ANALYSIS OF A MODIFIED MASS LUMPING METHOD FOR THE STABILIZATION OF FRICTIONAL CONTACT PROBLEMS

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Abstract. A common approach for the numerical simulation of nonlinear multibody contact problems is the use of Lagrange multipliers to model the contact conditions. The stability of standard time discretization algorithms is improved by introducing a modified mass matrix which assigns no mass to the potential contact nodes. By this, the spurious oscillations in the multiplier can be significantly reduced which facilitates the application of a primal-dual active set strategy to dynamical contact problems. The new mass matrix is calculated via a non-standard quadrature formula that requires no extra computational effort. In addition, the conservation properties of the underlying algorithm are carried over to the modified method, and the standard optimal a priori estimates are still satisfied. Several numerical examples show the optimality of the approach and illustrate the improvement in the results for the contact stresses.

Key words. dynamical contact problem, mass lumping, quadrature formula, energy conservation, Lagrange multiplier, a priori error estimate

AMS subject classifications. 65D32, 65M60, 70E55, 74B10, 74M15, 74S05

1. Introduction. The numerical simulation of multibody contact problems for linear elasticity plays an important role in many applications in mechanics. The interest in such type of problems led to extensive research activities both from the numerical and the theoretical point of view (see [8, 19, 21, 26, 29] and the references therein for an overview of the topic). A common approach for solving frictional contact problems is the use of (dual) Lagrange multipliers [27] to ensure the weak fulfillment of the contact and friction conditions. However, the values of these multipliers which represent the contact stresses often show oscillative behavior due to the time discretization (see, e.g., [3, 12]). Recently, different approaches have been discussed how to overcome this numerical artefact [11, 16, 17, 18, 20]. In [17, 18] a modified mass matrix is computed by redistributing the mass from the slave contact nodes to the inner nodes such that the original PDE decouples into an algebraic equation in time for the contact nodes and a PDE in time for the other nodes. But the new mass matrix is defined as the solution of a global constrained minimization problem which makes its computation quite expensive. This drawback is overcome in [11], where the mass matrix is only locally modified using special quadrature formulas. Thus, no extra computational cost comes into play by the modification.

In this paper, we discuss two different methods of defining suitable non-standard quadrature rules. The first approach, already presented in [11], needs certain conditions on the triangulation of the underlying mesh, whereas the second variant can be applied to completely unstructured shape-regular triangulations in 2D and 3D. We also derive an a priori estimate for the mass redistribution method under some assumptions on the regularity of the displacement.

The work is organized as follows: In Section 2, we describe the formulation of the nonlinear frictional contact problem we want to solve, give some preliminary definitions and motivate the use of a modified mass matrix. In Section 3, we interprete the modified quadrature rule as a combination of a stable interpolation operator and

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standard quadrature. Two different approaches to define this operator are discussed and conditions on the resulting quadrature formula are derived; further, we present some examples of quadrature rules that meet these requirements.

In the next two sections, we restrict ourselves to the case that no contact occurs because we need the displacement function to be smooth enough with respect to time. With these preliminaries, we prove a priori estimates for the semi-discrete system in Section 4 and for the fully discrete system in Section 5.

Section 6 contains some numerical examples. We confirm the error reduction properties proved in the previous sections for the case that no contact occurs. For two-body contact problems, we show that the oscillations of the contact stresses are significantly reduced by the mass redistribution. A 3D two-body contact example with friction is also included. Conclusions are drawn in Section 7.

2. Motivation and problem formulation. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be an open, bounded domain with piecewise smooth boundary $\partial\Omega$. We consider the following contact problem: Find the displacement vector $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) \subset \mathbb{R}^d$ satisfying

$$\begin{aligned} \varrho\, \ddot{\mathbf{u}} - \operatorname{div} \sigma(\mathbf{u}) &= \mathbf{l} && \text{in} \quad (0,T] \times \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{in} \quad (0,T] \times \Gamma_D, \\ \sigma(\mathbf{u})\mathbf{n} &= \mathbf{g}_N && \text{in} \quad (0,T] \times \Gamma_N, \\ \mathbf{u}(0,\cdot) &= \mathbf{u}_0 && \text{in} \quad \Omega, \\ \dot{\mathbf{u}}(0,\cdot) &= \mathbf{v}_0 && \text{in} \quad \Omega. \end{aligned} \tag{2.1}$$

For the theoretical background on contact problems see, e.g., [8, 21, 29]. The definition of the time-dependent spaces and norms used here can be found in [9].

The boundary $\partial\Omega$ is partitioned into three mutually disjoint subsets $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N \cup \bar{\Gamma}_C$ with $\operatorname{meas}(\Gamma_D) > 0$. As we can always reduce nonhomogeneous Dirichlet boundary conditions to the homogeneous case, the value of $\mathbf{u}$ on the Dirichlet boundary Γ_D is set to zero, whereas the external load on Γ_N is given by $\mathbf{g}_N$. The outward normal on the boundary $\partial\Omega$ is denoted by $\mathbf{n}$, $\mathbf{l}$ stands for the volume forces acting on Ω. The stress tensor $\sigma(\mathbf{u})$ is given by the constitutive equations of linear elasticity

$$\varepsilon(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right), \qquad \sigma(\mathbf{u}) := \lambda \operatorname{tr}(\varepsilon(\mathbf{u}))\mathrm{Id} + 2\mu\varepsilon(\mathbf{u}),$$

where we assume constant material parameters ϱ, μ, λ for ease of notation.

As we can treat a two-body contact problem in the same way as a one-body problem from the computational point of view (see [15, 28]), we restrict ourselves to the presentation of a one-body contact problem with a rigid obstacle for simplicity. On the potential contact boundary Γ_C, we decompose the displacement and the stress into its normal parts $u_n := \mathbf{u}\cdot\mathbf{n}$, $\sigma_n(\mathbf{u}) := (\sigma(\mathbf{u})\mathbf{n})\cdot\mathbf{n}$ and its tangential components $\mathbf{u}_\tau := \mathbf{u} - u_n\mathbf{n}$, $\boldsymbol{\sigma}_\tau(\mathbf{u}) := \sigma(\mathbf{u})\mathbf{n} - \sigma_n(\mathbf{u})\mathbf{n}$. Then we impose the following unilateral contact conditions on Γ_C [12, 23, 24]:

$$u_n \le g, \quad \sigma_n(\mathbf{u}) \le 0, \quad (u_n - g)\sigma_n(\mathbf{u}) = 0, \tag{2.2}$$

where the gap function g measures the distance between a point on Γ_C and its projection onto the obstacle. If we neglect friction, we set $\boldsymbol{\sigma}_\tau(\mathbf{u}) = \mathbf{0}$ on Γ_C, whereas in the case of Coulomb friction with friction coefficient $\mathfrak{F}$ [21, 23, 24], the friction condition is given by

$$\|\boldsymbol{\sigma}_\tau(\mathbf{u})\| - \mathfrak{F}|\sigma_n(\mathbf{u})| \le 0, \quad \dot{\mathbf{u}}_\tau + \alpha^2\sigma_\tau(\mathbf{u}) = 0, \quad \dot{\mathbf{u}}_\tau\big(\|\sigma_\tau(\mathbf{u})\| - \mathfrak{F}|\sigma_n(\mathbf{u})|\big) = 0. \tag{2.3}$$

From now on, we denote vector valued spaces by a bold letter, e.g., $\mathbf{H}^1(\Omega) := [H^1(\Omega)]^d$. To obtain the weak formulation of (2.1), we define $\mathbf{H}^1_D(\Omega) := \{\boldsymbol{\chi} \in \mathbf{H}^1(\Omega) : \boldsymbol{\chi}|_{\Gamma_D} = \mathbf{0}\}$ as the test space for the displacements. Next we introduce the Lagrange multiplier space $\mathbf{M}$ as the dual space of the trace space $\mathbf{W}$ of $\mathbf{H}^1_D(\Omega)$ restricted to Γ_C. The Lagrange multiplier $\boldsymbol{\lambda} \in \mathbf{M}$ represents $-\sigma(\mathbf{u})\mathbf{n}$. Thus, we arrive at the following weak mixed formulation (for further details see, e.g., [8, 14, 15, 19]): find $\mathbf{u} \in L^2((0,T), \mathbf{H}^1_D(\Omega))$ and $\boldsymbol{\lambda} \in L^2((0,T), \mathbf{M}(\boldsymbol{\lambda}))$ such that $\dot{\mathbf{u}} \in L^2((0,T), \mathbf{H}^1(\Omega))$, $\ddot{\mathbf{u}} \in L^2((0,T), (\mathbf{H}^1_D(\Omega))')$ and

$$\begin{aligned} m(\ddot{\mathbf{u}}, \boldsymbol{\chi}) + a(\mathbf{u}, \boldsymbol{\chi}) + b(\boldsymbol{\chi}, \boldsymbol{\lambda}) &= (\mathbf{f}, \boldsymbol{\chi}), && \boldsymbol{\chi} \in \mathbf{H}^1_D(\Omega), \quad t \in (0,T], \\ b_n(\mathbf{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) + b_\tau(\dot{\mathbf{u}}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\leq \langle g, \mu_n - \lambda_n \rangle, && \boldsymbol{\mu} \in \mathbf{M}(\boldsymbol{\lambda}), \quad t \in (0,T], \\ (\mathbf{u}(0,\cdot), \boldsymbol{\chi}) &= (\mathbf{u}_0, \boldsymbol{\chi}), && \boldsymbol{\chi} \in \mathbf{H}^1_D(\Omega), \\ (\dot{\mathbf{u}}(0,\cdot), \boldsymbol{\chi}) &= (\mathbf{v}_0, \boldsymbol{\chi}), && \boldsymbol{\chi} \in \mathbf{H}^1_D(\Omega), \end{aligned} \tag{2.4}$$

with the convex set of admissible multipliers

$$\mathbf{M}(\boldsymbol{\lambda}) := \Big\{ \boldsymbol{\mu} \in \mathbf{M} : \langle \boldsymbol{\mu}, \boldsymbol{\chi} \rangle \leq \langle \mathfrak{F}\, \lambda_n, \|\boldsymbol{\chi}_\tau\| \rangle, \quad \boldsymbol{\chi} \in \mathbf{W} \text{ with } \chi_n \leq 0 \Big\}$$

and the definitions $m(\mathbf{u}, \boldsymbol{\chi}) := \int_\Omega \varrho\, \mathbf{u} \cdot \boldsymbol{\chi}\, d\mathbf{x}$, $a(\mathbf{u}, \boldsymbol{\chi}) := \int_\Omega \sigma(\mathbf{u}) : \varepsilon(\boldsymbol{\chi})\, d\mathbf{x}$, $b_n(\boldsymbol{\chi}, \boldsymbol{\mu}) := \langle \chi_n, \mu_n \rangle := \int_{\Gamma_C} \chi_n\, \mu_n\, ds$, $b_\tau(\boldsymbol{\chi}, \boldsymbol{\mu}) := \langle \boldsymbol{\chi}_\tau, \boldsymbol{\mu}_\tau \rangle$ and $(\mathbf{f}, \boldsymbol{\chi}) := \int_\Omega \mathbf{l} \cdot \boldsymbol{\chi}\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{g}_N \cdot \boldsymbol{\chi}\, ds$. We remark that by these definitions $m(\cdot,\cdot)$ and $a(\cdot,\cdot)$ are symmetric and continuous bilinear forms [6]. Further, meas $(\Gamma_D) > 0$ implies that $a(\cdot,\cdot)$ is $\mathbf{H}^1_D(\Omega)$-elliptic.

Next we discretize (2.4) in space using a shape-regular triangulation $\mathcal{T}_h$ of Ω into quadrilaterals/hexahedrals or simplices. The discretization is such that the Dirichlet boundary $\bar{\Gamma}_D$ as well as the contact boundary $\bar{\Gamma}_C$ can be written as $\bigcup_{K \in \mathcal{T}_D} \bar{K}|_{\partial\Omega}$ and $\bigcup_{K \in \mathcal{T}_C} \bar{K}|_{\partial\Omega}$, respectively, for subsets $\mathcal{T}_D, \mathcal{T}_C \subset \mathcal{T}_h$. Let ϕ_p be the nodal finite element basis function associated with the node $p \in \mathcal{N}_h$, where $\mathcal{N}_h$ is the set of vertices of the triangulation $\mathcal{T}_h$. By $\mathcal{C}_h \subset \mathcal{N}_h$ we denote the set of all possible contact nodes on Γ_C.

For $\omega \subset \mathbb{R}^d$ and $k \in \mathbb{N}_0$, we define $\boldsymbol{\mathcal{P}}_k(\omega)$ as the space of polynomials with the basis functions $\mathbf{x}^{\boldsymbol{\alpha}}$ with $\mathbf{x} \in \omega$, $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ and $|\boldsymbol{\alpha}| \leq k$. Furthermore $\boldsymbol{\mathcal{Q}}_k(\omega)$ is the polynomial space spanned by the functions $\mathbf{x}^{\boldsymbol{\alpha}}$ with $\max_{1 \leq i \leq d}(\boldsymbol{\alpha}_i) \leq k$. Let $\mathbf{S}_h \subset \mathbf{H}^1_D(\Omega)$ be the space of lowest order ($k = 1$) conforming finite element functions on the triangulation $\mathcal{T}_h$ and $\mathbf{M}_h$ be the span of the dual Lagrange multiplier basis functions ψ_p, $p \in \mathcal{C}_h$. The latter are discontinuous piecewise polynomial functions of order k such that the following biorthogonality relation holds (for details see, e.g., [15, 27]):

$$\int_{\Gamma_C} \psi_p\, \phi_q\, ds = \delta_{pq} \int_{\Gamma_C} \phi_q\, ds, \quad p, q \in \mathcal{C}_h.$$

We obtain the following semi-discrete problem: find $\mathbf{u}_h \in L^2((0,T), \mathbf{S}_h)$, $\boldsymbol{\lambda}_h \in L^2((0,T), \mathbf{M}_h(\boldsymbol{\lambda}_h))$ with $\dot{\mathbf{u}}_h, \ddot{\mathbf{u}}_h$ as before and

$$\begin{aligned} m(\ddot{\mathbf{u}}_h, \boldsymbol{\chi}_h) + a(\mathbf{u}_h, \boldsymbol{\chi}_h) + b(\boldsymbol{\chi}_h, \boldsymbol{\lambda}_h) &= (\mathbf{f}, \boldsymbol{\chi}_h), && \boldsymbol{\chi}_h \in \mathbf{S}_h, \quad t \in (0,T], \\ b_n(\mathbf{u}_h, \boldsymbol{\mu}_h - \boldsymbol{\lambda}_h) + b_\tau(\dot{\mathbf{u}}_h, \boldsymbol{\mu}_h - \boldsymbol{\lambda}_h) &\leq \langle g, \mu_{hn} - \lambda_{hn} \rangle, && \boldsymbol{\mu}_h \in \mathbf{M}_h(\boldsymbol{\lambda}_h), \quad t \in (0,T], \\ (\mathbf{u}_h(0), \boldsymbol{\chi}_h) &= (\mathbf{u}_{0h}, \boldsymbol{\chi}_h), && \boldsymbol{\chi}_h \in \mathbf{S}_h, \\ (\dot{\mathbf{u}}_h(0), \boldsymbol{\chi}_h) &= (\mathbf{v}_{0h}, \boldsymbol{\chi}_h), && \boldsymbol{\chi}_h \in \mathbf{S}_h, \end{aligned} \tag{2.5}$$

with

$$\mathbf{M}_h(\boldsymbol{\lambda}_h) := \Big\{ \boldsymbol{\mu}_h \in \mathbf{M}_h : \langle \boldsymbol{\mu}_h, \boldsymbol{\chi}_h \rangle \le \langle \mathfrak{F}\, \lambda_{hn}, \|\boldsymbol{\chi}_{h\tau}\|_h \rangle, \quad \boldsymbol{\chi}_h \in \mathbf{W}_h \text{ with } \chi_{hn} \le 0 \Big\}$$

and the discrete absolute value $\|\boldsymbol{\chi}_{h\tau}\|_h := \sum_{p\in\mathcal{N}_h} \|\boldsymbol{\chi}_{p\tau}\|\, \phi_p$ for $\boldsymbol{\chi}_{h\tau} = \sum_{p\in\mathcal{N}_h} \boldsymbol{\chi}_{p\tau}\phi_p$. The boundary values $\mathbf{u}_{0h}$ and $\mathbf{v}_{0h}$ are suitable approximations of the continuous initial data. Writing the first equation of (2.5) in matrix notation gives

$$M_h \ddot{\mathbf{u}}_h + A_h \mathbf{u}_h = \mathbf{f}_h - B_h \boldsymbol{\lambda}_h. \tag{2.6}$$

We hereby use the notation $\mathbf{u}_h$ and $\boldsymbol{\lambda}_h$ for the functions as well as their vector representations.

Solving problem (2.5), we observe that the Lagrange multiplier exhibits spurious oscillations that makes the employment of, e.g., a primal-dual active set strategy [13, 15] significantly more difficult. Figure 2.1 presents the numerical results for the normal component λ_{hn} of the Lagrange multiplier, computed with the standard mass matrix M_h.

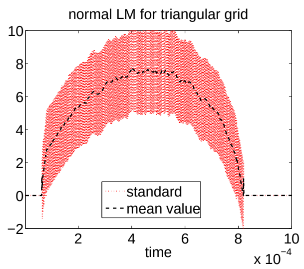


FIG. 2.1. *Normal Lagrange multiplier and its mean value for the problem given in Section 6.2.*

To stabilize this approach, we replace M_h by a modified mass matrix M_H which has no entries associated to the contact nodes $\mathcal{C}_h$. By this, we can decouple (2.6) into a purely algebraic equation in time for the contact nodes and a differential equation in time for the other nodes. This simplifies the solution of (2.5) and results in a higher regularity in time for the contact stresses $\boldsymbol{\lambda}_h$, as shown in [16]. Thereby no spurious oscillations in the numerical computation of $\boldsymbol{\lambda}_h$ occur.

In the rest of this work, c and C denote generic constants independent of the mesh size $h := \max_{K\in\mathcal{T}_h} h_K$ or time variables. Further, we use the abbreviations $\|\cdot\|_{k,\omega} := \|\cdot\|_{\mathbf{H}^k(\omega)}$ for the static and $\|\cdot\|_{l,k,\omega} := \|\cdot\|_{H^l((0,T),\mathbf{H}^k(\omega))}$ for the time-dependent norms. In addition, we set $\|\cdot\|_{\infty,k,\omega} := \|\cdot\|_{L^\infty((0,T),\mathbf{H}^k(\omega))}$ and define $\mathbf{V}_D := \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_D(\Omega)$.

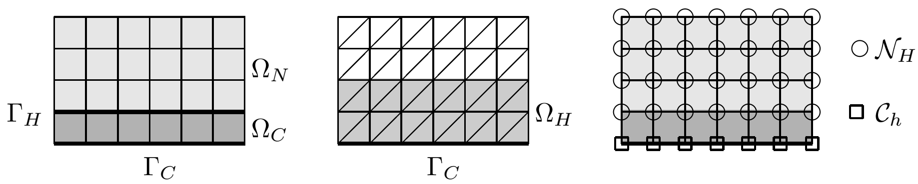


FIG. 3.1. *Examples for Ω_C and Ω_N.*

3. Construction of M_H. In order to describe the construction of M_H, we need some preliminary definitions. As illustrated in Figure 3.1, we denote the union of all elements along Γ_C and its complement by

$$\bar{\Omega}_C := \bigcup_{p\in\mathcal{C}_h} (\operatorname{supp}\phi_p), \qquad \Omega_N := \Omega\backslash\bar{\Omega}_C.$$

The interface $\partial\Omega_C \cap \partial\Omega_N$ is called Γ_H. Further we set $\bar{\Omega}_H := \bigcup_{p\subset\bar{\Gamma}_H}(\operatorname{supp}\phi_p)$ and $\mathcal{N}_H := \mathcal{N}_h\backslash\mathcal{C}_h$. We also introduce a triangulation $\bar{\mathcal{T}}_h$ of Ω possibly with hanging nodes such that the diameter of its elements is bounded by Ch. ($\bar{\mathcal{T}}_h$ will be specified in Sections 3.1 and 3.2).

We now define a modified bilinear form

$$m_H(\mathbf{u},\boldsymbol{\chi}) := \sum_{K\in\bar{\mathcal{T}}_h} Q_K(\varrho\,\mathbf{u}\cdot\boldsymbol{\chi}) \tag{3.1}$$

by choosing a quadrature formula Q_K such that the following condition is satisfied:

$Q1$) No quadrature point is placed in $\bar{\Omega}_C\backslash\bar{\Gamma}_H$.

By this we make sure that for the unit vector $\mathbf{e}_i \in \mathbb{R}^d$ the relation

$$m_H(\mathbf{e}_i\phi_p,\mathbf{e}_j\phi_q) = 0, \quad \text{for} \quad p\in\mathcal{C}_h \text{ or } q\in\mathcal{C}_h, \quad 1\le i,j\le d, \tag{3.2}$$

holds. Using $m_H(\cdot,\cdot)$ for the definition of the mass matrix (now denoted by M_H), we obtain the modified version of (2.6):

$$M_H\ddot{\mathbf{u}}_h + A_h\mathbf{u}_h = \mathbf{f}_h - B_h\boldsymbol{\lambda}_h. \tag{3.3}$$

The solutions of (2.6) and (3.3) are not the same; but from now on $(\mathbf{u}_h,\boldsymbol{\lambda}_h)$ only refers to the solution of (3.3) along with the corresponding contact conditions in order to keep the notation simple.

In the rest of this section, we are going to specify further conditions on the quadrature formula Q_K such that the modified bilinear form $m_H(\cdot,\cdot)$ is a suitable approximation of $m(\cdot,\cdot)$ (details are explained in Section 4). To this end, we assume that we can interprete $m_H(\cdot,\cdot)$ as a combination of some interpolation operator I_H defined on $\mathbf{S}_h$ and the bilinear form $m(\cdot,\cdot)$:

$$m_H(\boldsymbol{\chi}_h,\boldsymbol{\eta}_h) = m(I_H\boldsymbol{\chi}_h, I_H\boldsymbol{\eta}_h), \quad \boldsymbol{\chi}_h,\boldsymbol{\eta}_h\in\mathbf{S}_h. \tag{3.4}$$

We further impose the following three conditions on the operator I_H:

$P1$) I_H is $\mathbf{L}^2$-stable on Ω_H, i.e., $\|I_H\boldsymbol{\chi}_h\|_{0,\Omega_H} \le c\|\boldsymbol{\chi}_h\|_{0,\Omega_H}, \quad \boldsymbol{\chi}_h\in\mathbf{S}_h$.

$P2$) $I_H\boldsymbol{\chi}_h|_{\bar{\Omega}_N} = \boldsymbol{\chi}_h|_{\bar{\Omega}_N}, \quad \boldsymbol{\chi}_h\in\mathbf{S}_h$.

$P3$) $I_H\boldsymbol{\chi}_h|_K = \boldsymbol{\chi}_h|_K, \quad \boldsymbol{\chi}_h|_K\in\boldsymbol{\mathcal{P}}_0(K), \;\; K\in\bar{\mathcal{T}}_h$.

REMARK 3.1. *From $P1$ and $P2$, it follows that I_H is also $\mathbf{L}^2$-stable on Ω.*

REMARK 3.2. *Conditions $Q1$, $P1$, $P2$ and the relation (3.4) imply the following inequality (as shown in [11]):*

$$c\|\boldsymbol{\chi}_h\|^2_{0,\Omega_N} \le m_H(\boldsymbol{\chi}_h,\boldsymbol{\chi}_h) \le C\|\boldsymbol{\chi}_h\|^2_{0,\Omega_N}, \quad \boldsymbol{\chi}_h\in\mathbf{S}_h. \tag{3.5}$$

Hence we can state that $|\cdot|_H := \sqrt{m_H(\cdot,\cdot)}$ is equivalent to the $\mathbf{L}^2(\Omega_N)$-norm and thus is a seminorm on Ω. The $\mathbf{L}^2(\Omega)$-stability of I_H and (3.4) then implicate that $m_H(\cdot,\cdot)$ is continuous on $\mathbf{S}_h$.

By condition $Q1$ all quadrature points of Q_K are placed in $\bar{\Omega}_N$, and thus we get by (3.1) and $P2$:

$$m_H(\boldsymbol{\chi}_h,\boldsymbol{\eta}_h) = \sum_{K\in\bar{\mathcal{T}}_h} Q_K(\varrho\boldsymbol{\chi}_h\cdot\boldsymbol{\eta}_h) = \sum_{K\in\bar{\mathcal{T}}_h} Q_K(\varrho I_H\boldsymbol{\chi}_h\cdot I_H\boldsymbol{\eta}_h) = m_H(I_H\boldsymbol{\chi}_h, I_H\boldsymbol{\eta}_h).$$

Hence (3.4) is equivalent to

$$\sum_{K\in\bar{\mathcal{T}}_h} Q_K(\varrho I_H\boldsymbol{\chi}_h\cdot I_H\boldsymbol{\eta}_h) = \sum_{K\in\bar{\mathcal{T}}_h}\int_K \varrho I_H\boldsymbol{\chi}_h\cdot I_H\boldsymbol{\eta}_h\, d\mathbf{x}. \tag{3.6}$$

This formula holds if the quadrature formula Q_K is exact on each element $K\in\bar{\mathcal{T}}_h$ for the product of the functions $I_H\boldsymbol{\chi}_h\cdot I_H\boldsymbol{\eta}_h$. Thus, the choice of suitable formulas Q_K depends on the definition of the interpolation operator I_H, which is now given in terms of modified finite element basis functions $\{\phi_p^H\}_{p\in\mathcal{N}_H}$ with the span $\mathbf{S}_H$:

$$\mathbf{S}_H := \left\{\sum_{p\in\mathcal{N}_H}\boldsymbol{\alpha}_p\phi_p^H : \boldsymbol{\alpha}_p\in\mathbb{R}^d\right\}. \tag{3.7}$$

According to this, the interpolation operator I_H is defined via

$$I_H : \mathbf{S}_h\to\mathbf{S}_H : \left(\sum_{p\in\mathcal{N}_H}\boldsymbol{\alpha}_p\phi_p + \sum_{p\in\mathcal{C}_h}\boldsymbol{\beta}_p\phi_p\right)\mapsto\sum_{p\in\mathcal{N}_H}\boldsymbol{\alpha}_p\phi_p^H. \tag{3.8}$$

Relation (3.6) then implies the following additional condition on the quadrature formula Q_K:

$Q2)$ Q_K is exact on each $K\in\bar{\mathcal{T}}_h$ for all functions $\boldsymbol{\chi}_H\cdot\boldsymbol{\eta}_H$ with $\boldsymbol{\chi}_H,\boldsymbol{\eta}_H\in\mathbf{S}_H$.

The functions $\{\phi_p^H\}_{p\in\mathcal{N}_H}$ are defined elementwise on each element $K\in\bar{\mathcal{T}}_h$ such that I_H satisfies conditions $P1$ to $P3$. E.g., $P2$ is fulfilled if we set $\phi_p^H|_K=\phi_p|_K$ for $K\subset\Omega_N$, for $P1$ we need $I_H\boldsymbol{\chi}_h|_{\Omega_H}=\mathbf{0}|_{\Omega_H}$ if $\boldsymbol{\chi}_h|_{\Omega_H\setminus\Omega_C}=\mathbf{0}|_{\Omega_H\setminus\Omega_C}$.

In the following subsections, two possible choices for $\bar{\mathcal{T}}_h$ and $\{\phi_p^H\}_{p\in\mathcal{N}_H}$ are discussed and examples of corresponding quadrature formulas Q_K are given. The first construction can be applied to completely unstructured meshes which often consist of simplicial elements, the second is suitable for meshes satisfying some minor regularity requirements (as quadrilateral/hexahedral or uniformly refined meshes often do) which will be specified in Subsection 3.2.

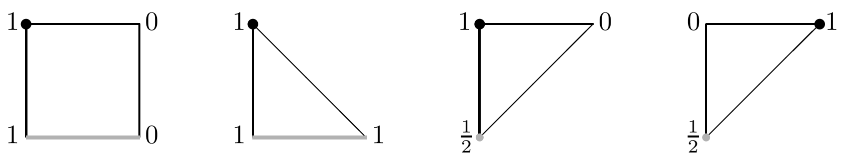


FIG. 3.2. *Nodal values of the modified basis functions ϕ_p^H on $\widehat{K}$ for $K\subset\Omega_C$ for the 2D case.*

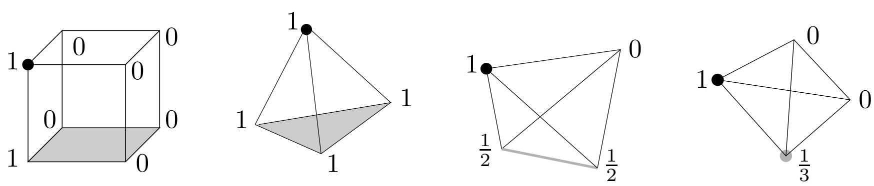


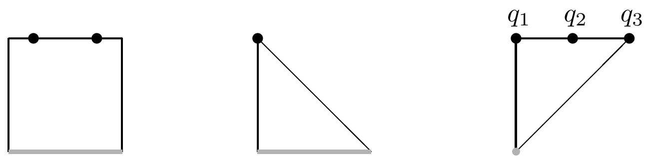
FIG. 3.3. *Nodal values of the modified basis functions ϕ_p^H on $\widehat{K}$ for $K\subset\Omega_C$ for the 3D case.*

3.1. Constant functions on Ω_C. We consider the case $\bar{\mathcal{T}}_h=\mathcal{T}_h$. For each element $K\subset\Omega_C$ and a vertex $p\subset\bar{K}\cap\bar{\Gamma}_H$, we define ϕ_p^H on K as shown in Figures

3.2 and 3.3 for the corresponding reference elements $\widehat{K}$. The intersection of $\partial\widehat{K}$ with the contact boundary Γ_C is on the bottom side of each element and marked in gray, whereas the vertex p is indicated by a filled black bullet. For $K \subset \Omega_N$ and $p \subset \bar{K}$, we set $\phi_p^H|_K = \phi_p|_K$ as stated above. The fulfillment of conditions $P1$ to $P3$ follows by construction.

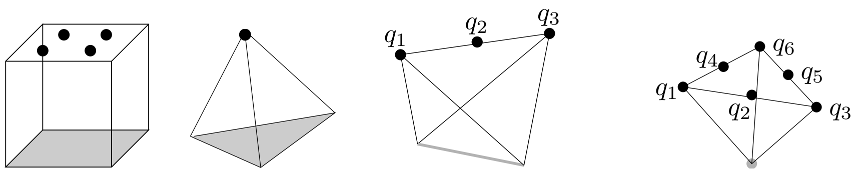
REMARK 3.3. *In Figures 3.2 and 3.3 the most common types of reference basis functions are shown – other possible situations can be handled in a similar way.*

REMARK 3.4. *For strongly anisotropic meshes the nodal values of the modified basis functions can be adapted according to the anisotropy (compare Remark 3.11).*



$w_1 = w_2 = \frac{1}{2}|\widehat{K}|$ (Gauß nodes) $\qquad w = |\widehat{K}| \qquad w_1 = w_3 = \frac{1}{12}|\widehat{K}|,\ w_2 = \frac{5}{6}|\widehat{K}|$

FIG. 3.4. *Quadrature rules in 2D on $\widehat{K}$ for $K \subset \Omega_C$ satisfying $Q1$ and $Q2$.*



$w_i = \frac{1}{4}|\widehat{K}|$ (Gauß nodes) $\qquad w = |\widehat{K}| \qquad w_1 = w_3 = \frac{1}{12}|\widehat{K}|,\ w_2 = \frac{5}{6}|\widehat{K}| \qquad w_1 = w_3 = w_6 = -\frac{2}{45}|\widehat{K}|,\ w_2 = w_4 = w_5 = \frac{17}{45}|\widehat{K}|$

FIG. 3.5. *Quadrature rules in 3D on $\widehat{K}$ for $K \subset \Omega_C$ satisfying $Q1$ and $Q2$.*

As we have $\phi_p^H|_K = \phi_p|_K$ for each element $K \subset \Omega_N$, the quadrature rules used for the calculation of the standard mass matrix M_h remain unaltered except for the elements $K \subset \Omega_C$. For the latter we need a modified Q_K such that all quadrature nodes are situated on Γ_H. Examples for suitable formulas on the corresponding reference elements $\widehat{K}$ are presented in Figures 3.4 and 3.5.

REMARK 3.5. *The negative weights do not disturb the computation as the local mass matrix for $\widehat{K}$ is positive semidefinite for each of the quadrature rules.*

The following lemma is a direct consequence of conditions $Q2$ and $P3$:

LEMMA 3.6. *If the quadrature formula Q_K is chosen according to conditions $Q1$ and $Q2$, the new mass matrix M_H in (3.3) conserves the total mass, i.e., $\mathbf{1}^T M_H \mathbf{1} = \mathbf{1}^T M_h \mathbf{1}$ holds with the vector $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^{d|\mathcal{N}_h|}$.*

REMARK 3.7. *Quadrature rules are usually implemented by defining the nodes and weights on the corresponding reference element $\widehat{K}$ and using the transformation $F_K : \widehat{K} \to K$ to obtain the nodes and weights on K. This implies that the function $F_K^{-1}(\boldsymbol{\chi}_h \cdot \boldsymbol{\eta}_h)$ for $\boldsymbol{\chi}_h, \boldsymbol{\eta}_h \in \mathbf{S}_H$ has to be integrated exactly on $\widehat{K}$ in order to ensure that $Q2$ is satisfied. If F_K^{-1} is an affine transformation, the determinant det(F_K^{-1}) is constant and the quadrature formulas given in Figures 3.4 and 3.5 can be defined on $\widehat{K}$. But for a quadrilateral/hexahedral element K that is not a parallelogram/parallelepiped we get det$(F_K^{-1}) \in \boldsymbol{\mathcal{P}}_1(K)$. In this case, which occurs for example for domains with curved boundaries, we have to calculate the weights on K itself.*

Hence the operator I_H is better suitable for simplicial meshes – which are usually prefered when a domain with a complicated boundary has to be discretized. If on the other hand quadrilateral/hexahedral meshes are used for the triangulation, the approach discussed in the next subsection might be a more suitable choice. It is motivated by the objective to conserve the first and second order moments of the original problem as well as the total mass [17, 18] (compare Lemmas 3.6 and 3.9).

3.2. Linear functions on macro-elements. In order to describe a second possibility to define the modified basis functions, we need the notion of so-called macro-elements (see [11] for a more detailed explanation).

In the rest of this subsection, we assume that $\mathcal{T}_h$ has a macro-element structure in the following sense: There exists a second triangulation $\mathcal{T}_H$ possibly with hanging nodes such that each element of $\mathcal{T}_H$ can be written as the union of elements in $\mathcal{T}_h$. Moreover if $K \in \mathcal{T}_H$ with $K \subset \Omega_N$ then $K \in \mathcal{T}_h$. Each $K \in \mathcal{T}_H \backslash \mathcal{T}_h$ contains at least one element of $\mathcal{T}_h$ being in Ω_C and exactly one element $K_1 \in \mathcal{T}_h$ with $K_1 \subset \Omega_N$. We note that $\mathcal{T}_H$ is not uniquely defined by these conditions. If $\mathcal{T}_h$ is obtained from $\mathcal{T}_{2h}$ by uniform refinement, such a $\mathcal{T}_H$ can easily be constructed, as shown in Figure 3.6 where possible reference macro-elements $\widehat{K}$ for quadrilateral and simplicial triangulations in 2D are displayed.

We denote the set of modified macro-elements K ($K \in \mathcal{T}_H$, but $K \notin \mathcal{T}_h$) with $\mathcal{D}_H$ and set $\bar{\mathcal{T}}_h = \mathcal{T}_H$. The degrees of freedom of each element $K \in \mathcal{T}_H$ are associated with the nodes of $\mathcal{N}_H$.

REMARK 3.8. *Such a triangulation $\mathcal{T}_H$ exists for any $\mathcal{T}_h$ as no further conditions on the shape of the macro-elements are imposed – but its implementation may be rather cumbersome.*

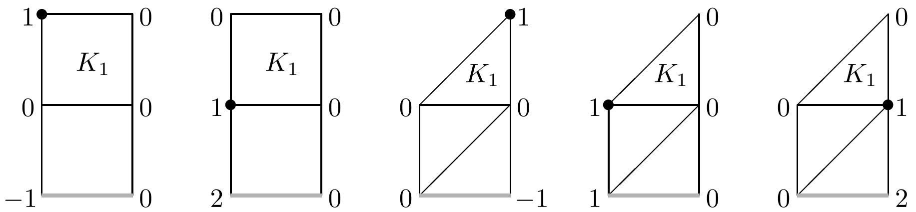


FIG. 3.6. *Nodal values of the modified basis functions $\widetilde{\phi}_p^H$ on $\widehat{K}$ for $K \in \mathcal{D}_H$ for the 2D case.*

A second set of modified basis functions $\{\widetilde{\phi}_p^H\}_{p \in \mathcal{N}_H}$ is defined in the following way: For each $K \in \mathcal{T}_H$ let $K_1 \in \mathcal{T}_h$, be the unique element satisfying $K_1 \subset K \cap \Omega_N$ (see Figure 3.6). For each vertex $p \subset \bar{K}_1$, we define $\widetilde{\phi}_p^H$ as the polynomial extension of $\phi_p|_{K_1}$ onto K. This leads to $\widetilde{\phi}_p^H|_{K'} = \phi_p|_{K'}$ for $K' \in \mathcal{T}_h \cap \mathcal{T}_H$.

The validity of conditions $P1$ to $P3$ is easily verified. We are even able to obtain a stronger version of $P3$:

$\widetilde{P3}$) $I_H \boldsymbol{\chi}_h = \boldsymbol{\chi}_h, \quad \boldsymbol{\chi}_h|_K \in \boldsymbol{\mathcal{P}}_1(K), \ \ K \in \mathcal{T}_H.$

Conditions $\widetilde{P3}$ and $Q2$ imply the following lemma (see also [11]):

LEMMA 3.9. *If the quadrature formula Q_K is chosen according to $Q1$ and $Q2$, the new mass matrix M_H in (3.3) conserves the zeroth, first and second order moments of the original system (2.6).*

In Figure 3.7, we give one example of a suitable quadrature formula for hexahedral and simplicial reference macro-elements in 2D, respectively. The rule for the hexahedral element exhibits a tensor product structure; thus, it can easily be generalized to the 3D case.

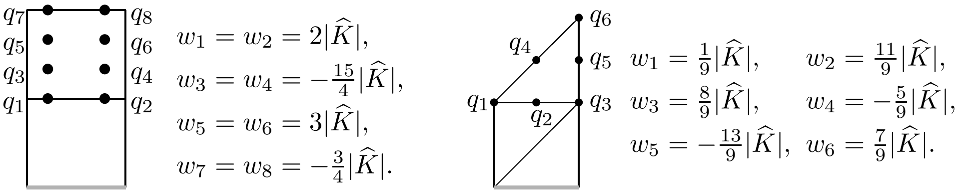


FIG. 3.7. *Quadrature rules on macro-elements $\widehat{K}$ for $K \in \mathcal{D}_H$ in 2D satisfying Q1 and Q2.*

REMARK 3.10. *For a simplicial grid, we get $(\boldsymbol{\chi}_H \cdot \boldsymbol{\eta}_H)|_K \in \boldsymbol{\mathcal{P}}_2(K)$ for $K \in \mathcal{T}_H$ and a constant determinant $\det(F_K^{-1})$. Hence, Q2 is equivalent to $Q_{\widehat{K}}$ being exact on $\boldsymbol{\mathcal{P}}_2(\widehat{K})$ in this case. But for quadrilateral/hexahedral elements, we have to take care of the additional factor $\det(F_K^{-1}) \in \boldsymbol{\mathcal{P}}_1(K)$ (see Remark 3.7); thus, the tensor product quadrature formula given on the left of Figure 3.7 is chosen such that it is exact for any function in $\boldsymbol{\mathcal{Q}}_3(\widehat{K})$. This is sufficient for the fulfillment of Q2.*

REMARK 3.11. *The modified basis functions ϕ_p^H or $\widetilde{\phi}_p^H$ for $p \subset \Gamma_H$ are in general not globally continuous because of their elementwise definition, which leads to $\mathbf{S}_H \not\subset \mathbf{S}_h$. This could be avoided by constructing the basis functions in the following way: For $q \in \mathcal{C}_h$, let $E_q \subset \mathcal{N}_H$ denote the set of nodes $p \subset \Gamma_H$ that are joined with q by an inner edge. We set $n_q := |E_q|$ and define the continuous function*

$$\phi_p^H := \phi_p + \sum_{q \in \mathcal{C}_h} \gamma_{pq} \phi_q, \quad \gamma_{pq} = \begin{cases} 0 & \text{if } p \notin E_q, \\ \frac{1}{n_q} & \text{if } p \in E_q. \end{cases}$$

This ensures $S_H \subset S_h$, but in general results in $\operatorname{supp} \phi_p^H \neq \operatorname{supp} \phi_p$. Hence an elementwise definition of appropriate quadrature formulas would not be possible.

4. Error estimate for the semi-discrete system. In the following, we restrict ourselves to the case that the body Ω is not in contact, i.e., we assume $\boldsymbol{\lambda}_h = \mathbf{0}$. Then the weak formulation of the contact problem reduces to

$$m(\ddot{\mathbf{u}}, \boldsymbol{\chi}) + a(\mathbf{u}, \boldsymbol{\chi}) = (\mathbf{f}, \boldsymbol{\chi}), \quad \boldsymbol{\chi} \in \mathbf{H}_D^1(\Omega), \tag{4.1}$$

along with the initial conditions of (2.4). The approximate solution $\mathbf{u}_h$ is given by

$$m_H(\ddot{\mathbf{u}}_h, \boldsymbol{\chi}_h) + a(\mathbf{u}_h, \boldsymbol{\chi}_h) = (\mathbf{f}, \boldsymbol{\chi}_h), \quad \boldsymbol{\chi}_h \in \mathbf{S}_h. \tag{4.2}$$

In order to gain an error estimate for the solution of (4.2), we proceed similar to the proof of the lumped mass matrix method in [25] which can be interpreted as a variational crime introduced by a quadrature formula (for details see [1]). By $R_h : \mathbf{H}_D^1(\Omega) \to \mathbf{S}_h$, we denote the Ritz projection that is defined via the equality

$$a(R_h \mathbf{v}, \boldsymbol{\chi}_h) = a(\mathbf{v}, \boldsymbol{\chi}_h), \quad \boldsymbol{\chi}_h \in \mathbf{S}_h, \quad \mathbf{v} \in \mathbf{H}_D^1(\Omega). \tag{4.3}$$

Then we get the following estimate for $1 \le s \le 2$ (for the proof see, e.g., [25]):

$$\|R_h \mathbf{v} - \mathbf{v}\|_{0,\Omega} + h \|\nabla(R_h \mathbf{v} - \mathbf{v})\|_{0,\Omega} \le C h^s \|\mathbf{v}\|_{s,\Omega}, \quad \mathbf{v} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}_D^1(\Omega). \tag{4.4}$$

We start with a lemma about the approximation properties of the interpolation operator I_H:

Lemma 4.1. *Let I_H satisfy conditions $P1$ to $P3$ (given in Section 3). We define the projection $R_H : \mathbf{H}^1_D(\Omega) \to S_H$ by*

$$R_H\mathbf{u} := I_H(R_h\mathbf{u}), \quad \mathbf{u} \in \mathbf{H}^1_D(\Omega). \tag{4.5}$$

Then the following inequality holds:

$$\|R_H\mathbf{v} - \mathbf{v}\|_{0,\Omega} \le Ch\|\mathbf{v}\|_{1,\Omega}, \quad \mathbf{v} \in \mathbf{H}^1_D(\Omega). \tag{4.6}$$

Proof. Let $\bar{\mathcal{T}}_h$ denote either $\mathcal{T}_h$ for the case described in Section 3.1 or $\mathcal{T}_H$ like in Section 3.2. The estimate (4.6) follows from

$$\|R_H\mathbf{v} - \mathbf{v}\|^2_{0,\Omega} \le 2\sum_{K\in\bar{\mathcal{T}}_h} \left(\|(I_H - \mathrm{Id})R_h\mathbf{v}\|^2_{0,K} + \|R_h\mathbf{v} - \mathbf{v}\|^2_{0,K}\right),$$

the transformation rule, the Bramble–Hilbert lemma (see, e.g., [4]) and condition $P3$ which leads to

$$\|(I_H - \mathrm{Id})R_h\mathbf{v}\|^2_{0,K} \le ch^2|R_h\mathbf{v}|^2_{1,K}.$$

Using (4.4) then concludes the proof. □

Remark 4.2. *Whereas an error estimate with respect to the $\mathbf{H}^1$-norm (see Theorem 4.9) is possible for any interpolation operator I_H satisfying conditions $P1$ to $P3$ without further assumptions on Ω, the proof of an $\mathbf{L}^2$-estimate (Theorem 4.13) requires the $\mathbf{H}^2$-regularity of problem (4.1) as stated in [4]. To ensure this regularity, we later require Ω to be convex and the solution $\mathbf{u}$ of (4.1) to be in $\mathbf{V}_0 := \mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega) \subset \mathbf{V}_D$, i.e., we assume homogeneous Dirichlet boundary conditions on $\partial\Omega$. These assumptions are quite common in the literature whenever $\mathbf{L}^2$-estimates are to be derived [4, 10]. However, the numerical results in Section 6.1 indicate that the boundary conditions are not necessary for optimal error decay.*

We define the quadrature error of the modified bilinear form using (3.4):

$$\varepsilon_H(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) := m_H(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) - m(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) = m(I_H\boldsymbol{\xi}_h, I_H\boldsymbol{\eta}_h) - m(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h),$$

for functions $\boldsymbol{\xi}_h, \boldsymbol{\eta}_h \in \mathbf{S}_h$. The next lemma gives a bound on this bilinear form:

Lemma 4.3. *Let $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$, $\boldsymbol{\chi}_h \in \mathbf{S}_h$ and assume that I_H meets conditions $P1$ to $P3$. Then the following estimate holds:*

$$|\varepsilon_H(R_h\mathbf{u}, \boldsymbol{\chi}_h)| \le Ch\|\mathbf{u}\|_{1,\Omega}\|\boldsymbol{\chi}_h\|_{1,\Omega}. \tag{4.7}$$

For $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1_0(\Omega)$, we obtain

$$|\varepsilon_H(R_h\mathbf{u}, R_h\mathbf{v})| \le Ch^2\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}. \tag{4.8}$$

Proof. First we remark that by extension with respect to (3.1), $m_H(I_H\boldsymbol{\xi}_h, I_H\boldsymbol{\eta}_h)$ is also well defined for any $\boldsymbol{\xi}_h, \boldsymbol{\eta}_h \in \mathbf{S}_h$. Then we have by (3.1) and (3.6)

$$\varepsilon_H(I_H\boldsymbol{\xi}_h, I_H\boldsymbol{\eta}_h) = m_H(I_H\boldsymbol{\xi}_h, I_H\boldsymbol{\eta}_h) - m(I_H\boldsymbol{\xi}_h, I_H\boldsymbol{\eta}_h) = 0.$$

Condition $P2$ implies further that $(\boldsymbol{\xi}_h - I_H\boldsymbol{\xi}_h)$ vanishes on $\bar{\Omega}_N$, such that we obtain $m_H(\boldsymbol{\xi}_h - I_H\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) = 0$ by $Q1$. With this in mind, we consider (4.7):

$$\begin{aligned}|\varepsilon_H(R_h\mathbf{u}, \boldsymbol{\chi}_h)| &\le |\varepsilon_H(R_h\mathbf{u} - R_H\mathbf{u}, \boldsymbol{\chi}_h)| + |\varepsilon_H(R_H\mathbf{u}, \boldsymbol{\chi}_h - R_H\boldsymbol{\chi}_h)| \\ &\le C\Big(\|R_h\mathbf{u} - R_H\mathbf{u}\|_{0,\Omega_C}\|\boldsymbol{\chi}_h\|_{0,\Omega_C} + \|R_H\mathbf{u}\|_{0,\Omega_C}\|\boldsymbol{\chi}_h - R_H\boldsymbol{\chi}_h\|_{0,\Omega_C}\Big) \\ &\le Ch\|\mathbf{u}\|_{1,\Omega}\|\boldsymbol{\chi}_h\|_{1,\Omega},\end{aligned}$$

where we made use of (4.4) and (4.6).

Now we turn to (4.8); for $K \in \mathcal{T}_h$ with $\partial K \cap \Gamma_C \neq \emptyset$ and $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ which implies $\mathbf{v}|_{\Gamma_C} = \mathbf{0}$, we can state:

$$\|\mathbf{v}\|^2_{0,K} \le Ch^2|\mathbf{v}|^2_{1,K} \quad \Rightarrow \quad \|\mathbf{v}\|^2_{0,\Omega_H} \le Ch^2|\mathbf{v}|^2_{1,\Omega_H}. \tag{4.9}$$

Condition $P1$, (4.4) and (4.9) finally lead to:

$$\begin{aligned}|\varepsilon_H(R_h\mathbf{u}, R_h\mathbf{v})| &\le |\varepsilon_H(R_h\mathbf{u} - R_H\mathbf{u}, R_h\mathbf{v})| + |\varepsilon_H(R_H\mathbf{u}, R_h\mathbf{v} - R_H\mathbf{v})| \\ &\le C(\|R_h\mathbf{u} - R_H\mathbf{u}\|_{0,\Omega_C}\|R_h\mathbf{v}\|_{0,\Omega_C} + \|R_H\mathbf{u}\|_{0,\Omega_C}\|R_h\mathbf{v} - R_H\mathbf{v}\|_{0,\Omega_C}) \\ &\le C(h^2\|\mathbf{u}\|_{1,\Omega}|R_h\mathbf{v}|_{1,\Omega_H} + h\|R_H\mathbf{u}\|_{0,\Omega_C}\|\mathbf{v}\|_{1,\Omega}) \\ &\le Ch^2\|\mathbf{u}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}.\end{aligned}$$

□

We further state two auxiliary results; firstly, we cite a lemma about the time derivative of the Ritz approximation (for the proof see, e.g., [1]):

LEMMA 4.4. *For* $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$ *we have* $R_h\dot{\mathbf{u}} = (\dot{R_h\mathbf{u}})$.

Secondly, we quote the following lemma given in [7] which is related to the Sobolev embedding theorem:

LEMMA 4.5. *Let* $\mathbf{u} \in L^2((0,T), X)$ *and* $\dot{\mathbf{u}} \in L^2((0,T), X')$ *for some Banach space* X. *Then we have* $\mathbf{u} \in C^0([0,T], X)$ *and*

$$\sup_{t\in[0,T]} \|\mathbf{u}(t)\|_X \le C\Big(\|\mathbf{u}\|_{L^2((0,T),X)} + \|\dot{\mathbf{u}}\|_{L^2((0,T),X')}\Big).$$

Now we are able to derive a priori bounds on the difference $(\mathbf{u}_h - \mathbf{u})$. We start with a $\mathbf{H}^1$-estimate which can be shown for I_H satisfying conditions $P1$ to $P3$ without additional assumptions on Ω or the boundary conditions.

To this end, we introduce $\mathbf{w}_h \in \mathbf{S}_h^{(0,T)} := L^2((0,T), \mathbf{S}_h)$ as a suitable elliptic projection of $\mathbf{u}$ onto $\mathbf{S}_h$:

$$a(\mathbf{w}_h, \boldsymbol{\chi}_h) = (\mathbf{f}, \boldsymbol{\chi}_h) - m_H(R_h\ddot{\mathbf{u}}, \boldsymbol{\chi}_h), \quad \boldsymbol{\chi}_h \in \mathbf{S}_h, \quad 0 \le t \le T. \tag{4.10}$$

For $\mathbf{S}_h \subset \mathbf{H}^1_D(\Omega)$ and a $\mathbf{H}^1_D(\Omega)$-coercive bilinear form $a(\cdot,\cdot)$ this equation uniquely defines $\mathbf{w}_h$. Now we bound the difference between $\mathbf{u}$ and $\mathbf{w}_h$:

LEMMA 4.6. *Let* $\mathbf{u} \in L^2((0,T), \mathbf{V}_D)$, $\ddot{\mathbf{u}} \in L^2((0,T), \mathbf{H}^1_D(\Omega))$ *and choose* I_H *such that conditions* $P1$ *to* $P3$ *hold. Then the solution* $\mathbf{w}_h$ *of (4.10) satisfies*

$$\|\mathbf{w}_h - \mathbf{u}\|_{0,1,\Omega} \le Ch\left(\|\mathbf{u}\|_{0,2,\Omega} + \|\ddot{\mathbf{u}}\|_{0,1,\Omega}\right). \tag{4.11}$$

Proof. We start with (4.1), (4.3) and (4.10) for a test function $\boldsymbol{\chi}_h \in \mathbf{S}_h$:

$$a(\mathbf{w}_h - R_h\mathbf{u}, \boldsymbol{\chi}_h) = m(\ddot{\mathbf{u}} - R_h\ddot{\mathbf{u}}, \boldsymbol{\chi}_h) + m(R_h\ddot{\mathbf{u}}, \boldsymbol{\chi}_h) - m_H(R_h\ddot{\mathbf{u}}, \boldsymbol{\chi}_h). \tag{4.12}$$

We choose $\boldsymbol{\chi}_h = \mathbf{w}_h - R_h\mathbf{u}$; then we get with the coercivity of $a(\cdot,\cdot)$, the continuity of $m(\cdot,\cdot)$, integration from 0 to T and the division of both sides by $\|\mathbf{w}_h - R_h\mathbf{u}\|_{0,1,\Omega}$:

$$\|\mathbf{w}_h - R_h\mathbf{u}\|_{0,1,\Omega} \le C\,\|\ddot{\mathbf{u}} - R_h\ddot{\mathbf{u}}\|_{0,0,\Omega} + \sup_{\substack{\boldsymbol{\chi}_h\in S_h^{(0,T)}\\ \boldsymbol{\chi}_h\neq 0}} \frac{C}{\|\boldsymbol{\chi}_h\|_{0,1,\Omega}} \int_0^T |\varepsilon_H(R_h\ddot{\mathbf{u}},\boldsymbol{\chi}_h)|dt.$$

With Lemma 4.3 and the Cauchy–Schwarz inequality, we obtain

$$\int_0^T |\varepsilon_H(R_h\ddot{\mathbf{u}},\boldsymbol{\chi}_h)|dt \le Ch\|\ddot{\mathbf{u}}\|_{0,1,\Omega}\|\boldsymbol{\chi}_h\|_{0,1,\Omega}.$$

The approximation property (4.4) of the Ritz projection R_h gives

$$\|\mathbf{u} - R_h\mathbf{u}\|_{0,1,\Omega} \le Ch\|\mathbf{u}\|_{0,2,\Omega},$$
$$\|\ddot{\mathbf{u}} - R_h\ddot{\mathbf{u}}\|_{0,0,\Omega} \le Ch\|\ddot{\mathbf{u}}\|_{0,1,\Omega}.$$

Thus we arrive at (4.11):

$$\|\mathbf{w}_h - \mathbf{u}\|_{0,1,\Omega} \le \|\mathbf{w}_h - R_h\mathbf{u}\|_{0,1,\Omega} + \|R_h\mathbf{u} - \mathbf{u}\|_{0,1,\Omega} \le Ch\left(\|\mathbf{u}\|_{0,2,\Omega} + \|\ddot{\mathbf{u}}\|_{0,1,\Omega}\right).$$

□

For the difference of the time derivatives of $\mathbf{u}$ and $\mathbf{w}_h$, we get:

LEMMA 4.7. *Let $\dot{\mathbf{u}} \in L^2((0,T),\mathbf{V}_D)$, $\frac{\partial^3\mathbf{u}}{\partial t^3} \in L^2((0,T),\mathbf{H}^1_D(\Omega))$ and choose I_H such that conditions $P1$ to $P3$ hold. Then the solution $\mathbf{w}_h$ of (4.10) satisfies*

$$\|\dot{\mathbf{w}}_h - \dot{\mathbf{u}}\|_{0,1,\Omega} \le Ch\left(\|\dot{\mathbf{u}}\|_{0,2,\Omega} + \left\|\frac{\partial^3\mathbf{u}}{\partial t^3}\right\|_{0,1,\Omega}\right). \tag{4.13}$$

Proof. If we differentiate (4.1) and (4.10) with respect to time and use Lemma 4.4, we get for any $\boldsymbol{\chi}_h \in \mathbf{S}_h$:

$$a(\dot{\mathbf{u}},\boldsymbol{\chi}_h) = (\dot{\mathbf{f}},\boldsymbol{\chi}_h) - m_H\left(\frac{\partial^3\mathbf{u}}{\partial t^3},\boldsymbol{\chi}_h\right),$$
$$a(\dot{\mathbf{w}}_h,\boldsymbol{\chi}_h) = (\dot{\mathbf{f}},\boldsymbol{\chi}_h) - m_H\left(R_h\frac{\partial^3\mathbf{u}}{\partial t^3},\boldsymbol{\chi}_h\right).$$

Then we proceed as in Lemma 4.6. □

Analogeously, we obtain for the second derivative with respect to time:

LEMMA 4.8. *Let $\ddot{\mathbf{u}} \in L^2((0,T),\mathbf{V}_D)$, $\frac{\partial^4\mathbf{u}}{\partial t^4} \in L^2((0,T),\mathbf{H}^1_D(\Omega))$ and choose I_H such that conditions $P1$ to $P3$ hold. Then the solution $\mathbf{w}_h$ of (4.10) satisfies*

$$\|\ddot{\mathbf{w}}_h - \ddot{\mathbf{u}}\|_{0,1,\Omega} \le Ch\left(\|\ddot{\mathbf{u}}\|_{0,2,\Omega} + \left\|\frac{\partial^4\mathbf{u}}{\partial t^4}\right\|_{0,1,\Omega}\right). \tag{4.14}$$

With these lemmas, we are able to prove the following result for the $\mathbf{H}^1$-norm of the error:

THEOREM 4.9. *Let $\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}} \in L^2((0,T),\mathbf{V}_D)$ and $\frac{\partial^3\mathbf{u}}{\partial t^3}, \frac{\partial^4\mathbf{u}}{\partial t^4} \in L^2((0,T),\mathbf{H}^1_D(\Omega))$. Let I_H satisfy $P1$ to $P3$. Take the boundary conditions $\mathbf{u}_h(0) = R_h\mathbf{u}_0$ and $\dot{\mathbf{u}}_h(0) =$*

$R_h\mathbf{v}_0$. *Then the following estimate holds:*

$$\sup_{0\le t\le T}\Big(\|\dot{\mathbf{u}}_h-\dot{\mathbf{u}}\|_{0,\Omega_N}+\|\mathbf{u}_h-\mathbf{u}\|_{1,\Omega}\Big)\le Ch\left(\sum_{s=0}^{2}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,2,\Omega}+\sum_{s=3}^{4}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,1,\Omega}\right). \tag{4.15}$$

Proof. Let $\mathbf{u}_h,\mathbf{w}_h$ be defined by (4.2) and (4.10). Setting $\boldsymbol{\theta}_h:=\mathbf{u}_h-\mathbf{w}_h$ gives for any test function $\boldsymbol{\chi}_h\in\mathbf{S}_h^{(0,T)}$:

$$m_H(\ddot{\boldsymbol{\theta}}_h,\boldsymbol{\chi}_h)+a(\boldsymbol{\theta}_h,\boldsymbol{\chi}_h)=m_H(R_h\ddot{\mathbf{u}},\boldsymbol{\chi}_h)-m_H(\ddot{\mathbf{w}}_h,\boldsymbol{\chi}_h). \tag{4.16}$$

We choose $\boldsymbol{\chi}_h=\dot{\boldsymbol{\theta}}_h$ in (4.16) and thus obtain

$$\frac{1}{2}\frac{d}{dt}\Big(|\dot{\boldsymbol{\theta}}_h|_H^2+a(\boldsymbol{\theta}_h,\boldsymbol{\theta}_h)\Big)=m_H(R_h\ddot{\mathbf{u}}-\ddot{\mathbf{w}}_h,\dot{\boldsymbol{\theta}}_h).$$

Using Cauchy–Schwarz' and Young's inequality and adding the positive term $a(\boldsymbol{\theta}_h,\boldsymbol{\theta}_h)$ on the right hand side, we arrive at

$$\frac{d}{dt}E\le C|R_h\ddot{\mathbf{u}}-\ddot{\mathbf{w}}_h|_H^2+E, \tag{4.17}$$

where we define

$$E=|\dot{\boldsymbol{\theta}}_h|_H^2+a(\boldsymbol{\theta}_h,\boldsymbol{\theta}_h).$$

From (4.17) we conclude with Gronwall's lemma, the seminorm equivalence (3.5) and the coercivity of $a(\cdot,\cdot)$ with a constant $C(T)$ that may depend on T:

$$\begin{aligned}\|\dot{\boldsymbol{\theta}}_h\|_{0,\Omega_N}^2+\|\boldsymbol{\theta}_h\|_{1,\Omega}^2&\le CE \qquad (4.18)\\ &\le C(T)\Big(\|\dot{\boldsymbol{\theta}}_h(0)\|_{0,\Omega_N}^2+\|\boldsymbol{\theta}_h(0)\|_{1,\Omega}^2+\|R_h\ddot{\mathbf{u}}-\ddot{\mathbf{w}}_h\|_{0,0,\Omega_N}^2\Big).\end{aligned}$$

This inequality holds for all $t\in(0,T]$ and hence also for the supremum. Now we have to estimate the terms on the right hand side. Firstly, we can state

$$\begin{aligned}\|R_h\ddot{\mathbf{u}}-\ddot{\mathbf{w}}_h\|_{0,0,\Omega_N}&\le\|R_h\ddot{\mathbf{u}}-\ddot{\mathbf{u}}\|_{0,0,\Omega_N}+\|\ddot{\mathbf{u}}-\ddot{\mathbf{w}}_h\|_{0,0,\Omega_N}\\ &\le Ch\left(\|\ddot{\mathbf{u}}\|_{0,2,\Omega}+\left\|\frac{\partial^4\mathbf{u}}{\partial t^4}\right\|_{0,1,\Omega}\right)\end{aligned}$$

where we made use of (4.4) and Lemma 4.8. Secondly, we get

$$\begin{aligned}\|\dot{\boldsymbol{\theta}}_h(0)\|_{0,\Omega_N}&\le\|R_h\mathbf{v}_0-\mathbf{v}_0\|_{0,\Omega_N}+\|\mathbf{v}_0-\dot{\mathbf{w}}_h(0)\|_{0,\Omega_N}\\ &\le Ch\|\mathbf{v}_0\|_{1,\Omega}+\|\dot{\mathbf{u}}-\dot{\mathbf{w}}_h\|_{0,0,\Omega_N}+\|\ddot{\mathbf{u}}-\ddot{\mathbf{w}}_h\|_{0,0,\Omega_N}\\ &\le Ch\left(\sum_{s=1}^{2}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,2,\Omega}+\sum_{s=3}^{4}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,1,\Omega}\right),\end{aligned}$$

where we applied Lemmas 4.5, 4.7 and 4.8. Finally, we estimate similarly:

$$\begin{aligned}\|\boldsymbol{\theta}_h(0)\|_{1,\Omega}&\le\|R_h\mathbf{u}_0-\mathbf{u}_0\|_{1,\Omega}+\|\mathbf{u}_0-\mathbf{w}_h(0)\|_{1,\Omega}\\ &\le Ch\|\mathbf{u}_0\|_{2,\Omega}+\|\mathbf{u}-\mathbf{w}_h\|_{0,1,\Omega}+\|\dot{\mathbf{u}}-\dot{\mathbf{w}}_h\|_{0,1,\Omega}\\ &\le Ch\left(\sum_{s=0}^{1}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,2,\Omega}+\sum_{s=2}^{3}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,1,\Omega}\right).\end{aligned}$$

If we insert all these results into (4.18), we conclude

$$\sup_{0\le t\le T}\left(\|\dot{\boldsymbol\theta}_h\|_{0,\Omega_N}+\|\boldsymbol\theta_h\|_{1,\Omega}\right)\le C(T)h\left(\sum_{s=0}^{2}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,2,\Omega}+\sum_{s=3}^{4}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,1,\Omega}\right). \quad (4.19)$$

Now we recall $\mathbf{u}_h-\mathbf{u}=\boldsymbol\theta_h+\mathbf{w}_h-\mathbf{u}$ and get

$$\|\dot{\mathbf{u}}_h-\dot{\mathbf{u}}\|_{0,\Omega_N}+\|\mathbf{u}_h-\mathbf{u}\|_{1,\Omega}\le\|\dot{\boldsymbol\theta}_h\|_{0,\Omega_N}+\|\boldsymbol\theta_h\|_{1,\Omega}+\|\dot{\mathbf{u}}-\dot{\mathbf{w}}_h\|_{0,\Omega_N}+\|\mathbf{u}-\mathbf{w}_h\|_{1,\Omega}.$$

Due to Lemmas 4.5, 4.6 and 4.7, we can estimate $\|\mathbf{u}-\mathbf{w}_h\|_{\infty,1,\Omega}$ and $\|\dot{\mathbf{u}}-\dot{\mathbf{w}}_h\|_{\infty,0,\Omega_N}$ such that we arrive at (4.15). □

Next, we turn to the estimate in the $\mathbf{L}^2$-norm, where we need to impose further restrictions on Ω and the boundary conditions.

Remark 4.10. *Homogeneous Dirichlet boundary conditions on $\partial\Omega$ lead to*

$$c\|\boldsymbol\chi_h\|_{0,\Omega}^2\le|\boldsymbol\chi_h|_H^2\le C\|\boldsymbol\chi_h\|_{0,\Omega}^2,\quad \boldsymbol\chi_h\in\mathbf{S}_h\cap\mathbf{H}_0^1(\Omega); \quad (4.20)$$

hence $|\cdot|_H$ now is a norm on Ω instead of only a seminorm (compare Remark 3.2).

Lemma 4.11. *Let Ω be convex, $\mathbf{u}\in L^2((0,T),\mathbf{V}_0)$, $\ddot{\mathbf{u}}\in L^2((0,T),\mathbf{H}_0^1(\Omega))$ and let the interpolation operator I_H satisfy conditions $P1$ to $P3$. Then we obtain:*

$$\|\mathbf{w}_h-\mathbf{u}\|_{0,0,\Omega}\le Ch^2\left(\|\mathbf{u}\|_{0,2,\Omega}+\|\ddot{\mathbf{u}}\|_{0,1,\Omega}\right). \quad (4.21)$$

Proof. We start with the observation that for $\boldsymbol\psi\in L^2((0,T),\mathbf{L}^2(\Omega))$, we get

$$\|\boldsymbol\psi\|_{0,0,\Omega}=\sup_{\mathbf{g}\in L^2((0,T),\mathbf{L}^2(\Omega))\setminus\{0\}}\frac{|\int_0^T(\boldsymbol\psi,\mathbf{g})dt|}{\|\mathbf{g}\|_{0,0,\Omega}}, \quad (4.22)$$

as $L^2((0,T),\mathbf{L}^2(\Omega))$ is its own dual space. Now we choose $\boldsymbol\varphi\in L^2((0,T),\mathbf{H}_0^1(\Omega))$ as the solution of the following auxiliary problem for any fixed $\mathbf{g}\in L^2((0,T),\mathbf{L}^2(\Omega))$:

$$a(\boldsymbol\varphi,\mathbf{v})=(\mathbf{g},\mathbf{v}),\quad \mathbf{v}\in\mathbf{H}_0^1(\Omega),\quad 0\le t\le T. \quad (4.23)$$

By our assumptions on the domain Ω, we obtain $\boldsymbol\varphi\in L^2((0,T),\mathbf{V}_0)$ (see [4]) and

$$\|\boldsymbol\varphi\|_{0,2,\Omega}\le C\|\mathbf{g}\|_{0,0,\Omega}. \quad (4.24)$$

If we choose $\mathbf{v}=\mathbf{w}_h-\mathbf{u}$ in (4.23) and use (4.3), (4.12), we get for $\boldsymbol\chi_h\in\mathbf{S}_h^{(0,T)}\cap\mathbf{H}_0^1(\Omega)$:

$$(\mathbf{w}_h-\mathbf{u},\mathbf{g})=a(\mathbf{w}_h-\mathbf{u},\boldsymbol\varphi-\boldsymbol\chi_h)+m(\ddot{\mathbf{u}}-R_h\ddot{\mathbf{u}},\boldsymbol\chi_h)+\varepsilon_H(R_h\ddot{\mathbf{u}},\boldsymbol\chi_h).$$

Now we take $\boldsymbol\chi_h=R_h\boldsymbol\varphi$ and integrate the above equality from 0 to T:

$$\left|\int_0^T(\mathbf{w}_h-\mathbf{u},\mathbf{g})dt\right|\le\int_0^T|a(\mathbf{w}_h-\mathbf{u},\boldsymbol\varphi-R_h\boldsymbol\varphi)|dt+\int_0^T|m(\ddot{\mathbf{u}}-R_h\ddot{\mathbf{u}},R_h\boldsymbol\varphi)|dt$$
$$+\int_0^T|\varepsilon_H(R_h\ddot{\mathbf{u}},R_h\boldsymbol\varphi)|dt. \quad (4.25)$$

For the first term on the right hand side, we get with the continuity of $a(\cdot,\cdot)$, (4.4) and Lemma 4.6:

$$\int_0^T|a(\mathbf{w}_h-\mathbf{u},\boldsymbol\varphi-R_h\boldsymbol\varphi)|dt\le Ch^2\left(\|\mathbf{u}\|_{0,2,\Omega}+\|\ddot{\mathbf{u}}\|_{0,1,\Omega}\right)\|\boldsymbol\varphi\|_{0,2,\Omega}.$$

The second and third term are estimated by means of (4.4), (4.9) and Lemma 4.3:

$$\int_0^T |m(\ddot{\mathbf{u}} - R_h\ddot{\mathbf{u}}, R_h\boldsymbol{\varphi})|dt + \int_0^T |\varepsilon_H(R_h\ddot{\mathbf{u}}, R_h\boldsymbol{\varphi})|dt \le Ch^2\|\ddot{\mathbf{u}}\|_{0,1,\Omega}\|\boldsymbol{\varphi}\|_{0,1,\Omega}.$$

Using (4.22), (4.24) and these estimates, we can now conclude

$$\begin{aligned}\|\mathbf{w}_h - \mathbf{u}\|_{0,0,\Omega} &= \sup_{\mathbf{g}\in L^2((0,T),\mathbf{L}^2(\Omega))\setminus\{0\}} \frac{|\int_0^T (\mathbf{w}_h - \mathbf{u}, \mathbf{g})dt|}{\|\mathbf{g}\|_{0,0,\Omega}} \\ &\le Ch^2\left(\|\mathbf{u}\|_{0,2,\Omega} + \|\ddot{\mathbf{u}}\|_{0,1,\Omega}\right).\end{aligned}$$

□

Similar to Lemmas 4.7 and 4.8, differentiation in time leads to

LEMMA 4.12. *Let* $\dot{\mathbf{u}} \in L^2((0,T),\mathbf{V}_0)$, $\frac{\partial^3\mathbf{u}}{\partial t^3} \in L^2((0,T),\mathbf{H}_0^1(\Omega))$ *and let the conditions of Lemma 4.11 on* Ω *and* I_H *be fulfilled. Then we obtain:*

$$\|\dot{\mathbf{w}}_h - \dot{\mathbf{u}}\|_{0,0,\Omega} \le Ch^2\left(\|\dot{\mathbf{u}}\|_{0,2,\Omega} + \left\|\frac{\partial^3\mathbf{u}}{\partial t^3}\right\|_{0,1,\Omega}\right). \tag{4.26}$$

These lemmas imply the following estimate:

THEOREM 4.13. *Let* Ω *be convex and assume* $\mathbf{u}, \dot{\mathbf{u}} \in L^2((0,T),\mathbf{V}_0)$, $\ddot{\mathbf{u}}, \frac{\partial^3\mathbf{u}}{\partial t^3} \in L^2((0,T),\mathbf{H}_0^1(\Omega))$. *Choose* I_H *according to P1 to P3. Take the initial conditions* $\mathbf{u}_h(0) = R_h\mathbf{u}_0$ *and* $\dot{\mathbf{u}}_h(0) = R_h\mathbf{v}_0$. *Then the following inequality holds:*

$$\|\mathbf{u}_h - \mathbf{u}\|_{\infty,0,\Omega} \le Ch^2\left(\sum_{s=0}^{1}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,2,\Omega} + \sum_{s=2}^{3}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,1,\Omega}\right). \tag{4.27}$$

Proof. The proof is done as in [1, Theorem 4.1], due to Lemmas 4.11, 4.12 and Remark 4.10. □

5. Error estimate for the fully discrete system. In this section, we look at the time-discretized version of (4.2): Let $J\tau := T$ for some integer J and the stepwidth τ. Set $\mathbf{w}^n := \mathbf{w}(t_n)$ for $\mathbf{w} \in C(\bar{\Omega} \times [0,T])$, $t_n := n\tau$, and define

$$\partial_t\mathbf{w}^n := \frac{1}{\tau}(\mathbf{w}^{n+1} - \mathbf{w}^n), \qquad \mathbf{w}^{n+1/2} := \frac{1}{2}(\mathbf{w}^{n+1} + \mathbf{w}^n).$$

The fully discrete problem we now consider reads: find sequences $(\mathbf{u}_h^n)_{n=0}^J$, $(\mathbf{v}_h^n)_{n=0}^J$ in $\mathbf{S}_h$, corresponding to the displacement and the velocity, such that

$$\begin{aligned}\mathbf{u}_h^0 &= R_h\mathbf{u}_0, \\ \mathbf{v}_h^0 &= R_h\mathbf{v}_0, \\ m_H(\partial_t\mathbf{v}_h^n, \boldsymbol{\chi}_h) + a(\mathbf{u}_h^{n+1/2}, \boldsymbol{\chi}_h) &= (\mathbf{f}^{n+1/2}, \boldsymbol{\chi}_h), \quad \boldsymbol{\chi}_h \in \mathbf{S}_h, \quad 0 \le n \le J-1, \\ \partial_t\mathbf{u}_h^n &= \mathbf{v}_h^{n+1/2}.\end{aligned} \tag{5.1}$$

In order to prove error estimates in terms of h and τ, we need a few lemmas for preparation. We set $\boldsymbol{\theta}_h := R_h\mathbf{u} - \mathbf{w}_h$ and further write

$$\boldsymbol{\xi}_h^n := \mathbf{u}_h^n - \mathbf{w}_h^n, \quad 0 \le n \le J, \tag{5.2}$$

$$\mathbf{p}_h^n := \mathbf{v}_h^n - \dot{\mathbf{w}}_h^n, \quad 0 \le n \le J. \tag{5.3}$$

LEMMA 5.1. *If we define sequences* $(\boldsymbol{\rho}_h^n)_{n=0}^{J-1}$, $(\boldsymbol{\pi}_h^n)_{n=0}^{J-1}$ *and* $(\boldsymbol{\epsilon}_h^n)_{n=0}^{J-1}$ *in* $\mathbf{S}_h$ *by*

$$\boldsymbol{\rho}_h^n := R_h \ddot{\mathbf{u}}^{n+1/2} - \partial_t \left(R_h \dot{\mathbf{u}}^n \right), \tag{5.4}$$

$$\boldsymbol{\pi}_h^n := \partial_t (R_h \mathbf{u}^n) - R_h \dot{\mathbf{u}}^{n+1/2}, \tag{5.5}$$

$$\boldsymbol{\epsilon}_h^0 := \partial_t \boldsymbol{\theta}_h^0 - \boldsymbol{\pi}_h^0 + \frac{\tau}{2} \boldsymbol{\rho}_h^0, \tag{5.6}$$

$$\boldsymbol{\epsilon}_h^n := \partial_t \boldsymbol{\theta}_h^n - \boldsymbol{\pi}_h^n + \frac{\tau}{2} \left(\sum_{k=0}^{n} \boldsymbol{\rho}_h^k + \sum_{k=0}^{n-1} \boldsymbol{\rho}_h^k \right), \quad 1 \le n \le J-1, \tag{5.7}$$

the following estimate holds with a constant $C(T)$ *depending on* T*:*

$$\max_{0 \le n \le J} \|\boldsymbol{\xi}_h^n\|_{0,\Omega_N}^2 \le C(T) \left(\tau \sum_{n=0}^{J-1} \|\boldsymbol{\epsilon}_h^n\|_{0,\Omega_N}^2 + \|\boldsymbol{\xi}_h^0\|_{0,\Omega_N}^2 \right). \tag{5.8}$$

Proof. Using (3.5) instead of the norm equivalence with respect to $\|\cdot\|_{0,\Omega}$, the proof is done as in [1, Lemma 5.1]. □

In order to prove the discrete equivalent of Theorem 4.9, we introduce another sequence $(\boldsymbol{\nu}_h^n)_{n=0}^{J-2}$ by

$$\boldsymbol{\nu}_h^n := \partial_t (\partial_t \boldsymbol{\theta}_h^n) - \partial_t \boldsymbol{\pi}_h^n + \boldsymbol{\rho}_h^{n+1/2}. \tag{5.9}$$

LEMMA 5.2. *The following estimate holds:*

$$\begin{aligned} &\max_{0 \le n \le (J-1)} \left(\|\partial_t \boldsymbol{\xi}_h^n\|_{0,\Omega_N}^2 + \|\boldsymbol{\xi}_h^{n+1/2}\|_{1,\Omega}^2 \right) \\ &\le C(T) \left(\tau \sum_{n=0}^{J-2} \|\boldsymbol{\nu}_h^n\|_{0,\Omega_N}^2 + \|\partial_t \boldsymbol{\xi}_h^0\|_{0,\Omega_N}^2 + \|\boldsymbol{\xi}_h^{1/2}\|_{1,\Omega}^2 \right). \end{aligned} \tag{5.10}$$

Proof. Using the definitions, we obtain (for details see [1, proof of Lemma 5.1]):

$$m_H \left(\partial_t (\partial_t \boldsymbol{\xi}_h^n), \boldsymbol{\chi}_h \right) + \frac{1}{2} a \left(\boldsymbol{\xi}_h^{n+1/2} + \boldsymbol{\xi}_h^{n+3/2}, \boldsymbol{\chi}_h \right) = m_H \left(\boldsymbol{\nu}_h^n, \boldsymbol{\chi}_h \right). \tag{5.11}$$

We choose the test function $\boldsymbol{\chi}_h = \partial_t \boldsymbol{\xi}_h^{n+1/2}$ in (5.11) and arrive at

$$\begin{aligned} &|\partial_t \boldsymbol{\xi}_h^{n+1}|_H^2 - |\partial_t \boldsymbol{\xi}_h^n|_H^2 + a \left(\boldsymbol{\xi}_h^{n+3/2}, \boldsymbol{\xi}_h^{n+3/2} \right) - a \left(\boldsymbol{\xi}_h^{n+1/2}, \boldsymbol{\xi}_h^{n+1/2} \right) \\ &\le 2\tau \, m_H \left(\boldsymbol{\nu}_h^n, \partial_t \boldsymbol{\xi}_h^{n+1/2} \right). \end{aligned}$$

Summing from 0 to $(l-1)$ for $1 \le l \le (J-1)$ and the use of Schwarz's inequality gives for any $\alpha > 0$:

$$\begin{aligned} &|\partial_t \boldsymbol{\xi}_h^l|_H^2 - |\partial_t \boldsymbol{\xi}_h^0|_H^2 + a \left(\boldsymbol{\xi}_h^{l+1/2}, \boldsymbol{\xi}_h^{l+1/2} \right) - a \left(\boldsymbol{\xi}_h^{1/2}, \boldsymbol{\xi}_h^{1/2} \right) \\ &\le 2\tau \left(\alpha \sum_{n=0}^{l-1} |\boldsymbol{\nu}_h^n|_H^2 + \frac{1}{\alpha} \sum_{n=0}^{l-1} \left| \partial_t \boldsymbol{\xi}_h^{n+1/2} \right|_H^2 \right). \end{aligned} \tag{5.12}$$

We set $A := \max_{0 \le n \le (J-1)} |\partial_t \boldsymbol{\xi}_h^n|_H$ which leads to

$$\sum_{n=0}^{l-1} \left| \partial_t \boldsymbol{\xi}_h^{n+1/2} \right|_H^2 \le A^2 J. \tag{5.13}$$

Choosing $\alpha = 4T$ in (5.12) and using (5.13), we get

$$|\partial_t \boldsymbol{\xi}_h^l|_H^2 + a\left(\boldsymbol{\xi}_h^{l+1/2}, \boldsymbol{\xi}_h^{l+1/2}\right) \le 8\tau T \sum_{n=0}^{J-2} |\boldsymbol{\nu}_h^n|_H^2 + \frac{1}{2}A^2 + |\partial_t \boldsymbol{\xi}_h^0|_H^2 + a\left(\boldsymbol{\xi}_h^{1/2}, \boldsymbol{\xi}_h^{1/2}\right).$$

This holds for each l with $1 \le l \le (J-1)$, hence we can take the maximum of the left hand side. Subtracting $\frac{1}{2}A^2$ on both sides as well as using the coercivity, the continuity of $a(\cdot,\cdot)$ and Remark 3.2, we finally arrive at (5.10). □

The estimation of $\sum_{n=0}^{J-1} \|\boldsymbol{\nu}_h^n\|_{0,\Omega_N}^2$ is done in the next lemma:

LEMMA 5.3. *Let $\ddot{\mathbf{u}} \in L^2((0,T), \mathbf{V}_D)$, $\frac{\partial^4 \mathbf{u}}{\partial t^4} \in L^2((0,T), \mathbf{H}_D^1(\Omega))$ and let I_H satisfy conditions $P1$ to $P3$. Then the following inequality holds:*

$$\tau \sum_{n=0}^{J-2} \|\boldsymbol{\nu}_h^n\|_{0,\Omega_N}^2 \le C\left(h\,\|\ddot{\mathbf{u}}\|_{0,2,\Omega} + (h+\tau^2)\left\|\frac{\partial^4 \mathbf{u}}{\partial t^4}\right\|_{0,1,\Omega}\right)^2. \tag{5.14}$$

Proof. By the definition of $\boldsymbol{\nu}_h^n$ (5.9) we get

$$\|\boldsymbol{\nu}_h^n\|_{0,\Omega_N}^2 \le \|\partial_t(\partial_t \boldsymbol{\theta}_h^n)\|_{0,\Omega_N}^2 + \|\boldsymbol{\rho}_h^{n+1/2} - \partial_t \boldsymbol{\pi}_h^n\|_{0,\Omega_N}^2. \tag{5.15}$$

Using Taylor expansion, we obtain for the first term on the right hand side (see [1, proof of Lemma 5.2] for details):

$$\begin{aligned}\partial_t(\partial_t \boldsymbol{\theta}_h^n) &= \frac{1}{\tau^2}\left(\boldsymbol{\theta}_h^{n+2} - 2\boldsymbol{\theta}_h^{n+1} + \boldsymbol{\theta}_h^n\right) \\ &= \frac{1}{\tau^2}\left(\int_{(n+1)\tau}^{(n+2)\tau} ((n+2)\tau - t)\ddot{\boldsymbol{\theta}}_h(t)dt + \int_{n\tau}^{(n+1)\tau} (t-n\tau)\ddot{\boldsymbol{\theta}}_h(t)dt\right).\end{aligned}$$

Hence we get

$$\|\partial_t(\partial_t \boldsymbol{\theta}_h^n)\|_{0,\Omega_N}^2 \le C\frac{1}{\tau}\int_{n\tau}^{(n+2)\tau} \left\|\ddot{\boldsymbol{\theta}}_h(t)\right\|_{0,\Omega_N}^2 dt.$$

Recalling the definition of $\boldsymbol{\theta}_h$, we conclude with Lemma 4.8 and (4.4):

$$\tau \sum_{n=0}^{J-2} \|\partial_t(\partial_t \boldsymbol{\theta}_h^n)\|_{0,\Omega_N}^2 \le Ch^2\left(\|\ddot{\mathbf{u}}\|_{0,2,\Omega}^2 + \left\|\frac{\partial^4 \mathbf{u}}{\partial t^4}\right\|_{0,1,\Omega}^2\right). \tag{5.16}$$

The second term on the right hand side of (5.15) gives with (5.4) and (5.5):

$$\boldsymbol{\rho}_h^{n+1/2} - \partial_t \boldsymbol{\pi}_h^n = R_h\left(\frac{1}{4}\ddot{\mathbf{u}}^{n+2} + \frac{1}{2}\ddot{\mathbf{u}}^{n+1} + \frac{1}{4}\ddot{\mathbf{u}}^n - \frac{1}{\tau^2}\mathbf{u}^{n+2} + \frac{2}{\tau^2}\mathbf{u}^{n+1} - \frac{1}{\tau^2}\mathbf{u}^n\right).$$

Taylor expansion at $t = (n+1)\tau$ and summation over n leads to

$$\tau \sum_{n=0}^{J-2} \|\boldsymbol{\rho}_h^{n+1/2} - \partial_t \boldsymbol{\pi}_h^n\|_{0,\Omega_N}^2 \le C\tau^4\left\|\frac{\partial^4 \mathbf{u}}{\partial t^4}\right\|_{0,1,\Omega}^2. \tag{5.17}$$

(5.16) and (5.17) conclude the proof. □

Now we are able to formulate the following:
THEOREM 5.4. *Under the assumptions of Theorem 4.9, we have*

$$\max_{0\le n\le J-1}\Big(\|\partial_t(\mathbf{u}_h^n-\mathbf{u}^n)\|_{0,\Omega_N}+\|\mathbf{u}_h^{n+1/2}-\mathbf{u}^{n+1/2}\|_{1,\Omega}\Big) \tag{5.18}$$
$$\le C\left(h\left(\sum_{s=0}^{2}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,2,\Omega}+\sum_{s=3}^{4}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,1,\Omega}\right)+\tau^2\left\|\frac{\partial^4\mathbf{u}}{\partial t^4}\right\|_{0,1,\Omega}\right).$$

Proof. With $\mathbf{u}_h^n-\mathbf{u}^n=\boldsymbol{\xi}_h^n+(\mathbf{w}_h^n-\mathbf{u}^n)$ we can estimate

$$\max_{0\le n\le J-1}\Big(\|\partial_t(\mathbf{u}_h^n-\mathbf{u}^n)\|_{0,\Omega_N}+\|\mathbf{u}_h^{n+1/2}-\mathbf{u}^{n+1/2}\|_{1,\Omega}\Big) \tag{5.19}$$
$$\le C\max_{0\le n\le J-1}\Big(\|\partial_t(\mathbf{w}_h^n-\mathbf{u}^n)\|_{0,\Omega_N}+\|\mathbf{w}_h^{n+1/2}-\mathbf{u}^{n+1/2}\|_{1,\Omega}$$
$$+\|\partial_t\boldsymbol{\xi}_h^n\|_{0,\Omega_N}+\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,\Omega}\Big).$$

For the last two terms we employ Lemmas 5.2 and 5.3 to obtain

$$\max_{0\le n\le (J-1)}\Big(\|\partial_t\boldsymbol{\xi}_h^n\|_{0,\Omega_N}^2+\|\boldsymbol{\xi}_h^{n+1/2}\|_{1,\Omega}^2\Big)\le C\Big(\|\partial_t\boldsymbol{\xi}_h^0\|_{0,\Omega_N}^2+\|\boldsymbol{\xi}_h^{1/2}\|_{1,\Omega}^2\Big)$$
$$+C\left(h\,\|\ddot{\mathbf{u}}\|_{0,2,\Omega}+(h+\tau^2)\left\|\frac{\partial^4\mathbf{u}}{\partial t^4}\right\|_{0,1,\Omega}\right)^2. \tag{5.20}$$

Only the initial terms remain to be estimated. The first one gives

$$\|\partial_t\boldsymbol{\xi}_h^0\|_{0,\Omega_N}\le\|\partial_t\boldsymbol{\xi}_h^0-\dot{\boldsymbol{\xi}}_h^{1/2}\|_{0,\Omega_N}+\|\dot{\boldsymbol{\xi}}_h^{1/2}\|_{0,\Omega_N}. \tag{5.21}$$

Here we employ again a Taylor expansion to obtain

$$\|\partial_t\boldsymbol{\xi}_h^0-\dot{\boldsymbol{\xi}}_h^{1/2}\|_{0,\Omega_N}\le\tau\int_0^\tau\|\ddot{\boldsymbol{\xi}}_h(t)\|_{0,\Omega_N}^2dt\le C\|\ddot{\boldsymbol{\xi}}_h\|_{0,0,\Omega_N}^2.$$

With this result we get with Lemmas 4.5, 4.7, 4.8:

$$\|\partial_t\boldsymbol{\xi}_h^0\|_{0,\Omega_N}\le C\Big(\|(R_h-\mathrm{Id})\ddot{\mathbf{u}}\|_{0,0,\Omega_N}+\|(R_h-\mathrm{Id})\dot{\mathbf{u}}^{1/2}\|_{0,\Omega_N}$$
$$+\|\dot{\mathbf{u}}-\dot{\mathbf{w}}_h\|_{0,0,\Omega_N}+\|\ddot{\mathbf{u}}-\ddot{\mathbf{w}}_h\|_{0,0,\Omega_N}\Big)$$
$$\le Ch\left(\sum_{s=1}^{2}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,2,\Omega}+\sum_{s=3}^{4}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,1,\Omega}\right).$$

The term $\|\partial_t(\mathbf{w}_h^n-\mathbf{u}^n)\|_{0,\Omega_N}$ in (5.19) is estimated in a similar way and gives the same upper bound.

For $\|\boldsymbol{\xi}_h^{1/2}\|_{1,\Omega}$ in (5.20) (and analogeously $\|\mathbf{w}_h^{n+1/2}-\mathbf{u}^{n+1/2}\|_{1,\Omega}$ in (5.19)) we get:

$$\|\boldsymbol{\xi}_h^{1/2}\|_{1,\Omega}\le\|(R_h-\mathrm{Id})\mathbf{u}^{1/2}\|_{1,\Omega}+\|\mathbf{u}^{1/2}-\mathbf{w}_h^{1/2}\|_{1,\Omega}$$
$$\le Ch\|\mathbf{u}^{1/2}\|_{2,\Omega}+\|\mathbf{u}-\mathbf{w}_h\|_{0,1,\Omega_N}+\|\dot{\mathbf{u}}-\dot{\mathbf{w}}_h\|_{0,1,\Omega_N}$$
$$\le Ch\left(\sum_{s=0}^{1}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,2,\Omega}+\sum_{s=2}^{3}\left\|\frac{\partial^s\mathbf{u}}{\partial t^s}\right\|_{0,1,\Omega}\right).$$

Hence the desired inequality (5.18) is shown. □

For the estimates in the $\mathbf{L}^2$-norm we again need to impose restrictions as in Theorem 4.13. We begin with bounding the right hand side of (5.8):

Lemma 5.5. *Assume that Ω is convex, $\frac{\partial^s \mathbf{u}}{\partial t^s} \in L^2((0,T), \mathbf{V}_0)$ for $1 \le s \le 4$. Let I_H satisfy conditions $P1$ to $P3$, $\mathbf{u}$ be the solution of (4.1) and $(\mathbf{u}_h^n)_{n=0}^J$ the solution of (5.1). Then we have the following estimate for the sequence $(\boldsymbol{\epsilon}_h^n)_{N=0}^J$ defined by (5.7):*

$$\tau \sum_{n=0}^{J-1} \|\boldsymbol{\epsilon}_h^n\|_{0,\Omega_N}^2 \le C \left(h^2 \sum_{s=1}^{3} \left\| \frac{\partial^s \mathbf{u}}{\partial t^s} \right\|_{0,2,\Omega} + h^2 \tau^2 \left\| \frac{\partial^4 \mathbf{u}}{\partial t^4} \right\|_{0,2,\Omega} + \tau^2 \sum_{s=3}^{4} \left\| \frac{\partial^s \mathbf{u}}{\partial t^s} \right\|_{0,0,\Omega} \right)^2 . \tag{5.22}$$

Proof. We proceed along the lines of [1, Lemma 5.2], using Lemma 4.12 and the norm equivalence (4.20). □

We can now prove the equivalent of Theorem 4.13 for the discrete case:

Theorem 5.6. *Under the conditions of Lemma 5.5 and with $\mathbf{u} \in L^2((0,T), \mathbf{V}_0)$, we obtain*

$$\max_{0 \le n \le J} \|\mathbf{u}_h^n - \mathbf{u}^n\|_{0,\Omega} \le C \left(h^2 \sum_{s=0}^{3} \left\| \frac{\partial^s \mathbf{u}}{\partial t^s} \right\|_{0,2,\Omega} + h^2 \tau^2 \left\| \frac{\partial^4 \mathbf{u}}{\partial t^4} \right\|_{0,2,\Omega} + \tau^2 \sum_{s=3}^{4} \left\| \frac{\partial^s \mathbf{u}}{\partial t^s} \right\|_{0,0,\Omega} \right) . \tag{5.23}$$

Proof. With the given initial conditions, Lemma 4.5, (5.2) and the triangle inequality, we have

$$\|\boldsymbol{\xi}_h^0\|_{0,\Omega_N} \le \|R_h \mathbf{u}_0 - \mathbf{u}_0\|_{0,\Omega_N} + \|\mathbf{u} - \mathbf{w}\|_{0,0,\Omega_N} + \|\dot{\mathbf{u}} - \dot{\mathbf{w}}\|_{0,0,\Omega_N} .$$

Using (4.4) as well as Lemmas 4.11 and 4.12, we get

$$\|\boldsymbol{\xi}_h^0\|_{0,\Omega_N} \le C h^2 \left(\sum_{s=0}^{1} \left\| \frac{\partial^s \mathbf{u}}{\partial t^s} \right\|_{0,2,\Omega} + \sum_{s=2}^{3} \left\| \frac{\partial^s \mathbf{u}}{\partial t^s} \right\|_{0,1,\Omega} \right) . \tag{5.24}$$

Combining the results of Lemma 5.1, Lemma 5.5 and (5.24), we obtain

$$\max_{0 \le n \le J} \|\boldsymbol{\xi}_h^n\|_{0,\Omega_N} \le C \left(h^2 \sum_{s=0}^{3} \left\| \frac{\partial^s \mathbf{u}}{\partial t^s} \right\|_{0,2,\Omega} + h^2 \tau^2 \left\| \frac{\partial^4 \mathbf{u}}{\partial t^4} \right\|_{0,2,\Omega} + \tau^2 \sum_{s=3}^{4} \left\| \frac{\partial^s \mathbf{u}}{\partial t^s} \right\|_{0,0,\Omega} \right) .$$

With Lemma 4.5 we finally get

$$\max_{0 \le n \le J} \|\mathbf{u}_h^n - \mathbf{u}^n\|_{0,\Omega_N} \le \max_{0 \le n \le J} \|\boldsymbol{\xi}_h^n\|_{0,\Omega_N} + \|\mathbf{u} - \mathbf{w}_h\|_{0,0,\Omega_N} + \|\dot{\mathbf{u}} - \dot{\mathbf{w}}_h\|_{0,0,\Omega_N}$$

which concludes the proof using Lemmas 4.11, 4.12 and (4.20). □

6. Numerical results. For the numerical tests, we use the finite element toolbox UG [2]. All problems are discretized in time using the midpoint rule as investigated in Section 5 which can be regarded as a HHT scheme with the parameters $2\alpha = 2\beta = \gamma = 1$ (see [21]). We remark that we do not use exactly the contact conditions (2.2) for our computations, but rather follow the approach given in [22] and fulfill the persistency condition $\sigma_n(\mathbf{u}) \frac{\partial}{\partial t}(u_n - g) = 0$ in a discrete way in order to get an energy-conserving algorithm. For the discrete version of the friction conditions

(2.3), we follow [5]. The time discretized version of problem (2.6) can be written as follows: given the discrete displacement $\mathbf{u}_h^n$ and the velocity $\mathbf{v}_h^n$ at time t_n, compute the increment of the displacement $\Delta\mathbf{u}_h^n := \mathbf{u}_h^{n+1} - \mathbf{u}_h^n$ and the velocity $\mathbf{v}_h^{n+1}$ satisfying

$$\begin{aligned} \left(\frac{2}{\tau^2}M_h + \frac{1}{2}A_h\right)\Delta\mathbf{u}_h^n &= \frac{2}{\tau}M_h\mathbf{v}_h^n - A_h\mathbf{u}_h^n - B_h\boldsymbol{\lambda}_h^{n+1/2} + \mathbf{f}_h^{n+1/2}, \\ \mathbf{v}_h^{n+1} &= -\mathbf{v}_h^n + \frac{2}{\tau}\Delta\mathbf{u}_h^n. \end{aligned} \tag{6.1}$$

For $\boldsymbol{\lambda}_h = \mathbf{0}$ this corresponds to the matrix version of system (5.1). The numerical treatment of the contact conditions is explained in more detail in [11].

The energy plots present the discrete kinetic energy at time t_n given by $\mathcal{E}_{\text{kin}}^n := \frac{1}{2}(\mathbf{v}_h^n)^T M_H \mathbf{v}_h^n$, the potential energy $\mathcal{E}_{\text{pot}}^n := \frac{1}{2}(\mathbf{u}_h^n)^T A_h \mathbf{u}_h^n$ and the total energy $\mathcal{E}_{\text{tot}}^n := \mathcal{E}_{\text{kin}}^n + \mathcal{E}_{\text{pot}}^n$. The pictures with the distorted bodies show the effective stress $\sigma_{\text{eff}} := \sum_{i,j=1}^d |\sigma_{ij} - \delta_{ij}p|^2$ with the pressure $p := \frac{1}{d}\operatorname{tr}(\sigma)$. The contact work is defined by $\mathcal{E}_{\text{cont}}^n := \sum_{k=0}^{n-1}\sum_{p\in\mathcal{C}_h} \Delta\mathbf{u}_p^k \cdot \boldsymbol{\lambda}_p^{k+1/2}$.

6.1. Linear beam in 2D. Firstly, we validate our error estimates proved in Sections 4 and 5 via numerical computation. To this end, we consider the cross section of an elastic beam of length 10 and height 1.

As material parameters for the linear elasticity law, we use Young's modulus $E = 200$, Poisson's ratio $\nu = 0.3$ and mass density $\varrho = 1.0$. We choose the right hand side $\mathbf{f}$ and Neumann boundary conditions on $\partial\Omega$ such that the exact solution is given by

$$\mathbf{u}(\mathbf{x}, t) = \big(0,\ \ 0.02\,x_1 \cdot (10 - x_1) \cdot x_2^2 \cdot \sin(2\pi t)\big)^T.$$

This leads to homogeneous Neumann boundary conditions on the bottom boundary given by $x_2 = 0$, which we define as Γ_C. In the spatial domain, we use a uniform quadrilateral grid with mesh widths from $h = \frac{1}{4}$ to $h = \frac{1}{32}$, for the time discretization we compute 30 time steps with $\tau = 10^{-5}$.

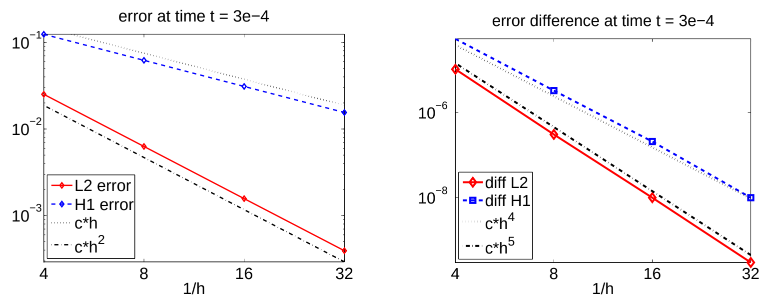


FIG. 6.1. *Error reduction with respect to the mesh size h.*

The left picture of Figure 6.1 presents the exact error at time t_{30} with respect to the mesh size h. As the computation with the linear and the constant modification (presented in Subsection 3.2 and 3.1, respectively) leads to almost exactly the same error, only one value is plotted. We obtain an error reduction of $\mathcal{O}(h^2)$ in the $\mathbf{L}^2$-norm and of $\mathcal{O}(h)$ in the $\mathbf{H}^1$-norm which corresponds to the theoretical results of Sections 4 and 5. On the right picture of Figure 6.1, the absolute value of the difference between the error for the modified and the standard computation is plotted. Here we observe numerically that the modified error converges to the standard error with a convergence order of 4 in the $\mathbf{H}^1$-norm and 5 in the $\mathbf{L}^2$-norm.

REMARK 6.1. *As we do not impose homogeneous Dirichlet boundary conditions on $\partial\Omega$, the result with respect to the $\mathbf{L}^2$-error is not covered by our proof in Section 4. This indicates that the regularity of $\mathbf{u}$ is the most important assumption for optimal error decay.*

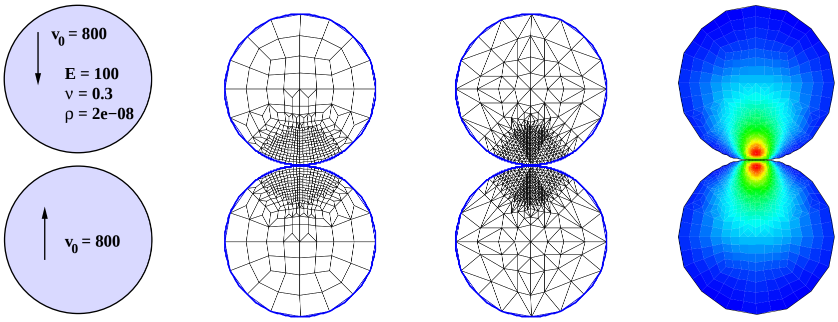


FIG. 6.2. *Initial grids and effective stresses for the two circle contact problem without friction.*

6.2. Frictionless contact in 2D. Further, we investigate the contact of two elastic discs, each with the same radius $R = 8$ and the data $E = 100$, $\nu = 0.3$, $\varrho = 2.0 \cdot 10^{-8}$. The initial distance is 0.1, and both circles move at a speed of 800 towards each other. We only consider the setting without friction as the frictional work is very small for this example. The grid is refined near the contact zone as shown in Figure 6.2, the size of the time step is $\tau = 10^{-6}$.

We assemble the mass matrix in five different ways with two distinct grids: The standard quadrature and the constant modification is applied to a quadrilateral as well as a triangular mesh; the linear modification is computed for the quadrilateral grid only. We use the quadrature formulas described in Subsections 3.1 and 3.2 on the appropriate reference elements and transform them to the actual elements. The computed energies are shown in Figure 6.3.

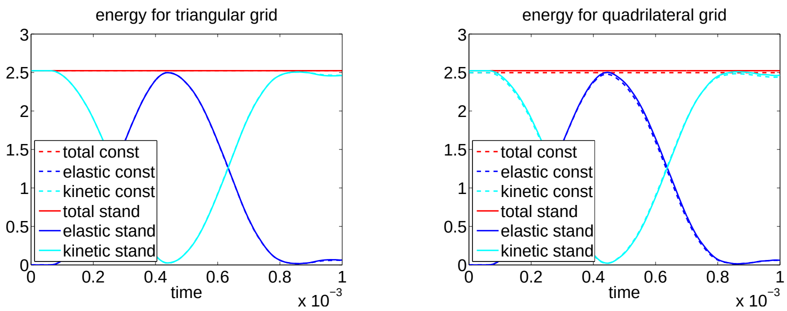


FIG. 6.3. *Energy results for the two circle contact problem without friction.*

We obtain the same total energy for each computation except for the constant modification on the quadrilateral grid. This is due to the linear determinant of the transformation that is not necessarily integrated exactly (see Remark 3.7). Hence we rather use the quadrature formula proposed in Subsection 3.2 if we are dealing with a quadrilateral mesh and curvilinear boundaries. The results for the linear modification are almost exactly the same as for the standard computation and thus only the standard value is shown in Figure 6.3.

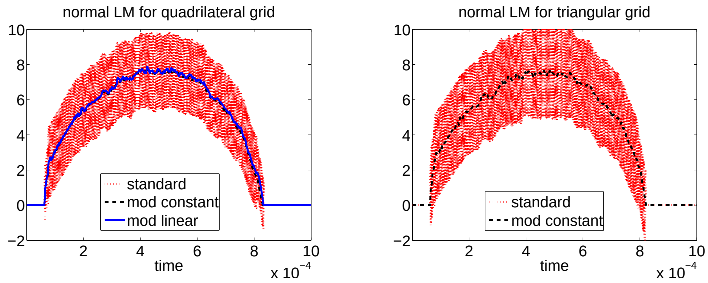


FIG. 6.4. *LM for quadrilateral and simplicial grid at the bottom slave node.*

Figure 6.4 shows the improvement in the contact stresses due to the modified mass matrix computed for the quadrilateral as well as the simplicial grid. Again we can see that the standard method exhibits unphysical oscillations whereas the modification of the mass matrix leads to smoother results. We observe that the beginning and the end of the contact period are the same for all calculations.

REMARK 6.2. *The macro-triangulation $\mathcal{T}_H$ necessary for the linear modification is easily constructed in this example.*

6.3. Dynamical two-body contact in 3D with Coulomb friction. In this subsection, we show a numerical example for a dynamical two-body contact problem with Coulomb friction ($\mathfrak{F} = 0.4$) in the three-dimensional case. We consider a ball falling onto a brick, each discretized with an unstructured tetrahedral mesh. For the ball with radius $r = 0.6$, assumed to be the slave side, we use Young's modulus $E_s = 3 \cdot 10^5$, Poisson's ratio $\nu_s = 0.3$ and mass density $\varrho_s = 1.0$. Similarly, the parameters for the brick of length 2, width 2 and height 0.3 are given by $E_m = 3 \cdot 10^4$, $\nu_m = 0.3$ and $\varrho_m = 1.0$. The bottom side of the brick is fixed, and the ball has an initial velocity of $\mathbf{v}_0 = (0, 0, -50)^T$ downwards; in addition, the ball is rotating about the x_1-direction with a rotational speed of $\omega = 50$. The time step is $\Delta t = 10^{-4}$.

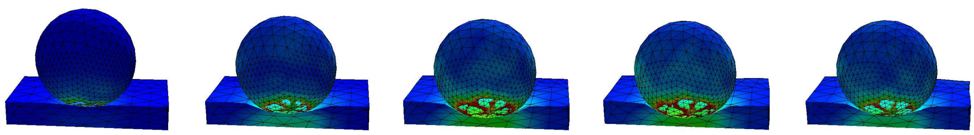


FIG. 6.5. *Effective stresses and deformation at times t_{60}, t_{100}, t_{140}, t_{180} and t_{220}.*

Figure 6.5 shows the computed effective stresses at different time steps. We remark that the calculation with the modified mass matrix using the constant modification leads to reasonable results for the evolution of the contact and the distribution of the stresses.

The computed contact stresses are further investigated in Figure 6.6, where the norm of the normal and the tangential part of the Lagrange multiplier associated to various slave nodes are shown. The nodes come into contact at different times, depending on their position on the sphere of the ball. The node is sliding if the value of $\mathfrak{F}\|\boldsymbol{\lambda}_{h\tau}\|$ is equal to $|\lambda_{hn}|$, otherwise it is sticking.

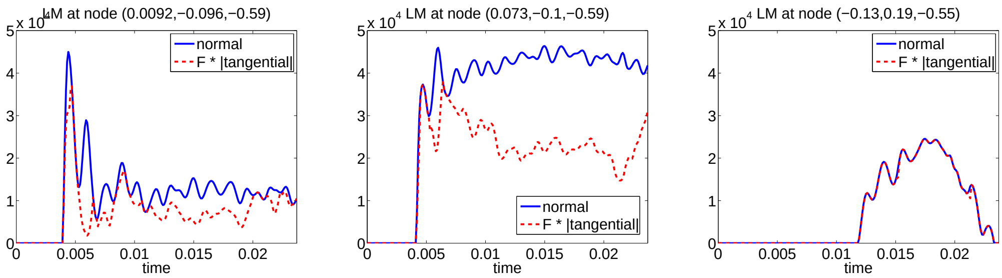


FIG. 6.6. *Values of* $|\lambda_{hn}|$ *and* $\mathfrak{F}\|\boldsymbol{\lambda}_{h\tau}\|$ *for the slave nodes* $(0.0092, -0.096, -0.59)$, $(0.073, -0.10, -0.59)$ *and* $(-0.13, 0.19, -0.55)$.

7. Conclusion. We have constructed a modified mass matrix that assigns no mass and therefore no inertia to the potential contact nodes by use of non-standard quadrature formulas. Conditions on these formulas have been obtained via the analysis of appropriate modified basis functions and a corresponding interpolation operator. Two ways of choosing these basis functions have been investigated – the first can be defined on any shape-regular grid in 2D and 3D, the second needs a mesh with a macro-element structure. Examples of suitable quadrature rules have been given.

Further, we have shown a priori estimates for the semi-discrete and for the fully discrete system with the modified mass matrix. Optimal estimates with respect to the $\mathbf{H}^1$-norm have been proved for both kinds of interpolation operator without further restrictions on the domain. We also have shown an $\mathbf{L}^2$-norm estimate under some more restrictive but common assumptions.

We have applied our modified method to different problems in 2D and 3D including friction. The computations have shown that the oscillations in the contact stresses are in fact substantially reduced, without destroying desirable properties of the basic algorithm such as energy conservation. The good error reduction properties of the method and its applicability to unstructured simplicial meshes have been demonstrated.

We conclude that combining an energy conserving scheme such as proposed by [22] with the modification of the mass matrix gives stable and smooth results without increasing the computational work.

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