Universität Stuttgart

A Framework for Maxwell's Equations in Non-Inertial Frames Based on Differential Forms

Bernd Flemisch, Stefan Kurz, Barbara Wohlmuth

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Preprint 2006/011

Universität Stuttgart

A Framework for Maxwell's Equations in Non-Inertial Frames Based on Differential Forms

Bernd Flemisch, Stefan Kurz, Barbara Wohlmuth

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Preprint 2006/011

Institut für Angewandte Analysis und Numerische Simulation (IANS)
Fakultät Mathematik und Physik
Fachbereich Mathematik
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: ians-preprints@mathematik.uni-stuttgart.de
WWW: http://preprints.ians.uni-stuttgart.de

ISSN **1611-4176**

A Framework for Maxwell's Equations in Non-Inertial Frames Based on Differential Forms

S. Kurz*, ETAS GmbH, Stuttgart, Germany
B. Flemisch† and B. Wohlmuth, IANS, Universität Stuttgart, Germany

Abstract

We set up a consistent framework for the Lagrangian view of (3+1)-dimensional electro-dynamics using the language of differential forms with no need for coordinate systems or reference frames. A natural decomposition mechanism admits the construction of this framework with a minimum of overhead. Employing two observers, one holonomic and the other one locally inertial, opens the possibility to use the simple form of both the Maxwell equations and the constitutive relations simultaneously. Connections to standard results are provided, and the feasibility is further demonstrated by means of a classical application.

INTRODUCTION

In many engineering applications the interaction between the electromagnetic field and moving bodies is of great interest. It is natural to use a Lagrangian description, where the unknowns are defined on a mesh which moves and deforms together with the considered objects. What is the correct form of Maxwell's equations and the constitutive laws under such circumstances? The aim of the present paper is to tackle this question by using the language of differential forms.

We start from Maxwell's equations and constitutive relations in four-dimensional flat Minkowskian space-time. While this essentially constitutes an Eulerian point of view, we proceed to a four-dimensional Lagrangian description by transformation to the canonical reference space. The next step is to introduce a (3+1)-decomposition mechanism which is based exclusively on the pair of a vector field and a 1-form, describing an observer in space-time, [4, 7]. With the help of this mechanism, all fundamental operators like exterior derivative, Hodge, and contraction can be easily decomposed. An application to four-dimensional electro-dynamics yields the common notions of three-velocity and contraction factor, defines the three-dimensional field components, and provides general (3+1)-Maxwell and constitutive equations. It becomes obvious that if the observer responsible for the decomposition is holonomic or locally inertial, then Maxwell's equations or constitutive laws adopt their simple form, respectively. This observation becomes utilized by introducing two observers to the Lagrangian setting, one holonomic and the other locally inertial. In order to profit from their individual advantages, a transformation law connecting the two observers is established, again simply by plugging in the decomposition mechanism. This concludes a convenient description of (3+1)-dimensional electro-dynamics.

The remainder of the paper connects our approach to previouly known results. In particular, the notion of a Lorentz boost is introduced, exhibiting the fundamental property that its decomposition identifies physically equivalent field components inherited from different (3+1)-decompositions. Employing this Lorentz boost yields transformation laws for differential forms and vector fields coinciding with the well-known relations for the connection of inertial frames. As an application of our approach, we consider the classical paradoxon by Schiff, [8]. The paper is completed by concluding remarks and an appendix introducing general concepts and providing the proofs of some results employed in the text.

SPACE-TIME ELECTRO-DYNAMICS

Our model for physical spacetime is that of a four-dimensional affine space M_4 equipped with a metric g of signature (+ - - -), referred to as Minkowski space. The metric is represented by the mapping $g : \Lambda\mathcal{X}^1(M_4) \to \Lambda\mathcal{F}^1(M_4)$ from the space of smooth (multi-)vector fields to the space of smooth differential forms, defining the extent of a p-form $\boldsymbol{\omega}$ as $\|\boldsymbol{\omega}\| = \sqrt{|(\boldsymbol{\omega}|g^{-1}(\boldsymbol{\omega}))|}$, where $\cdot|\cdot$ is the usual duality product. In M_4, electro-dynamic phenoma are described by Maxwell's equations, namely,

$$\mathrm{d}\underline{F} = 0, \quad \text{and} \quad \mathrm{d}\underline{G} = \underline{\mathcal{J}}, \tag{1}$$

where d stands for the exterior differential operator, $\underline{F}, \underline{G} \in \mathcal{F}^2(M_4)$ are the electromagnetic field and excitation, respectively, and $\underline{\mathcal{J}} \in \mathcal{F}^3(M_4)$ the four-current density. The field $\underline{F}$ and the excitation $\underline{G}$ are linked by the constitutive laws, [2],

$$\mathbf{i_u}\big(* \underline{G} + c_0\varepsilon\underline{F}\big) = 0, \tag{2a}$$

$$\mathbf{i_u}\big(* \underline{F} - c_0\mu\underline{G}\big) = 0, \tag{2b}$$

where $\mathbf{i_u}$ denotes the contraction by the four-velocity $\mathbf{u}$, c_0 the vacuum velocity of light, ε and μ the electric and magnetic permeabilities, and $*$ is the Hodge operator associated with the metric g. The four-velocity vector field $\mathbf{u}$ is time-like and of unit length with respect to g, $\mathbf{i_u}g(\mathbf{u}) = 1$. It is tangent to the world-lines of the volume elements in M_4. Maxwell's equations (1) and the constitutive laws (2) constitute a complete four-dimensional Eulerian description of electro-dynamics in M_4.

* stefan.kurz@gmx.de
† flemisch@ians.uni-stuttgart.de

LAGRANGIAN PERSPECTIVE

For the description of dynamical processes, it is common to distinguish between two fundamentally different perspectives. Within the *spatial* or *Eulerian* description, the observer is fixed with respect to a chosen laboratory frame and keeps track of the motion of the bodies or particles and of the variation of the associated fields. The velocity $\mathbf{v}$ describing the motion of the volume elements relative to the laboratory observer, enters the equations in an explicit way. In contrast to the Eulerian perspective stands the *material* or *Lagrangian* description. Here, the observer is fixed with respect to the body or particle under consideration and describes all events from this point of view. This actual configuration is created by a placement mapping Φ, which we will define in the following. The Lagrangian perspective offers a central advantage compared to the Eulerian one for numerical formulations, namely, the observer is connected to the mesh and moves and deforms together with it.

The Lagrangian observer describes the events from a reference space M_4^0, which is the product of a one-dimensional oriented affine space M_1^0 and a three-dimensional oriented affine space M_3^0, the configuration space, as illustrated in Figure 1. As mentioned above, the reference space M_4^0 and the physical space M_4 are linked by a placement mapping in form of a diffeomorphism $\Phi : M_4^0 \to M_4$. The physical interpretation of Φ

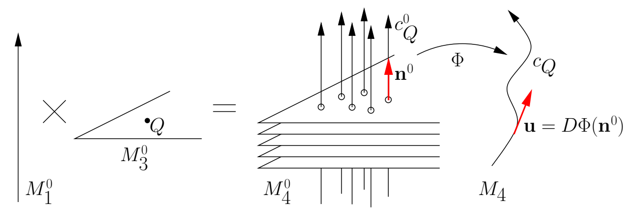

Figure 1: Reference space M_4^0, placement mapping Φ.

is deduced from the observation of a point $Q \in M_3^0$, which defines a curve

$$c_Q^0 = M_1^0 \times Q \tag{3}$$

in M_4^0. Then the mapped curve $c_Q = \Phi(c_Q^0)$ in M_4 is exactly the world-line of the volume element labelled by Q. The Lagrangian description of electro-dynamics is nothing more than the reformulation of Maxwell's equations (1) and the constitutive laws (2) within the reference space M_4^0. This is accomplished by pull-back of the involved field quantities and operators via the placement mapping Φ. After setting

$$\underline{F}^0 = \Phi^*\underline{F}, \quad \underline{G}^0 = \Phi^*\underline{G}, \quad \underline{\mathcal{J}}^0 = \Phi^*\underline{\mathcal{J}},$$

the transformed equations are of the same kind as the original ones, namely,

$$\mathrm{d}^0\underline{F}^0 = 0, \quad \text{and} \quad \mathrm{d}^0\underline{G}^0 = \underline{\mathcal{J}}^0, \tag{4a}$$

$$\mathbf{i}_{\mathbf{u}^0}\left(*^0\,\underline{G}^0 + c_0\varepsilon\underline{F}^0\right) = 0, \tag{4b}$$

$$\mathbf{i}_{\mathbf{u}^0}\left(*^0\,\underline{F}^0 - c_0\mu\underline{G}^0\right) = 0, \tag{4c}$$

where $*^0$ indicates the Hodge operator of the pulled-back metric g^0. With (4), a complete four-dimensional Lagrangian description has been derived, whose properties will be elaborated in the sequel.

(3+1) DECOMPOSITION MECHANISM

Definition

In order to keep as much flexibility as possible, we employ the techniques presented in [4, p. 117]. The setting is illustrated in Figure 2. For a general space-time manifold

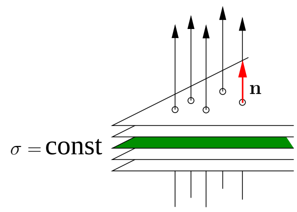

Figure 2: Foliation and fibration of M described by $\mathbf{n}$ and $\sigma =$ const.

M, let a fibration of M be described by a three-parameter vector field $\mathbf{n}$, introducing the notion of relative space. Furthermore, let a foliation of M be described by a one-parameter family of hypersurfaces $\sigma =$ const., introducing the notion of relative time. Setting $\boldsymbol{\sigma} = \mathrm{d}\sigma$, it is possible to scale $\mathbf{n}$ and $\boldsymbol{\sigma}$ such that

$$\mathbf{i}_{\mathbf{n}}\boldsymbol{\sigma} = 1. \tag{5}$$

The pair $(\mathbf{n}, \boldsymbol{\sigma})$ constructed in this manner constitutes an observer in space-time. In the special case of an inertial observer, the fibration consists of parallel time-like lines with the leafs of the foliation being orthogonal, and we have $\boldsymbol{\sigma} = g(\mathbf{n})$. Without the parallelism of the fibres, a locally inertial observer is obtained provided that $\boldsymbol{\sigma} = g(\mathbf{n})$ in every point. If a frame of reference was attached to an observer $(\mathbf{n}, \boldsymbol{\sigma})$ satisfying (5), $\mathbf{n}$ and $\boldsymbol{\sigma}$ would describe the temporal basis elements of this frame. However, the following (3+1) decomposition mechanism does not rely on the existence of a reference frame. In particular, we construct a projection operator $P = P_{\mathbf{n},\boldsymbol{\sigma}}$ depending only on the two constitutive parameters $\mathbf{n}$ and $\boldsymbol{\sigma}$.

We first consider the space $\mathcal{X}^p = \mathcal{X}^p(M)$ of smooth p-vector fields on M. Setting

$$\mathcal{X}^p_{\boldsymbol{\omega}} = \left\{\mathbf{w} \in \mathcal{X}^p : \boldsymbol{\omega}\big|\mathbf{w} = 0\right\},$$

the projection of a (1-)vector field is defined by

$$P : \mathcal{X}^1 \to \mathcal{X}^1_{\boldsymbol{\sigma}} \times \mathcal{X}^0_{\boldsymbol{\sigma}}, \quad \mathbf{w} \mapsto (\mathbf{a}, b),$$
$$b = \boldsymbol{\sigma}\big|\mathbf{w}, \quad \mathbf{a} = \mathbf{w} - b\mathbf{n}.$$

In Figure 3, the action of P on a vector $\mathbf{w}$ is sketched in 2D.

We proceed analogously for decomposing forms. Let $\mathcal{F}^p = \mathcal{F}^p(M)$ denote the space of smooth p-forms on M,

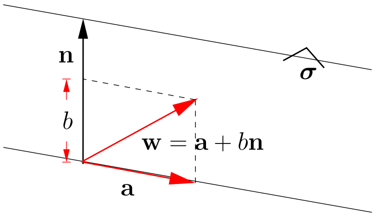

Figure 3: Projection of a vector $\mathbf{w}$.

and define

$$\mathcal{F}^p_{\mathbf{w}} = \{\boldsymbol{\omega} \in \mathcal{F}^p : \mathbf{i}_{\mathbf{w}}\boldsymbol{\omega} = 0\}\,.$$

Then, the projection of a 1-form is given by

$$P : \mathcal{F}^1 \to \mathcal{F}^1_{\mathbf{n}} \times \mathcal{F}^0_{\mathbf{n}}, \quad \boldsymbol{\omega} \mapsto (\boldsymbol{\alpha}, \beta),$$
$$\beta = \boldsymbol{\omega}\big|\mathbf{n}, \quad \boldsymbol{\alpha} = \boldsymbol{\omega} - \beta\boldsymbol{\sigma}.$$

In Figure 4, the action of P on a covector $\boldsymbol{\omega}$ is sketched in 2D. The domain of P can be extended to $\mathcal{F}^p$ by setting

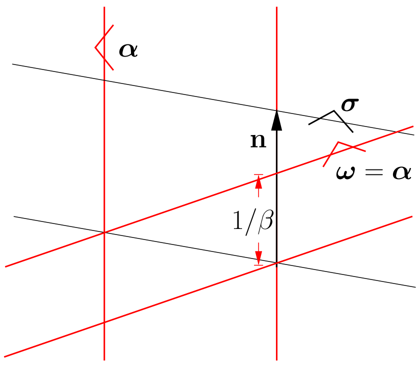

Figure 4: Projection of a covector $\boldsymbol{\omega}$.

$$P : \mathcal{F}^p \to \mathcal{F}^p_{\mathbf{n}} \times \mathcal{F}^{p-1}_{\mathbf{n}}, \quad \boldsymbol{\omega} \mapsto (\boldsymbol{\alpha},\, \boldsymbol{\beta}),$$
$$\boldsymbol{\beta} = \mathbf{i}_{\mathbf{n}}\boldsymbol{\omega}, \quad \boldsymbol{\alpha} = \boldsymbol{\omega} - \boldsymbol{\sigma}\wedge\boldsymbol{\beta} = \mathbf{i}_{\mathbf{n}}(\boldsymbol{\sigma}\wedge\boldsymbol{\omega}). \tag{6}$$

From the definition above, it is obvious that the projection P has the inverse

$$P^{-1} : \mathcal{F}^p_{\mathbf{n}} \times \mathcal{F}^{p-1}_{\mathbf{n}} \to \mathcal{F}^p, \quad (\boldsymbol{\alpha},\, \boldsymbol{\beta}) \mapsto \boldsymbol{\omega} = \boldsymbol{\alpha} + \boldsymbol{\sigma}\wedge\boldsymbol{\beta}.$$

Properties

We show some basic properties and use the projection P to decompose the fundamental operations.

Basic relations For two forms $\boldsymbol{\omega} \in \mathcal{F}^p$ and $\boldsymbol{\eta} \in \mathcal{F}^q$, we examine the action of the projection P on the exterior product $\boldsymbol{\omega}\wedge\boldsymbol{\eta}$. Let $P\boldsymbol{\omega} = ({}^{\parallel}\boldsymbol{\omega}, \boldsymbol{\omega}_{\perp})$ and $P\boldsymbol{\eta} = ({}^{\parallel}\boldsymbol{\eta}, \boldsymbol{\eta}_{\perp})$. Following [4, p. 117], we obtain

$$P(\boldsymbol{\omega}\wedge\boldsymbol{\eta}) = \begin{pmatrix} {}^{\parallel}\boldsymbol{\omega}\wedge{}^{\parallel}\boldsymbol{\eta} \\ \boldsymbol{\omega}_{\perp}\wedge{}^{\parallel}\boldsymbol{\eta} + (-1)^p\,{}^{\parallel}\boldsymbol{\omega}\wedge\boldsymbol{\eta}_{\perp} \end{pmatrix}$$
$$= \begin{pmatrix} {}^{\parallel}\boldsymbol{\omega}\wedge{}^{\parallel}\boldsymbol{\eta} \\ \boldsymbol{\omega}_{\perp}\wedge\boldsymbol{\eta} + (-1)^p\boldsymbol{\omega}\wedge\boldsymbol{\eta}_{\perp} \end{pmatrix}.$$

Especially useful will be the case $\boldsymbol{\eta} \in \mathcal{F}^q_{\mathbf{n}}$. Then ${}^{\parallel}\boldsymbol{\eta} = \boldsymbol{\eta}$ and $\boldsymbol{\eta}_{\perp} = 0$, which gives

$$P(\boldsymbol{\omega}\wedge\boldsymbol{\eta}) = (P\boldsymbol{\omega})\wedge\boldsymbol{\eta}, \quad \boldsymbol{\eta} \in \mathcal{F}^q_{\mathbf{n}}. \tag{7}$$

Decomposition of the exterior derivative We indicate the Lie derivative along the vector field $\mathbf{n}$ as "temporal" derivative

$$\bullet = \mathcal{L}_{\mathbf{n}} = \mathbf{i}_{\mathbf{n}}\circ\mathrm{d} + \mathrm{d}\circ\mathbf{i}_{\mathbf{n}}.$$

For the exterior differentiation of a projected pair $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{F}^p_{\mathbf{n}} \times \mathcal{F}^{p-1}_{\mathbf{n}}$, it is natural to use the composition $P\circ\mathrm{d}\circ P^{-1}$. Following the definition of the projection P, it is easy to see that

$$P\circ\mathrm{d} = \begin{pmatrix} \mathbf{i}_{\mathbf{n}}\circ(\boldsymbol{\sigma}\wedge\mathrm{d}) \\ \mathbf{i}_{\mathbf{n}}\circ\mathrm{d} \end{pmatrix} = \begin{pmatrix} \mathrm{d}_3 \\ \bullet - \mathrm{d}\circ\mathbf{i}_{\mathbf{n}} \end{pmatrix},$$

which defines the exterior three-derivative d_3. Considering

$$P\circ\mathrm{d}\circ P^{-1}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} = P\circ\mathrm{d}(\boldsymbol{\alpha} + \boldsymbol{\sigma}\wedge\boldsymbol{\beta})$$
$$= \begin{pmatrix}\mathrm{d}_3\\ \bullet\end{pmatrix}\boldsymbol{\alpha} + P(\mathrm{d}\boldsymbol{\sigma}\wedge\boldsymbol{\beta} - \boldsymbol{\sigma}\wedge\mathrm{d}\boldsymbol{\beta})$$
$$= \begin{pmatrix}\mathrm{d}_3\\ \bullet\end{pmatrix}\boldsymbol{\alpha} + (P\circ\mathrm{d}\boldsymbol{\sigma})\wedge\boldsymbol{\beta} - P(\boldsymbol{\sigma}\wedge\mathrm{d}\boldsymbol{\beta})$$
$$= \begin{pmatrix}\mathrm{d}_3\\ \bullet\end{pmatrix}\boldsymbol{\alpha} + \left(\begin{pmatrix}\mathrm{d}_3\\ \bullet\end{pmatrix}\boldsymbol{\sigma}\wedge - \begin{pmatrix}0\\ \mathrm{d}_3\end{pmatrix}\right)\boldsymbol{\beta}$$
$$= \begin{pmatrix}\mathrm{d}_3 & \mathrm{d}_3\boldsymbol{\sigma}\wedge\\ \bullet & -\mathrm{d}_3 + \dot{\boldsymbol{\sigma}}\wedge\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix},$$

we conclude that

$$P\circ\mathrm{d}\circ P^{-1} = \begin{pmatrix}\mathrm{d}_3 & \boldsymbol{\eta}\wedge\\ \bullet & -\mathrm{d}_3 + \boldsymbol{\delta}\wedge\end{pmatrix}, \tag{8}$$

where $\boldsymbol{\delta} = \dot{\boldsymbol{\sigma}}$ is the acceleration 1-form and $\boldsymbol{\eta} = \mathrm{d}_3\boldsymbol{\sigma}$ the vorticity 2-form, [3]. In other words, we have that $\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{\eta} + \boldsymbol{\sigma}\wedge\boldsymbol{\delta}$ with $(\boldsymbol{\eta}, \boldsymbol{\delta}) = P(\mathrm{d}\boldsymbol{\sigma})$. In the case that $(\mathbf{n}, \boldsymbol{\sigma})$ is seen as the temporal basis vector and covector of a reference frame, one obtains a geodesic frame for $\boldsymbol{\delta} = 0$, and an irrotational frame for $\boldsymbol{\eta} = 0$. Whenever $\boldsymbol{\delta} \neq 0$ or $\boldsymbol{\eta} \neq 0$, one speaks of an anholonomic frame. We remark that the composition $\mathrm{d}_3\circ\mathrm{d}_3 = -\boldsymbol{\eta}\wedge^{\bullet}$. This does not contradict Stoke's theorem, since $\boldsymbol{\eta} \neq 0$ yields $\boldsymbol{\sigma}\wedge\mathrm{d}\boldsymbol{\sigma} \neq 0$ which violates the Frobenius integrability condition. In this case, a three-dimensional integral submanifold does not exist.

Of special interest will be the case $\mathrm{d}\boldsymbol{\sigma} = 0$ yielding $\boldsymbol{\delta} = \boldsymbol{\eta} = 0$. Then, the decomposition of the exterior derivative is given by the simple form

$$P\circ\mathrm{d}\circ P^{-1} = \begin{pmatrix}\mathrm{d}_3 & 0\\ \bullet & -\mathrm{d}_3\end{pmatrix}.$$

A direct consequence of these considerations is that the canonical form of the (3+1)D Maxwell equations is only guaranteed for $\boldsymbol{\delta} = \boldsymbol{\eta} = 0$, as will become obvious in Section "Decomposition of Maxwell's equations".

Decomposition of the Hodge operator Let

$$s : \Lambda(\mathcal{F}) \to \Lambda(\mathcal{F}), \quad \boldsymbol{\omega} \mapsto (-1)^{\deg \boldsymbol{\omega}} \boldsymbol{\omega}.$$

The metric isomorphism $g^{-1} : \mathcal{F}^p \to \mathcal{X}^p$ can be decomposed into

$$Pg^{-1} = (g_3^{-1}s,\ .), \quad g_3^{-1} : \mathcal{F}^p_{\mathbf{n}} \to \mathcal{X}^p_{\boldsymbol{\sigma}}, \tag{9}$$

with a positive definite metric g_3. In particular, we have

$$g^{-1}(\boldsymbol{\omega}) = g_3^{-1}(s\boldsymbol{\omega}) + \mathbf{n} \wedge \chi(s\boldsymbol{\omega}), \tag{10}$$

for a mapping $\chi : \mathcal{F}^p_{\mathbf{n}} \to \mathcal{X}^{p-1}_{\boldsymbol{\sigma}}$. A justification for (10) is given in the appendix. It is possible to express the induced Hodge $*_3$ for the 3-metric g_3 in terms of the 4D Hodge $*$ and the vector field $\mathbf{n}$. In particular, using (46) and (41), we obtain for the oriented volume 3-form $\boldsymbol{\Omega}_3 \in \mathcal{F}^3_{\mathbf{n}}$ that

$$\begin{aligned} *_3\boldsymbol{\alpha} &= \boldsymbol{\Omega}_3 \big| g_3^{-1}(\boldsymbol{\alpha}) = \boldsymbol{\Omega}_3 \big| (g^{-1}(s\boldsymbol{\alpha}) + \mathbf{n} \wedge \chi(s\boldsymbol{\alpha})) \\ &= \boldsymbol{\Omega}_3 \big| g^{-1}(s\boldsymbol{\alpha}) + (\boldsymbol{\Omega}_3 \big| \mathbf{n}) \big| \chi(s\boldsymbol{\alpha}) = \boldsymbol{\Omega}_3 \big| g^{-1}(s\boldsymbol{\alpha}). \end{aligned}$$

By choosing $\boldsymbol{\Omega}_3 = -\rho \mathbf{i_n} \boldsymbol{\Omega}$ with $\rho > 0$, where $\boldsymbol{\Omega}$ is the oriented unit volume 4-form, the positive orientation of $\mathcal{F}^3_{\mathbf{n}}$ is determined. We derive

$$\begin{aligned} *_3\boldsymbol{\alpha} &= -\rho(\mathbf{i_n}\boldsymbol{\Omega}) \big| g^{-1}(s\boldsymbol{\alpha}) = -\rho(\boldsymbol{\Omega} \big| \mathbf{n}) \big| g^{-1}(s\boldsymbol{\alpha}) \\ &= -\rho\boldsymbol{\Omega} \big| (\mathbf{n} \wedge g^{-1}(s\boldsymbol{\alpha})) = -\rho\boldsymbol{\Omega} \big| (g^{-1}(\boldsymbol{\alpha}) \wedge \mathbf{n}) \\ &= \rho \mathbf{i_n} * \boldsymbol{\alpha}. \end{aligned}$$

In order to determine the scaling factor ρ, we use the fact that the volume 3-form $\boldsymbol{\Omega}_3$ is of unit volume as well as the properties (49) and (47) of the Hodge operator, and observe

$$\begin{aligned} 1 = \|\boldsymbol{\Omega}_3\|^2 &= \boldsymbol{\Omega}_3 \big| g_3^{-1}(\boldsymbol{\Omega}_3) = *_3\boldsymbol{\Omega}_3 \\ &= \rho \mathbf{i_n} * \boldsymbol{\Omega}_3 = -\rho^2 \mathbf{i_n} * \mathbf{i_n}\boldsymbol{\Omega} \\ &= \rho^2 \mathbf{i_n}(g(\mathbf{n}) \wedge *\boldsymbol{\Omega}) = \rho^2 g(\mathbf{n}) \big| \mathbf{n} = \rho^2 \|\mathbf{n}\|^2, \end{aligned}$$

yielding $\rho = \|\mathbf{n}\|^{-1}$. This allows to express the induced Hodge $*_3$ as

$$*_3 = \|\mathbf{n}\|^{-1} \mathbf{i_n} * \ .$$

Now, everything is available to decompose the Hodge $*$ with respect to the projection P. To this end, define $\lambda \in \mathbb{R}$ and $\mathbf{w} \in \mathcal{X}^1_{\boldsymbol{\sigma}}$ by $(\mathbf{w},\ \lambda) = Pg^{-1}(\boldsymbol{\sigma})$, namely

$$\begin{aligned} \mathbf{w} &= g^{-1}(\boldsymbol{\sigma}) - (\boldsymbol{\sigma} \big| g^{-1}(\boldsymbol{\sigma}))\mathbf{n}, \\ \lambda &= \boldsymbol{\sigma} \big| g^{-1}(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}\|^2. \end{aligned} \tag{11}$$

Considering

$$\begin{pmatrix} \boldsymbol{\alpha}' \\ \boldsymbol{\beta}' \end{pmatrix} = P * P^{-1} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = P * (\boldsymbol{\alpha} + \boldsymbol{\sigma} \wedge \boldsymbol{\beta}),$$

and using (48) and (49) gives

$$\begin{aligned} \boldsymbol{\beta}' &= \mathbf{i_n} * (\boldsymbol{\alpha} + \boldsymbol{\sigma} \wedge \boldsymbol{\beta}) = \mathbf{i_n} * \boldsymbol{\alpha} + \mathbf{i_n}\mathbf{i}_{g^{-1}(\boldsymbol{\sigma})} * s\boldsymbol{\beta} \\ &= \mathbf{i_n} * \boldsymbol{\alpha} - \mathbf{i}_{g^{-1}(\boldsymbol{\sigma})}\mathbf{i_n} * s\boldsymbol{\beta} = \mathbf{i_n} * \boldsymbol{\alpha} - \mathbf{i_w}\mathbf{i_n} * s\boldsymbol{\beta} \\ &= \|\mathbf{n}\| *_3 \boldsymbol{\alpha} - \|\mathbf{n}\| \mathbf{i_w} *_3 s\boldsymbol{\beta}, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\alpha}' &= \mathbf{i_n}(\boldsymbol{\sigma} \wedge *(\boldsymbol{\alpha} + \boldsymbol{\sigma} \wedge \boldsymbol{\beta})) \\ &= \mathbf{i_n}(- * \mathbf{i}_{g^{-1}(\boldsymbol{\sigma})} s(\boldsymbol{\alpha} + \boldsymbol{\sigma} \wedge \boldsymbol{\beta})) \\ &= -\mathbf{i_n} * \mathbf{i_w} s\boldsymbol{\alpha} + \mathbf{i_n} * \mathbf{i}_{g^{-1}(\boldsymbol{\sigma})}(\boldsymbol{\sigma} \wedge s\boldsymbol{\beta}) \\ &= -\mathbf{i_n} * \mathbf{i_w} s\boldsymbol{\alpha} + \mathbf{i_n} * ((\mathbf{i}_{g^{-1}(\boldsymbol{\sigma})}\boldsymbol{\sigma}) \wedge s\boldsymbol{\beta} - \boldsymbol{\sigma} \wedge \mathbf{i}_{g^{-1}(\boldsymbol{\sigma})} s\boldsymbol{\beta}) \\ &= -\mathbf{i_n} * \mathbf{i_w} s\boldsymbol{\alpha} + \mathbf{i_n} * (\lambda s\boldsymbol{\beta} - \boldsymbol{\sigma} \wedge \mathbf{i_w} s\boldsymbol{\beta}) \\ &= -\mathbf{i_n} * \mathbf{i_w} s\boldsymbol{\alpha} + \lambda \mathbf{i_n} * s\boldsymbol{\beta} - \mathbf{i_n} * (\boldsymbol{\sigma} \wedge \mathbf{i_w} s\boldsymbol{\beta}) \\ &= -\mathbf{i_n} * \mathbf{i_w} s\boldsymbol{\alpha} + \lambda \mathbf{i_n} * s\boldsymbol{\beta} - \mathbf{i_n}\mathbf{i}_{g^{-1}(\boldsymbol{\sigma})} * s\mathbf{i_w} s\boldsymbol{\beta} \\ &= -\mathbf{i_n} * \mathbf{i_w} s\boldsymbol{\alpha} + \lambda \mathbf{i_n} * s\boldsymbol{\beta} - \mathbf{i_w}\mathbf{i_n} * \mathbf{i_w}\boldsymbol{\beta} \\ &= -\|\mathbf{n}\| *_3 \mathbf{i_w} s\boldsymbol{\alpha} + \|\mathbf{n}\| \lambda *_3 s\boldsymbol{\beta} - \|\mathbf{n}\| \mathbf{i_w} *_3 \mathbf{i_w}\boldsymbol{\beta}. \end{aligned}$$

Thus, we have shown that the Hodge $*$ decomposes to

$$P * P^{-1} = \|\mathbf{n}\| \begin{pmatrix} - *_3 \mathbf{i_w} s & \lambda *_3 s - \mathbf{i_w} *_3 \mathbf{i_w} \\ *_3 & -\mathbf{i_w} *_3 s \end{pmatrix}. \tag{12}$$

Of special interest is the case $\mathbf{w} = 0$. If $(\mathbf{n}, \boldsymbol{\sigma})$ is extended to a reference frame, this frame is said to be time-orthogonal. Then, the decomposition simplifies to

$$P * P^{-1} = \|\mathbf{n}\| \begin{pmatrix} 0 & \lambda *_3 s \\ *_3 & 0 \end{pmatrix}.$$

Moreover, from (11) it follows for $\mathbf{w} = 0$ that $g^{-1}(\boldsymbol{\sigma}) = \lambda \mathbf{n}$, thus,

$$\lambda = \boldsymbol{\sigma} \big| g^{-1}(\boldsymbol{\sigma}) = \lambda g(\mathbf{n}) \big| \lambda \mathbf{n} = \lambda^2 \|\mathbf{n}\|^2,$$

yielding $\lambda = \|\mathbf{n}\|^{-2}$. Therefore, the decomposition of the Hodge operator in time-orthogonal frames is given by

$$P * P^{-1} = \begin{pmatrix} 0 & \|\mathbf{n}\|^{-1} *_3 s \\ \|\mathbf{n}\| *_3 & 0 \end{pmatrix}. \tag{13}$$

The consequence of these observations is that simple (3+1)D constitutive laws only occur for $\mathbf{w} = 0$, as will be shown in the next section.

Decomposition of the contraction The projection P also enables us to decompose the contraction of a p-form $\boldsymbol{\omega} = \boldsymbol{\alpha} + \boldsymbol{\sigma} \wedge \boldsymbol{\beta}$ by a vector field $\mathbf{u} \in \mathcal{X}$. We can write $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{a} + b\mathbf{n}, \qquad \mathbf{a} \in \mathcal{X}^1_{\boldsymbol{\sigma}},\ b \in \mathcal{X}^0.$$

Observing that

$$\mathbf{i_u}\boldsymbol{\omega} = \mathbf{i_a}\boldsymbol{\alpha} - \boldsymbol{\sigma} \wedge \mathbf{i_a}\boldsymbol{\beta} + b\boldsymbol{\beta},$$

it is easy to see that the contraction $\mathbf{i_u}$ decomposes to

$$P\mathbf{i_u}P^{-1} = \begin{pmatrix} \mathbf{i_a} & b \\ 0 & -\mathbf{i_a} \end{pmatrix}. \tag{14}$$

(3+1) DECOMPOSITION OF ELECTRO-DYNAMICS

Decomposition of the four-velocity

Projecting the four-velocity $\mathbf{u}$ defines the three-velocity $\mathbf{v}$ measuring the velocity in three-space $\mathcal{X}^1_{\boldsymbol{\sigma}}$ of the observer given by $(\mathbf{u}, \boldsymbol{\mu})$ relative to $(\mathbf{n}, \boldsymbol{\sigma})$. Additionally, one obtains the contraction factor γ_1 by setting

$$\gamma_1 \begin{pmatrix} \mathbf{v}/c_0 \\ 1 \end{pmatrix} = P\mathbf{u}. \tag{15}$$

In particular, we have

$$P\mathbf{u} = \begin{pmatrix} \mathbf{u} - (\boldsymbol{\sigma}|\mathbf{u})\mathbf{n} \\ \boldsymbol{\sigma}|\mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{u} - \gamma_1 \mathbf{n} \\ \gamma_1 \end{pmatrix}. \tag{16}$$

Given the 1-form $\boldsymbol{\mu}$ with $\boldsymbol{\mu}|\mathbf{u} = 1$, we define the factor γ_2 and the 1-form $\boldsymbol{\nu}$ by

$$\gamma_2 \begin{pmatrix} \boldsymbol{\nu}/c_0 \\ 1 \end{pmatrix} = P\boldsymbol{\mu}. \tag{17}$$

yielding analogously

$$P\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu} - (\boldsymbol{\mu}|\mathbf{n})\boldsymbol{\sigma} \\ \boldsymbol{\mu}|\mathbf{n} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu} - \gamma_2 \boldsymbol{\sigma} \\ \gamma_2 \end{pmatrix}. \tag{18}$$

Setting $-\beta^2 = c_0^{-2}\boldsymbol{\nu}|\mathbf{v}$, the relations (15)–(18) imply that

$$-\gamma_1\gamma_2\beta^2 = 1 - \gamma_1\gamma_2.$$

For the contraction factor γ defined by $\gamma^2 = \gamma_1\gamma_2$, we observe that

$$\gamma = \left(1 - \beta^2\right)^{-1/2}. \tag{19}$$

In our setting, we will consider the special case of locally inertial observers,

$$\mathbf{n} = g^{-1}(\boldsymbol{\sigma}), \quad \mathbf{u} = g^{-1}(\boldsymbol{\mu}). \tag{20}$$

Then we have

$$\gamma_1 = \boldsymbol{\sigma}|\mathbf{u} = \boldsymbol{\sigma}|g^{-1}(\boldsymbol{\mu}) = \boldsymbol{\sigma}\cdot\boldsymbol{\mu} = \boldsymbol{\mu}|g^{-1}(\boldsymbol{\sigma}) = \boldsymbol{\mu}|\mathbf{n} = \gamma_2,$$

thus, $\gamma = \gamma_1 = \gamma_2$. From (15) and (16), we conclude

$$\mathbf{u} = \gamma(\mathbf{n} + \mathbf{v}/c_0). \tag{21}$$

With (19) and (21), we have reconstructed the conventional definitions of $\mathbf{u}$ and γ which are based on $\mathbf{v}$ as constitutive parameter.

Decomposition of Maxwell's equations

With the projection operator P, it is possible to decompose the four-dimensional electro-dynamic quantities into their three-dimensional components. The four-potential $\underline{\Phi}$, the excitation $\underline{G}$, the electromagnetic field $\underline{F}$, and the four-current density $\underline{\mathcal{J}}$ decompose to

$$\begin{pmatrix} \underline{A} \\ -\underline{\varphi}/c_0 \end{pmatrix} = P\underline{\Phi}, \quad \begin{pmatrix} \underline{D} \\ \underline{H}/c_0 \end{pmatrix} = P\underline{G}, \tag{22a}$$

$$\begin{pmatrix} \underline{B} \\ -\underline{E}/c_0 \end{pmatrix} = P\underline{F}, \quad \begin{pmatrix} \underline{\rho} \\ -\underline{J}/c_0 \end{pmatrix} = P\underline{\mathcal{J}}. \tag{22b}$$

These projections define the conventional three-dimensional electromagnetic field quantities, [4, p. 118ff.]. Projecting the four-dimensional Maxwell equations (1) and substituting $P^{-1}P$ for the identity yields

$$P\,\mathrm{d}P^{-1}P\underline{F} = 0, \quad \text{and} \quad P\,\mathrm{d}P^{-1}P\underline{G} = P\underline{\mathcal{J}}.$$

Using the decompositions (8) and (22) of the four-differential and of the field quantities, respectively, we obtain the (3+1)-Maxwell equations

$$\begin{aligned} \mathrm{d}_3\underline{H} &= \underline{J} + \dot{\underline{D}} + \boldsymbol{\delta}\wedge\underline{H}, & \mathrm{d}_3\underline{B} &= 0 + \boldsymbol{\eta}\wedge\underline{E}, \\ \mathrm{d}_3\underline{E} &= -\dot{\underline{B}} \quad + \boldsymbol{\delta}\wedge\underline{E}, & \mathrm{d}_3\underline{D} &= \underline{\rho} - \boldsymbol{\eta}\wedge\underline{H}, \end{aligned}$$

the continuity equation $\mathrm{d}\underline{\mathcal{J}} = 0$ yields

$$\mathrm{d}_3\underline{J} + \dot{\underline{\rho}} = \boldsymbol{\delta}\wedge\underline{J},$$

and the definition of the four-potential $\mathrm{d}\underline{\Phi} = \underline{F}$ gives the potential equations

$$\underline{E} = -\,\mathrm{d}_3\underline{\varphi} - \dot{\underline{A}} + \underline{\varphi}\wedge\boldsymbol{\delta}, \qquad \underline{B} = \mathrm{d}_3\underline{A} - \underline{\varphi}\wedge\boldsymbol{\eta}.$$

It becomes obvious that it will not be convenient to operate in anholonomic settings, i.e. whenever $\boldsymbol{\delta}$ or $\boldsymbol{\eta}$ are different from zero.

Decomposition of the constitutive laws

We proceed in the same manner to derive (3+1)-dimensional constitutive laws. The projection of (2) gives

$$P\mathbf{i}_{\mathbf{u}}P^{-1}\left(P * P^{-1}P\underline{G} + c_0\varepsilon P\underline{F}\right) = 0, \tag{23a}$$

$$P\mathbf{i}_{\mathbf{u}}P^{-1}\left(P * P^{-1}P\underline{F} - c_0\mu P\underline{G}\right) = 0. \tag{23b}$$

Using the decompositions (14), (12), and (22) of the contraction, the Hodge star, and the field quantities, respectively, yields (3+1)-dimensional constitutive laws. A particularly easy situation is obtained by introducing a locally inertial observer, in which the considered material element is instantaneously at rest. Then, the three-velocity $\mathbf{v}$ is zero, hence, $\mathbf{u} = \mathbf{n}$ by (21) and $\mathbf{n} = g^{-1}(\boldsymbol{\sigma})$. As a consequence, we have $\mathbf{a} = 0$, $b = 1$ in (14) and, due to the time-orthogonality, the Hodge star decomposes to the simple form (13), with $\|\mathbf{n}\| = 1$. Therefore, indicating the such projected field components with a prime, relation (23a) implies that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & *_3 s \\ *_3 & 0 \end{pmatrix} \begin{pmatrix} \underline{D}' \\ \underline{H}'/c_0 \end{pmatrix} + c_0\varepsilon \begin{pmatrix} \underline{B}' \\ -\underline{E}'/c_0 \end{pmatrix} \right] = 0,$$

yielding

$$*_3\underline{D}' = \varepsilon\underline{E}'. \tag{24a}$$

Similarly, we obtain from (23b) that

$$*_3\underline{B}' = \mu\underline{H}'. \tag{24b}$$

In the more general setting of (20) and (21), we have $\mathbf{a} = \gamma\mathbf{v}/c_0$ and $b = \gamma$ in (14). The constitutive laws (23) yield

$$\frac{\gamma}{c_0}\begin{pmatrix}\mathbf{i_v} & c_0\\ 0 & -\mathbf{i_v}\end{pmatrix}\begin{pmatrix}-*_3\underline{H}/c_0 + c_0\varepsilon\underline{B}\\ *_3\underline{D} - \varepsilon\underline{E}\end{pmatrix} = 0,$$
$$\frac{\gamma}{c_0}\begin{pmatrix}\mathbf{i_v} & c_0\\ 0 & -\mathbf{i_v}\end{pmatrix}\begin{pmatrix}*_3\underline{E}/c_0 - c_0\mu\underline{D}\\ *_3\underline{B} - \mu\underline{H}\end{pmatrix} = 0.$$

The first line of each system implies

$$*_3\underline{D} - \mathbf{i_v}(*_3\underline{H})/c_0^2 = \varepsilon(\underline{E} - \mathbf{i_v}\underline{B}),$$
$$*_3\underline{B} + \mathbf{i_v}(*_3\underline{E})/c_0^2 = \mu(\underline{H} + \mathbf{i_v}\underline{D}).$$

Applying $\mathbf{i_v}*_3$ gives

$$\underline{D} \;=\; *_3\varepsilon\left(\underline{E} - (1 - c^2/c_0^2)\mathbf{i_v}\underline{B}\right) + O(\beta^2),$$
$$\underline{B} \;=\; *_3\mu\left(\underline{H} + (1 - c^2/c_0^2)\mathbf{i_v}\underline{D}\right) + O(\beta^2),$$

which are the well-known relations confirmed by Wilson and Röntgen/Eichwald, respectively. If the decomposition was not time-orthogonal, we would have to deal with the more involved expression (12) for the decomposed Hodge operator, which would lead to a quite complicated form of the constitutive laws. The advantage of our approach is that we will not have to consider this situation.

APPLICATION TO THE LAGRANGIAN PERSPECTIVE

In the following, we establish two (3+1)-decompositions of the reference space M_4°. To this end, we parametrize the curves c_Q° defined in (3) by arc-length with respect to the pulled-back metric g°. This admits the introduction of a coordinate t_Q° on c_Q° and, proceeding like this for all Q in M_3°, of a coordinate t° in M_4°. The set described by $t_Q^{\circ} = 0$, $Q \in M_3^{\circ}$, constitutes a leaf of the foliation of M_4°. The pair $(\mathbf{n}^{\circ}, \boldsymbol{\sigma}^{\circ})$ given by

$$\mathbf{n}^{\circ} = \partial_{c_0 t^{\circ}}, \quad \boldsymbol{\sigma}^{\circ} = \mathrm{d}(c_0 t^{\circ}),$$

represents the natural foliation of M_4°. By construction, we have $\boldsymbol{\sigma}^{\circ}\big|\mathbf{n}^{\circ} = 1$, thus, it is possible to apply the projection formalism. We remark that the choice of $(\mathbf{n}^{\circ}, \boldsymbol{\sigma}^{\circ})$ is unique. Although for any non-zero $\alpha \in \mathcal{F}^0(M_4^{\circ})$, the pair $(\alpha\mathbf{n}^{\circ}, \alpha^{-1}\boldsymbol{\sigma}^{\circ})$ describes the same foliation, the choice $\alpha = 1$ is fixed by the fact that the arc-length parametrization requires $|\alpha| = 1$, while the orientation of M_1° sets the sign.

By construction, we have that $D\Phi(\mathbf{n}^{\circ}) = \mathbf{u}$, i.e., $\mathbf{n}^{\circ} = \mathbf{u}^{\circ}$ resulting in $\mathbf{v}^{\circ} = 0$, as it is suggested by a Lagrangian description. Moreover, we have $\mathrm{d}\boldsymbol{\sigma}^{\circ} = 0$ by construction, thus, the projected Maxwell equations are of their simple form. We remark that the condition $g(\mathbf{n}^{\circ})\big|\mathbf{n}^{\circ} > 0$ reduces the possibilities for admissible placement mappings Φ, requiring that the curves c_Q° and c_Q have to be inside the light cone defined by the metric g° and g, respectively.

The metric admits the definition of a second, metric-compatible observer, by setting

$$\mathbf{n}' = \mathbf{n}^{\circ}, \quad \boldsymbol{\sigma}' = g^{\circ}(\mathbf{n}').$$

We immediately observe

$$\boldsymbol{\sigma}'\big|\mathbf{n}' = g(\mathbf{n}')\big|\mathbf{n}' = g(\mathbf{n}^{\circ})\big|\mathbf{n}^{\circ} = \|\mathbf{n}^{\circ}\|^2 = 1,$$

thus, the projection mechanism may be applied again. Moreover, since $\mathbf{v}^{\circ} = 0$, $\|\mathbf{n}'\| = 1$, and due to the time-orthogonality, the projected constitutive laws are of their simple form (24). The push-forward $D\Phi(\mathbf{n}', \boldsymbol{\sigma}')$ defines a locally inertial observer in M_4. As a consequence, if $\boldsymbol{\omega}' \in \mathcal{F}^p_{\mathbf{n}'}(M_4^{\circ})$ is a field quantity with respect to the projection defined by $(\mathbf{n}', \boldsymbol{\sigma}')$, then $\Lambda D\Phi(\boldsymbol{\omega}') \in \mathcal{F}^p_{\mathbf{u}}(M_4)$ is the corresponding measurable physical quantity, according to the hypothesis of locality, [5].

In order to suitably relate the two foliations, one has to formulate a transformation law which establishes a one-to-one correspondence between the spaces $\mathcal{F}^p_{\mathbf{n}^{\circ}} \times \mathcal{F}^{p-1}_{\mathbf{n}^{\circ}}$ and $\mathcal{F}^p_{\mathbf{n}'} \times \mathcal{F}^{p-1}_{\mathbf{n}'}$. We remark that both spaces are identical, since $\mathbf{n}^{\circ} = \mathbf{n}'$. The correspondence is simply given by the mapping $P^{\circ} \circ (P')^{-1}$. We first observe that

$$P^{\circ}\boldsymbol{\sigma}' = \begin{pmatrix}\boldsymbol{\sigma}' - (\mathbf{i}_{\mathbf{n}^{\circ}}\boldsymbol{\sigma}')\boldsymbol{\sigma}^{\circ}\\ \mathbf{i}_{\mathbf{n}^{\circ}}\boldsymbol{\sigma}'\end{pmatrix} = \begin{pmatrix}\boldsymbol{\sigma}' - \boldsymbol{\sigma}^{\circ}\\ 1\end{pmatrix}.$$

Using (7), one obtains

$$P^{\circ}(\boldsymbol{\alpha}' + \boldsymbol{\sigma}' \wedge \boldsymbol{\beta}') = P^{\circ}\boldsymbol{\alpha}' + P^{\circ}(\boldsymbol{\sigma}') \wedge \boldsymbol{\beta}',$$

which gives

$$\begin{pmatrix}\boldsymbol{\alpha}^{\circ}\\ \boldsymbol{\beta}^{\circ}\end{pmatrix} = \begin{pmatrix}1 & (\boldsymbol{\sigma}' - \boldsymbol{\sigma}^{\circ})\wedge\\ 0 & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}'\\ \boldsymbol{\beta}'\end{pmatrix}. \tag{25}$$

As pointed out before, the advantage of employing both decompositions is that we can employ simple $(3+1)$-Maxwell equations with respect to $(\mathbf{n}^{\circ}, \boldsymbol{\sigma}^{\circ})$ and simple constitutive laws with respect to $(\mathbf{n}', \boldsymbol{\sigma}')$. The transformation law (25) completes a convenient description of (3+1)-dimensional electro-dynamics.

CONNECTION TO THE EULERIAN PERSPECTIVE

In this section, we connect our results to the usual text-book version of the field transformations.

Lorentz boost

In order to stress the symmetric nature of our observations, we change the nomenclature of the two constitutive pairs $(\mathbf{n},\boldsymbol{\sigma})$ and $(\mathbf{u},\boldsymbol{\mu})$ to $(\mathbf{u},\boldsymbol{\mu})$ and $(\mathbf{u}',\boldsymbol{\mu}')$, respectively, in contrast to previous sections. Setting $\underline{\mathbf{u}} = (\mathbf{u},\mathbf{u}')^{\mathrm{T}}$, $\underline{\boldsymbol{\mu}} = (\boldsymbol{\mu},\boldsymbol{\mu}')^{\mathrm{T}}$, and assuming $\mathbf{u} = g^{-1}(\boldsymbol{\mu})$, we have

$$\underline{\boldsymbol{\mu}}\underline{\mathbf{u}}^{\mathrm{T}} = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix},$$

where the left hand side has to be understood as component-wise duality product. In order to simplify our notations, we make use of the kernel matrix

$$M = -\frac{1}{1+\gamma}\begin{pmatrix} 1 & 1 \\ 1-2(1+\gamma) & 1 \end{pmatrix}.$$

Elementary calculations reveal that

$$M^{\mathrm{T}}\underline{\boldsymbol{\mu}}\underline{\mathbf{u}}^{\mathrm{T}}M = M\underline{\boldsymbol{\mu}}\underline{\mathbf{u}}^{\mathrm{T}}M^{\mathrm{T}}, \tag{26a}$$
$$M + M^{\mathrm{T}} + M^{\mathrm{T}}\underline{\boldsymbol{\mu}}\underline{\mathbf{u}}^{\mathrm{T}}M = 0, \tag{26b}$$

namely,

$$\begin{aligned} &M + M^{\mathrm{T}} + M^{\mathrm{T}}\underline{\boldsymbol{\mu}}\underline{\mathbf{u}}^{\mathrm{T}}M \\ &\quad = -\frac{1}{1+\gamma}\begin{pmatrix} 2 & -2\gamma \\ -2\gamma & 2 \end{pmatrix} \\ &\qquad + \frac{1}{(1+\gamma)^2}\begin{pmatrix} \boldsymbol{\mu}-(1+2\gamma)\boldsymbol{\mu}' \\ \boldsymbol{\mu}+\boldsymbol{\mu}' \end{pmatrix}\begin{pmatrix} \mathbf{u}-(1+2\gamma)\mathbf{u}' \\ \mathbf{u}+\mathbf{u}' \end{pmatrix}^{\mathrm{T}}, \end{aligned}$$

and the second term equals

$$\begin{aligned} &\frac{1}{(1+\gamma)^2}\begin{pmatrix} 1-2(1+2\gamma)\gamma+(1+2\gamma)^2 & -2\gamma(1+\gamma) \\ -2\gamma(1+\gamma) & 2(1+\gamma) \end{pmatrix} \\ &\qquad = \frac{1}{1+\gamma}\begin{pmatrix} 2 & -2\gamma \\ -2\gamma & 2 \end{pmatrix}, \end{aligned}$$

which gives (26b). We define the Lorentz boost of a vector field $\mathbf{w}$ and of a 1-form $\boldsymbol{\omega}$ as

$$\begin{aligned} B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}) &= \mathbf{w} + \underline{\mathbf{u}}^{\mathrm{T}}M\underline{\boldsymbol{\mu}}\big|\mathbf{w}, \\ B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega}) &= \boldsymbol{\omega} + \boldsymbol{\omega}\big|\underline{\mathbf{u}}^{\mathrm{T}}M^{\mathrm{T}}\underline{\boldsymbol{\mu}}, \end{aligned}$$

and its inverse as

$$\begin{aligned} B^{-1}_{\mathbf{u},\mathbf{u}'}(\mathbf{w}) &= \mathbf{w} + \underline{\mathbf{u}}^{\mathrm{T}}M^{\mathrm{T}}\underline{\boldsymbol{\mu}}\big|\mathbf{w} = B_{\mathbf{u}',\mathbf{u}}(\mathbf{w}), \\ B^{-1}_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega}) &= \boldsymbol{\omega} + \boldsymbol{\omega}\big|\underline{\mathbf{u}}^{\mathrm{T}}M\underline{\boldsymbol{\mu}} \quad = B_{\mathbf{u}',\mathbf{u}}(\boldsymbol{\omega}), \end{aligned}$$

respectively. It is easy to see that

$$B_{\mathbf{u},\mathbf{u}'}(\mathbf{u}) = \mathbf{u}'.$$

To see that the mapping $B^{-1}_{\mathbf{u},\mathbf{u}'}$ is indeed the inverse of $B_{\mathbf{u},\mathbf{u}'}$, we consider the compositions $B_{\mathbf{u},\mathbf{u}'} \circ B^{-1}_{\mathbf{u},\mathbf{u}'}$ and $B^{-1}_{\mathbf{u},\mathbf{u}'} \circ B_{\mathbf{u},\mathbf{u}'}$ and show by making use of (26) that the identity mapping is obtained. For example, applying the composition to vector fields, we have

$$\begin{aligned} &B_{\mathbf{u},\mathbf{u}'} \circ B^{-1}_{\mathbf{u},\mathbf{u}'}(\mathbf{w}) \\ &\quad = \mathbf{w} + \underline{\mathbf{u}}^{\mathrm{T}}M^{\mathrm{T}}\underline{\boldsymbol{\mu}}\big|\mathbf{w} + \underline{\mathbf{u}}^{\mathrm{T}}M\underline{\boldsymbol{\mu}}\big|(\mathbf{w} + \underline{\mathbf{u}}^{\mathrm{T}}M^{\mathrm{T}}\underline{\boldsymbol{\mu}}\big|\mathbf{w}) \\ &\quad = \mathbf{w} + \underline{\mathbf{u}}^{\mathrm{T}}(M + M^{\mathrm{T}} + M\underline{\boldsymbol{\mu}}\underline{\mathbf{u}}^{\mathrm{T}}M^{\mathrm{T}})\underline{\boldsymbol{\mu}}\big|\mathbf{w} \\ &\quad = \mathbf{w}. \end{aligned}$$

The Lorentz boost has been defined such that the duality product of a 1-form $\boldsymbol{\omega}$ and a vector field $\mathbf{w}$ is preserved under the boost. In particular, again employing the identity (26), we obtain

$$\begin{aligned} &B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega})\big|B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}) \\ &\quad = (\boldsymbol{\omega} + \boldsymbol{\omega}\big|\underline{\mathbf{u}}^{\mathrm{T}}M^{\mathrm{T}}\underline{\boldsymbol{\mu}})\big|(\mathbf{w} + \underline{\mathbf{u}}^{\mathrm{T}}M\underline{\boldsymbol{\mu}}\big|\mathbf{w}) \\ &\quad = \boldsymbol{\omega}\big|\mathbf{w} + \boldsymbol{\omega}\big|\underline{\mathbf{u}}^{\mathrm{T}}(M + M^{\mathrm{T}} + M^{\mathrm{T}}\underline{\boldsymbol{\mu}}\underline{\mathbf{u}}^{\mathrm{T}}M)\underline{\boldsymbol{\mu}}\big|\mathbf{w} \\ &\quad = \boldsymbol{\omega}\big|\mathbf{w}. \end{aligned} \tag{27}$$

An immediate consequence of this observation is the fact that the scalar product of two vector fields $\mathbf{w}_1$ and $\mathbf{w}_2$ is also preserved under the Lorentz boost, i.e.,

$$B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}_1) \cdot B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}_2) = \mathbf{w}_1 \cdot \mathbf{w}_2. \tag{28}$$

In order to see this, we first check a commutative property for the Riesz isomorphism g. Assuming that $\underline{\boldsymbol{\mu}} = g(\underline{\mathbf{u}})$, we have for a vector field $\mathbf{w}$

$$\begin{aligned} g(\mathbf{w})\big|\underline{\mathbf{u}}^{\mathrm{T}}M^{\mathrm{T}}\underline{\boldsymbol{\mu}} &= g(\underline{\mathbf{u}}^{\mathrm{T}})\big|\mathbf{w}M^{\mathrm{T}}\underline{\boldsymbol{\mu}} = \underline{\boldsymbol{\mu}}^{\mathrm{T}}\big|\mathbf{w}M^{\mathrm{T}}\underline{\boldsymbol{\mu}} \\ &= \underline{\boldsymbol{\mu}}^{\mathrm{T}}M\underline{\boldsymbol{\mu}}\big|\mathbf{w} = g(\underline{\mathbf{u}}^{\mathrm{T}})M\underline{\boldsymbol{\mu}}\big|\mathbf{w} \\ &= g(\underline{\mathbf{u}}^{\mathrm{T}}M\underline{\boldsymbol{\mu}}\big|\mathbf{w}), \end{aligned}$$

yielding

$$\begin{aligned} B_{\mathbf{u},\mathbf{u}'}g(\mathbf{w}) &= g(\mathbf{w}) + g(\mathbf{w})\big|\underline{\mathbf{u}}^{\mathrm{T}}M^{\mathrm{T}}\underline{\boldsymbol{\mu}} \\ &= g(\mathbf{w} + \underline{\mathbf{u}}^{\mathrm{T}}M\underline{\boldsymbol{\mu}}\big|\mathbf{w}) = g(B_{\mathbf{u},\mathbf{u}'}(\mathbf{w})). \end{aligned}$$

Using (27), one obtains (28) by

$$\begin{aligned} B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}_1) \cdot B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}_2) &= g(B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}_1))\big|B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}_2) \\ &= B_{\mathbf{u},\mathbf{u}'}g(\mathbf{w}_1)\big|B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}_2) \\ &= g(\mathbf{w}_1)\big|\mathbf{w}_2 = \mathbf{w}_1 \cdot \mathbf{w}_2. \end{aligned}$$

It is possible to extend the Lorentz boost $B_{\mathbf{u},\mathbf{u}'}$ to p-forms and p-vector fields by means of the exterior p-compound, namely

$$\begin{aligned} B_{\mathbf{u},\mathbf{u}'} &: \Lambda\mathcal{F}^1(M) \to \Lambda\mathcal{F}^1(M), \\ B_{\mathbf{u},\mathbf{u}'} &: \Lambda\mathcal{X}^1(M) \to \Lambda\mathcal{X}^1(M). \end{aligned}$$

As we will demonstrate in the appendix, it is possible to show that (27) also holds for the extension by using the properties of the p-compound. Elementary, but lengthy calculations reveal that

$$B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega}) = \boldsymbol{\omega} + \underline{\boldsymbol{\mu}}^{\mathrm{T}} \wedge M\underline{\mathbf{i}}\boldsymbol{\omega} - \det M\boldsymbol{\mu}\wedge\boldsymbol{\mu}'\wedge\mathbf{i}_{\mathbf{u}}\mathbf{i}_{\mathbf{u}'}\boldsymbol{\omega}, \tag{29}$$

where $\underline{\mathbf{i}} = (\mathbf{i}_{\mathbf{u}}, \mathbf{i}_{\mathbf{u}'})^{\mathrm{T}}$. In the following, the Lorentz boost will be decomposed with respect to $(\mathbf{u},\boldsymbol{\mu})$. To this end, let $\boldsymbol{\omega} = \boldsymbol{\alpha} + \boldsymbol{\mu}\wedge\boldsymbol{\beta}$, and observe that

$$\begin{aligned} \mathbf{i}_{\mathbf{u}}\boldsymbol{\omega} &= \boldsymbol{\beta}, \\ \mathbf{i}_{\mathbf{u}'}\boldsymbol{\omega} &= \mathbf{i}_{\mathbf{u}'}\boldsymbol{\alpha} + \gamma\boldsymbol{\beta} - \boldsymbol{\mu}\wedge\mathbf{i}_{\mathbf{u}'}\boldsymbol{\beta}, \\ \mathbf{i}_{\mathbf{u}}\mathbf{i}_{\mathbf{u}'}\boldsymbol{\omega} &= -\mathbf{i}_{\mathbf{u}'}\mathbf{i}_{\mathbf{u}}\boldsymbol{\omega} = -\mathbf{i}_{\mathbf{u}'}\boldsymbol{\beta}. \end{aligned}$$

Following (29), we first see that

$$M\underline{\mathbf{i}}\boldsymbol{\omega} = -\frac{1}{1+\gamma}\begin{pmatrix}\mathbf{i}_{\mathbf{u}'}\boldsymbol{\alpha} + (1+\gamma)\boldsymbol{\beta} - \boldsymbol{\mu}\wedge\mathbf{i}_{\mathbf{u}'}\boldsymbol{\beta}\\ \mathbf{i}_{\mathbf{u}'}\boldsymbol{\alpha} - (1+\gamma)\boldsymbol{\beta} - \boldsymbol{\mu}\wedge\mathbf{i}_{\mathbf{u}'}\boldsymbol{\beta}\end{pmatrix},$$

yielding

$$\begin{aligned}\underline{\boldsymbol{\mu}}^{\mathrm{T}}\wedge M\underline{\mathbf{i}}\boldsymbol{\omega} = &-\frac{1}{1+\gamma}\left(\boldsymbol{\mu}\wedge(\mathbf{i}_{\mathbf{u}'}\boldsymbol{\alpha} + (1+\gamma)\boldsymbol{\beta})\right.\\ &\left.+\,\boldsymbol{\mu}'\wedge(\mathbf{i}_{\mathbf{u}'}\boldsymbol{\alpha} - (1+\gamma)\boldsymbol{\beta} - \boldsymbol{\mu}\wedge\mathbf{i}_{\mathbf{u}'}\boldsymbol{\beta})\right).\end{aligned}$$

Thus, using the fact that $\det M = 2/(1+\gamma)$, (29) amounts to

$$\begin{aligned}\overline{\boldsymbol{\omega}} = B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega}) &= \boldsymbol{\alpha} - \frac{1}{1+\gamma}(\boldsymbol{\mu}+\boldsymbol{\mu}')\wedge\mathbf{i}_{\mathbf{u}'}\boldsymbol{\alpha}\\ &\quad + \boldsymbol{\mu}'\wedge\boldsymbol{\beta} + \frac{1}{1+\gamma}\boldsymbol{\mu}\wedge\boldsymbol{\mu}'\wedge\mathbf{i}_{\mathbf{u}'}\boldsymbol{\beta}\\ &= \boldsymbol{\alpha} + \boldsymbol{\mu}'\wedge\boldsymbol{\beta}\\ &\quad - \frac{1}{1+\gamma}\left((\boldsymbol{\mu}+\boldsymbol{\mu}')\wedge\mathbf{i}_{\mathbf{u}'}\boldsymbol{\alpha} - \boldsymbol{\mu}\wedge\boldsymbol{\mu}'\wedge\mathbf{i}_{\mathbf{u}'}\boldsymbol{\beta}\right).\end{aligned}$$

Application of the projection P to $\overline{\boldsymbol{\omega}}$ gives

$$\begin{aligned}\overline{\boldsymbol{\beta}} &= \mathbf{i}_{\mathbf{u}}\overline{\boldsymbol{\omega}}\\ &= \gamma\boldsymbol{\beta} - \mathbf{i}_{\mathbf{u}'}\boldsymbol{\alpha} + \frac{1}{1+\gamma}(\boldsymbol{\mu}'-\gamma\boldsymbol{\mu})\wedge\mathbf{i}_{\mathbf{u}'}\boldsymbol{\beta},\\ \overline{\boldsymbol{\alpha}} &= \mathbf{i}_{\mathbf{u}}(\boldsymbol{\mu}\wedge\overline{\boldsymbol{\omega}})\\ &= \boldsymbol{\alpha} + (\boldsymbol{\mu}'-\gamma\boldsymbol{\mu})\wedge(\boldsymbol{\beta} - \frac{1}{1+\gamma}\mathbf{i}_{\mathbf{u}'}\boldsymbol{\alpha}).\end{aligned}$$

It is important to note that both pairs $(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $(\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\beta}})$ are members of the same subspace $\mathcal{F}^p_{\mathbf{u}}\times\mathcal{F}^{p-1}_{\mathbf{u}}$. We intend to rewrite the transformation formulas above in terms of the three-velocity $\mathbf{v}$. From (21), we have by an application of g that

$$\boldsymbol{\mu}' - \gamma\boldsymbol{\mu} = \gamma\boldsymbol{\nu}/c_0 \tag{30}$$

for the Riesz dual $\boldsymbol{\nu} = g(\mathbf{v})$. Moreover,

$$\mathbf{i}_{\mathbf{u}'}\boldsymbol{\omega} = \gamma c_0^{-1}\mathbf{i}_{\mathbf{v}}\boldsymbol{\omega}, \quad \boldsymbol{\omega}\in\mathcal{F}^p_{\mathbf{u}}. \tag{31}$$

In matrix notation, we can now write the Lorentz boost as

$$P\circ B_{\mathbf{u},\mathbf{u}'}\circ P^{-1} = \begin{pmatrix}1-\sigma\boldsymbol{\nu}\wedge\mathbf{i}_{\mathbf{v}} & \gamma c_0^{-1}\boldsymbol{\nu}\wedge\\ -\gamma c_0^{-1}\mathbf{i}_{\mathbf{v}} & \gamma+\sigma\boldsymbol{\nu}\wedge\mathbf{i}_{\mathbf{v}}\end{pmatrix}, \tag{32}$$

where $\sigma = \gamma^2/(c_0^2(1+\gamma))$.

The Lorentz boost and the decomposition mechanism are commutative in the sense that

$$P'\circ B_{\mathbf{u},\mathbf{u}'} = B_{\mathbf{u},\mathbf{u}'}\circ P, \tag{33}$$

where $B_{\mathbf{u},\mathbf{u}'}$ is applied component-wise to projected quantities. We will show (33) in the appendix. An immediate consequence of (33) is the important property of the Lorentz boost being an isomorphism between the subspaces $\Lambda\mathcal{F}^1_{\mathbf{u}}(M)$ and $\Lambda\mathcal{F}^1_{\mathbf{u}'}(M)$.

Lorentz transformation

In this section, we answer the question of how to transfer relative electromagnetic field components from one observer $(\mathbf{u},\boldsymbol{\mu})$ to another observer $(\mathbf{u}',\boldsymbol{\mu}')$. It is clear that the pair $(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{F}^p_{\mathbf{u}}\times\mathcal{F}^{p-1}_{\mathbf{u}}$ has a natural image $(\boldsymbol{\alpha}',\boldsymbol{\beta}')\in\mathcal{F}^p_{\mathbf{u}'}\times\mathcal{F}^{p-1}_{\mathbf{u}'}$ obtained by

$$\begin{pmatrix}\boldsymbol{\alpha}'\\ \boldsymbol{\beta}'\end{pmatrix} = P'\circ P^{-1}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}.$$

The relation above expresses the fact that $(\boldsymbol{\alpha},\boldsymbol{\beta})$ are the components of $\boldsymbol{\omega} = \boldsymbol{\alpha} + \boldsymbol{\mu}\wedge\boldsymbol{\beta}$ relative to the observer characterized by $(\mathbf{u},\boldsymbol{\mu})$, while $(\boldsymbol{\alpha}',\boldsymbol{\beta}')$ are the components of the same object $\boldsymbol{\omega}$ relative to the observer $(\mathbf{u}',\boldsymbol{\mu}')$. By construction, the pairs $(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $(\boldsymbol{\alpha}',\boldsymbol{\beta}')$ are not members of the same subspace. However, the usual Lorentz transformation compares objects of only one subspace, namely $\Lambda\mathcal{F}^1_{\mathbf{u}}$. The crucial step is now to identify a pair $(\boldsymbol{\alpha}',\boldsymbol{\beta}')\in\mathcal{F}^p_{\mathbf{u}'}\times\mathcal{F}^{p-1}_{\mathbf{u}'}$ with a physically equivalent pair $(\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\beta}})\in\mathcal{F}^p_{\mathbf{u}}\times\mathcal{F}^{p-1}_{\mathbf{u}}$. In special relativity, two forms $\boldsymbol{\omega}'\in\mathcal{F}^p_{\mathbf{u}'}$ and $\overline{\boldsymbol{\omega}}\in\mathcal{F}^p_{\mathbf{u}}$ are declared to be physically equivalent if they are related by a Lorentz boost, [7, p. 203], i.e.,

$$\boldsymbol{\omega}'\sim\overline{\boldsymbol{\omega}} \;\Leftrightarrow\; \boldsymbol{\omega}' = B_{\mathbf{u},\mathbf{u}'}\overline{\boldsymbol{\omega}}.$$

Therefore, the pair $(\boldsymbol{\alpha}',\boldsymbol{\beta}')$ is equivalent to $(\overline{\boldsymbol{\alpha}},\overline{\boldsymbol{\beta}})$ given by

$$\begin{aligned}\begin{pmatrix}\overline{\boldsymbol{\alpha}}\\ \overline{\boldsymbol{\beta}}\end{pmatrix} &= B_{\mathbf{u},\mathbf{u}'}\circ P'\circ P^{-1}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}\\ &= P\circ B_{\mathbf{u},\mathbf{u}'}\circ P^{-1}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix},\end{aligned}$$

where we have used the commutativity (33). This means that the transformation is given by the decomposed Lorentz boost (32), up to the sign of $(\mathbf{v},\boldsymbol{\nu})$.

The application of (32) to the field $\underline{F}$ and the current $\underline{\mathcal{J}}$ yields

$$\begin{aligned}\overline{\underline{B}} &= \underline{B} - \sigma\boldsymbol{\nu}\wedge\mathbf{i}_{\mathbf{v}}\underline{B} + \gamma\boldsymbol{\nu}\wedge\underline{E}/c_0^2,\\ \overline{\underline{E}} &= \gamma\underline{E} + \sigma\boldsymbol{\nu}\wedge\mathbf{i}_{\mathbf{v}}\underline{E} - \gamma\mathbf{i}_{\mathbf{v}}\underline{B},\\ \overline{\underline{\rho}} &= \underline{\rho} - \sigma\boldsymbol{\nu}\wedge\mathbf{i}_{\mathbf{v}}\underline{\rho} + \gamma\boldsymbol{\nu}\wedge\underline{J}/c_0^2,\\ \overline{\underline{J}} &= \gamma\underline{J} + \sigma\boldsymbol{\nu}\wedge\mathbf{i}_{\mathbf{v}}\underline{J} - \gamma\mathbf{i}_{\mathbf{v}}\underline{\rho}.\end{aligned}$$

In the following, we reformulate the transformation laws above in terms of vector fields. In three-space $\mathcal{X}^1_{\boldsymbol{\mu}}$ with its induced Euclidean metric g_3 and Hodge $*_3$, a differential form is uniquely identified by its vector proxy obtained by the Riesz isomorphism. It is possible to identify

field	p	**form**
f	0	${}^0f = f$
$\mathbf{a}$	1	${}^1\mathbf{a} = g(\mathbf{a})$
$\mathbf{b}$	2	${}^2\mathbf{b} = *_3 g(\mathbf{b})$
g	3	${}^3g = *_3 g$

The operators $\wedge$ and $\mathbf{i}$ translate to

$$
\begin{aligned}
{}^2(\mathbf{a}\times\mathbf{b}) &= {}^1\mathbf{a}\wedge{}^1\mathbf{b},\\
{}^3(\mathbf{a}\cdot\mathbf{b}) &= {}^1\mathbf{a}\wedge{}^2\mathbf{b},\\
\mathbf{i}_\mathbf{v}{}^1\mathbf{a} &= {}^0(\mathbf{a}\cdot\mathbf{v}),\\
\mathbf{i}_\mathbf{v}{}^2\mathbf{b} &= {}^1(\mathbf{b}\times\mathbf{v}),\\
\mathbf{i}_\mathbf{v}{}^3 g &= {}^2(g\mathbf{v}).
\end{aligned}
$$

This yields the usual transformations

$$
\begin{aligned}
\vec{B}\,' &= \vec{B}_\parallel + \gamma(\vec{B}_\perp - \mathbf{v}\times\vec{E}/c_0^2),\\
\vec{E}\,' &= \vec{E}_\parallel + \gamma(\vec{E}_\perp + \mathbf{v}\times\vec{B}),\\
\rho' &= \gamma(\rho - \mathbf{v}\cdot\vec{J}/c_0^2),\\
\vec{J}\,' &= \vec{J}_\perp + \gamma(\vec{J}_\parallel - \mathbf{v}\rho).
\end{aligned}
$$

EXAMPLE

We present an application of the above formalism to the classical paradoxon by Schiff [8]. Originally, the setting is given in terms of two conducting spheres. For the sake of simplicity and in order to illustrate the idea, we investigate an analogue in terms of an infinitely long cylinder, see [1, p. 320]. The following field quantities are chosen with respect

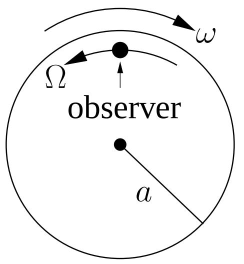

Figure 5: Paradoxon by Schiff (simplified): homogeneously charged cylinder, two different situations.

to an inertial observer attached to the axis of the cylinder. The cylinder of radius a is homogeneously charged with a surface charge distribution $\sigma = q/(2\pi a)$. It may rotate at an angular speed ω, causing an azimuthal convection current $j_\varphi = -\omega q\delta(r-a)/(2\pi)$, where δ denotes the delta distribution. We install an observer which may rotate at an angular speed Ω. We consider two different situations in which the observer measures the field inside the cylinder. In the first one, only the cylinder moves, whereas in the second one, only the observer moves:

- $\Omega = 0,\ \omega = \omega_0$: As a result of the convection current, the interior of the cylinder is filled by an electromagnetic field with radial displacement $D_r = 0$ and axial magnetic field $H_z = -\omega q/(2\pi)$.
- $\Omega = \omega_0,\ \omega = 0$: The interior is free of an electromagnetic field, which has to hold for arbitrary observers. Therefore, the observer should find $D_r' = H_z' = 0$. However, should not this case be identical to the first one from the observer's point of view? He always experiences the same kinematics: a cylinder moving at the same angular speed.

The resolution of this apparent paradox results from considering the correct measurable fields for the second case. One of the benefits of our approach is now that we consider Maxwell's equations with respect to the observer $(\mathbf{n}^0, \boldsymbol{\sigma}^0)$ where they basically keep their usual form. Nevertheless, they have to be transformed into rotating cylinder coordinates, as found for example in [1, p. 268]. In M_4, we choose coordinates $(c_0 t, r, \varphi, z)$ yielding the natural bases $(c_0^{-1}\partial_t, \partial_r, \partial_\varphi, \partial_z)$ and $(c_0\,\mathrm{d}t, \mathrm{d}r, \mathrm{d}\varphi, \mathrm{d}z)$ of the tangent and cotangent spaces, respectively. The metric g in matrix notation is given by $\mathrm{diag}\,(1, -1, -r^2, -1)$. For the reference space M_4^0, the coordinates and bases are completely analogous, namely $(c_0 t^0, r^0, \varphi^0, z^0)$, $(c_0^{-1}\partial_{t^0}, \partial_{r^0}, \partial_{\varphi^0}, \partial_{z^0})$, and $(c_0\,\mathrm{d}t^0, \mathrm{d}r^0, \mathrm{d}\varphi^0, \mathrm{d}z^0)$.

The placement mapping $\Phi : M_4^0 \to M_4$ is given by

$$
\begin{aligned}
&c_0 t = \gamma c_0 t^0, \quad r = r^0, z = z^0\\
&\varphi = \varphi^0 + \frac{\Omega}{c_0}\gamma c_0 t^0 = \varphi^0 + \Omega t,
\end{aligned}
$$

with $\gamma = (1 - (\Omega r^0/c_0)^2)^{-1/2}$. The Jacobian $D\Phi$ establishes the connection between the bases of the cotangent and tangent spaces,

$$
\begin{aligned}
c_0\,\mathrm{d}t &= \gamma c_0\,\mathrm{d}t^0, & c_0^{-1}\partial_t &= \gamma^{-1}c_0^{-1}\partial_{t^0} - \Omega c_0^{-1}\partial_{\varphi^0},\\
\mathrm{d}r &= \mathrm{d}r^0, & \partial_r &= \partial_{r^0},\\
\mathrm{d}\varphi &= \mathrm{d}\varphi^0 + \frac{\gamma\Omega}{c_0}c_0\,\mathrm{d}t^0, & \partial_\varphi &= \partial_{\varphi^0},\\
\mathrm{d}z &= \mathrm{d}z^0, & \partial_z &= \partial_{z^0}.
\end{aligned}
$$

For the pulled-back metric g^0, the coefficients can be calculated in the standard manner. In particular, we obtain

$$
\begin{aligned}
c_0^{-1}\partial_{t^0}\cdot c_0^{-1}\partial_{t^0} &= \gamma^2 c_0^{-1}\partial_t\cdot c_0^{-1}\partial_t + (\gamma\Omega/c_0)^2\partial_\varphi\cdot\partial_\varphi\\
&= \gamma^2(1-(r\Omega/c_0)^2) = 1,
\end{aligned}
$$

which confirms the arc-length parametrization by t^0. The full matrix representation of g^0 is given by

$$
g^0 = \begin{pmatrix} 1 & 0 & -\gamma r^2\Omega/c_0 & 0\\ 0 & -1 & 0 & 0\\ -\gamma r^2\Omega/c_0 & 0 & -r^2 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix},
$$

which is obviously not time-orthogonal, as expected. The foliation-induced observer is simply given by $(\mathbf{n}^0, \boldsymbol{\sigma}^0) = (c_0^{-1}\partial_{t^0}, c_0\,\mathrm{d}t^0)$. For the local inertial observer, we have $\mathbf{n}' = \mathbf{n}^0 = c_0^{-1}\partial_{t^0}$ and

$$
\begin{aligned}
\boldsymbol{\sigma}' = g^0(\mathbf{n}') = g^0\begin{pmatrix}1\\0\\0\\0\end{pmatrix} &= \begin{pmatrix}1\\0\\-\gamma r^2\Omega/c_0\\0\end{pmatrix}\\
&= c_0\,\mathrm{d}t^0 - \gamma r^2\Omega/c_0\,\mathrm{d}\varphi^0 = \boldsymbol{\sigma}^0 - \gamma r^2\Omega/c_0\,\mathrm{d}\varphi^0.
\end{aligned}
$$

Thus, the difference $\boldsymbol{\sigma}'-\boldsymbol{\sigma}^0$ required for the transformation (25) is given by

$$\boldsymbol{\sigma}' - \boldsymbol{\sigma}^0 = -\gamma r^2 \Omega / c_0 \, \mathrm{d}\varphi^0 .$$

We now turn to the source data of the given problem. In M_4, we have

$$\underline{\rho} = \frac{q}{2\pi} \delta(r - a) \; \mathrm{d}r \wedge \mathrm{d}\varphi \wedge \mathrm{d}z, \qquad \underline{J} = 0,$$

with respect to the projection $P = P_{c_0^{-1}\partial t,\, c_0\, \mathrm{d}t}$. Therefore, the four-current density is $\underline{\mathcal{J}} = P^{-1}(\underline{\rho}, -\underline{J}/c_0) = \underline{\rho}$. By pull-back, we have in M_4^0

$$\underline{\mathcal{J}}^0 = \Phi^* \underline{\mathcal{J}} = \frac{q}{2\pi} \delta(r^0 - a) \, \mathrm{d}r^0 \wedge (\mathrm{d}\varphi^0 + \gamma \Omega \, \mathrm{d}t^0) \wedge \mathrm{d}z^0 .$$

Decomposition with respect to P^0 yields

$$\begin{pmatrix} \underline{\rho}^0 \\ -\underline{J}^0 / c_0 \end{pmatrix} = P^0 \underline{\mathcal{J}}^0 = \frac{q}{2\pi} \delta(r^0 - a) \begin{pmatrix} \mathrm{d}r^0 \wedge \mathrm{d}\varphi^0 \wedge \mathrm{d}z^0 \\ \gamma \Omega c_0^{-1} \, \mathrm{d}z^0 \wedge \mathrm{d}r^0 \end{pmatrix} . \tag{34}$$

We remark that due to the axial symmetry of the domain and the data, all involved quantities depend only on r^0, not on φ^0 or z^0. Therefore, the three-derivative is given by $\mathrm{d}_3 = \mathrm{d}r^0 \wedge \partial_{r^0}$. Moreover, currents are purely azimuthal, the fields $\underline{E}, \underline{D}$ purely radial, and the fields $\underline{H}, \underline{B}$ purely axial. Maxwell's equations for $\underline{D}$ and $\underline{H}$ reduce to

$$-\partial_{r^0} H_z^0 \;=\; j_\varphi^0, \tag{35a}$$
$$\partial_{r^0} D_r^0 \;=\; \rho^0 . \tag{35b}$$

From (34), we deduce for the solution of (35a)

$$H_z^0 = \frac{q}{2\pi} \begin{cases} c_2 & r^0 < a, \\ c_2 + \gamma \Omega & r^0 > a, \end{cases} \tag{36}$$

and for the solution of (35b)

$$D_r^0 = \frac{q}{2\pi} \begin{cases} c_1 & r^0 < a, \\ c_1 + 1 & r^0 > a. \end{cases} \tag{37}$$

In order to derive physically observable pulled-back quantities, we employ the transformation (25). This results in

$$\underline{D}' = (D_r^0 + \frac{\gamma r^2 \Omega}{c_0^2} H_z^0) \, \mathrm{d}\varphi^0 \wedge \mathrm{d}z^0 ,$$
$$\underline{H}' = \underline{H}^0 = H_z^0 \, \mathrm{d}z^0 .$$

If $\underline{D}'$ and $\underline{H}'$ are written in coordinates with respect to the coordinate basis $(c_0 \, \mathrm{d}t^0, \mathrm{d}r^0, \mathrm{d}\varphi^0, \mathrm{d}z^0)$, we have

$$D_r' = D_r^0 + \frac{\gamma r^2 \Omega}{c_0^2} H_z^0, \qquad H_z' = H_z^0 .$$

From (36) and (37), we obtain

$$D_r' = \frac{q}{2\pi} \begin{cases} c_1 + c_2 \frac{\gamma (r^0)^2 \Omega}{c_0^2} & r^0 < a, \\ c_1 + c_2 \frac{\gamma (r^0)^2 \Omega}{c_0^2} + \gamma^2 & r^0 > a. \end{cases} \tag{38}$$

Using the constitutive laws (24) requires the calculation of the three-dimensional Hodge $*_3^0$. To this end, we need the metric for the subspace $\Lambda \mathcal{F}^1_{\mathbf{n}^0}$ induced from g^0. The matrix representation of g^0 as a mapping $\mathcal{F}^1 \to \mathcal{X}^1$ is given by

$$(g^0)^{-1} = \begin{pmatrix} \gamma^{-2} & 0 & -\Omega/(\gamma c_0) & 0 \\ 0 & -1 & 0 & 0 \\ -\Omega/(\gamma c_0) & 0 & -(\gamma r)^{-2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which, together with (9), yields $(g_3^0)^{-1} = \mathrm{diag}\,(1, (\gamma r)^{-2}, 1)$, such that the coefficients of the metric (g_3^0) are given by $\mathrm{diag}\,(1, (\gamma r)^2, 1)$. This motivates the explicit construction of the Hodge $*_3^0$ by replacing r by γr^0 in the expression for the three-Hodge in cylindrical coordinates, namely

$$*_3^0 : \begin{cases} 1 \mapsto \gamma r^0 \, \mathrm{d}r^0 \wedge \mathrm{d}\varphi^0 \wedge \mathrm{d}z^0 \mapsto 1, \\ \mathrm{d}r^0 \mapsto \gamma r^0 \, \mathrm{d}\varphi^0 \wedge \mathrm{d}z^0 \mapsto \mathrm{d}r^0, \\ \mathrm{d}\varphi^0 \mapsto (\gamma r^0)^{-1} \, \mathrm{d}z^0 \wedge \mathrm{d}r^0 \mapsto \mathrm{d}\varphi^0, \\ \mathrm{d}z^0 \mapsto \gamma r^0 \, \mathrm{d}r^0 \wedge \mathrm{d}\varphi^0 \mapsto \mathrm{d}z^0 . \end{cases}$$

From (24), we observe

$$D_r' \, \mathrm{d}\varphi^0 \wedge \mathrm{d}z^0 = \varepsilon E_r' *_3^0 \mathrm{d}r^0 = \varepsilon E_r' \gamma r^0 \, \mathrm{d}\varphi^0 \wedge \mathrm{d}z^0 ,$$

yielding

$$D_r' = \varepsilon \gamma r^0 E_r' .$$

Now, we can deduce the integration constants c_1 and c_2 in (36) and (38), respectively. Since D_r' has to remain bounded for $r^0 \to \infty$, we have $c_2 = 0$. Since $E_r' = D_r'/(\varepsilon \gamma r^0)$ also has to remain bounded for $r^0 \to 0$, we have $c_1 = 0$. This finally gives the solution

$$H_z' = \frac{\gamma q}{2\pi} \begin{cases} 0 & r^0 < a, \\ \Omega & r^0 > a, \end{cases} \tag{39}$$

and for the solution of (35b)

$$D_r' = \frac{\gamma^2 q}{2\pi} \begin{cases} 0 & r^0 < a, \\ 1 & r^0 > a. \end{cases} \tag{40}$$

Thus, the interior is free of an electromagnetic field.

In order to validate our result, we push-forward our solution from M_4^0 to M_4, and decompose the result with respect to the inertial observer $(c_0^{-1} \partial \, \mathrm{d}t, c_0 \, \mathrm{d}t)$. As result, we expect the electro-static field of a line charge on the cylinder axis for $r > a$. In M_4^0, the excitation $\underline{G}^0$ is given by

$$\begin{aligned} \underline{G}^0 &= (P')^{-1} \begin{pmatrix} \underline{D}' \\ \underline{H}'/c_0 \end{pmatrix} = \underline{D}' + \boldsymbol{\sigma}' \wedge \underline{H}'/c_0 \\ &= \frac{\gamma q}{2\pi} \left(\gamma \, \mathrm{d}\varphi^0 \wedge \mathrm{d}z^0 + (c_0 \, \mathrm{d}t^0 - \gamma \frac{r^2 \Omega}{c_0} \, \mathrm{d}\varphi^0) \wedge \Omega \, \mathrm{d}z^0 \right) \\ &= \frac{\gamma q}{2\pi} \left(\gamma^{-1} \, \mathrm{d}\varphi^0 \wedge \mathrm{d}z^0 + \Omega \, \mathrm{d}t^0 \wedge \mathrm{d}z^0 \right), \end{aligned}$$

for $r^{0} > a$. The pushed-forward excitation $\underline{G}$ in M_4 amounts to

$$\underline{G} = D\Phi\underline{G}^{0} = \frac{\gamma q}{2\pi}\left(\gamma^{-1}(\mathrm{d}\varphi - \Omega\,\mathrm{d}t)\wedge\mathrm{d}z + \Omega\gamma^{-1}\,\mathrm{d}t\wedge\mathrm{d}z\right)$$
$$= \frac{q}{2\pi}\,\mathrm{d}\varphi\wedge\mathrm{d}z.$$

Applying the projection P with respect to $(c_0^{-1}\partial\,\mathrm{d}t, c_0\,\mathrm{d}t)$ yields

$$\begin{pmatrix}\underline{D}\\ \underline{H}/c_0\end{pmatrix} = P\underline{G} \quad\Rightarrow\quad \underline{D} = \frac{q}{2\pi}\,\mathrm{d}\varphi\wedge\mathrm{d}z,\;\; \underline{H} = 0,$$

as has been expected.

CONCLUSION

Summing up, our approach sets up a consistent framework for the Lagrangian view of (3+1)-dimensional electro-dynamics using the language of differential forms with no need for coordinate systems or reference frames. The decomposition mechanism, [4], admits the construction of this framework with a minimum of overhead, only relying on the notion of an observer. Employing two observers, one holonomic and the other locally inertial, opens the possibility to use the simple form of both the Maxwell equations and the constitutive relations simultaneously. The construction of a Lorentz boost yields connections to the standard results. The feasibility and usefulness of the approach is further demonstrated by means of a classical example, [8].

APPENDIX: ELEMENTS OF DIFFERENTIAL GEOMETRY

Pre-metric concepts

Generalized duality product We first introduce some pre-metric concepts. The duality product $\cdot\big|\cdot : \mathcal{F}^p \times \mathcal{X}^p$, $(\boldsymbol{\omega}, \mathbf{w}) \mapsto \boldsymbol{\omega}\big|\mathbf{w} = \boldsymbol{\omega}(\mathbf{w})$, can be generalized to the domain $\mathcal{F}^p \times \mathcal{X}^q$ for $p \geq q$, [6, p. 21]. The generalized product $\boldsymbol{\omega}\big|\mathbf{w} \in \mathcal{F}^{p-q}$ of a p-form $\boldsymbol{\omega}$ and a q-vector field $\mathbf{w}$ is defined by

$$(\boldsymbol{\omega}\big|\mathbf{w})\big|\mathbf{v} = \boldsymbol{\omega}\big|(\mathbf{w}\wedge\mathbf{v}), \qquad \forall\,\mathbf{v}\in\mathcal{X}^{p-q}.$$

An immediate consequence of the definition is the rule

$$(\boldsymbol{\omega}\big|\mathbf{w})\big|\mathbf{v} = \boldsymbol{\omega}\big|(\mathbf{w}\wedge\mathbf{v}), \qquad \forall\,\mathbf{v}\in\mathcal{X}^{r},\; r\leq p-q. \quad (41)$$

Given the notion of the generalized duality product, the contraction of a p-form $\boldsymbol{\omega}$ by a vector field $\mathbf{w}$ is simply defined as

$$\mathbf{i} : \mathcal{X}^1 \times \mathcal{F}^p \to \mathcal{F}^{p-1}, \quad (\mathbf{w}, \boldsymbol{\omega}) \mapsto \mathbf{i}_{\mathbf{w}}\boldsymbol{\omega} = \boldsymbol{\omega}\big|\mathbf{w}.$$

Exterior p-compound For a linear mapping $L : V \to W$ between two vector spaces V and W, the extension by the exterior p-compound to $L : \Lambda V \to \Lambda W$ is defined by

$$L(\boldsymbol{\omega}) = L(\boldsymbol{\omega}_1\wedge\ldots\wedge\boldsymbol{\omega}_p) = L(\boldsymbol{\omega}_1)\wedge\ldots\wedge L(\boldsymbol{\omega}_p) \quad (42)$$

for $\boldsymbol{\omega} = \boldsymbol{\omega}_1\wedge\ldots\wedge\boldsymbol{\omega}_p \in W^p$.

In the following, we prove the preservation (27) of the duality product under the Lorentz boost $B = B_{\mathbf{u},\mathbf{u}'}$ of general p-forms and p-vector fields, $p > 1$. Since every p-form (vector field) can be written as a sum of decomposable forms (vector fields), it is sufficient to show the relation for decomposable forms and vector fields. To this end, let $\boldsymbol{\omega} = \boldsymbol{\omega}_1\wedge\ldots\wedge\boldsymbol{\omega}_p \in \mathcal{F}^p$ and $\mathbf{w} = \mathbf{w}_1\wedge\ldots\wedge\mathbf{w}_p \in \mathcal{X}^p$ be given, and observe by using the defining property (42) of the p-compound and preservation (27), that

$$\begin{aligned}
B(\boldsymbol{\omega})\big|B(\mathbf{w}) &= B(\boldsymbol{\omega}_1\wedge\ldots\wedge\boldsymbol{\omega}_p)\big|B(\mathbf{w}_1\wedge\ldots\wedge\mathbf{w}_p)\\
&= (B(\boldsymbol{\omega}_1)\wedge\ldots\wedge B(\boldsymbol{\omega}_p))\,\big|\,(B(\mathbf{w}_1)\wedge\ldots\wedge B(\mathbf{w}_p))\\
&= \det\begin{pmatrix} B(\boldsymbol{\omega}_1)\big|B(\mathbf{w}_1) & \ldots & B(\boldsymbol{\omega}_1)\big|B(\mathbf{w}_p)\\ \vdots & & \vdots\\ B(\boldsymbol{\omega}_p)\big|B(\mathbf{w}_1) & \ldots & B(\boldsymbol{\omega}_p)\big|B(\mathbf{w}_p)\end{pmatrix}\\
&= \det\begin{pmatrix} \boldsymbol{\omega}_1\big|\mathbf{w}_1 & \ldots & \boldsymbol{\omega}_1\big|\mathbf{w}_p\\ \vdots & & \vdots\\ \boldsymbol{\omega}_p\big|\mathbf{w}_1 & \ldots & \boldsymbol{\omega}_p\big|\mathbf{w}_p\end{pmatrix}\\
&= (\boldsymbol{\omega}_1\wedge\ldots\wedge\boldsymbol{\omega}_p)\big|(\mathbf{w}_1\wedge\ldots\wedge\mathbf{w}_p)\\
&= \boldsymbol{\omega}\big|\mathbf{w}.
\end{aligned}$$

Further properties of the Lorentz boost Next, we will show the commutativity (33). From the definition of the projection (6), it becomes obvious that it is sufficient to show

$$B_{\mathbf{u},\mathbf{u}'}\circ\mathbf{i}_{\mathbf{u}} = \mathbf{i}_{\mathbf{u}'}\circ B_{\mathbf{u},\mathbf{u}'}. \quad (43)$$

The proof only relies on the fact that $B_{\mathbf{u},\mathbf{u}'}$ is a linear operator. We first show that for a p-form $\boldsymbol{\omega}$ and a q-vector field $\mathbf{w}$, $q < p$, it holds that

$$B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega})\big|B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}) = B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega}\big|\mathbf{w}). \quad (44)$$

This follows immediately from (41) and (27) by observing that

$$\begin{aligned}
(B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega})\big|B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}))\big|B_{\mathbf{u},\mathbf{u}'}(\mathbf{v})&\\
&= B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega})\big|(B_{\mathbf{u},\mathbf{u}'}(\mathbf{w})\wedge B_{\mathbf{u},\mathbf{u}'}(\mathbf{v}))\\
&= B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega})\big|B_{\mathbf{u},\mathbf{u}'}(\mathbf{w}\wedge\mathbf{v})\\
&= \boldsymbol{\omega}\big|(\mathbf{w}\wedge\mathbf{v}) = (\boldsymbol{\omega}\big|\mathbf{w})\big|\mathbf{v}\\
&= B_{\mathbf{u},\mathbf{u}'}^{-1}\circ B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega}\big|\mathbf{w})\big|B_{\mathbf{u},\mathbf{u}'}^{-1}\circ B_{\mathbf{u},\mathbf{u}'}(\mathbf{v})\\
&= B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega}\big|\mathbf{w})\big|B_{\mathbf{u},\mathbf{u}'}(\mathbf{v}),
\end{aligned}$$

which holds for any $\mathbf{v}\in\mathcal{X}^r$, $r \leq p-q$. From (44), it follows that

$$\begin{aligned}
\mathbf{i}_{\mathbf{u}'}B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega}) &= B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega})\big|B_{\mathbf{u},\mathbf{u}'}(\mathbf{u})\\
&= B_{\mathbf{u},\mathbf{u}'}(\boldsymbol{\omega}\big|\mathbf{u}) = B_{\mathbf{u},\mathbf{u}'}(\mathbf{i}_{\mathbf{u}}\boldsymbol{\omega}),
\end{aligned}$$

which yields (43).

Metric

The Hodge operator $*: \mathcal{F}^p \to \mathcal{F}^{n-p}$, $\boldsymbol{\omega} \mapsto *\boldsymbol{\omega}$ is given by

$$(-1)^{s(N)} * \boldsymbol{\omega} \cdot \boldsymbol{\eta} = (\boldsymbol{\omega} \wedge \boldsymbol{\eta}) \cdot \boldsymbol{\Omega}, \quad \forall\, \boldsymbol{\eta} \in \mathcal{F}^{n-p}, \tag{45}$$

where $\boldsymbol{\Omega}$ is the oriented volume n-form with $\|\boldsymbol{\Omega}\| = 1$ and $s(N)$ stands for the signature of the metric g, i.e., $(-1)^{s(N)} = \boldsymbol{\Omega}\big|g^{-1}(\boldsymbol{\Omega})$. One observes that

$$\begin{aligned}(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) \cdot \boldsymbol{\Omega} &= \boldsymbol{\Omega}\big|g^{-1}(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) = \boldsymbol{\Omega}\big|(g^{-1}(\boldsymbol{\omega}) \wedge g^{-1}(\boldsymbol{\eta})) \\ &= (\boldsymbol{\Omega}\big|g^{-1}(\boldsymbol{\omega}))\big|g^{-1}(\boldsymbol{\eta}) = \boldsymbol{\eta} \cdot \boldsymbol{\Omega}\big|g^{-1}(\boldsymbol{\omega}).\end{aligned}$$

Together with the definition (45), this observation yields an explicit representation of the Hodge $*$ as

$$*\boldsymbol{\omega} = (-1)^{s(N)} \boldsymbol{\Omega}\big|g^{-1}(\boldsymbol{\omega}). \tag{46}$$

We immediately observe that

$$*\boldsymbol{\Omega} = (-1)^{s(N)} \boldsymbol{\Omega}\big|g^{-1}(\boldsymbol{\Omega}) = 1. \tag{47}$$

For the inverse $*^{-1}$, it holds that

$$*^{-1}\boldsymbol{\omega} = (-1)^{p(n-p)+s(N)} * \boldsymbol{\omega} = (-1)^{p+1} * \boldsymbol{\omega}$$

for all $\boldsymbol{\omega} \in \mathcal{F}^p(M)$, since $n = 4$ and $s(N) = 3$. Setting $\widetilde{\boldsymbol{\Omega}} = (-1)^{s(N)}\boldsymbol{\Omega}$, we can relate the Hodge operator applied to the exterior product to the metric by observing that

$$\begin{aligned}*(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) &= \widetilde{\boldsymbol{\Omega}}\big|g^{-1}(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) = \widetilde{\boldsymbol{\Omega}}\big|(g^{-1}(\boldsymbol{\omega}) \wedge g^{-1}(\boldsymbol{\eta})) \\ &= (\widetilde{\boldsymbol{\Omega}}\big|g^{-1}(\boldsymbol{\omega}))\big|g^{-1}(\boldsymbol{\eta}) = *\boldsymbol{\omega}\big|g^{-1}(\boldsymbol{\eta}) \\ &= \mathbf{i}_{g^{-1}(\boldsymbol{\eta})} * \boldsymbol{\omega}.\end{aligned}$$

In particular, we have that

$$*(\boldsymbol{\eta} \wedge \boldsymbol{\omega}) = \mathbf{i}_{g^{-1}(\boldsymbol{\eta})} * s\boldsymbol{\omega}, \quad \forall\, \boldsymbol{\omega} \in \mathcal{F}^p,\ \boldsymbol{\eta} \in \mathcal{F}^1, \tag{48}$$

where $s\boldsymbol{\omega} = (-1)^{\deg \boldsymbol{\omega}}\boldsymbol{\omega}$. It follows that

$$\begin{aligned}&(-1)^{(n-p+1)(p-1)+s(N)} \boldsymbol{\eta} \wedge *\boldsymbol{\omega} \\ &\quad = * * (\boldsymbol{\eta} \wedge *\boldsymbol{\omega}) \\ &\quad = *(\mathbf{i}_{g^{-1}(\boldsymbol{\eta})} * (s * \boldsymbol{\omega})) \\ &\quad = (-1)^{n-p} * (\mathbf{i}_{g^{-1}(\boldsymbol{\eta})} * * \boldsymbol{\omega}) \\ &\quad = (-1)^{(p+1)(n-p)+s(N)} * (\mathbf{i}_{g^{-1}(\boldsymbol{\eta})} \boldsymbol{\omega}),\end{aligned}$$

which yields

$$\boldsymbol{\eta} \wedge *\boldsymbol{\omega} = (-1)^k * \mathbf{i}_{g^{-1}(\boldsymbol{\eta})}\boldsymbol{\omega},$$

with

$$k = p - 1 + 2(s(N) + (n-p)p),$$

such that

$$\boldsymbol{\eta} \wedge *\boldsymbol{\omega} = (-1)^{p-1} * \mathbf{i}_{g^{-1}(\boldsymbol{\eta})}\boldsymbol{\omega} = - * \mathbf{i}_{g^{-1}(\boldsymbol{\eta})} s\boldsymbol{\omega}. \tag{49}$$

In order to relate the four-metric g and the three-metric g_3 by (10), we proceed inductively. For $\boldsymbol{\omega} \in \mathcal{F}^1_{\mathbf{n}}$, we require that

$$g^{-1}(\boldsymbol{\omega}) = -g_3^{-1}(\boldsymbol{\omega}) + \mathbf{n} \wedge \chi_0(\boldsymbol{\omega}),$$

for some function $\chi_0 : \mathcal{F}^1_{\mathbf{n}} \to \mathcal{X}^0 = \mathcal{X}^0_{\boldsymbol{\sigma}}$. The extension to 2-forms by the exterior p-compound yields for $\boldsymbol{\omega}, \boldsymbol{\eta} \in \mathcal{F}^1_{\mathbf{n}}$

$$\begin{aligned}g^{-1}(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) &= g^{-1}(\boldsymbol{\omega}) \wedge g^{-1}(\boldsymbol{\eta}) \\ &= (-g_3^{-1}(\boldsymbol{\omega}) + \mathbf{n} \wedge \chi_0(\boldsymbol{\omega})) \\ &\quad \wedge (-g_3^{-1}(\boldsymbol{\eta}) + \mathbf{n} \wedge \chi_0(\boldsymbol{\eta})) \\ &= +g_3^{-1}(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) + \mathbf{n} \wedge \chi_1(\boldsymbol{\omega} \wedge \boldsymbol{\eta}),\end{aligned}$$

with

$$\begin{aligned}\chi_1(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) &= \widetilde{\chi}_1(\boldsymbol{\omega}, \boldsymbol{\eta}) \\ &= g_3^{-1}(\boldsymbol{\omega}) \wedge \chi_0(\boldsymbol{\eta}) - g_3^{-1}(\boldsymbol{\eta}) \wedge \chi_0(\boldsymbol{\omega}),\end{aligned}$$

this definition being justified by the fact that $\widetilde{\chi}_1$ is bilinear and skew-symmetric. By induction, we conclude that for $\boldsymbol{\omega} \in \mathcal{F}^p_{\mathbf{n}}$

$$g^{-1}(\boldsymbol{\omega}) = g_3^{-1}(s\boldsymbol{\omega}) + \mathbf{n} \wedge \chi_{p-1}(s\boldsymbol{\omega})$$

for $\chi_{p-1} : \mathcal{F}^p_{\mathbf{n}} \to \mathcal{X}^{p-1}_{\boldsymbol{\sigma}}$, with

$$s : \Lambda(\mathcal{F}) \to \Lambda(\mathcal{F}), \quad \boldsymbol{\omega} \mapsto (-1)^{\deg \boldsymbol{\omega}}\boldsymbol{\omega}.$$

REFERENCES

[1] J. V. Bladel, *Relativity and Engineering*, ser. Springer Series in Electrophysics. Berlin: Springer, 1984.

[2] W. Burke, *Applied differential geometry*. Cambridge, UK: Cambridge University Press, 1985.

[3] M. Fecko, "On 3+1 decompositions with respect to an observer field via differential forms," *J. Math. Phys.*, vol. 38, no. 9, pp. 4542–4560, 1997.

[4] F. Hehl and Y. Obukhov, *Foundations of Classical Electrodynamics*. Boston: Birkhäuser, 2003.

[5] B. Mashhoon, "The hypothesis of locality in relativistic physics," *Physics Letters A*, vol. 145, no. 4, pp. 147–153, Apr. 1990.

[6] K. Meetz and W. Engl, *Elektromagnetische Felder*. Berlin: Springer, 1980.

[7] T. Matolcsi, *Spacetime Without Reference Frames*. Akadémiai Kiaó (Publishing House of the Hungarian Academy of Sciences), Budapest, 1993.

[8] L. Schiff, "A question in general relativity," *Proc. Nat. Acad. Sci. USA*, vol. 25, pp. 391–395, 1939.

Erschienene Preprints ab Nummer 2006/001

Komplette Liste: `http://preprints.ians.uni-stuttgart.de`

2006/001 *Klimke, A.:* Sparse Grid Interpolation Toolbox - User's Guide

2006/002 *Klimke, A., Wohlmuth, B.:* Constructing Dimension-Adaptive Sparse Grid Interpolants using Parallel Function Evaluations

2006/003 *Hartmann, S., Brunssen, S., Ramm, E., Wohlmuth, B.:* Application of a primal-dual active set strategy for unilateral non-linear dynamic contact problems of thin-walled structures

2006/004 *Sändig, A.-M.:* Partielle Differentialgleichungen Vorlesung im Wintersemester 2005/2006

2006/005 *Nicaise, S., Witowski, K., Wohlmuth, B.:* An a posteriori error estimator for the Lame equation based on H(div)-conforming stress approximations

2006/006 *Geis, W. (ed.), Sändig, A.-M. (ed.):* Second International Workshop - Direct and Inverse Problems in Piezoelectricity,Hirschegg (Kleinwalsertal), Austria, July 16-19, 2006

2006/007 *Klimke, A.:* Efficient Construction of Hierarchical Polynomial Sparse Grid Interpolants using the Fast Discrete Cosine Transform

2006/008 *Buchukuri, T., Chkadua, O., Natroshvili, D.:* Mathematical modelling and analysis of interaction problems for metallic-piezoelectric composite structures with regard to thermal stresses

2006/009 *Merkle, C., Rohde, C.:* The Sharp-Interface Approach for Fluids with Phase Change: Riemann Problems and Ghost Fluid Techniques

2006/010 *Sändig, A.-M.:* Regularity results for linear elliptic boundary value problems in polygons. Lectures at the Charles University Prague, Oct.05

2006/011 *Flemisch, B., Kurz, S., Wohlmuth, B.:* A Framework for Maxwell's Equations in Non-Inertial Frames Based on Differential Forms