenvalues for the N-N problem

Theorem 8 *(i) If $\varphi_0 \neq \pi, \varphi_0 \neq 2\pi$ and $\alpha_\nu = \alpha_\nu(\varphi_0)$ is a simple zero of $D_{N-N}(\alpha)$, then the following singular functions occur:*

$$\mathbf{u}_\nu(r,\varphi) = r^{\alpha_\nu}[c_1(\alpha_\nu)\boldsymbol{h}^1_{N-N}(\alpha_\nu,\varphi)+c_2(\alpha_\nu)\mathbf{h}^2_{N-N}(\alpha_\nu,\varphi)] = r^{\alpha_\nu}\boldsymbol{h}_{N-N}(\alpha_\nu,\varphi) \quad (6.14)$$

where

$$c_1(\alpha) = \mu[(2-\alpha_\nu)\sin\alpha_\nu\varphi_0 + \alpha_\nu\sin(\alpha_\nu-2)\varphi_0] \quad (6.15)$$

$$c_2(\alpha) = \alpha_\nu(\cos\alpha_\nu\varphi_0 - \cos(\alpha_\nu-2)\varphi_0) \quad (6.16)$$

$$\mathbf{h}^1_{N-N}(\alpha_\nu,\varphi) = \begin{pmatrix} \left(\frac{-2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1\right)\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi \\ \left(\frac{-2\mu}{(\lambda+\mu)\alpha_\nu} + 1\right)\sin\alpha_\nu\varphi - \sin(\alpha_\nu-2)\varphi \end{pmatrix} \quad (6.17)$$

$$\mathbf{h}^2_{N-N}(\alpha_\nu,\varphi) = \begin{pmatrix} \left(\frac{-2\mu}{(\lambda+\mu)\alpha_\nu} - 1\right)\sin\alpha_\nu\varphi + \sin(\alpha_\nu-2)\varphi \\ \left(\frac{-2(\lambda+2\mu)}{(\lambda+\mu)\alpha_\nu} - 1\right)\cos\alpha_\nu\varphi + \cos(\alpha_\nu-2)\varphi \end{pmatrix}. \quad (6.18)$$

(ii) If $\varphi_0 \neq \pi, \varphi_0 \neq 2\pi$ and α_ν is a double zero of $D_{N-N}(\alpha)$, then the singular functions occur:

$$\mathbf{u}^1_\nu(r,\varphi) = r^{\alpha_\nu}\boldsymbol{h}_{N-N}(\alpha_\nu,\varphi), \quad (6.19)$$

$$\mathbf{u}^2_\nu(r,\varphi) = r^{\alpha_\nu}(\mathbf{H}_{N-N}(\alpha_\nu,\varphi) + \ln r\,\mathbf{h}_{N-N}(\alpha_\nu,\varphi)) \quad (6.20)$$

with

$$\mathbf{H}_{N-N}(\alpha_\nu,\varphi) = \frac{d}{d\alpha}h_{N-N}(\alpha,\varphi)|_{\alpha=\alpha_\nu}. \quad (6.21)$$

(iii) If $\varphi_0 = \pi$, then no proper singular functions exist.

(iv) If $\varphi_0 = 2\pi$ then $\alpha_\nu = \frac{\nu}{2}$, and we get two linearly independent functions :

$$\mathbf{u}^1_\nu(r,\varphi) = \begin{pmatrix} \left(\frac{-4(\lambda+2\mu)}{(\lambda+\mu)\nu} - 1\right)\cos\frac{\nu\varphi_0}{2} + \cos\left(\frac{\nu}{2}-2\right)\varphi_0 \\ \left(\frac{-4\mu}{(\lambda+\mu)\nu} + 1\right)\sin\frac{\nu\varphi_0}{2} - \sin\left(\frac{\nu}{2}-2\right)\varphi_0) \end{pmatrix} \quad (6.22)$$

$$\mathbf{u}^2_\nu(r,\varphi) = \begin{pmatrix} \left(\frac{-4\mu)}{(\lambda+\mu)\nu} - 1\right)\sin\frac{\nu\varphi_0}{2} + \sin\left(\frac{\nu}{2}-2\right)\varphi_0 \\ \left(\frac{4(\lambda+2\mu)}{(\lambda+\mu)\nu} - 1\right)\cos\frac{\nu}{2}\varphi_0 + \cos\left(\frac{\nu}{2}-2\right)\varphi_0 \end{pmatrix} \quad (6.23)$$

where $\nu = 1, 2, \ldots$.

Index

Bibliography

[1] Agranovich, M.S. *Elliptic Boundary Problems* in Encyclopedia of Mathematical Science, Vol. 79. Partial Differential Equations IX, 2–143, 1997.

[2] Agranovich, M.S., Visik, M.I. *Elliptic Boundary value Problems depending of a parameter*. Doklady Akademii Nauk SSSR, Band 149, Nr.2, 223–227, 1963.

[3] Dauge, M. *Elliptic Boundary Value Problems on Corner Domains. Smoothness and Asymptotic Expansion*. vol. 1341 of *Lecture Notes in Mathematics*. Springer Verlag, 1988.

[4] Grisvard, P. *Elliptic Problems in Nonsmooth Domains*. Pitman, 1985.

[5] Kondrat'ev, V.A. *Boundary value problems for elliptic equations in domains with conical or angular points*. *Trans. Moscow Math. Soc. 10* (1967), 227 - 313.

[6] Kozlov, V.A., Maz'ya, V.G., and Rossmann, J. *Elliptic Boundary Value Problems in Domains with Point Singularities*. American Mathematical Society, 1997.

[7] Mazya, V.G., Plamenevsky, B.A.; *The coefficients in the asymptotics of solutions of elliptic boundary value problems with conical points*. Mathematische Nachrichten 76, S. 29-60, 1977.

[8] Nazarov, S.A., Plamenevsky, B.A. *Elliptic Problems in Domains with Piecewise Smooth Boundaries*. Walter de Gruyter, Berlin, 1994.

[9] Sändig, A.-M., Richter, U., Sändig, R. *The regularity of boundary value problems for the Lamé equations in a polygonal domain*. Rostocker Math. Kolloq. 36, S. 21 - 50, 1989.

[10] Wloka, J. *Partielle Differentialgleichungen*. Stuttgart: Teubner Verlag, 1982.

Prof. Dr. Anna-Margarete Sändig

E-Mail: `saendig@ians.uni-stuttgart.de`

WWW: `http://ians.uni-stuttgart.de ˜ saendig/`

Erschienene Preprints ab Nummer 2006/001

Komplette Liste: `http://preprints.ians.uni-stuttgart.de`

2006/001 *Klimke, A.*: Sparse Grid Interpolation Toolbox - User's Guide

2006/002 *Klimke, A., Wohlmuth, B.*: Constructing Dimension-Adaptive Sparse Grid Interpolants using Parallel Function Evaluations

2006/003 *Hartmann, S., Brunssen, S., Ramm, E., Wohlmuth, B.*: Application of a primal-dual active set strategy for unilateral non-linear dynamic contact problems of thin-walled structures

2006/004 *Sändig, A.-M.*: Partielle Differentialgleichungen Vorlesung im Wintersemester 2005/2006

2006/005 *Nicaise, S., Witowski, K., Wohlmuth, B.*: An a posteriori error estimator for the Lame equation based on H(div)-conforming stress approximations

2006/006 *Geis, W. (ed.), Sändig, A.-M. (ed.)*: Second International Workshop - Direct and Inverse Problems in Piezoelectricity,Hirschegg (Kleinwalsertal), Austria, July 16-19, 2006

2006/007 *Klimke, A.*: Efficient Construction of Hierarchical Polynomial Sparse Grid Interpolants using the Fast Discrete Cosine Transform

2006/008 *Buchukuri, T., Chkadua, O., Natroshvili, D.*: Mathematical modelling and analysis of interaction problems for metallic-piezoelectric composite structures with regard to thermal stresses

2006/009 *Merkle, C., Rohde, C.*: The Sharp-Interface Approach for Fluids with Phase Change: Riemann Problems and Ghost Fluid Techniques

2006/010 *Anna-Margarete Sändig*: Regularity results for linear elliptic boundary value problems in polygons. Lectures at the Charles University Prague, Oct.05