Universität Stuttgart

Variational Methods for nonlinear boundary value problems in elasticity

Lectures at the Charles University Prague, Febr.07

Prof. Dr. Anna-Margarete Sändig

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Vorlesungsskript 2007/004

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Chapter 1

Introduction

This 6 hours course was delivered at the Charles University Prague in February 2007 in the framework of the European Socrates (Erasmus) Program. The goal was to give a short and compact introduction into different elastic material models and to discuss their mathematical treatment with direct variational methods. The course is divided into two sections, the first one is called *Elastic materials*, the second one *Variational methods*. Preparing these both sections I have used the well known books of P. Ciarlet *Mathematical Elasticity* and of L.C.Evans *Partial Differential Equations*. Further literature is to find in the bibliography.

The main items are:

- Basic notations from kinematics (C^1-deformation, displacement, strain)
- Stresses and the equations of equilibrium
- Elastic materials and their constututive equations (St.Venant-Kirchhoff materials, linear elastic materials, Ramberg/Osgood models, p-Laplacian, quasilinear systems of p-structure, hyperelastic materials)
- Existence of minimizers of the corresponding energy functionals (coercitivity, convexity)

Chapter 2

Elastic materials

A material body Ω deforms when it is subjected to external forces. In classical continuum mechanics a deformation is introduced as follows:

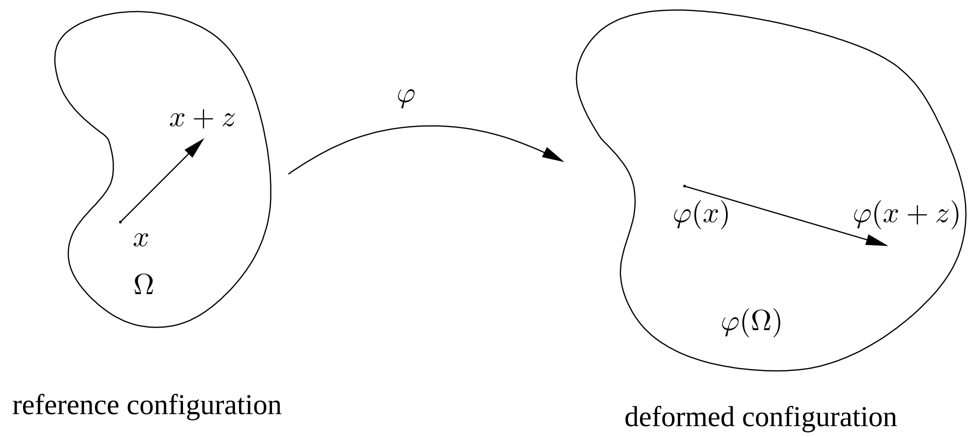
Definition 1 *Let be $\Omega \subset \mathbb{R}^3$ a piecewise smooth domain, that means Ω is an open, bounded and connected subset of $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$; $\overline{\Omega}$ denotes its closure.*
A mapping $\varphi\colon\ \overline{\Omega} \to \mathbb{R}^3$ is a C^1–deformation, if:

- *φ is differentiable in Ω, the deformation gradient*

$$D\varphi = \nabla\varphi = \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} & \frac{\partial\varphi_1}{\partial x_3} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_2} & \frac{\partial\varphi_2}{\partial x_3} \\ \frac{\partial\varphi_3}{\partial x_1} & \frac{\partial\varphi_3}{\partial x_2} & \frac{\partial\varphi_3}{\partial x_3} \end{pmatrix} \tag{2.1}$$

  is a continuous mapping. $D\varphi\colon\ \overline{\Omega} \to \mathbb{R}^{3\times 3}$.

- *φ is bijective on Ω, i.e. φ^{-1} exists on $\Omega^\varphi = \varphi(\Omega)$.*
- *φ is orientation–preserving: $\det(D\varphi(x)) > 0 \quad \forall x \in \Omega$.*



The mapping $u\colon \overline{\Omega} \to I\!R^3$, defined by the relation

$$\varphi = \mathrm{id} + u \tag{2.2}$$

is the **displacement**. Note, that $D\varphi = I + Du$.
In order to introduce the **strain tensor**, we consider the Euclidean distance between two points x and $x+z$ in the reference configuration and the distance of their image points $\varphi(x)$ and $\varphi(x+z)$:

$$\|x + z - x\|^2_{\boldsymbol{E}^3} = (z, z) = z^\top z\,.$$

$$\begin{aligned}
\|\varphi(x+z) - \varphi(x)\|^2_{\boldsymbol{E}^3} &= \|D\,\varphi(x)z + o(z)\|^2_{\boldsymbol{E}^3} \\
&= (D\,\varphi(x)z, D\,\varphi(x)z) + (o(z), D\,\varphi(x)z) + (D\,\varphi(x)z, o(z)) + (o(z), o(z))^2 \\
&= z^\top (D\,\varphi)^\top D\,\varphi\, z + o(\|z\|^2_{\boldsymbol{E}^3})\,.
\end{aligned}$$

The matrix $D\varphi^T D\varphi$ is a local measure ($\|z\|$ has to be small) for the strain in respect to the deformation φ.

Definition 2 *Let $\Omega \subset I\!R^3$ be a domain, $\varphi\colon \overline{\Omega} \to I\!R^3$ a C^1–deformation. The symmetric tensor*

$$C = (D\varphi)^T D\varphi : \overline{\Omega} \to I\!R^{3\times 3} \tag{2.3}$$

is called the right Cauchy–Green strain tensor, whereas

$$S = D\,\varphi(D\,\varphi)^\top$$

is the left Cauchy–Green strain tensor. The symmetric tensor

$$E = \frac{1}{2}(C - I) : \overline{\Omega} \to I\!R^{3\times 3} \tag{2.4}$$

is called the Green–St. Venant strain tensor.

It follows from (2.2) that:

$$\begin{aligned}
C &= I + (Du)^T + Du + (Du)^T Du \qquad (2.5)\\
E &= \frac{1}{2}((Du)^T + Du + (Du)^T Du). \qquad (2.6)
\end{aligned}$$

After this short introduction into kinematics we will come to the force balance equations in the deformed and reference configuration.

2.1 Stresses and the equations of equilibrium

Continuum mechanics for static problems is based on the following stress principle, named after the fundamental contributions of Euler [1757, 1771] and Cauchy [1823, 1827].

Deformed configuration

Axiom (stress principle of Euler and Cauchy):

Consider a body occupying a deformed configuration $\overline{\Omega}^\varphi$, subjected to applied forces which are represented by densities $f^\varphi\colon\ \Omega^\varphi \to I\!R^3$, $g^\varphi\colon\ \Gamma_1^\varphi \to I\!R^3$, $\Gamma_1^\varphi \subset \partial\Omega^\varphi$.

Then there exists a vector field

$$t^\varphi\colon\ \overline{\Omega}^\varphi \times S^3 \to I\!R^3, \tag{2.7}$$

where S^3 is the unit sphere in $I\!R^3$, such that

1. **force balance holds:**
   For any subdomain $A^\varphi \subset \overline{\Omega}^\varphi$ and any point $x^\varphi \in \Gamma_1^\varphi \cap \partial A^\varphi$, where the unit outer normal vector n^φ to $\Gamma_1^\varphi \cap \partial A^\varphi$ exists, we have

   $$\int\limits_{A^\varphi} f^\varphi(x^\varphi)\, dx^\varphi + \int\limits_{\partial A^\varphi} t^\varphi(x^\varphi, n^\varphi)\, da^\varphi = 0,$$

   and

   $$t^\varphi(x^\varphi, n^\varphi) = g^\varphi(x^\varphi).$$

2. **moment balance holds:**
   For any subdomain $A^\varphi \subset \Omega^\varphi$ it is

   $$\int\limits_{A^\varphi} x^\varphi \times f^\varphi(x^\varphi)\, dx^\varphi + \int\limits_{\partial A^\varphi} x^\varphi \times t^\varphi(x^\varphi, n^\varphi)\, da^\varphi = 0.$$

Theorem 1 *(Cauchy's theorem)*

There exists a continuously differentiable tensor field

$$T^\varphi\colon\ \varphi(\overline{\Omega}) \to I\!R^{3\times 3}$$

with

(a) $T^\varphi(x^\varphi)n = t^\varphi(x^\varphi, n)$ *for all* $x^\varphi \in \overline{\Omega}^\varphi$ *and for all* $n \in S^3$,

(b) $-div^\varphi\, T^\varphi(x^\varphi) = f^\varphi(x^\varphi)$ *for all* $x^\varphi \in \Omega^\varphi$,

(c) $T^\varphi(x^\varphi) = T^\varphi(x^\varphi)^T$ *for all* $x^\varphi \in \overline{\Omega}^\varphi$,

(d) $T^\varphi(x^\varphi)n^\varphi = g^\varphi(x^\varphi)$ *for all* $x^\varphi \in \Gamma_1^\varphi \subset \partial\Omega$.

Equations in the reference configuration

The boundary value problem in the deformed configuration reads:

$$\begin{aligned} -\mathrm{div}^{\varphi}\, T^{\varphi}(x^{\varphi}) &= f^{\varphi}(x^{\varphi}) && \text{in } \Omega^{\varphi} \\ T^{\varphi}(x^{\varphi})n^{\varphi} &= g^{\varphi}(x^{\varphi}) && \text{on } \Gamma_1^{\varphi} \subset \partial\Omega^{\varphi}. \end{aligned}$$

Our goal is to formulate the field equations in the reference configuration. We use the Piola transformation $P\colon\ \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ defined by

$$\begin{aligned} PT^{\varphi}(x^{\varphi}) &= T(x) = T^{\varphi}(x^{\varphi})\mathrm{Cof}(D\varphi) \\ &= T^{\varphi}(x^{\varphi}) \det(D\varphi)[D\varphi(x)]^{-T} \end{aligned}$$

$T(x)$ is called the **first Piola–Kirchhoff stress tensor**.

Since

$$-\mathrm{div}\, T(x) = -\det D\varphi\, \mathrm{div}^{\varphi} T^{\varphi}(x^{\varphi}),$$

we get the following boundary value problem in the reference configuration:

$$\begin{aligned} -\mathrm{div}\, T(x) &= f(x) = (\det D\varphi(x)) f^{\varphi} \circ \varphi(x) && \text{in } \Omega && (2.8) \\ T(x)n(x) &= g(x) = (\det D\varphi(x))\, |D\varphi(x)^{-T} n|\, g^{\varphi} \circ \varphi(x) && \text{on } \Gamma_1 && (2.9) \end{aligned}$$

with

$$D\varphi(x)T(x)^T = T(x)D\varphi(x)^T. \qquad (2.10)$$

Instead of the nonsymmetric first Piola–Kirchhoff stress tensor the **symmetric second Piola–Kirchhoff stress tensor** is defined as

$$\Sigma(x) = \det(D\varphi) D\varphi(x)^{-1} T^{\varphi}(x^{\varphi}) D\varphi(x)^{-T} = D\varphi(x)^{-1} T(x).$$

It satisfies the following boundary value problem in the reference configuration:

$$\begin{aligned} -\mathrm{div}\,(D\varphi\Sigma)(x) &= f(x) && \text{in } \Omega && (2.11) \\ D\varphi(x)\Sigma(x)n(x) &= g(x) && \text{on } \Gamma_1, && (2.12) \end{aligned}$$

Remarks:

In the equations (2.11) and (2.12) , the **9 unknowns** Σ and φ appear. (Note that f and g also depend on φ.)
For simplification the notation *dead loads* is used. The density $f\colon\ \Omega \to \mathbb{R}^3$, or the density $g\colon\ \Gamma_1 \to \mathbb{R}^3$ is associated with a dead load, if it is considered independent of the particular deformation φ. This is meaningful, if $\varphi(x)$ is close to the identity (small displacements). Examples for dead loads are: gravity, water- and air pressure.

Until now, no material properties are used!

2.2 Elastic materials and their constitutive equations

Because the three equations of equilibrium are valid for each macroscopic continuum (gas, liquid, solid), the missing six equations should give informations about the constituting materials. Elastic materials are characterized by the following property: The deformation vanishes instantaneously, when the forces are removed. This can be mathematically described with the help of a response function.

Definition 3 *A material occupying a domain $\Omega \subset I\!R^3$ is elastic, if there exists a mapping*

$$R\colon \overline{\Omega} \times I\!R_+^{3\times 3} \to I\!R^{3\times 3},$$

such that in any deformed configuration (that means for every C^1–deformation $\varphi\colon \overline{\Omega} \to I\!R^3$) the Cauchy stress tensor can be expressed on the reference configuration:

$$T^{\varphi}(\varphi(x)) = R(x, D\varphi(x)) \qquad \forall x \in \overline{\Omega}. \tag{2.13}$$

The mapping R is called response function *for the Cauchy stress tensor, the equation (2.13) is called the* constitutive equation *of the material.*

An elastic material is isotropic, if the response function is independent of rotations. It is objective (frame indifferent), if the Cauchy stress vector is rotated by the same matrix as the configuration.

Theorem 2 *(Rivlin–Ericksen representation theorem, 1955)*
A response function of an elastic material is objective and isotropic if and only if

$$R(x, F) = \tilde{R}(x, FF^T) = \sum_{i=0}^{2} \beta_i(x, I_S, II_S, III_S) S^i \tag{2.14}$$

for all $F \in R_+^{3\times 3}$. Here, $S = FF^T$, $I_S = tr\ S, II_S = tr\ Cof\ S$, $III_S = \det S$ are the invariants of S. We remind that $S = D\varphi D\varphi^T$ is the left Cauchy–Green strain tensor.

It follows the constitutive relation:

Theorem 3 *(stress-strain relation)*
Let $\Omega \subset I\!R^3$ be a domain occupied by an isotropic and objective elastic material. Then there is a relation between the second Piola–Kirchhoff stress tensor Σ and the right Cauchy–Green strain tensor $C = (\nabla\varphi)^T \nabla\varphi$.

$$\Sigma(x) = \sum_{i=0}^{2} \gamma_i(x, I_C, II_C, III_C)[C(x)]^i. \tag{2.15}$$

Here, I_C, II_C, III_C are the invariants of the matrix $C(x)$, $\gamma_i\colon \overline{\Omega} \times I\!R^3 \to I\!R$.

The constitutive equations near the reference configuration

Now, we study the situation that we are close to the reference configuration. We have

$$\Sigma(x) = \tilde{\Sigma}(x, C) = \tilde{\Sigma}(x, I + 2E),$$

where $E = \frac{1}{2}(C - I)$ is the Green strain tensor, $C = D\varphi^T D\varphi$.
We consider the mapping

$$f(E) = \tilde{\Sigma}(x, I + 2E) = \sum_{i=0}^{2} \gamma_i(x, I_C, II_C, III_C)(I + 2E)^i \tag{2.16}$$

and take the Taylor expansion (Fréchet derivative) in the point I for small E and fixed x.

Theorem 4 *Let an elastic material with a objective and isotropic response function at a point $x \in \overline{\Omega}$ be given. Assume that the coefficients γ_i of the representation (2.16) are continuously differentiable. Then functions $p, \lambda, \mu \colon \overline{\Omega} \to \mathbb{R}$ exist, such that*

$$\Sigma(x) = \tilde{\Sigma}(x, C) = -p(x)I + \lambda(x)(tr\, E(x))I + 2\mu(x)E(x) + o(E, x). \tag{2.17}$$

Corollary:

If the reference configuration is in natural state, then $-p(x) = 0$ and

$$\Sigma(x) = \tilde{\Sigma}(x, C) = \lambda(x)\ \mathrm{tr}E(x)I + 2\mu(x)E(x) + o(E(x)).$$

If the material in addition is homogeneous, then $\lambda(x) = \lambda$ and $\mu(x) = \mu$ are constants. In this case they are called **Lamé constants**.

Definition 4 *(St. Venant–Kirchhoff materials)*
An isotropic and objective elastic material is a St. Venant–Kirchhoff material, if

$$\Sigma(x) = \hat{\Sigma}(x, D\varphi) = \tilde{\Sigma}(x, C) = \tilde{\Sigma}(x, 2E + I) = \lambda tr\, E(x)I + 2\mu E(x), \tag{2.18}$$

where λ and μ are constants.

The corresponding nonlinear boundary value problem reads:

$$-\mathrm{div}\,\{(I + \nabla u(x))[\lambda \mathrm{tr}\, E(u)(x) + 2\mu E(u)(x)]\} = f(x) \tag{2.19}$$

$$u(x) = 0 \qquad \text{for } x \in \Gamma_0 \tag{2.20}$$

$$(I + Du(x))[\lambda \mathrm{tr}\, E(u)(x) + 2\mu E(u)(x)]n(x) = g(x) \qquad \text{for } x \in \Gamma_1 \tag{2.21}$$

for given f and g.

Linearization. Instead of the nonlinear strain tensor E we consider the linearized strain tensor $\varepsilon(u) = \frac{1}{2}(Du^T + Du)$ and the linear stress-strain relation (Hooke's law)

$$\sigma(\varepsilon) = \lambda \mathrm{tr}\, \varepsilon I + 2\mu\varepsilon = \sigma(\varepsilon(u))$$

and get the linear Lamé equation system.

$$-\mathrm{div}\sigma = f.$$

2.3 Further models of elastic materials

We have introduced St. Venant–Kirchhoff materials which belong to the class of geometrical nonlinear elastic materials, where the strain is nonlinear with respect to Du. Besides this class physically nonlinear elastic materials are studied, that means the stress-strain relation is nonlinear, but the strain is a linear relation with respect to the deformation gradient. We start with the physically nonlinear Ramberg/Osgood mode, compare also [6].

Ramberg-Osgood model. In this model the constitutive law is described by a power-law like relationship which was first suggested by W. Ramberg and W.R. Osgood in 1943 for aluminium alloys [8]. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with Lipschitz boundary which is split into a Dirichlet part $\Gamma_D = \Gamma_0$ and a Neumann part $\Gamma_N = \Gamma_1$. The field equations for the determination of the displacement field $u : \Omega \to \mathbb{R}^d$ and stress field $\sigma : \Omega \to \mathbb{R}^{d\times d}_{\text{sym}}$ read as follows:

$$\operatorname{div}\sigma + f = 0 \quad \text{in } \Omega, \tag{2.22}$$

$$\varepsilon(u) - A\sigma - \frac{3\tilde{\alpha}}{2E}\left(\frac{\sigma_e}{\sigma_y}\right)^{q-2}\sigma^D = 0 \quad \text{in } \Omega, \tag{2.23}$$

$$u = g \quad \text{on } \Gamma_D, \tag{2.24}$$

$$\sigma n = h \quad \text{on } \Gamma_N. \tag{2.25}$$

Here, f denotes the volume force density, h the surface force density and g the displacement on Γ_D. Furthermore $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the linearised strain tensor, $\sigma^D = \sigma - \frac{1}{d}\text{tr}\sigma I$ the deviatoric part of the stress tensor and $\sigma_e = \sqrt{3/2}|\sigma^D| = \sqrt{3/2\sum_{i,j=1}^d(\sigma^D_{ij})^2}$ the von Mises effective stress. The material constants in the constitutive law (2.23) have the following meaning: $A \in \text{Lin}(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d})$ is a symmetric and positive definite elasticity tensor with $(A\sigma)_{ij} = \sum_{k,l=1}^d A_{ijkl}\sigma_{kl}$; E is the Young modulus, σ_y the yield stress and $\tilde{\alpha}$ a further material parameter. **The exponent q is called strain hardening coefficient** and describes the hardening behaviour of the material. Assuming that A corresponds to an isotropic material and that $\sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, relation (2.23) reduces to

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} + \frac{\tilde{\alpha}\sigma_y}{E}\left|\frac{\sigma_{11}}{\sigma_y}\right|^{q-1}. \tag{2.26}$$

Typical graphs for relation (2.26) are plotted in figure 2.1 for different strain hardening parameters q. If $q = 2$ then relation (2.23) describes a linear elastic material. For $q > 2$ the material is strain hardening and for $q \to \infty$ the Ramberg/Osgood model is an approximation of the linear elastic, perfect-plastic Hencky model. A mathematical proof of this assertion is given in [11, 2]. Equations (2.22)-(2.25) are the field equations of a physically nonlinear and geometrically linearised elastic material model. However, these equations are applied mainly for the description of aluminium alloys and stainless steel alloys, [9, 10], which show in reality an elasto-plastic behaviour: if a quasi-static cycle of loading and unloading is applied to these materials then in general there remains a small permanent plastic strain after the cycle is finished. This phenomenon cannot be correctly

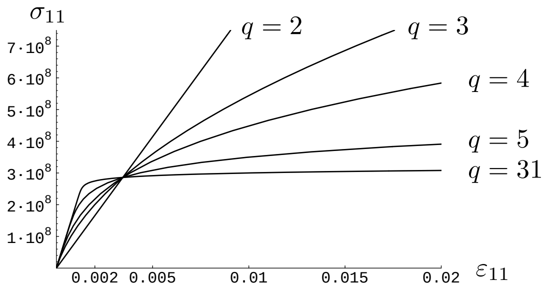


Figure 2.1: Relation (2.26) with $E = 197$GPa, $\sigma_y = 286$MPa, $\tilde{\alpha} = 1.378$. Note that $\frac{\tilde{\alpha}\sigma_y}{E} \approx 0.002$. The parameters are taken from [9].

described by the elastic Ramberg/Osgood model which predicts vanishing strains after unloading. Thus the Ramberg/Osgood model can be applied for the description of the metals mentioned above only under the assumption that the applied loading is quasi-static and monotone. Accepting this condition the terms in the constitutive relation (2.23) can be interpreted as follows:

$$\varepsilon = \underbrace{A\sigma}_{\varepsilon_{\rm el}} + \underbrace{\frac{3\tilde{\alpha}}{2E}\left(\frac{\sigma_e}{\sigma_y}\right)^{q-2}\sigma^D}_{\varepsilon_{\rm pl}} = \varepsilon_{\rm el} + \varepsilon_{\rm pl}. \tag{2.27}$$

The strain tensor ε is split into an elastic strain $\varepsilon_{\rm el}$, which depends linearly on the stresses, and into a plastic strain $\varepsilon_{\rm pl}$, which depends nonlinearly on the deviatoric part of the stresses. The material behaves nearly linear elastic if the von Mises effective stress σ_e is less than the constant σ_y. If σ_e is larger than σ_y then the plastic strains $\varepsilon_{\rm pl}$ from (2.27) dominate the strain tensor. This justifies the name "yield stress" for the constant σ_y. One should note that the considered materials have no yield plateau and therefore the yield stress σ_y is not uniquely determinable. Usually σ_y is chosen as the 0.2% proof stress $\sigma_{0.2}$, see e.g. [9]. Some typical values are listed in table 2.1 with $\tilde{\alpha} = 0.002E/\sigma_y$.

	austenitic steel alloys [9]	aluminium alloys [10], [7]
E	180-200 GPa	66-75 GPa
$\sigma_y = \sigma_{0.2}$	300-600 MPa	160-300 MPa
q	5.45-8.9	20-45

Table 2.1: Typical values of the material parameters.

Materials of p-Laplace type. In electrostatics, in modified Newton-Stokes equations, in the description of the plane Mode III case (shearing) in fracture mechanics the following boundary value problem appears: Find $u : \Omega \to \mathbb{R}$ such that:

$$\begin{aligned} -\mu\mathrm{div}\left((\kappa + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u\right) &= f \qquad \text{in } \Omega, \\ \mu(\kappa + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u \cdot \mathrm{n} &= 0 \qquad \text{on } \Gamma_N, \\ u &= u_D \qquad \text{on } \Gamma_D. \end{aligned}$$

Some relevant values of the parameters μ and p are:

Table 1:

Alloy	μ [MPa]	p
Copper Cu (annealed)	315	1.54
Brass, 70 Cu-30 Zn (annealed)	895	1.49
Stainless steel 14301 (annealed)	1275	1.45

The parameter $\kappa \geq 0$ is often introduced to stabilise numerical methods. In some papers κ is chosen as 0 or 1.

Materials of p-structure. The p-Laplacian belongs to the class of p-structures, that means to a system of quasilinear elliptic equations with growth conditions depending on p, $p \in (1, \infty)$. These models are usually introduced with the help of associate energy densities W.
Let $\overline{\Omega} \subset I\!R^d$ and assume that the function $W : I\!R^{m\times d} \to I\!R$ satisfies the following conditions:

H1 $W \in \mathcal{C}^1(I\!R^{m\times d}, I\!R) \cap \mathcal{C}^2(I\!R^{m\times d}\backslash\{0\}, I\!R)$.

H2 There exist $c_0 \in I\!R$, $c_1, c_2 > 0$ such that for every $A \in I\!R^{m\times d}$

$$c_0 + c_1|A|^p \leq W(A) \leq c_2(1 + |A|^p) \; .$$

H3 There exist $c > 0$ such that for every $A \in I\!R^{m\times d}$:

$$|DW(A)| \leq c(1 + |A|^{p-1}) \; , \quad |D^2W(A)| \leq c|A|^{p-2} \; .$$

H4 There exist $c > 0$ such that for every $A, B \in I\!R^{m\times d}$, $A \neq 0$:

$$D^2W(A)[B,B] = \sum_{k,j=1}^{m} \sum_{r,s=1}^{d} \frac{\partial^2 W(A)}{\partial A_{ks} \partial A_{jr}} B_{ks} B_{jr} \geq c|A|^{p-2}|B|^2 \; .$$

Here, we use the notation $DW(A) = \left(\frac{\partial W(A)}{\partial A_{kl}}\right)_{kl} \in I\!R^{m\times d}$ and $A : B = \sum_{k=1}^m \sum_{l=1}^d A_{kl} B_{kl}$, $|A| = \sqrt{A : A}$ for $A, B \in I\!R^{m\times d}$. Note, that A can be identified with ∇u.

The boundary-transmission problem reads find $u : \Omega \to I\!R^m$ such that

$$\operatorname{div}(DW(\nabla u)) + f = 0 \quad \text{in } \Omega \; , \tag{2.28}$$
$$u = 0 \quad \text{on } \Gamma_D \; , \tag{2.29}$$
$$DW(\nabla u) n = g \quad \text{on } \Gamma_N \; . \tag{2.30}$$

The p-Laplace equation is included here with $W(\nabla u) = \frac{\mu}{p}(\kappa + |\nabla u|^2)^{\frac{p}{2}}$ for $u : \Omega \to I\!R$.

We see, that this definition is in agreement with the definition of elasticity and the equilibrium equations, if $D_A W(\nabla u) = \sigma$. This leads us to the definition of hyperelastic materials.

2.4 Hyperelastic materials

We start with the definition:

Definition 5 *An elastic material is hyperelastic, if there exists a stored energy function (also called elastic strain energy density)*

$$W : \bar{\Omega} \times I\!R^{d\times d}_{+} \to I\!R$$

such that

$$T(x) = \hat{T}(x, F) = \frac{\partial W}{\partial F}(x, F) = D_F W(x, F) \quad \forall\, x \in \overline{\Omega}, \forall\, F \in I\!R^{d\times d}_{+} \,. \tag{2.31}$$

Here is $I\!R^{d\times d}_{+} = \{F \in I\!R^{d\times d} : \ detF > 0\}$. *The functional*

$$J^{\varphi}_{el} = \int\limits_{\Omega} W(x, D\varphi(x))\,\mathrm{d}x$$

is the elastic energy (or strain energy) for a deformation φ,

$$J(\varphi) = \int\limits_{\Omega} W(x, D\varphi(x))\,\mathrm{d}x - \left(\int\limits_{\Omega} f \cdot \varphi \,\mathrm{d}x + \int\limits_{\Gamma_1} g \cdot \varphi \,\mathrm{d}a \right)$$

is the total energy.

We remark, if we consider Du instead of $D\varphi$, then $F \in I\!R^{d\times d}$ should be taken.

Examples:

1° **Homogeneous, isotropic, linear elastic materials**
For homogeneous, isotropic, linear elastic materials the elastic strain energy density is given by

$$\begin{aligned} W(\varepsilon) &= \frac{\lambda}{2}(\mathrm{tr}\varepsilon)^2 + \mu \mathrm{tr}(\varepsilon^2) \\ &= \frac{\lambda}{2}\mathrm{tr}\varepsilon\, I : \varepsilon + \mu\, \varepsilon : \varepsilon, \end{aligned}$$

where $\varepsilon = \frac{1}{2}(Du + Du^\top) = \frac{1}{2}(D\varphi + D\varphi^\top) - I$. Setting $F = D\varphi$, $F_S = \frac{1}{2}(D\varphi + D\varphi^\top)$ we get

$$W(\varepsilon) = \hat{W}(F) = \frac{\lambda}{2}(\mathrm{tr}F - d)^2 + \mu(F_S - I) : (F_S - I).$$

We calculate the Fréchet derivative $D_F\hat{W}(F)$: Let be $H \in \mathbb{R}^{d\times d}_{+}$. Then

$$\begin{aligned} \hat{W}(F + H) - \hat{W}(F) &= \frac{\lambda}{2}\left[(\mathrm{tr}F - d + \mathrm{tr}H)^2 - (\mathrm{tr}F - d)^2\right] \\ &\quad + \mu\left[(F_S - I + H_S) : (F_S - I + H_S) - (F_S - I) : (F_S - I)\right] \\ &= \frac{\lambda}{2}\left[2(\mathrm{tr}F - d)\mathrm{tr}H\right] + 2\mu(F_S - I) : H_S + \mathrm{o}(H) \\ &= \lambda(\mathrm{tr}F - d)I : H + \mu(F_S - I) : (H + H^\top) + \mathrm{o}(H) \\ &= \lambda(\mathrm{tr}F - d)I : H + 2\mu(F_S - I) : H + \mathrm{o}(H). \end{aligned}$$

It follows

$$D_F\hat{W}(F) = \lambda \text{tr}\varepsilon\, I + 2\mu\varepsilon = D_\varepsilon W(\varepsilon) = \sigma = \underline{\underline{C}}\varepsilon,$$

where $\underline{\underline{C}}$ is the corresponding material tensor.This means that these materials are hyperelastic.

Remark: We have

$$W(\varepsilon) = \frac{1}{2}\underline{\underline{C}}\varepsilon : \varepsilon = \frac{1}{2}\sigma : \varepsilon,$$

for anisotropic homogeneous elastic materials with symmetric material tensor $\underline{\underline{C}}$ of fourth order.

2° **Geometrical nonlinear elastic materials**
The elastic energy density is in this case given by

$$W(E) = \frac{\lambda}{2}(\text{tr}E)^2 + \mu\text{tr}(E^2)$$

where $E = \frac{1}{2}(C - I) = \frac{1}{2}((D\varphi)^\top D\varphi - I)$.

Setting $F = D\varphi$ we have

$$\begin{aligned} W(E) = \hat{W}(F) &= \frac{\lambda}{2\cdot 4}(\text{tr}F^\top F - d)^2 + \frac{\mu}{4}(F^\top F - I) : (F^\top F - I) \\ = \tilde{W}(C) &= \frac{\lambda}{8}(\text{tr}(C - I))^2 + \frac{\mu}{4}(C - I) : (C - I). \end{aligned}$$

For the computation of $D_F\hat{W}(F)$ we use the following lemma:

Lemma 1
The first Piola stress tensor T is given by

$$D_F\hat{W}(F) = 2FD_C\tilde{W}(C) = FD_EW(E) = T.$$

For the second Piola stress tensor Σ it holds

$$D_EW(E) = \Sigma.$$

Proof ([3] p. 149/150)
1. step.
We show that $D_F\hat{W} = 2FD_C\tilde{W}(C)$. Let be $H \in \mathbb{R}_+^{d\times d}$.

$$\begin{aligned} \hat{W}(F + H) - \hat{W}(F) &= \tilde{W}\left((F + H)^\top (F + H)\right) - \tilde{W}(F^\top F) \\ &= \tilde{W}(F^\top F + H^\top F + F^\top H + H^\top H) - \tilde{W}(F^\top F) \\ &= D_C\tilde{W}(C) : (H^\top F + F^\top H) + \text{o}(H). \end{aligned}$$

Since $A : BC = CA^\top : B^\top = B^\top A : C$ for arbitrary matrices, we get

$$\begin{aligned} \hat{W}(F + H) - \hat{W}(F) &= F(D_C\tilde{W})^\top : H + F(D_C\tilde{W}) : H + \text{o}(H) \\ &= 2F(D_C\tilde{W}) : H + \text{o}(H). \end{aligned}$$

Here, we have used that $D_C\tilde{W}(C)$ is symmetric.

It follows that

$$D_F\hat{W}(F) = 2FD_C\tilde{W}(C).$$

Since $E = \frac{1}{2}(C - I)$, we get immediately from the chain rule

$$2D_C\tilde{W}(C) = D_E W(E).$$

2. step.
We claculate $D_E W(E)$ analogously to the first example:

$$W(E + H) - W(E) = \lambda \mathrm{tr} E\, I : H + \mu \mathrm{tr}(EH + HE) + \mathrm{o}(H).$$

Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\mathrm{tr}(A^\top B) = A : B$ for arbitrary matrices, we get $(E^\top = E)$.

$$W(E + H) - W(E) = \lambda \mathrm{tr} E\, I : H + 2\mu E : H + \mathrm{o}(H).$$

It follows

$$D_E W(E) = \lambda \mathrm{tr} E I + 2\mu E = \Sigma$$

and $FD_E W(E) = T$. Consequently, this kind of materials are hyperelastic.

3° **Materials of p-Laplace type**
The elastic strain energy $W_{el}(\cdot) : \mathbb{R}^d \to \mathbb{R}$ has the form

$$W_{el}(a) = \frac{\mu}{p}(\kappa + |a|^2)^{\frac{p}{2}}.$$

Since

$$D_a W_{el}(a) = \mu(\kappa + |a|^2)^{\frac{p-2}{p}} a,$$

we get for $a = \nabla u$

$$D_a W_{el}(\nabla u) = \mu(\kappa + |\nabla u|^2)^{\frac{p-2}{p}} \nabla u = \sigma.$$

Thus, we have shown that these materials are hyperelastic.

4° **Ramberg-Osgood Materials**
Here the situation is more complicated. The stress-strain relation is given from

$$\varepsilon = \varepsilon(\sigma) = A\sigma + \frac{3\tilde{a}}{2E}\left(\frac{\sigma_e}{\sigma_y}\right)^{q-2} \sigma^D = A\sigma + \alpha|\sigma^D|^{q-2}\sigma^D$$

and not, as usually in the form

$$\sigma = \sigma(\varepsilon).$$

Therefore, a complementary energy density is introduced

$$\begin{aligned} W_{comp} :&\mathbb{R}^{d\times d} \to \mathbb{R}, \\ &W_{comp}(\sigma) = \frac{1}{2}(A\sigma : \sigma) + \frac{\alpha}{q}|\sigma^D|^q. \end{aligned}$$

Note, that

$$D_\sigma W_{comp}(\sigma) = A\sigma + \alpha|\sigma^D|^{q-2}\sigma^D = \varepsilon.$$

The elastic strain energy density is defined as [13]

$$W_{el}(\varepsilon) = \sup_{\sigma \in \mathbb{R}^{d\times d}_{sym}} (\varepsilon : \sigma - W_{comp}(\sigma)).$$

It holds [13]

$$\begin{aligned} \sigma = D_\varepsilon W_{el}(\varepsilon) &\Leftrightarrow \varepsilon = D_\sigma W_{comp}(\sigma), \\ W_{el}(\varepsilon) + W_{comp}(\sigma) = \sigma : \varepsilon &\Leftrightarrow \varepsilon = D_\sigma W_{comp}(\sigma). \end{aligned}$$

Thus, Ramberg/Osgood Materials belong to the class of hyperelastic materials.

Chapter 3

Variational Methods

Let us start with the description of the basic idea ([5], pp.431-432). Given is a boundary value problem in an abstract form: Find $u \in M \subset X$ such that

$$A[u] = 0. \tag{3.1}$$

If $A[\cdot]$ is the *derivative (first variation)* of an appropriate energy functional $J[\cdot]$

$$A[\cdot] = J'[\cdot],$$

then the problem (3.1) reads: Find $u \in M \subset X$ such that

$$J'[u] = 0. \tag{3.2}$$

That means, u is a critical point of J. If J has a minimum $u_{min} \in M$, then (3.2) is satisfied and u_{min} solves (3.1). Thus, the problem (3.1) is reduced to assure that minimizers (or critical points) of J exist and to compute them.
If the boundary value problem describes the behaviour of a hyperelastic material (see (2.28, (2.29), (2.30), (2.31)), then the existence of such a functional $J_{el} = \int_\Omega W(x, Du)\ dx$ is ensured.

Direct methods in the calculus of variations ([4], p.4) deal directly with the investigation of the functional J. That means, we have to show that $J = J(u)$ has a mimimizer $u_{min} \in M$ and that u_{min} solves the corresponding weak Euler-Lagrange equations

$$\langle A[u], v\rangle = a(u, v) = \int_\Omega D_F W(x, Du) : Dv\, dx = 0\ \forall v \in M^0 \subset X. \tag{3.3}$$

Note, that the weak formulation (3.3) of the boundary value problem is also called variational formulation.

Indirect methods start with the weak or strong formulated boundary value problems in form of Euler Lagrange equations. Then we should find solutions which minimize the corresponding energy functionals.

We will study direct methods in what follows.

3.1 Coercivity

We investigate under which assumptions to the energy density W the functional $J[\cdot]$ has a minimizer in a subset M of a Sobolev space X. For nonlinear problems it is typically that $X = W^{1,q}(\Omega), 1 < q < \infty$. The exponent q is given by a growth condition for W.

We remind that even a smooth function $f : \mathbb{R} \to \mathbb{R}$, bounded below, needs not attain its infimum. Consider, for instance, $f(x) = \mathrm{e}^x$ oder $f(x) = \frac{1}{1+x^2}$.
It is

$$
\begin{aligned}
0 &\leq \inf_{x\in\mathbb{R}} \mathrm{e}^x = \lim_{x\to-\infty} \mathrm{e}^x = 0\,, \text{see Figure 3.1} \\
0 &\leq \inf_{x\in\mathbb{R}} \frac{1}{1+x^2} = \lim_{x\to\pm\infty} \frac{1}{1+x^2} = 0\,, \text{see Figure 3.2.}
\end{aligned}
$$

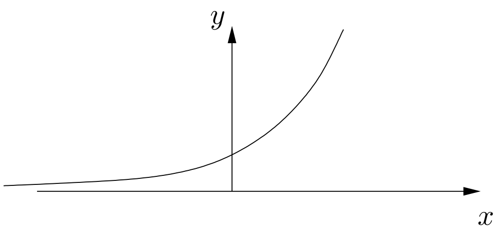


Figure 3.1: The function $f(x) = \mathrm{e}^x$

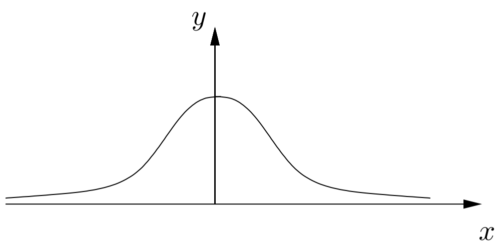


Figure 3.2: The function $f(x) = \frac{1}{1+x^2}$

Therefore, we need some growth condition which controls f if $|x| \to \infty$. Assume, for instance, $f(x) \geq c_1|x|^q - c_2$, then the infimum is realized by a finite value.

Analogously we demand that $J[u]$ grows rapidly as $|u| \to \infty$ in a sense specified below.

Definition 6 (Coercivity condition) *Let be $q \in \mathbb{R}, 1 < q < \infty$. The energy density $W = W(F,x)$ satisfies a coercitivity condition, if there exist constants $\alpha > 0, \beta \geq 0$ with*

$$W(F,x) \geq \alpha|F|^q - \beta \quad \forall\, F \in \mathbb{R}^{d\times d}, x \in \Omega. \tag{3.4}$$

Here is $|F| := (F : F)^{\frac{1}{2}}, F$ is identified with ∇u. The resulting inequality

$$J[u] = \int\limits_{\Omega} W(Du,x)\,\mathrm{d}x \geq \alpha\|Du\|^q_{L_q(\Omega)} - \beta \tag{3.5}$$

with

$$\|Du\|^q_{L_q(\Omega)} = \sum_{i=1}^{d}\sum_{j=1}^{d}\|\partial_j u_i\|^q_{L_q(\Omega)}\,,$$

is called coercivity condition to $J[\cdot]$.

It follows $J[u] \to \infty$ as $\|Du\|_{L_q(\Omega)} \to \infty$.

In agreement with the coercivity condition we introduce the set of admissible vectorfields u for Dirichlet boundary conditions:

$$M := \{u \in (W^{1,q}(\Omega))^d : u = g \in (W^{1-\frac{1}{q},q}(\partial\Omega))^d\} .$$

Examples

1° **Linear elastic, homogeneous materials**
We have

$$W(\varepsilon) = \frac{1}{2}\underline{\underline{C}}\varepsilon : \varepsilon$$

The corresponding energy functional is coercive for $q = 2$ due to the positive definiteness of $\underline{\underline{C}}$ and inequalities of Young and Korn.

2° **Materials of p-Laplace type**
Here is $F = a \in R^{d\times 1}$ and

$$W_{el}(a) = \frac{\mu}{p}(\kappa + |a|^2)^{\frac{p}{2}} \geq \alpha|a|^p.$$

3° **Ramberg-Osgood Materials and p-structures**
The associate energy functionals are coercive.

Remark: There are sharper and weaker versions of coercivity.

3.2 Convexity

Besides the coercivity of the energy density we need its convexity in order to assure a minimizer of the corresponding functional. Analogously to classical convex functions we define:

Definition 7 *The energy density W is convex if*

$$W(\tau F + (1-\tau)G, x) \leq \tau W(F,x) + (1-\tau)W(G,x) \quad \text{for all } F, G \in I\!R^{d\times d}, x \in \Omega, 0 \leq \tau \leq 1 . \tag{3.6}$$

W is strict convex, if

$$W(\tau F + (1-\tau)G, x) < \tau W(F,x) + (1-\tau)W(G,x) \quad \text{for all } F, G \in I\!R^{d\times d}, F \neq G, x \in \Omega, 0 < \tau < 1 . \tag{3.7}$$

The convexity can be characterized as follows:

Lemma 2

a) *Let be* $W : \mathbb{R}^{d\times d} \to \mathbb{R}$ *be convex. Then exists for every* $G \in \mathbb{R}^{d\times d}$ *an element* $Q \in \mathbb{R}^{d\times d}$ *such that for all* $F \in \mathbb{R}^{d\times d}$

$$W(F) \geq W(G) + Q : (F - G)$$

b) *If* W *is Frechet-differentiable with respect to* F*, then* W *is convex if and only if*

$$W(F) \geq W(G) + D_F W(G) : (F - G) \; \forall F, G \in \mathbb{R}^{d\times d}.$$

Remark: Nearly all energy densities of our examples are convex, excepted the corresponding density function for geometric nonlinear elastic materials.

Now, we are able to formulate the main theorem:

Theorem 5 (Existence of a minimizer) *Assume that the energy density function* W *is differentiable, convex and satisfies a coercitivity condition with* $q > 1$*. Then exist an element* $u \in M \subset W^{1,q}(\Omega)$ *such that*

$$J[u] = \min_{w\in M} I[w] \, .$$

Sketch of the proof([5], chapter 8.2)

1. Choice of a minimizing sequence.
   Let be $m = \inf_{w\in M} J[w]$. We choose a minimizing sequence $(u_k)_{k=1,2...}$
   $$J[u_k] \to m \, . \tag{3.8}$$
2. Boundedness of the minimizing sequence.
   It is possible to assume that $W(F,x) \geq \alpha |F|^q$, that means $\beta = 0$. If not,we consider $\tilde{W} = W + \beta$ instead of W. It is
   $$J[w] = \int_\Omega W(Dw,x)\,\mathrm{d}x \geq \alpha \int_\Omega |Dw|^q \,\mathrm{d}x. \tag{3.9}$$
   From (3.8) it follows
   $$\sup_k \|Du_k\|_{L^q(\Omega)} < \infty \, .$$
   Since $u_k \in M$, we can conclude that $\|u_k\|_{L^q(\Omega)} \leq c$, too. Therefore, the minimizing sequence $(u_k)_{k=1,2,...}$ is bounded in $W^{1,q}(\Omega)$.
3. Weak convergence of a subsequence
   A bounded sequence in a reflexive Banachspace has a weak convergent subsequence.
   $$u_{k_j} \rightharpoonup u \quad \text{in } W^{1,q}(\Omega) \, .$$
   Since for $w \in M, u_{k_j} - w \in \overset{\circ}{W}{}^{1,q}(\Omega)$ and this set is weakly closed, it follows that $u - w \in \overset{\circ}{W}{}^{1,q}(\Omega)$. Therefore, $u|_{\partial\Omega} = g$.

4. Weakly lower semicontinuity
   From the boundedness below and the convexity it follows that $J[\cdot]$ is weakly lower semicontinuous: $J[u] \leq \lim_{j\to\infty} \inf I[u_{k_j}] = m$. Thus we get

$$J[u] = m = \min_{w \in X} J[w] \,.$$

■

3.3 Uniqueness

In general, a minimizer is not uniquely defined. Let us consider an example:

$$J[u] = \int_0^1 (1 - (u')^2)^2 \,\mathrm{d}x \quad \text{in} \, \overset{\circ}{W}{}^{1,4}((0,1)) = M = X \,.$$

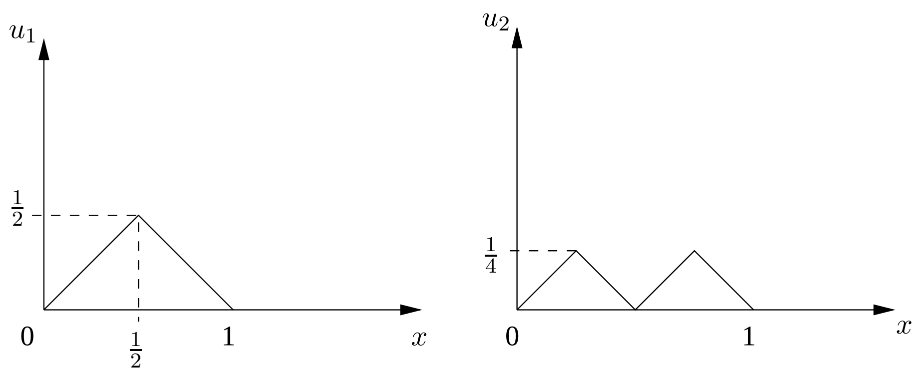


Figure 3.3: The functions u_1 und u_2

It is

$$\begin{aligned} \inf_{u \in M} J[u] &= 0 \,, \\ J[u_1] &= 0, \; u_1 \in \overset{\circ}{W}{}^{1,4}((0,1)) \quad \text{see Figure 3.3} \,, \\ J[u_2] &= 0, \; u_2 \in \overset{\circ}{W}{}^{1,4}((0,1)) \quad \text{see Figure 3.3} \,. \end{aligned}$$

Lemma 3 *If $W(\cdot, x)$ is strict convex $\forall x \in \Omega$ and $J[\cdot]$ has a minimizer in M, then the minimizer is uniquely determined.*

3.4 Euler-Lagrange equations

Until now, we have discussed, under which conditions a minimizer of the energy functional $J[\cdot] : M \to I\!R, J[u] = \int_\Omega W(Du,x)\,dx$ exists. In this section we investigate, when the corresponding weak Euler-Lagrange equations are well defined and when the minimizer is a weak solution.
We start with a formal derivation of the Euler-Lagrange equations, assuming that all quantities are well defined.

Lemma 4 *If a minimizer of the functional $J[\cdot]$ is sufficiently smooth and the energy density is differentiable, then it solves the Euler-Lagrange equations:*

$$-\mathrm{div} D_F W(Du,x) = 0\,.$$

Proof:
Let be $v \in (C_0^\infty(\Omega))^d$. We introduce the function $i = i(\tau), i : I\!R \to I\!R$

$$i(\tau) := J[u + \tau v]\,,$$

where u is a fixed minimizer of $J[\cdot]$. Then $\tau = 0$ realizes a minimum of i and therefore $i'(0) = 0$. Let us calculate $i'(\tau)$ (first variation) of $i(\tau) = \int\limits_\Omega W(Du + \tau Dv, x)\,\mathrm{d}x$. Applying the chain rule we get

$$i'(\tau) \quad = \quad \int\limits_\Omega D_F W(Du + \tau Dv, x) : Dv \,\mathrm{d}x$$

and

$$i'(0) = \int\limits_\Omega D_F W(Du,x) : Dv \,\mathrm{d}x = 0. \tag{3.10}$$

By partial integration it follows

$$\int\limits_\Omega -\mathrm{div} W_F(Du,x) \cdot v \,\mathrm{d}x = 0 \quad \forall v \in (C_0^\infty(\Omega))^d\,.$$

The fundamental lemma of the variational calculus yields

$$-\mathrm{div} W_F(Du,x) = 0\,. \tag{3.11}$$

Definition 8 *An element $u \in M = \{w \in (W^{1,q}(\Omega))^d : w = g \,auf\, \partial\Omega\}$ is weak solution of the boundary value problem*

$$\begin{aligned} -\mathrm{div} D_F W(Du,x) &= 0 \quad in\,\Omega \\ u &= g \quad auf\,\partial\Omega\,, \end{aligned} \tag{3.12}$$

if

$$\int\limits_\Omega D_F W(Du,x) : Dv \,dx = 0 \quad \forall\, v \in (\overset{\circ}{W}{}^{1,q}(\Omega))^d\,. \tag{3.13}$$

In order to guarantee, that the integral in (3.13) is well defined, we assume for the energy density function W a growth condition from above. Thus, we come to the following theorem:

Theorem 6 (Evans,p) *Assume, that for the differentiable energy function W holds: There is a constant $c > 0$, such that*

$$\begin{aligned} |W(F,x)| &\leq c(|F|^q + 1) && (3.14) \\ |D_F W(F,x)| &\leq c(|F|^{q-1} + 1) && (3.15) \\ & && (3.16) \end{aligned}$$

for all $F \in \mathbb{R}^{d\times d}, x \in \Omega$. Then a minimizer $u \in M = \{w \in (W^{1,q}(\Omega))^d : w = g \text{ on } \partial\Omega\}$ of $J[\cdot]$,

$$J[u] = \min_{w\in M} J[w]\,,$$

is a weak solution of (3.12)*, that means u satifies* (3.13)*.*

Remark: Vice versa, we cannot conclude, that every weak solution of the Euler-Lagrange equations is a minimum of the corresponding energy functional. But one can prove: If the differentiable energy density $W(F,x)$ is convex with respect to F, then a weak solution from M is a minimizer of the corresponding energy functional $J[\cdot]$.

Remark: Not all nonlinear elastic materials possesses convex energy densities, e.g. St.Venant-Kirchhoff materials (geometrical nonlinear). A weaker condition [1], the polyconvexity, ensures that a minimizer of $J[\cdot]$ exists also in this case.

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