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Boundary integral equations for a three-dimensional Brinkman flow problem

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Dedicated to Prof. Dr. Ernst P. Stephan on the occasion of his 60^{th} birthday

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Abstract

The purpose of this paper is to prove existence and uniqueness in Sobolev or Hölder spaces for a boundary value problem which describes the flow of a viscous incompressible fluid past a porous particle embedded in a second porous medium, by using the Brinkman model and potential theory. Some particular cases, which refer to Stokes flow past a porous particle, or to Brinkman's flow past a void, are also presented together with corresponding asymptotic results for the flow velocity field and the hydrodynamic force exerted on the particle.

Keywords: Brinkman model, potential theory, boundary integral equations, existence and uniqueness, Sobolev spaces, Hölder spaces.

AMS Subject classification: 76D, 76M.

1 Introduction

The problem of viscous incompressible fluid flow past porous particles embedded in different porous media has attained great interest due to its appearance in various biotechnological, chemical and geological applications, such as the treatment of transport and chemical reaction within catalyst particles, the modeling of polymer molecules as porous particles, immobilization of cells or enzymes and perfusion chromatography, the study of the flow of water or other fluids in the earthen soil, and also the flow of various kinds of fluids past porous rocks embedded in porous soil (see e.g. [2, 5, 40, 43, 44]).

The problem of creeping flow past a porous particle in terms of Brinkman's model has been firstly treated by Higdon and Kojima [12], by using a direct boundary integral

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approach. Qin and Kaloni [27] used the Stokes and Brinkman models for the Stokes flow past a porous spherical shell, and derived an explicit solution. Davis and Stone [5] studied the flow within and around a porous particle which is contained in a bed of many similar particles, and used two approximate models based on the Brinkman equations in order to describe the principal features of the flow. The first is a swarm model which considers an individual spherical particle to be a porous medium of specified permeability. The fluid and particles surrounding the individual test particle are considered as a second effective porous medium with different permeability (see also [14], [38]). The second model is a cell model in which the spherical porous particle is located in a spherical envelope of fluid. The flow within the particle is described by the Stokes equation and the flow in the spherical envelope is modeled by the Brinkman equation. For both models, Davis and Stone [5] expressed the pressure and velocity fields in terms of the fundamental solutions to the Laplacian and a modified Helmholtz equation. The Brinkman model has also been employed by Tsay and Weinbaum [40] in order to study flow through an array of fibers. The general solutions of Stokes and Brinkman equations have also been used to study viscous incompressible flow problems in the presence of porous spherical particles (see e.g. [25, 26, 29, 30]). Qin and Kaloni [28] developed a Cartesian tensor method for solving the isotropic Brinkman equation. Sano [37] and Raja Sekhar and Sano [35] discussed the problem of viscous incompressible flow around a 2D void that has a circular boundary with a slight but symmetrical deformation. The analysis corresponding to the 3D problem can be found in [34]. A more general problem, which refers to the 2D viscous incompressible fluid flow in a granular material past an arbitrary void, was studied by Raja Sekhar and Sano in [36].

It is well known that the boundary integral methods are very useful for solving boundary value problems for the Stokes system (see e.g. [13, 18, 31, 32, 42]). On the other hand, Costabel [3] showed for strongly elliptic second order systems that the corresponding boundary integral operators on a Lipschitz boundary still provide coerciveness on appropriate Sobolev–Slobodetski spaces on the boundary and regularity properties. The solvability of the Dirichlet and traction problem for the Stokes system on Lipschitz domains with L^2–boundary data was investigated by Fabes et al. in [6, 7]. They obtained existence, uniqueness and regularity results, and extended the classical results of potential theory from smooth to Lipschitz domains. Kohr and Wendland [22] presented the variational formulation and a corresponding fast multipole Galerkin boundary element method for a mixed boundary value problem of Dirichlet and Neumann type associated with the Stokes system in a Lipschitz domain in $\mathbb{R}^3$.

Recently, Kohr [15, 16, 17] used potential theory and employed indirect boundary integral methods in order to study boundary value problems of Dirichlet, Neumann or mixed type for the Stokes resolvent equation in bounded or exterior domains in $\mathbb{R}^n$ $(n \geq 2)$ (see also [18]). Since the Brinkman equation is mathematically equivalent to the Stokes resolvent equation, the potential theory for the Stokes resolvent system may be used to treat several porous media flow problems which involve the Brinkman model. In view of this property, Kohr and Raja Sekhar [19, 20] have obtained existence and uniqueness for the problem of Stokes flow in a granular material with a void, or for the problem of two-dimensional porous media flows with porous inclusions based on Brinkman's equation. Also, Kohr, Raja Sekhar and Wendland [21] have used an indirect boundary integral method for the Stokes flow past a porous body, and have obtained some asymptotic results in both cases of large and, respectively, of low permeability of the porous body.

In this paper we show existence and uniqueness in Hölder or Sobolev spaces for a boundary value problem which describes the flow of a viscous incompressible fluid past an arbitrary porous particle embedded in a second porous medium, by using the Brinkman model for both, external and internal flows. The boundary S of the porous particle is assumed to be a Lyapunov surface (i.e., $S \in C^{1,\alpha}$, $\alpha \in (0,1]$) or, more generally, a strong Lipschitz surface (i.e., $S \in C^{0,1}$). We use a boundary integral method that reduces the flow problem to the solution of a system of Fredholm integral equations that has a unique solution in some Hölder or Sobolev spaces. Certain particular cases, which describe Stokes flow past a porous particle, or Brinkman flow past a void, are also presented, together with the corresponding asymptotic results for the hydrodynamic force on the particle. In the particular case of a porous sphere embedded in a second porous medium, we recover some known asymptotic results that have been previously obtained in [20, 21]. Our approach is similar to the case of elasticity as was investigated by Costabel and Stephan in [4].

2 Formulation of the problem

Let $D_0 \subset \mathbb{R}^3$ be a bounded, simply connected domain whose boundary S is a simple, closed and connected surface of Lyapunov type, i.e., $S \in C^{1,\alpha}$, $\alpha \in (0,1]$, or, more generally, of strong Lipschitz type, i.e., $S \in C^{0,1}$, and let $D_e = \mathbb{R}^3 \setminus \overline{D}_0$ be the exterior domain with boundary S. Let us consider the following boundary value problem consisting of the Brinkman system in both domains D_e and D_0, respectively:

$$\nabla \cdot \mathbf{v}^e = 0 \quad \text{and} \tag{2.1}$$

$$-\nabla p^e + (\nabla^2 - \chi_1^2)\mathbf{v}^e = \mathbf{0} \quad \text{in } D_e; \tag{2.2}$$

$$\nabla \cdot \mathbf{v}^i = 0 \quad \text{and} \tag{2.3}$$

$$-\nabla p^i + (\nabla^2 - \chi_0^2)\mathbf{v}^i = \mathbf{0} \quad \text{in } D_0; \tag{2.4}$$

the transmission conditions:

$$\mathbf{v}^i = \mathbf{v}^e \quad \text{and} \tag{2.5}$$

$$\mathbf{t}(\mathbf{v}^i) = \mathbf{t}(\mathbf{v}^e) \quad \text{on } S; \tag{2.6}$$

and the decay conditions at infinity:

$$\begin{aligned} &(\mathbf{v}^e - \mathbf{U}^\infty)(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad (p^e - p^\infty)(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \quad \text{and} \\ &T_{j\ell}(\mathbf{v}^e - \mathbf{U}^\infty)(\mathbf{x}) = O(|\mathbf{x}|^{-2}) \quad \text{as} \quad |\mathbf{x}| \to \infty, \end{aligned} \tag{2.7}$$

where

$$T_{j\ell}(\mathbf{v}^e - \mathbf{U}^\infty) = -(p^e - p^\infty)\delta_{j\ell} + \left(\frac{\partial v_j^e}{\partial x_\ell} + \frac{\partial v_\ell^e}{\partial x_j} \right), \quad j, l = 1, 2, 3,$$

χ_0 and χ_1 are non-negative constants, $\mathbf{U}^\infty$ is a given constant velocity, and p^∞ is the corresponding pressure function given by $p^\infty = -\chi_1^2 \mathbf{U}^\infty \cdot \mathbf{x}$.

From the physical point of view the boundary value problem (2.1)-(2.7) describes the flow of a viscous incompressible fluid within and around a stationary porous particle that occupies the bounded domain D_0 and is embedded in a second porous medium. Therefore, $\mathbf{v}^e$ and p^e are the volume averaged fluid velocity and pressure fields of the flow outside the porous particle, and $\mathbf{v}^i$ and p^i are those of the inner flow, each of these fields being unknown. Both, external and internal flows are governed by the continuity and Brinkman equations, and the corresponding boundary velocity and traction fields, $(\mathbf{v}^e, \mathbf{t}(\mathbf{v}^e))$ and $(\mathbf{v}^i, \mathbf{t}(\mathbf{v}^i))$, are equal across the boundary S. In addition, the fields $\mathbf{v}^e - \mathbf{U}^\infty$ and $p^e - p^\infty$ vanish at infinity such that the asymptotic conditions (2.7) are satisfied. The constants χ_0 and χ_1 have the expressions:

$$\chi_0 = \frac{a}{\sqrt{k_0}}, \quad \chi_1 = \frac{a}{\sqrt{k_1}}, \tag{2.8}$$

where a is a characteristic length of the porous particle with permeability k_0; k_1 is the permeability of the exterior porous medium.

Note that the hydrodynamic traction $\mathbf{t}(\mathbf{v}^e) = (t_1(\mathbf{v}^e), t_2(\mathbf{v}^e), t_3(\mathbf{v}^e))$ at the boundary S is defined by the formula

$$t_j(\mathbf{v}^e)(\mathbf{x}) = T_{j\ell}(\mathbf{v}^e)(\mathbf{x}) n_\ell(\mathbf{x})\,, \tag{2.9}$$

where

$$T_{j\ell}(\mathbf{v}^e)(\mathbf{x}) = -p^e(\mathbf{x})\delta_{j\ell} + \left(\frac{\partial v_j^e(\mathbf{x})}{\partial x_\ell} + \frac{\partial v_\ell^e(\mathbf{x})}{\partial x_j}\right), \quad j, \ell = 1, 2, 3.$$

Also $T_{j\ell}(\mathbf{v}^e)$ are the components of the hydrodynamic stress tensor field $\mathbf{T}(\mathbf{v}^e)$ corresponding to the fields $\mathbf{v}^e$ and p^e, and $\mathbf{n} = (n_1, n_2, n_3)$ denotes the outward unit normal defined at almost all points $\mathbf{x} \in S$ in the case of a strong Lipschitz boundary, and, respectively, at all points of the boundary S if this is a Lyapunov surface.

In the relations (2.9) and in what follows we use Einstein's repeated-index summation convention.

3 Uniqueness of the solution to the transmission problem (2.1)-(2.7)

Next we show the uniqueness of the solution to the problem (2.1)-(2.7). Some additional results, needed in order to obtain this property, will be presented below.

3.1 Preliminary results

First, we assume that the boundary $S = \partial D_0$ is a strong Lipschitz surface in $\mathbb{R}^3$, and introduce some function spaces that are useful for further arguments.

3.1.1 Inner spaces

Let us denote by $H^s(D_0)$ and $H^s_{\rm loc}(D_e)$, $s \in \mathbb{R}$, the usual Sobolev spaces (see e.g. [11])[1]. For $r > 0$, the trace space $H^r(S)$ is well defined, and $H^{-r}(S)$ denotes its dual space with respect to the $L^2(S)$–duality,

$$\langle \mathbf{v}, \mathbf{w} \rangle := \int_S v_j(\mathbf{y}) w_j(\mathbf{y}) dS(\mathbf{y}) \ \text{ for } \mathbf{v} \in H^r(S),\ \mathbf{w} \in H^{-r}(S).$$

[1] We use the notation $H^s(S)$ instead of $\left(H^s(S)\right)^3$, and similarly for the other product Sobolev spaces.

Also let $H^1_{\rm div}(D_0)$ be the following space:

$$H^1_{\rm div}(D_0) = \big\{ \mathbf{w} \in H^1(D_0) \mid \nabla \cdot \mathbf{w} = 0 \ \text{ in } \ D_0 \big\},$$

The space $\widetilde{H}^{-1}(D_0)$ is defined by the completion of $L^2(D_0)$ with respect to the norm (see [13])

$$\|u\|_{\widetilde{H}^{-1}(D_0)} := \sup_{0 \neq \psi \in H^1(D_0)} \Big\{ |(\psi, u)_{L^2(D_0)}| / \|\psi\|_{H^1(D_0)} \Big\}.$$

This completion is denoted by $\widetilde{H}^{-1}(D_0) = (H^1(D_0))'$, which is the dual of $H^1(D_0)$.

We have the decomposition

$$\widetilde{H}^{-1}(D_0) = \widetilde{H}^{-1}_S(D_0) \oplus \widetilde{H}^{-1}_0(D_0),$$

where the spaces $\widetilde{H}^{-1}_S(D_0)$ and $\widetilde{H}^{-1}_0(D_0)$ are orthogonal in the Hilbert space $\widetilde{H}^{-1}(D_0)$, and

$$\widetilde{H}^{-1}_S(D_0) := \{ f \in \widetilde{H}^{-1}(D_0) \,|\, \text{supp f} \subset \text{S} \}.$$

The following result is known as Gagliardo's Trace Lemma (see [10]).

Lemma 3.1 *For $s \in \left(\frac{1}{2}, \frac{3}{2}\right]$ the trace mapping $\gamma_0 : H^s_{\rm loc}(\mathbb{R}^3) \to H^{s-\frac{1}{2}}(S)$ is continuous, where*

$$\gamma_0 : u \longmapsto \gamma_0 u = u\big|_S. \tag{3.1}$$

Now, let us introduce the function space $H^1(D_0, {\rm P_{St}})$, defined as follows:

$$H^1(D_0, {\rm P_{St}}) := \big\{ (\mathbf{u}, p) \in H^1_{\rm div}(D_0) \times L^2(D_0) | - \nabla^2 \mathbf{u} + \nabla p \in \widetilde{H}^{-1}_0(D_0) \big\},$$

where ${\rm P_{St}}$ is the Stokes projection operator given by

$${\rm P_{St}}(\mathbf{u}, p) := -\nabla^2 \mathbf{u} + \nabla p \ \text{ for } (\mathbf{u}, p) \in H^1_{\rm div}(D_0) \times L^2(D_0).$$

This space is equipped with the graph norm

$$\|(\mathbf{u}, p)\|_{H^1(D_0, P_{\rm St})} := \|\mathbf{u}\|_{H^1(D_0)} + \|p\|_{L^2(D_0)} + \|\nabla p - \nabla^2 \mathbf{u}\|_{\widetilde{H}^{-1}(D_0)}.$$

Then we use the following result (see [13], [22]):

Lemma 3.2 *For given fixed $(\mathbf{u}, p) \in H^1(D_0, {\rm P_{St}})$, the linear mapping*

$$\begin{aligned} \mathbf{v} \longmapsto \int_S \mathbf{v} \cdot \mathbf{t}(\mathbf{u}) dS : &= 2 \int_{D_0} E_{jk}(\mathbf{u}) E_{jk}(\mathbf{v}) d\mathbf{x} - \int_{D_0} p \nabla \cdot \mathcal{Z} \mathbf{v} d\mathbf{x} \\ &+ \int_{D_0} \mathcal{Z} \mathbf{v} \cdot (-\nabla p + \nabla^2 \mathbf{u}) d\mathbf{x} \end{aligned}$$

for $\mathbf{v} \in H^{1/2}(S)$, defines a continuous linear functional $\mathbf{t}(\mathbf{u}) \in H^{-1/2}(S)$, where $\mathcal{Z}$ is a right inverse to the trace operator γ_0 given in (3.1)*, and*

$$E_{kj}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right), \quad E_{kj}(\mathbf{v}) = \frac{1}{2} \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k} \right).$$

In addition, the linear mapping $\mathbf{t} : H^1(D_0, \mathrm{P_{St}}) \to H^{-\frac{1}{2}}(S)$,

$$H^1(D_0, \mathrm{P_{St}}) \ni (\mathbf{u}, p) \longmapsto \mathbf{t}(\mathbf{u})|_S \equiv \mathbf{t}\big(\mathbf{u}\big|_{D_0}\big) \in H^{-\frac{1}{2}}(S),$$

is continuous and the following identity holds:

$$\int_{D_0} (\nabla p - \nabla^2 \mathbf{u}) \cdot \mathbf{v} d\mathbf{x} + \int_S \mathbf{t}(\mathbf{u}) \cdot \mathbf{v} dS =$$

$$2 \int_{D_0} E_{jk}(\mathbf{u}) E_{jk}(\mathbf{v}) d\mathbf{x} - \int_{D_0} p \nabla \cdot \mathbf{v} d\mathbf{x} \quad \textit{for } (\mathbf{u}, p) \in H^1(D_0, \mathrm{P_{St}}),\ \mathbf{v} \in H^1(D_0). \tag{3.2}$$

Thus, $\mathbf{t}$ *is an extension of the classical linear mapping* $(C^2(D_0) \cap C^1(\overline{D}_0)) \times C^1(\overline{D}_0) \ni (\mathbf{u}, p) \longmapsto \mathbf{t}(\mathbf{u})|_S$ *to* $H^1(D_0, \mathrm{P_{St}})$.

3.1.2 Exterior spaces

For the exterior domain D_e we use the spaces (see [13]):

$$H^1_{\mathrm{div,loc}}(D_e) = \big\{ \mathbf{w} \in H^1_{\mathrm{loc}}(D_e) \mid \nabla \cdot \mathbf{w} = 0 \ \text{ in } \ D_e \big\},$$

$$H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}}) := \big\{ (\mathbf{u}, p) \in H^1_{\mathrm{div,loc}}(D_e) \times L^2_{\mathrm{loc}}(D_e) | -\nabla^2 \mathbf{u} + \nabla p \in \tilde{H}^{-1}_{\mathrm{comp},0}(D_e) \big\},$$

where, for $s > 0$, the space $\tilde{H}^{-s}_{\mathrm{comp}}(D_e)$ is the dual of the space $H^s_{\mathrm{loc}}(D_e)$ with respect to the bilinear pairing defined by the bilinear form

$$\langle v, w \rangle_{D_e} := \int_{D_e} v(\mathbf{y}) w(\mathbf{y}) d\mathbf{y} \ \text{ for } v \in H^s_{\mathrm{loc}}(D_e),\ w \in \tilde{H}^{-s}_{\mathrm{comp}}(D_e),$$

and

$$H^s_{\mathrm{loc}}(D_e) := \big\{ v \in \mathcal{D}'(D_e) \mid \text{for every } \ \varphi \in C_0^\infty(D_e) : \varphi v \in H^s(\mathbb{R}^3) \big\}.$$

The following result is a version of Lemma 3.2 and holds in the domain D_e (see [13]).

Lemma 3.3 *For given* $(\mathbf{u}, p) \in H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}})$ *there exists a uniquely determined boundary traction* $\mathbf{t}(\mathbf{u}) \in H^{-1/2}(S)$*, which defines a continuous extension of the classical linear mapping* $(C^2(D_e) \cap C^1(\overline{D}_e)) \times C^1(\overline{D}_e) \ni (\mathbf{u}, p) \longmapsto \mathbf{t}(\mathbf{u})|_S$ *to*

$$H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}}) \ni (\mathbf{u}, p) \longmapsto \mathbf{t}(\mathbf{u})|_S \equiv \mathbf{t}\big(\mathbf{u}\big|_{D_e}\big) \in H^{-\frac{1}{2}}(S)\,.$$

3.2 Uniqueness in the case of a strong Lipschitz boundary

Now, we are able to prove the desired uniqueness result in the case of a strong Lipschitz boundary.

Theorem 3.4 *If* S *is a strong Lipschitz surface, then the problem* (2.1)-(2.7) *has at most one solution* $((\mathbf{v}^e, p^e), (\mathbf{v}^i, p^i))$ *such that* $((\mathbf{v}^e - \mathbf{U}^\infty, p^e - p^\infty), (\mathbf{v}^i, p^i)) \in H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}}) \times H^1(D_0, \mathrm{P_{St}})$.

Proof. Let us assume that the problem (2.1)-(2.7) has two solutions and denote by $((\mathbf{v}_0^e, p_0^e), (\mathbf{v}_0^i, p_0^i))$ their difference. Consequently, the pairs $(\mathbf{v}_0^e, p_0^e)$ and $(\mathbf{v}_0^i, p_0^i)$ satisfy the following equations, transmission and far field conditions:

$$\nabla \cdot \mathbf{v}_0^e = 0, \quad -\nabla p_0^e + (\nabla^2 - \chi_1^2)\mathbf{v}_0^e = \mathbf{0} \ \text{ in } D_e, \tag{3.3}$$

$$\nabla \cdot \mathbf{v}_0^i = 0, \quad -\nabla p_0^i + (\nabla^2 - \chi_0^2)\mathbf{v}_0^i = \mathbf{0} \ \text{ in } D_0, \tag{3.4}$$

$$\mathbf{v}_0^i = \mathbf{v}_0^e, \quad \mathbf{t}(\mathbf{v}_0^i) = \mathbf{t}(\mathbf{v}_0^i) \ \text{ on } S \tag{3.5}$$

$$\mathbf{v}_0^e(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad T_{j\ell}(\mathbf{v}_0^e)(\mathbf{x}) = O(|\mathbf{x}|^{-2}) \quad \text{as } \ |\mathbf{x}| \to \infty, \tag{3.6}$$

where $\mathbf{v}_0^e\big|_S = \gamma_0\big(\mathbf{v}_0^e\big|_{D_e}\big)$ and $\mathbf{v}_0^i\big|_S = \gamma_0\big(\mathbf{v}_0^e\big|_{D_0}\big)$.

In addition, $\mathbf{t}(\mathbf{v}_0^i)\big|_S = \mathbf{t}\big(\mathbf{v}_0^i\big|_{D_0}\big) = (t_1(\mathbf{v}_0^i), t_2(\mathbf{v}_0^i), t_3(\mathbf{v}_0^i))\big|_S$ is the boundary traction in $H^{-1/2}(S)$, provided by $\mathbf{v}_0^i = (v_{0,1}^i, v_{0,2}^i, v_{0,3}^i)$ and p_0^i (see Lemma 3.2), and $\mathbf{t}(\mathbf{v}_0^e)\big|_S = \mathbf{t}_0\big(\mathbf{v}_0^e\big|_{D_0}\big) = (t_1(\mathbf{v}_0^e), t_2(\mathbf{v}_0^e), t_3(\mathbf{v}_0^e))\big|_S$ is the boundary traction provided by $\mathbf{v}_0^e = (v_{0,1}^e, v_{0,2}^e, v_{0,3}^e)$ and p_0^e (see Lemma 3.3).

Now, taking into account the identity (3.2) and the equations (3.4), we obtain the identity (see [18], p. 24-25)

$$\int_{D_0} (\chi_0^2 |\mathbf{v}_0^i|^2 + 2E_{kj}(\mathbf{v}_0^i)E_{kj}(\mathbf{v}_0^i))d\mathbf{x} = \int_S \mathbf{v}_0^i \cdot \mathbf{t}(\mathbf{v}_0^i) dS, \tag{3.7}$$

where

$$E_{kj}(\mathbf{v}_0^i) = \frac{1}{2}\left(\frac{\partial v_{0,k}^i}{\partial x_j} + \frac{\partial v_{0,j}^i}{\partial x_k}\right).$$

Next, we apply the identity (3.2) to the pair $(\mathbf{v}_0^e, p_0^e)$ on the bounded domain $D_R = D_e \cap B_R(0)$, where $B_R(0)$ is a ball with the center at the origin O, chosen inside D_0, and a large radius R such that $S \subset B_R(0)$, and then we let $R \to \infty$ and use the far field conditions (3.6), as well as the equations (3.3). In this manner we get the identity

$$\int_{D_e} (\chi_1^2 |\mathbf{v}_0^e|^2 + 2E_{kj}(\mathbf{v}_0^e)E_{kj}(\mathbf{v}_0^e))d\mathbf{x} = -\int_S \mathbf{v}_0^e \cdot \mathbf{t}(\mathbf{v}_0^e) dS. \tag{3.8}$$

In view of the transmission conditions (3.5), the identities (3.7) and (3.8) yield the equality

$$\int_{D_e} (\chi_1^2 |\mathbf{v}_0^e|^2 + 2E_{jk}(\mathbf{v}_0^e)E_{jk}(\mathbf{v}_0^e))d\mathbf{x} = -\int_{D_0} (\chi_0^2 |\mathbf{v}_0^i|^2 + 2E_{jk}(\mathbf{v}_0^i)E_{jk}(\mathbf{v}_0^i))d\mathbf{x},$$

where the left-hand side has opposite sign to the right-hand side. Therefore, both sides vanish, i.e.,

$$\int_{D_e} (\chi_1^2 |\mathbf{v}_0^e|^2 + 2E_{jk}(\mathbf{v}_0^e)E_{jk}(\mathbf{v}_0^e))d\mathbf{x} = 0, \quad \int_{D_0} (\chi_0^2 |\mathbf{v}_0^i|^2 + 2E_{jk}(\mathbf{v}_0^i)E_{jk}(\mathbf{v}_0^i))d\mathbf{x} = 0,$$

and, hence,

$$\mathbf{v}_0^e = \mathbf{0} \ \text{ in } D_e, \tag{3.9}$$

$$\mathbf{v}_0^i = \mathbf{0} \ \text{ in } D_0. \tag{3.10}$$

The second equation in (3.3) and the property (3.9) yield that $p_0^e = c_0^e$ in D_e, where c_0^e is a constant. However, p_0^e vanishes at infinity, and, hence, $c_0^e = 0$. Consequently, $\mathbf{v}_0^e$ and p_0^e vanish in the exterior domain D_e, i.e.,

$$\mathbf{v}_0^e = \mathbf{0}, \quad p_0^e = 0 \ \text{ in } D_e. \tag{3.11}$$

Similarly, from the second equation in (3.4) and the property (3.10) we get

$$p_0^i = c_0^i \ \text{ in } D_0,$$

where $c_0^i \in \mathbb{R}$.

In addition, the continuity of the linear mapping $\mathbf{t} : H^1_{\text{loc}}(D_e, \mathrm{P}_{\mathrm{St}}) \to H^{-\frac{1}{2}}(S)$ and the relations (3.11) yield $\mathbf{t}(\mathbf{v}_0^e) = \mathbf{0}$ on S, and from the boundary conditions (3.5) we find that $\mathbf{t}(\mathbf{v}_0^i) = -c_0^i\mathbf{n} = \mathbf{0}$ on S, and, hence, $c_0^i = 0$. Therefore, $\mathbf{v}_0^i$ and p_0^i vanish in the bounded domain D_0, i.e.,

$$\mathbf{v}_0^i = \mathbf{0}, \quad p_0^i = 0 \ \text{ in } D_0. \tag{3.12}$$

The relations (3.11) and (3.12) show the desired uniqueness result. □

3.3 Uniqueness in the case of a Lyapunov boundary

In the case of a Lyapunov boundary, we have the following uniqueness result:

Theorem 3.5 *If $S = \partial D_0$ is a closed Lypaunov surface, then the transmission problem* (2.1)-(2.7) *has at most one classical solution* $((\mathbf{v}^e, p^e), (\mathbf{v}^i, p^i)) \in ((C^2(D_e) \cap C^1(\overline{D}_e)) \times C^1(\overline{D}_e)) \times ((C^2(D_0) \cap C^1(\overline{D}_0)) \times C^1(\overline{D}_0))$.

Since the proof of this result is similar to that of Theorem 3.4, we omit it.

4 Surface potentials associated with the Brinkman equation

In this section we present the main properties of the surface potentials corresponding to the Brinkman equation.

4.1 The fundamental solution of the Brinkman system

Let us now refer to the Brinkman system due to the continuity and Brinkman equations:

$$\nabla \cdot \mathbf{v} = 0, \quad -\nabla p + (\nabla^2 - \chi^2)\mathbf{v} = \mathbf{0}, \tag{4.1}$$

where $\chi > 0$.

Let us denote by $\mathcal{G}^{\chi^2}$ and $\mathbf{\Pi}^{\chi^2}$ the fundamental Brinkman tensor and its associated pressure vector, which determine the fundamental solution $(\mathcal{G}^{\chi^2}, \mathbf{\Pi}^{\chi^2})$ of the Brinkman system in $\mathbb{R}^3$. The components of the tensor $\mathcal{G}^{\chi^2}$ and those of the vector $\mathbf{\Pi}^{\chi^2}$ have the expressions

$$\begin{aligned} \mathcal{G}_{jk}^{\chi^2}(\mathbf{x} - \mathbf{y}) &= \frac{\delta_{jk}}{|\mathbf{x} - \mathbf{y}|} A_1(\chi|\mathbf{x} - \mathbf{y}|) + \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x} - \mathbf{y}|^3} A_2(\chi|\mathbf{x} - \mathbf{y}|) \\ \Pi_j^{\chi^2}(\mathbf{x} - \mathbf{y}) &= 2\frac{x_j - y_j}{|\mathbf{x} - \mathbf{y}|^3}, \end{aligned} \tag{4.2}$$

where

$$A_1(z) = 2e^{-z}(1 + z^{-1} + z^{-2}) - 2z^{-2},$$
$$A_2(z) = -2e^{-z}(1 + 3z^{-1} + 3z^{-2}) + 6z^{-2} \tag{4.3}$$

(see e.g. [18], p. 81).

The components of the stress and pressure tensors $\mathbf{S}^{\chi^2}$ and $\mathbf{\Lambda}^{\chi^2}$, associated with the fundamental solution $(\mathcal{G}^{\chi^2}, \mathbf{\Pi}^{\chi^2})$, are given by

$$S_{ijk}^{\chi^2}(\mathbf{x}-\mathbf{y}) = -\Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y})\delta_{ik} + \frac{\partial \mathcal{G}_{ij}^{\chi^2}(\mathbf{x}-\mathbf{y})}{\delta x_k} + \frac{\partial \mathcal{G}_{kj}^{\chi^2}(\mathbf{x}-\mathbf{y})}{\partial x_i}$$
$$= -2\left\{\delta_{ik}\frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^3}D_1(\chi|\mathbf{x}-\mathbf{y}|) + \left(\delta_{kj}\frac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|^3} + \delta_{ij}\frac{x_k - y_k}{|\mathbf{x}-\mathbf{y}|^3}\right)D_2(\chi|\mathbf{x}-\mathbf{y}|)\right.$$
$$\left. + \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^5}D_3(\chi|\mathbf{x}-\mathbf{y}|)\right\}, \tag{4.4}$$

$$\Lambda_{ik}^{\chi^2}(\mathbf{x}-\mathbf{y}) = 2\frac{\delta_{ik}}{|\mathbf{x}-\mathbf{y}|^3}\left(\chi^2|\mathbf{x}-\mathbf{y}|^2 - 2\right) + 12\frac{(x_i - y_i)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^5}, \tag{4.5}$$

where

$$\begin{aligned} D_1(z) &= 2e^{-z}(1 + 3z^{-1} + 3z^{-2}) - 6z^{-2} + 1 \\ D_2(z) &= e^{-z}(z + 3 + 6z^{-1} + 6z^{-2}) - 6z^{-2} \\ D_3(z) &= e^{-z}(-2z - 12 - 30z^{-1} - 30z^{-2}) + 30z^{-2} \end{aligned} \tag{4.6}$$

(see e.g. [18] p. 82, 190).

The expressions (4.3) and (4.6) provide in particular the kernel's behaviour for $z = \chi|\mathbf{y}-\mathbf{x}| \to \infty$.

4.2 Potential theory for the Brinkman system

Let us now assume that $\mathbf{g} \in H^{-1/2+\sigma}(S)$ and $\mathbf{h} \in H^{1/2+\sigma}(S)$, where $\sigma \in \mathbb{R}$ with $|\sigma| \leq 1/2$.

The *single-* and *double-layer potentials*, $\mathbf{V}_{\chi^2}(\cdot, \mathbf{g})$ and $\mathbf{W}_{\chi^2}(\cdot, \mathbf{h})$, associated with the Brinkman system and having the densities $\mathbf{g}$ and $\mathbf{h}$ respectively, are defined for each $\mathbf{x} \in \mathbb{R}^3 \setminus S$ by

$$(\mathbf{V}_{\chi^2})_k(\mathbf{x}, \mathbf{g}) = \frac{1}{8\pi}\int_S \mathcal{G}_{kj}^{\chi^2}(\mathbf{x}-\mathbf{y})g_j(\mathbf{y})dS(\mathbf{y}) \tag{4.7}$$

$$(\mathbf{W}_{\chi^2})_k(\mathbf{x}, \mathbf{h}) = \frac{1}{8\pi}\int_S S_{jk\ell}^{\chi^2}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}), \quad k = 1, 2, 3. \tag{4.8}$$

In addition, with the pressure terms $P_{\chi^2}^s(\cdot, \mathbf{g})$ and $P_{\chi^2}^d(\cdot, \mathbf{h})$ given for each $\mathbf{x} \in \mathbb{R}^3 \setminus S$ by

$$P_{\chi^2}^s(\mathbf{x}, \mathbf{g}) = \frac{1}{8\pi}\int_S \Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y})g_j(\mathbf{y})dS(\mathbf{y}), \tag{4.9}$$

$$P_{\chi^2}^d(\mathbf{x}, \mathbf{h}) = \frac{1}{8\pi}\int_S \Lambda_{j\ell}^{\chi^2}(\mathbf{x}-\mathbf{y})n_\ell(\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}), \tag{4.10}$$

the pairs $(\mathbf{V}_{\chi^2}(\cdot, \mathbf{g}), P_{\chi^2}^s(\cdot, \mathbf{g}))$ and $(\mathbf{W}_{\chi^2}(\cdot, \mathbf{h}), P_{\chi^2}^d(\cdot, \mathbf{h}))$ will satisfy the Brinkman system of equations (4.1) in each of the domains D_e and D_0.

Also let $\mathbf{H}_{\chi^2}(\cdot,\mathbf{g})$ be the normal stress due to the single-layer potential $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$ and having the components defined in a neighborhood $U\subset\mathbb{R}^3$ of S by the relations

$$(\mathbf{H}_{\chi^2})_k(\mathbf{x},\mathbf{g})=(T_{k\ell}(\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})))(\mathbf{x})n_\ell(\tilde{\mathbf{x}})\quad \text{for }\ \mathbf{x}\in U\setminus S,\quad k=1,2,3,$$

where $\tilde{\mathbf{x}}$ denotes the orthogonal projection of $\mathbf{x}\in U$ onto S. Note that

$$(\mathbf{H}_{\chi^2})_k(\mathbf{x},\mathbf{g})=\frac{1}{8\pi}\int_S S^{\chi^2}_{kj\ell}(\mathbf{x}-\mathbf{y})n_\ell(\tilde{\mathbf{x}})g_j(\mathbf{y})dS(\mathbf{y})\quad \text{for } \mathbf{x}\in U\setminus S, \tag{4.11}$$

as follows from (4.7) and (4.9).

Next, we denote by $\mathbf{K}^{\chi^2}(\mathbf{y},\mathbf{x})$ and $\mathbf{D}^{\chi^2}(\mathbf{x},\mathbf{y})$ the kernels which define the double-layer potential $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$ and its normal stress $\mathbf{H}_{\chi^2}(\cdot,\mathbf{g})$. These kernels have the components

$$K^{\chi^2}_{jk}(\mathbf{y},\mathbf{x})=S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y}),\quad D^{\chi^2}_{jk}(\mathbf{x},\mathbf{y})=S^{\chi^2}_{kj\ell}(\mathbf{x}-\mathbf{y})n_\ell(\tilde{\mathbf{x}}),\quad j,k=1,2,3.$$

For further arguments, we decompose the Brinkman tensor $\mathcal{G}^{\chi^2}$ and its corresponding stress tensor $\mathbf{S}^{\chi^2}$ as follows:

$$\begin{gathered}\mathcal{G}^{\chi^2}_{kj}(\mathbf{x}-\mathbf{y})=\mathcal{G}_{kj}(\mathbf{x}-\mathbf{y})+\mathcal{G}^{\chi^2,0}_{kj}(\mathbf{x}-\mathbf{y}),\\ S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})=S_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})+S^{\chi^2,0}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y}),\end{gathered}$$

where $\mathcal{G}$ and $\mathbf{S}$ are the fundamental and stress tensors for the Stokes system, given by the formulas (4.2)-(4.6) with $\chi=0$. In addition, the matrix kernel $\mathcal{G}^{\chi^2,0}$ with the components $\mathcal{G}^{\chi^2,0}_{kj}$ is continuous, and the matrix kernel $\mathbf{S}^{\chi^2,0}\mathbf{n}$ with the components $S^{\chi^2,0}_{jk\ell}n_\ell$ is bounded. Consequently, the regularity of $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$, $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$ and $\mathbf{H}_{\chi^2}(\cdot,\mathbf{g})$ is provided by the tensors $\mathcal{G}$ and $\mathbf{S}$ for the Stokes system. In view of this property, one obtains the following result (see e.g. [18] p. 201, [22]):

Theorem 4.1 *Let us assume that $\mathbf{g}\in H^{-1/2+\sigma}(S)$ and $\mathbf{h}\in H^{1/2+\sigma}(S)$, where $\sigma\in\mathbb{R}$, $|\sigma|\le 1/2$, and define the surface potentials $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$, $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$ and $\mathbf{H}_{\chi^2}(\cdot,\mathbf{g})$ as in* (4.7), (4.8) *and* (4.11). *Then we have the following relations almost everywhere on S:*

$$\mathbf{V}^+_{\chi^2}(\cdot,\mathbf{g})=\mathbf{V}^-_{\chi^2}(\cdot,\mathbf{g})=\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})\quad \textit{and} \tag{4.12}$$

$$\mathbf{W}^+_{\chi^2}(\cdot,\mathbf{h})-\mathbf{W}^*_{\chi^2}(\cdot,\mathbf{h})=\mathbf{h}=\mathbf{W}^*_{\chi^2}(\cdot,\mathbf{h})-\mathbf{W}^-_{\chi^2}(\cdot,\mathbf{h})\quad \textit{in }\ H^{1/2}(S); \tag{4.13}$$

$$\mathbf{H}^+_{\chi^2}(\cdot,\mathbf{g})-\mathbf{H}^*_{\chi^2}(\cdot,\mathbf{g})=-\mathbf{g}=\mathbf{H}^*_{\chi^2}(\mathbf{x}_0,\mathbf{g})-\mathbf{H}^-_{\chi^2}(\cdot,\mathbf{g})\quad \textit{in }\ H^{-1/2}(S). \tag{4.14}$$

In addition, in $H^{-1/2}(S)$ there exist the limiting values of the boundary traction due to the double-layer potential $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$ on both sides of S, $\mathbf{T}^+(\mathbf{W}_{\chi^2}(\mathbf{h}))$ and $\mathbf{T}^-(\mathbf{W}_{\chi^2}(\mathbf{h}))$, and they are equal, i.e.,

$$\mathbf{T}^+(\mathbf{W}_{\chi^2}(\mathbf{h}))=\mathbf{T}^-(\mathbf{W}_{\chi^2}(\mathbf{h}))\equiv\mathbf{T}(\mathbf{W}_{\chi^2}(\mathbf{h}))\quad \textit{in }\ H^{-1/2}(S). \tag{4.15}$$

Note that the superscripts + and − apply for the limiting values evaluated from the external side and, respectively, the internal side of S, and the symbol $*$ refers to the direct value on the boundary S. For example, we have

$$\mathbf{V}^+_{\chi^2}(\cdot,\mathbf{g})=\gamma_0\big(\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})\big|_{D_e}\big),\quad \mathbf{V}^-_{\chi^2}(\cdot,\mathbf{g})=\gamma_0\big(\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})\big|_{D_0}\big),$$

$$\mathbf{T}^{+}(\mathbf{W}_{\chi^2}(\mathbf{h})) = \mathbf{t}\big(\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})\big|_{D_e}\big), \quad \mathbf{T}^{-}(\mathbf{W}_{\chi^2}(\mathbf{h})) = \mathbf{t}\big(\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})\big|_{D_0}\big).$$

Denoting by $\mathbf{D}_{\chi^2}$ the operator given in (4.15), we have $\mathbf{D}_{\chi^2} : H^{1/2}(S) \xrightarrow{\text{cont}} H^{-1/2}(S)$, where

$$(\mathbf{D}_{\chi^2}\mathbf{h})_j(\mathbf{x}) = \text{p.f.}\int_S D^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y})h_\ell(\mathbf{y})dS(\mathbf{y}), \tag{4.16}$$

and

$$D^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y}) = -\Lambda^{\chi^2}_{\ell k}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})n_j(\mathbf{x}) + \Big(\frac{\partial}{\partial x_j}S^{\chi^2}_{\ell ik}(\mathbf{y}-\mathbf{x}) + \frac{\partial}{\partial x_i}S^{\chi^2}_{\ell jk}(\mathbf{y}-\mathbf{x})\Big)n_i(\mathbf{x})n_k(\mathbf{y}).$$

is the hypersingular kernel. The symbol p.f. denotes Hadamard's finite part integral. This operator belongs to the class of hypersingular operators.

Taking into account the continuity of the Gagliardo trace mapping $\gamma_0 : H^s_{\text{loc}}(\mathbb{R}^3) \to H^{s-\frac{1}{2}}(S)$ for $s \in \big(\frac{1}{2},\frac{3}{2}\big]$ (see Lemma 3.1), and using the notation

$$[\gamma_0 u]\big|_S := \gamma_0\big(u\big|_{D_e}\big) - \gamma_0\big(u\big|_{D_0}\big)$$

for the jump of $u \in H^s_{\text{loc}}(\mathbb{R}^3)$ across S, we may write the relations (4.12)-(4.15) on S as

$$[\gamma_0\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})]\big|_S = \mathbf{0}, \quad \mathbf{t}\big(\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})\big|_{D_e}\big) - \mathbf{t}\big(\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})\big|_{D_0}\big) = -\mathbf{g} \quad \text{for } \mathbf{g} \in H^{-1/2}(S)$$

$$[\gamma_0\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})]\big|_S = \mathbf{h}, \quad \mathbf{t}\big(\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})\big|_{D_e}\big) - \mathbf{t}\big(\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})\big|_{D_0}\big) = \mathbf{0} \quad \text{for } \mathbf{h} \in H^{1/2}(S).$$

The surface potentials $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$, $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$, $P^s_{\chi^2}(\cdot,\mathbf{g})$, $P^d_{\chi^2}(\cdot,\mathbf{h})$ decay at infinity as follows:

$$\mathbf{V}_{\chi^2}(\mathbf{x},\mathbf{g}) = \mathcal{O}(|\mathbf{x}|^{-3}), \quad \mathbf{W}_{\chi^2}(\mathbf{x},\mathbf{h}) = \mathcal{O}(|\mathbf{x}|^{-2}) \text{ as } |\mathbf{x}| \to \infty, \tag{4.17}$$

$$P^s_{\chi^2}(\mathbf{x},\mathbf{g}) = \mathcal{O}(|\mathbf{x}|^{-2}), \quad P^d_{\chi^2}(\mathbf{x},\mathbf{h}) = \mathcal{O}(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty. \tag{4.18}$$

Now, let $\sigma \in \mathbb{R}$ such that $|\sigma| \le 1/2$, and let $\mathcal{V}_{\chi^2} : H^{-1/2+\sigma}(S) \to H^{1/2+\sigma}(S)$ and $\mathbf{K}_{\chi^2} : H^{1/2+\sigma}(S) \to H^{1/2+\sigma}(S)$ be the *single-* and *double-layer integral operators* for the Brinkman system. Also let $\mathcal{V} : H^{-1/2+\sigma}(S) \to H^{1/2+\sigma}(S)$ and $\mathbf{K} : H^{1/2+\sigma}(S) \to H^{1/2+\sigma}(S)$ be the integral operators corresponding to the Stokes system. The operators $\mathcal{V}_{\chi^2}$ and $\mathbf{K}_{\chi^2}$ are defined by

$$\mathcal{V}_{\chi^2}\mathbf{g} = \mathbf{V}_{\chi^2}(\cdot,\mathbf{g}), \quad \mathbf{K}_{\chi^2}\mathbf{h} = \mathbf{W}^*_{\chi^2}(\cdot,\mathbf{h}) \quad \text{for all } \mathbf{g} \in H^{-1/2+\sigma}(S), \mathbf{h} \in H^{1/2+\sigma}(S),$$

and the operators $\mathcal{V}$ and $\mathbf{K}$ can be defined in a similar manner.

Let us introduce the notations

$$\Lambda^{\chi^2,0}_{\ell k}(\mathbf{x}-\mathbf{y}) = \Lambda^{\chi^2}_{\ell k}(\mathbf{x}-\mathbf{y}) - \Lambda_{\ell k}(\mathbf{x}-\mathbf{y}),$$

$$K^{\chi^2,0}_{jk}(\mathbf{y},\mathbf{x}) = K^{\chi^2}_{jk}(\mathbf{y},\mathbf{x}) - K_{jk}(\mathbf{y},\mathbf{x}),$$

where $\mathbf{\Lambda}^{\chi^2}$ is the pressure tensor for the Brinkman system, and $\mathbf{\Lambda}$ the pressure tensor for the Stokes system. Also, $K^{\chi^2}_{jk}(\mathbf{y},\mathbf{x})$ are the components of the matrix kernel of $\mathbf{K}_{\chi^2}$, and $K_{jk}(\mathbf{y},\mathbf{x})$ the components of the matrix kernel of $\mathbf{K}$. Then we have the following result:

Theorem 4.2 *a) Let $S = \partial D_0$ be a strong Lipschitz surface. Then:*

• *The complementary single- and double-layer integral operators have the mapping properties*

$$\mathcal{V}_{\chi^2,0} : H^{-1/2}(S) \overset{\text{comp}}{\longrightarrow} H^1(S) \overset{\text{comp}}{\hookrightarrow} H^{1/2}(S), \quad \mathbf{K}_{\chi^2,0} : H^{1/2}(S) \overset{\text{comp}}{\longrightarrow} H^1(S) \overset{\text{comp}}{\hookrightarrow} H^{1/2}(S),$$

where

$$\mathcal{V}_{\chi^2,0} = \mathcal{V}_{\chi^2} - \mathcal{V}, \quad \mathbf{K}_{\chi^2,0} = \mathbf{K}_{\chi^2} - \mathbf{K}; \tag{4.19}$$

• *The adjoint to the complementary double-layer integral operator has the mapping properties*

$$\mathbf{K}'_{\chi^2,0} : H^{-1/2}(S) \overset{\text{comp}}{\longrightarrow} L^2(S) \overset{\text{comp}}{\hookrightarrow} H^{-1/2}(S),$$

where

$$(\mathbf{K}'_{\chi^2,0}\mathbf{g})_j(\mathbf{x}) = \int_S K^{\chi^2,0}_{j\ell}(\mathbf{x},\mathbf{y}) g_\ell(\mathbf{y}) dS(\mathbf{y}); \tag{4.20}$$

• *The complementary hypersingular boundary integral operator has the mapping properties*

$$\mathbf{D}_{\chi^2,0} : H^{1/2}(S) \overset{\text{comp}}{\longrightarrow} L^2(S) \overset{\text{comp}}{\hookrightarrow} H^{-1/2}(S),$$

where

$$(\mathbf{D}_{\chi^2,0}\mathbf{h})_j(\mathbf{x}) = \int_S D^{\chi^2,0}_{j\ell}(\mathbf{x},\mathbf{y}) h_\ell(\mathbf{y}) dS(\mathbf{y}), \tag{4.21}$$

with the kernel

$$\begin{aligned} D^{\chi^2,0}_{j\ell}(\mathbf{x},\mathbf{y}) = &- \Lambda^{\chi^2,0}_{\ell k}(\mathbf{x}-\mathbf{y}) n_k(\mathbf{y}) n_j(\mathbf{x}) \\ &+ \Big(\frac{\partial}{\partial x_j} S^{\chi^2,0}_{\ell i k}(\mathbf{y}-\mathbf{x}) + \frac{\partial}{\partial x_i} S^{\chi^2,0}_{\ell j k}(\mathbf{y}-\mathbf{x})\Big) n_i(\mathbf{x}) n_k(\mathbf{y}). \end{aligned}$$

b) If $S = \partial D_0$ is a closed Lyapunov surface of class $C^{1,\alpha}$ and $\lambda \in (0,\alpha)$, $\alpha \in (0,1]$, then the linear integral operators $\mathcal{V}_{\chi^2,0} : C^\lambda(S) \to C^{1,\lambda}(S)$, $\mathbf{K}_{\chi^2,0} : C^{1,\lambda}(S) \to C^{1,\lambda}(S)$, $\mathbf{K}'_{\chi^2,0} : C^\lambda(S) \to C^\lambda(S)$ and $\mathbf{D}_{\chi^2,0} : C^{1,\lambda}(S) \to C^\lambda(S)$ are compact.

c) If S is a closed Lyapunov surface of class $C^{1,\alpha}$ and $\mathbf{\Phi} \in C^{1,\lambda}(S)$ ($\lambda \in (0,\alpha)$), or, more generally, S is a strong Lipschitz surface and $\mathbf{\Phi} \in H^{1/2}(S)$, then there exist the limiting values of the boundary traction due to the complementary double-layer potential $\mathbf{W}_{\chi^2,0}(\cdot,\mathbf{\Phi})$ almost everywhere on both sides of S, $\mathbf{T}^+(\mathbf{W}_{\chi^2,0}(\mathbf{\Phi}))$ and $\mathbf{T}^-(\mathbf{W}_{\chi^2,0}(\mathbf{\Phi}))$, and they are equal in $L^2(S)$, i.e.,

$$\mathbf{T}^+(\mathbf{W}_{\chi^2,0}(\mathbf{\Phi})) = \mathbf{T}^-(\mathbf{W}_{\chi^2,0}(\mathbf{\Phi})) \equiv \mathbf{T}(\mathbf{W}_{\chi^2,0}(\mathbf{\Phi})) \in L^2(S). \tag{4.22}$$

The proof of Theorem 4.2 will be given in Appendix.

5 Boundary integral representations of the solution

Next, we prove existence and uniqueness of the solution to the boundary value problem (2.1)-(2.7) in the case of a strong Lipschitz boundary $S = \partial D_0$. The proof of the classical solution to the same boundary value problem, but in the case of a closed Lyapunov surface $S = \partial D_0$, may be obtained in a similar manner.

• First, we assume that $\chi_1 \neq \chi_0$, where χ_1 and χ_0 are the parameters given in (2.8).

In order to prove that the boundary value problem (2.1)-(2.7) has a unique solution $((\mathbf{v}^e, p^e), (\mathbf{v}^i, p^i))$ such that $((\mathbf{v}^e - \mathbf{U}^\infty, p^e - p^\infty), (\mathbf{v}^i, p^i)) \in H^1_{\text{loc}}(D_e, \mathrm{P_{St}}) \times H^1(D_0, \mathrm{P_{St}})$, we consider the following boundary integral representations:

$$v_k^e(\mathbf{x}) = U_k^\infty + \frac{1}{8\pi}\int_S K_{jk}^{\chi_1^2}(\mathbf{y}, \mathbf{x})\phi_j(\mathbf{y})dS(\mathbf{y}) + \frac{1}{8\pi}\int_S \mathcal{G}_{kj}^{\chi_1^2}(\mathbf{x}-\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}), \quad \mathbf{x} \in D_e, \tag{5.1}$$

$$p^e(\mathbf{x}) = p^\infty(\mathbf{x}) + \frac{1}{8\pi}\int_S \Lambda_{jk}^{\chi_1^2}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\phi_j(\mathbf{y})dS(\mathbf{y}) + \frac{1}{8\pi}\int_S \Pi_j^{\chi_1^2}(\mathbf{x}-\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}),\ \mathbf{x} \in D_e, \tag{5.2}$$

and

$$v_k^i(\mathbf{x}) = \frac{1}{8\pi}\int_S K_{jk}^{\chi_0^2}(\mathbf{y}, \mathbf{x})\phi_j(\mathbf{y})dS(\mathbf{y}) + \frac{1}{8\pi}\int_S \mathcal{G}_{kj}^{\chi_0^2}(\mathbf{x}-\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}), \quad \mathbf{x} \in D_0, \tag{5.3}$$

$$p^i(\mathbf{x}) = \frac{1}{8\pi}\int_S \Lambda_{jk}^{\chi_0^2}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\phi_j(\mathbf{y})dS(\mathbf{y}) + \frac{1}{8\pi}\int_S \Pi_j^{\chi_0^2}(\mathbf{x}-\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}),\ \mathbf{x} \in D_0, \tag{5.4}$$

where $\mathbf{\Phi} = (\phi_1, \phi_2, \phi_3) \in H^{1/2}(S)$ and $\mathbf{h} = (h_1, h_2, h_3) \in H^{-1/2}(S)$ are unknown densities. Therefore, the external velocity field $\mathbf{v}^e - \mathbf{U}^\infty$ is sought as a combination of double- and single-layer potentials, $\mathbf{W}_{\chi_1^2}(\cdot, \mathbf{\Phi})$ and $\mathbf{V}_{\chi_1^2}(\cdot, \mathbf{h})$, associated with the continuity and Brinkman equations (2.1) and (2.2). Similarly, we try to obtain the inner velocity field $\mathbf{v}^i$ as a combination of double- and single-layer potentials, $\mathbf{W}_{\chi_0^2}(\cdot, \mathbf{\Phi})$ and $\mathbf{V}_{\chi_0^2}(\cdot, \mathbf{h})$, corresponding to the continuity and Brinkman equations (2.3) and (2.4).

The fields $\mathbf{v}^e$ and p^e satisfy the Brinkman system of equations (2.1) and (2.2), and the fields $\mathbf{v}^i$ and p^i satisfy the Brinkman system of equations (2.3) and (2.4). In addition, we have $((\mathbf{v}^e - \mathbf{U}^\infty, p^e - p^\infty), (\mathbf{v}^i, p^i)) \in H^1_{\text{loc}}(D_e, \mathrm{P_{St}}) \times \mathrm{H}^1(\mathrm{D}_0, \mathrm{P_{St}})$. Also the asymptotic relations (4.17) and (4.18) yield that the exterior fields $\mathbf{v}^e$ and p^e satisfy the far field conditions (2.7).

On the other hand, making use of properties (4.12) and (4.13), which provide the continuity behavior of single- and double-layer potentials across the boundary S, we find that

$$\begin{aligned} v_k^{e+}(\mathbf{x}_0) = U_k^\infty + \frac{1}{2}\phi_k(\mathbf{x}_0) + \frac{1}{8\pi}\int_S K_{jk}^{\chi_1^2}(\mathbf{y}, \mathbf{x}_0)\phi_j(\mathbf{y})dS(\mathbf{y}) \\ + \frac{1}{8\pi}\int_S \mathcal{G}_{kj}^{\chi_1^2}(\mathbf{x}_0-\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}) \quad \text{for a.e. } \mathbf{x}_0 \in S, \end{aligned} \tag{5.5}$$

$$\begin{aligned} v_k^{i-}(\mathbf{x}_0) = -\frac{1}{2}\phi_k(\mathbf{x}_0) + \frac{1}{8\pi}\int_S K_{jk}^{\chi_0^2}(\mathbf{y}, \mathbf{x}_0)\phi_j(\mathbf{y})dS(\mathbf{y}) \\ + \frac{1}{8\pi}\int_S \mathcal{G}_{kj}^{\chi_0^2}(\mathbf{x}_0-\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}) \quad \text{for a.e. } \mathbf{x}_0 \in S, \end{aligned} \tag{5.6}$$

where

$$v_k^{e+} = \gamma_0(v_k^e|_{D_e}), \quad v_k^{i-} = \gamma_0(v_k^i|_{D_0}) \quad \text{on S}.$$

According to the formulas (5.5) and (5.6) and the boundary condition (2.5), we obtain the following equations:

$$\begin{aligned} \phi_k(\mathbf{x}_0) - \frac{1}{8\pi}\int_S K_{jk}^c(\mathbf{y}, \mathbf{x}_0)\phi_j(\mathbf{y})dS(\mathbf{y}) \\ - \frac{1}{8\pi}\int_S \mathcal{G}_{kj}^c(\mathbf{x}_0-\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}) = -U_k^\infty \quad \text{for a.e. } \mathbf{x}_0 \in S, \end{aligned} \tag{5.7}$$

where

$$K^c_{jk}(\mathbf{y},\mathbf{x}_0) = K^{\chi_0^2}_{jk}(\mathbf{y},\mathbf{x}_0) - K^{\chi_1^2}_{jk}(\mathbf{y},\mathbf{x}_0), \quad \mathcal{G}^c_{jk}(\mathbf{x}_0-\mathbf{y}) = \mathcal{G}^{\chi_0^2}_{jk}(\mathbf{x}_0-\mathbf{y}) - \mathcal{G}^{\chi_1^2}_{jk}(\mathbf{x}_0-\mathbf{y}).$$

Now, using the property (4.14), which provides the jump relations of the boundary traction due to a single-layer potential, the property (4.22), which yields the continuity of the normal stress due to a complementary double-layer potential across the boundary S, as well as the boundary integral representations (5.1)-(5.4), we get the following relations in $H^{-1/2}(S)$:

$$t_k(\mathbf{v}^e)(\mathbf{x}_0) = t_k^\infty(\mathbf{x}_0) - \frac{1}{2}h_k(\mathbf{x}_0) + \frac{1}{8\pi}\int_S K^{\chi_1^2}_{kj}(\mathbf{x}_0,\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}) + T_{kj}(\mathbf{W}_{\chi_1^2}(\mathbf{\Phi}))(\mathbf{x}_0)n_j(\mathbf{x}_0) \quad \text{on S}, \tag{5.8}$$

$$t_k(\mathbf{v}^i)(\mathbf{x}_0) = \frac{1}{2}h_k(\mathbf{x}_0) + \frac{1}{8\pi}\int_S K^{\chi_0^2}_{kj}(\mathbf{x}_0,\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}) + T_{kj}(\mathbf{W}_{\chi_0^2}(\mathbf{\Phi}))(\mathbf{x}_0)n_j(\mathbf{x}_0) \quad \text{on S}, \tag{5.9}$$

where

$$t_k(\mathbf{v}^e)\big|_S = t_k(\mathbf{v}^e|_{D_e}) = \gamma_0\big(T_{kj}\big|_{D_e}(\mathbf{v}^e)\big)n_j, \quad t_k(\mathbf{v}^i)\big|_S = t_k(\mathbf{v}^i|_{D_0}) = \gamma_0\big(T_{kj}\big|_{D_0}(\mathbf{v}^e)\big)n_j,$$

and $\mathbf{t}^\infty$ is the boundary traction due to the given fields $\mathbf{U}^\infty$ and p^∞, i.e.,

$$t_k^\infty(\mathbf{x}_0) = -p^\infty n_k(\mathbf{x}_0) = \chi_1^2(\mathbf{U}^\infty\cdot\mathbf{x}_0)n_k(\mathbf{x}_0).$$

Also $\mathbf{T}(\mathbf{W}_{\chi_1^2}(\mathbf{\Phi}))$ and $\mathbf{T}(\mathbf{W}_{\chi_0^2}(\mathbf{\Phi}))$ denote the stress fields due to the double-layer potentials $\mathbf{W}_{\chi_1^2}(\cdot,\mathbf{\Phi})$ and $\mathbf{W}_{\chi_0^2}(\cdot,\mathbf{\Phi})$, respectively.

Next, using the transmission condition (2.6) and the formulas (5.8) and (5.9), we get the boundary integral equations

$$h_k(\mathbf{x}_0) + \frac{1}{8\pi}\int_S K^c_{kj}(\mathbf{x}_0,\mathbf{y})h_j(\mathbf{y})dS(\mathbf{y}) + T_{kj}(\mathbf{W}^c(\mathbf{\Phi}))(\mathbf{x}_0)n_j(\mathbf{x}_0) = t_k^\infty(\mathbf{x}_0) \quad \text{on S}, \tag{5.10}$$

where

$$\mathbf{W}^c(\mathbf{\Phi}) = \mathbf{W}_{\chi_0^2}(\cdot,\mathbf{\Phi}) - \mathbf{W}_{\chi_1^2}(\cdot,\mathbf{\Phi}).$$

Note that $\mathbf{t}^\infty \in L^2(S) \overset{\text{comp}}{\hookrightarrow} H^{-1/2}(S)$ (since S is a strong Lipschitz boundary, $\mathbf{n}$ is defined almost everywhere on S and $\mathbf{n}\in L^\infty(S)$).

On the other hand, from Theorem 4.2 it follows that the linear integral operators

$$\mathcal{V}^c : H^{-1/2}(S) \overset{\text{comp}}{\longrightarrow} H^1(S) \overset{\text{comp}}{\hookrightarrow} H^{1/2}(S), \quad \mathbf{K}^c : H^{1/2}(S) \overset{\text{comp}}{\longrightarrow} H^1(S) \overset{\text{comp}}{\hookrightarrow} H^{1/2}(S),$$

given by

$$\mathcal{V}^c = \mathcal{V}_{\chi_0^2} - \mathcal{V}_{\chi_1^2} = \mathcal{V}_{\chi_0^2,0} - \mathcal{V}_{\chi_1^2,0}, \quad \mathbf{K}^c = \mathbf{K}_{\chi_0^2} - \mathbf{K}_{\chi_1^2} = \mathbf{K}_{\chi_0^2,0} - \mathbf{K}_{\chi_1^2,0}$$

are compact, as mappings into $H^{1/2}(S)$.

In addition, Theorem 4.2 yields that the linear integral operators

$$\mathbf{K}^{c'} : H^{-1/2}(S) \stackrel{\text{comp}}{\longrightarrow} L^2(S) \stackrel{\text{comp}}{\hookrightarrow} H^{-1/2}(S),$$

$$\mathbf{D}^c = \mathbf{T}(\mathbf{W}^c)\cdot\mathbf{n} : H^{1/2}(S) \stackrel{\text{comp}}{\longrightarrow} L^2(S) \stackrel{\text{comp}}{\hookrightarrow} H^{-1/2}(S),$$

given by

$$\mathbf{K}^{c'} = \mathbf{K}'_{\chi_0^2} - \mathbf{K}'_{\chi_1^2} = \mathbf{K}'_{\chi_0^2,0} - \mathbf{K}'_{\chi_1^2,0}, \quad \mathbf{D}^c = \mathbf{D}_{\chi_0^2} - \mathbf{D}_{\chi_1^2} = \mathbf{D}_{\chi_0^2,0} - \mathbf{D}_{\chi_1^2,0}$$

are also compact, as mappings into $H^{-1/2}(S)$.

Thus, the boundary value problem (2.1)-(2.7) reduces to the system of Fredholm integral equations of the second kind (5.7) and (5.10), which may be written in the form:

$$\mathbf{\Phi} - \mathbf{K}^c\mathbf{\Phi} - \mathcal{V}^c\mathbf{h} = -\mathbf{U}^\infty \quad \text{a.e. on } S,$$

$$\mathbf{h} + \mathbf{K}^{c\prime}\mathbf{h} + \mathbf{D}^c\mathbf{\Phi} = \mathbf{t}^\infty \quad \text{on } S.$$

In order to prove existence and uniqueness of the solution $(\mathbf{\Phi}, \mathbf{h})$ to this system in the Sobolev space $H^{1/2}(S) \times H^{-1/2}(S)$, we show that the corresponding homogeneous system has only the trivial solution. Then the desired existence and uniqueness result will be a direct consequence of Fredholm's alternative in the dual system $\langle H^{1/2}(S) \times H^{-1/2}(S), H^{1/2}(S) \times H^{-1/2}(S)\rangle$, where $\langle\cdot,\cdot\rangle = (\cdot,\cdot)_{L^2(S)}$ (see [23]).

For showing uniqueness, let us consider the homogeneous system of equations on S:

$$\mathbf{\Phi}^0 - \mathbf{K}^c\mathbf{\Phi}^0 - \mathcal{V}^c\mathbf{h}^0 = -\mathbf{U}^\infty \tag{5.11}$$

$$\mathbf{h}^0 + \mathbf{K}^{c\prime}\mathbf{h}^0 + \mathbf{D}^c\mathbf{\Phi}^0 = \mathbf{0}, \tag{5.12}$$

and let $(\mathbf{\Phi}^0, \mathbf{h}^0) \in H^{1/2}(S) \times H^{-1/2}(S)$ be an arbitrary solution to this system. With this solution we determine the fields $(\mathbf{u}^e, q^e)$ and $(\mathbf{u}^i, q^i)$ given by the boundary potentials for $\mathbf{x} \in \mathbb{R}^3 \setminus S$:

$$u_k^e(\mathbf{x}) = \frac{1}{8\pi}\int_S K_{jk}^{\chi_1^2}(\mathbf{y},\mathbf{x})\phi_j^0(\mathbf{y})dS(\mathbf{y}) + \frac{1}{8\pi}\int_S \mathcal{G}_{kj}^{\chi_1^2}(\mathbf{x}-\mathbf{y})h_j^0(\mathbf{y})dS(\mathbf{y}),$$

$$q^e(\mathbf{x}) = \frac{1}{8\pi}\int_S \Lambda_{jk}^{\chi_1^2}(\mathbf{x}-\mathbf{y})\phi_j^0(\mathbf{y})dS(\mathbf{y}) + \frac{1}{8\pi}\int_S \Pi_j^{\chi_1^2}(\mathbf{x}-\mathbf{y})h_j^0(\mathbf{y})dS(\mathbf{y}),$$

and

$$u_k^i(\mathbf{x}) = \frac{1}{8\pi}\int_S K_{jk}^{\chi_0^2}(\mathbf{y},\mathbf{x})\phi_j^0(\mathbf{y})dS(\mathbf{y}) + \frac{1}{8\pi}\int_S \mathcal{G}_{kj}^{\chi_0^2}(\mathbf{x}-\mathbf{y})h_j^0(\mathbf{y})dS(\mathbf{y}),$$

$$q^i(\mathbf{x}) = \frac{1}{8\pi}\int_S \Lambda_{jk}^{\chi_0^2}(\mathbf{x}-\mathbf{y})\phi_j^0(\mathbf{y})dS(\mathbf{y}) + \frac{1}{8\pi}\int_S \Pi_j^{\chi_0^2}(\mathbf{x}-\mathbf{y})h_j^0(\mathbf{y})dS(\mathbf{y}).$$

These potentials have the following properties:

$$\nabla\cdot\mathbf{u}^e = 0, \quad -\nabla q^e + (\nabla^2 - \chi_1^2)\mathbf{u}^e = \mathbf{0} \quad \text{and} \tag{5.13}$$

$$\nabla\cdot\mathbf{u}^i = 0, \quad -\nabla q^i + (\nabla^2 - \chi_0^2)\mathbf{u}^e = \mathbf{0} \text{ in } \mathbb{R}^3 \setminus S,$$

$$\mathbf{u}^e(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad p^e(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad T_{j\ell}(\mathbf{u}^e)(\mathbf{x}) = O(|\mathbf{x}|^{-2}) \text{ as } |\mathbf{x}| \to \infty,$$

in view of which we get the identities (see the identity (3.2))

$$\int_{D_e} (2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e) + \chi_1^2|\mathbf{u}^e|^2)d\mathbf{x} = -\int_S \gamma_0(\mathbf{u}^e|_{D_e}) \cdot \mathbf{t}(\mathbf{u}^e|_{D_e})dS, \tag{5.14}$$

$$\int_{D_0} (2E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i) + \chi_0^2|\mathbf{u}^i|^2)d\mathbf{x} = \int_S \gamma_0(\mathbf{u}^i|_{D_0}) \cdot \mathbf{t}(\mathbf{u}^i|_{D_0})dS, \tag{5.15}$$

where

$$E_{jk}(\mathbf{u}^e) = \frac{1}{2}\left(\frac{\partial u_j^e}{\partial x_k} + \frac{\partial u_k^e}{\partial x_j}\right), \quad E_{jk}(\mathbf{u}^i) = \frac{1}{2}\left(\frac{\partial u_j^i}{\partial x_k} + \frac{\partial u_k^i}{\partial x_j}\right), \quad j,k = 1,2,3.$$

Now, according the fact that $(\mathbf{\Phi}^0, \mathbf{h}^0)$ is a solution to the system of equations (5.11) and (5.12), and taking into account the properties (4.12)-(4.14), we obtain the relations

$$\gamma_0(\mathbf{u}^e|_{D_e}) = \gamma_0(\mathbf{u}^i|_{D_0}) \quad \text{and} \quad \mathbf{t}(\mathbf{u}^e|_{\mathrm{D_e}}) = \mathbf{t}(\mathbf{u}^i|_{\mathrm{D_0}}) \;\; \text{on S}, \tag{5.16}$$

and, hence, the equality

$$\int_S \gamma_0(\mathbf{u}^e|_{D_e}) \cdot \mathbf{t}(\mathbf{u}^e|_{D_e})dS = \int_S \gamma_0(\mathbf{u}^i|_{D_0}) \cdot \mathbf{t}(\mathbf{u}^i|_{D_0})dS. \tag{5.17}$$

Therefore, the properties (5.14), (5.15) and (5.17) lead to the identity

$$\int_{D_e} (2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e) + \chi_1^2|\mathbf{u}^e|^2)d\mathbf{x} = -\int_{D_0} (2E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i) + \chi_0^2|\mathbf{u}^i|^2)d\mathbf{x},$$

which implies that

$$\begin{aligned} \mathbf{u}^e &= \mathbf{0} \;\; \text{in } D_e, \\ \mathbf{u}^i &= \mathbf{0} \;\; \text{in } D_0. \end{aligned} \tag{5.18}$$

In addition, in view of the second equation in (5.13) and from the fact that the pressure field q^e vanishes at infinity, we obtain

$$q^e = 0 \;\; \text{in } D_e. \tag{5.19}$$

Similarly, we get $q^i = c \in \mathbb{R}$ in D_0. However, from the continuity of the linear mappings $\mathbf{t} : H^1(D_0, \mathrm{P_{St}}) \to H^{-\frac{1}{2}}(S)$ and $\mathbf{t} : H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}}) \to H^{-\frac{1}{2}}(S)$, as well as the relations (5.16), (5.18) and (5.19) we get

$$\mathbf{t}(\mathbf{u}^i|_{D_0}) = \mathbf{t}(\mathbf{u}^e|_{D_e}) = \mathbf{0} \;\; \text{in } H^{-1/2}(S),$$

and hence $c = 0$, i.e.,

$$\mathbf{u}^i = \mathbf{0} \quad \text{and} \quad \mathrm{q^i} = 0 \;\; \text{in } \mathrm{D_0}. \tag{5.20}$$

Now, taking into account the relations (4.12) and (4.13), we obtain the jump formula

$$\gamma_0(\mathbf{u}^e|_{D_e}) - \gamma_0(\mathbf{u}^e|_{D_0}) = \mathbf{\Phi}^0 \;\; \text{in} \;\; H^{1/2}(S).$$

This formula together with the property (5.18) show that

$$\gamma_0(\mathbf{u}^e|_{D_0}) = -\mathbf{\Phi}^0 \;\; \text{in} \;\; H^{1/2}(S). \tag{5.21}$$

Similarly, using the jump formula

$$\gamma_0(\mathbf{u}^i|_{D_e}) - \gamma_0(\mathbf{u}^i|_{D_0}) = \mathbf{\Phi}^0 \ \text{ in } \ H^{1/2}(S),$$

as well as the property (5.20), we find that

$$\gamma_0(\mathbf{u}^i|_{D_e}) = \mathbf{\Phi}^0 \ \text{ in } \ H^{1/2}(S). \tag{5.22}$$

On the other hand, using the jump relations for the boundary traction of a single-layer potential (i.e., the formulas (4.14)), as well as the continuity of the boundary traction of a double-layer potential, we deduce that the boundary traction due to the fields $\mathbf{u}^e$ and q^e has a jump across S given by the formula

$$\mathbf{t}(\mathbf{u}^e|_{D_e}) - \mathbf{t}(\mathbf{u}^e|_{D_0}) = -\mathbf{h}^0 \ \text{ in } H^{-1/2}(S).$$

Since $\mathbf{t}(\mathbf{u}^e|_{D_e}) = \mathbf{0}$ in $H^{-1/2}(S)$, we get

$$\mathbf{t}(\mathbf{u}^e|_{D_0}) = \mathbf{h}^0 \ \text{ in } H^{-1/2}(S). \tag{5.23}$$

Similarly, we find the relation

$$\mathbf{t}(\mathbf{u}^i|_{D_e}) = -\mathbf{h}^0 \ \text{ in } H^{-1/2}(S). \tag{5.24}$$

On the other hand, the fields $\mathbf{u}^e$ and q^e satisfy the identity

$$\int_{D_0} (2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e) + \chi_1^2|\mathbf{u}^e|^2)d\mathbf{x} = \int_S \gamma_0(\mathbf{u}^e|_{D_0}) \cdot \mathbf{t}(\mathbf{u}^e|_{D_0})dS,$$

which, in view of the properties (5.21) and (5.23), becomes

$$\int_{D_0} (2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e) + \chi_1^2|\mathbf{u}^e|^2)d\mathbf{x} = -\int_S \mathbf{\Phi}^0 \cdot \mathbf{h}^0 dS. \tag{5.25}$$

Also, from the identity

$$\int_{D_e} (2E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i) + \chi_0^2|\mathbf{u}^e|^2)d\mathbf{x} = -\int_S \gamma_0(\mathbf{u}^i|_{D_e}) \cdot \mathbf{t}(\mathbf{u}^i|_{D_e})dS$$

and the properties (5.22) and (5.24), we get

$$\int_{D_e} (2E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i) + \chi_0^2|\mathbf{u}^e|^2)d\mathbf{x} = \int_S \mathbf{\Phi}^0 \cdot \mathbf{h}^0 dS. \tag{5.26}$$

Consequently, the identities (5.25) and (5.26) imply the following equality:

$$\int_{D_0} (2E_{jk}(\mathbf{u}^e)E_{jk}(\mathbf{u}^e) + \chi_1^2|\mathbf{u}^e|^2)d\mathbf{x} = -\int_{D_e} (2E_{jk}(\mathbf{u}^i)E_{jk}(\mathbf{u}^i) + \chi_0^2|\mathbf{u}^e|^2)d\mathbf{x},$$

in view of which we find that

$$\mathbf{u}^e = \mathbf{0} \ \text{ in } D_0 \tag{5.27}$$

and

$$\mathbf{u}^i = \mathbf{0} \ \text{ in } D_e. \tag{5.28}$$

In addition, since the fields $\mathbf{u}^i$ and q^i satisfy the Brinkman equation

$$-\nabla q^i + (\nabla^2 - \chi_0^2)\mathbf{u}^i = \mathbf{0} \ \text{ in } D_e$$

and the pressure field q^i vanishes at infinity, we deduce that

$$q^i = 0 \ \text{ in } D_e. \tag{5.29}$$

Finally, using the relations (5.21) and (5.27) we find that

$$\mathbf{\Phi}^0 = \mathbf{0} \ \text{ in } H^{1/2}(S),$$

and from the properties (5.24), (5.28) and (5.29) we deduce that

$$\mathbf{h}^0 = \mathbf{0} \ \text{ in } H^{-1/2}(S).$$

Consequently, the homogeneous system of equations (5.11) and (5.12) has only the trivial solution in the space $H^{1/2}(S) \times H^{-1/2}(S)$ and thus, in view of Fredholm's alternative, the non-homogeneous system of equations (5.7) and (5.10) admits a unique solution $(\mathbf{\Phi}, \mathbf{h}) \in H^{1/2}(S) \times H^{-1/2}(S)$, as desired.

• In the case $\chi_1 = \chi_0$, the system of equations (5.7) and (5.10) has the solution

$$h_k(\mathbf{x}) = \chi_1^2(\mathbf{U}^\infty \cdot \mathbf{x})n_k(\mathbf{x}), \quad \phi_k(\mathbf{x}) = -U_k^\infty, \quad \mathbf{x} \in S, \quad k = 1, 2, 3,$$

and the boundary integral representations of the fields $\mathbf{v}^e$ and $\mathbf{v}^i$ reduce to

$$\mathbf{v}^e = \mathbf{U}^\infty \text{ in } D_e, \quad \mathbf{v}^i = \mathbf{U}^\infty \text{ in } D_0.$$

Concluding the previous arguments, we obtain the following existence and uniqueness result.

Theorem 5.1 *Let S be a strong Lipschitz surface. Then the system of Fredholm integral equations* (5.7) *and* (5.10) *has a unique solution* $(\mathbf{\Phi}, \mathbf{h}) \in H^{1/2}(S) \times H^{-1/2}(S)$*, and the boundary integral representations* (5.1)-(5.4)*, obtained with the vector densities* $\mathbf{\Phi}$ *and* $\mathbf{h}$*, determine the unique solution* $((\mathbf{v}^e, p^e), (\mathbf{v}^i, p^i))$*,* $((\mathbf{v}^e - \mathbf{U}^\infty, p^e - p^\infty), (\mathbf{v}^i, p^i)) \in H^1_{\text{loc}}(D_e, \mathrm{P_{St}}) \times H^1(D_0, \mathrm{P_{St}})$*, to the boundary value problem consisting of the equations* (2.1)-(2.4) *and the boundary and far field conditions* (2.5)-(2.7)*.*

In fact, we have $(\mathbf{\Phi}, \mathbf{h}) \in H^1(S) \times L^2(S)$ (see the results in Appendix).

Now, using similar arguments as in the case of a strong Lipschitz boundary, we can also obtain existence and uniqueness for the classical solution to our boundary value problem in the case of a Lypaunov surface:

Theorem 5.2 *Let S be a closed Lyapunov surface of class $C^{1,\alpha}$ in $\mathbb{R}^3$, $\alpha \in (0, 1]$, and let $\lambda \in (0, \alpha)$. Then the system of Fredholm integral equations* (5.7) *and* (5.10) *has a unique solution* $(\mathbf{\Phi}, \mathbf{h}) \in C^{1,\lambda}(S) \times C^\lambda(S)$*, and the boundary integral representations* (5.1)-(5.4)*, obtained with the vector densities* $\mathbf{\Phi}$ *and* $\mathbf{h}$*, determine the unique classical solution* $((\mathbf{v}^e, p^e), (\mathbf{v}^i, p^i)) \in ((C^2(D_e) \cap C^1(\overline{D}_e)) \times C^1(\overline{D}_e)) \times ((C^2(D_0) \cap C^1(\overline{D}_0)) \times C^1(\overline{D}_0))$ *to the boundary value problem consisting of the equations* (2.1)-(2.4) *and the boundary and far field conditions* (2.5)-(2.7)*.*

Remark. Every Lyapunov surface is also a strong Lipschitz surface.

6 Special flow cases

In this section we consider some special flow cases that correspond to particular values of the parameters χ_0 and χ_1, and obtain asymptotic formulas for the hydrodynamic force exerted on the porous particle in these special cases.

6.1 Stokes flow past a porous particle with large permeability

First, let us consider the case of exterior Stokes flow past a porous particle with small permeability, i.e., $\chi_1 = 0$ and $\chi_0 \ll 1$. Therefore, we may formally expand the inner flow fields $\mathbf{v}^i$ and p^i, as well as the exterior flow fields $\mathbf{v}^e$ and p^e by using the Taylor formula with respect to χ_0, as follows:

$$\begin{aligned} \mathbf{v}^i &= \mathbf{v}^i_{(0)} + \chi_0 \mathbf{v}^i_{(1)} + \chi_0^2 \mathbf{v}^i_{(2)} + \chi_0^3 \mathbf{v}^i_{(3)} + \cdots , \\ p^i &= p^i_{(0)} + \chi_0 p^i_{(1)} + \chi_0^2 p^i_{(2)} + \chi_0^3 p^i_{(3)} + \cdots , \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \mathbf{v}^e &= \mathbf{v}^e_{(0)} + \chi_0 \mathbf{v}^e_{(1)} + \chi_0^2 \mathbf{v}^e_{(2)} + \chi_0^3 \mathbf{v}^e_{(3)} + \cdots , \\ p^e &= p^e_{(0)} + \chi_0 p^e_{(1)} + \chi_0^2 p^e_{(2)} + \chi_0^3 p^e_{(3)} + \cdots . \end{aligned} \tag{6.2}$$

Substituting these expansions into the equations, the boundary and far field conditions (2.1)-(2.7), and collecting the zero order terms with respect to χ_0, we obtain the following transmission problem:

$$\begin{cases} \nabla \cdot \mathbf{v}^i_{(0)} = 0, \quad -\nabla p^i_{(0)} + \nabla^2 \mathbf{v}^i_{(0)} = \mathbf{0} \ \text{ in } D_0, \\ \nabla \cdot \mathbf{v}^e_{(0)} = 0, \quad -\nabla p^e_{(0)} + \nabla^2 \mathbf{v}^e_{(0)} = \mathbf{0} \ \text{ in } D_e, \\ \mathbf{v}^e_{(0)} = \mathbf{v}^i_{(0)}, \quad \mathbf{t}(\mathbf{v}^e_{(0)}) = \mathbf{t}(\mathbf{v}^i_{(0)}) \ \text{ on } S, \\ \mathbf{v}^e_{(0)}(\mathbf{x}) \to \mathbf{U}^\infty, \ \ p^e_{(0)}(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty. \end{cases}$$

Using the uniqueness result of the solution to this problem (see Theorem 3.5), we find that

$$\mathbf{v}^e_{(0)} = \mathbf{U}^\infty, \quad p^e_{(0)} = 0 \ \text{ in } D_e, \tag{6.3}$$

$$\mathbf{v}^i_{(0)} = \mathbf{U}^\infty, \quad p^i_{(0)} = 0 \ \text{ in } D_0. \tag{6.4}$$

Similarly, collecting the first order terms with respect to χ_0, we obtain the transmission problem

$$\begin{cases} \nabla \cdot \mathbf{v}^i_{(1)} = 0, \quad -\nabla p^i_{(1)} + \nabla^2 \mathbf{v}^i_{(1)} = \mathbf{0} \ \text{ in } D_0, \\ \nabla \cdot \mathbf{v}^e_{(1)} = 0, \quad -\nabla p^e_{(1)} + \nabla^2 \mathbf{v}^e_{(1)} = \mathbf{0} \ \text{ in } D_e, \\ \mathbf{v}^e_{(1)} = \mathbf{v}^i_{(1)}, \quad \mathbf{t}(\mathbf{v}^e_{(1)}) = \mathbf{t}(\mathbf{v}^i_{(1)}) \ \text{ on } S. \\ \mathbf{v}^e_{(1)}(\mathbf{x}) \to \mathbf{0}, \ \ p^e_{(1)}(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty, \end{cases}$$

which has the unique solution

$$\mathbf{v}^e_{(1)} = \mathbf{0}, \quad p^e_{(1)} = 0 \ \text{ in } D_e, \tag{6.5}$$

$$\mathbf{v}^i_{(1)} = \mathbf{0}, \quad p^i_{(1)} = 0 \ \text{ in } D_0. \tag{6.6}$$

Now, collecting the second and third order terms with respect to χ_0, we find the transmission problems

$$\begin{cases} \nabla\cdot\mathbf{u}^i_{(2)}=0, \quad -\nabla p^i_{(2)}+\nabla^2\mathbf{v}^i_{(2)}=\mathbf{v}^i_{(0)}=\mathbf{U}^\infty \ \text{ in } D_0,\\ \nabla\cdot\mathbf{v}^e_{(2)}=0, \quad -\nabla p^e_{(2)}+\nabla^2\mathbf{v}^e_{(2)}=\mathbf{0} \ \text{ in } D_e,\\ \mathbf{v}^e_{(2)}=\mathbf{v}^i_{(2)}, \quad \mathbf{t}(\mathbf{v}^e_{(2)})=\mathbf{t}(\mathbf{v}^i_{(2)}) \ \text{ on } S,\\ \mathbf{v}^e_{(2)}(\mathbf{x})\to\mathbf{0}, \ \ p^e_{(2)}(\mathbf{x})\to 0 \quad \text{as } |\mathbf{x}|\to\infty, \end{cases}$$

and

$$\begin{cases} \nabla\cdot\mathbf{v}^i_{(3)}=0, \quad -\nabla p^i_{(3)}+\nabla^2\mathbf{v}^i_{(3)}=\mathbf{v}^i_{(1)}=\mathbf{0}, \ \text{ in } D_0\\ \nabla\cdot\mathbf{v}^e_{(3)}=0, \quad -\nabla p^e_{(3)}+\nabla^2\mathbf{v}^e_{(3)}=\mathbf{0} \ \text{ in } D_e,\\ \mathbf{v}^e_{(3)}=\mathbf{u}^i_{(3)}, \quad \mathbf{t}(\mathbf{v}^e_{(3)})=\mathbf{t}(\mathbf{v}^i_{(3)}) \ \text{ on } S,\\ \mathbf{v}^e_{(3)}(\mathbf{x})\to\mathbf{0}, \ \ p^e_{(3)}(\mathbf{x})\to 0 \quad \text{as } |\mathbf{x}|\to\infty, \end{cases}$$

with the unique solutions

$$v^e_{(2)k}(\mathbf{x})=-\frac{U^\infty_j}{8\pi}\int_{D_0}\mathcal{G}_{kj}(\mathbf{x}-\mathbf{y})d\mathbf{y}, p^e_{(2)}(\mathbf{x})=-\frac{U^\infty_j}{8\pi}\int_{D_0}\Pi_j(\mathbf{x}-\mathbf{y})d\mathbf{y} \ \text{ for } \ \mathbf{x}\in D_e \tag{6.7}$$

$$v^i_{(2)k}(\mathbf{x})=-\frac{U^\infty_j}{8\pi}\int_{D_0}\mathcal{G}_{kj}(\mathbf{x}-\mathbf{y})d\mathbf{y}, p^i_{(2)}(\mathbf{x})=-\frac{U^\infty_j}{8\pi}\int_{D_0}\Pi_j(\mathbf{x}-\mathbf{y})d\mathbf{y} \ \text{ for } \ \mathbf{x}\in D_0 \tag{6.8}$$

respectively, and

$$\mathbf{v}^e_{(3)}=\mathbf{0}, \quad p^e_{(3)}=0 \ \text{ in } D_e \tag{6.9}$$

$$\mathbf{v}^i_{(3)}=\mathbf{0}, \quad p^i_{(3)}=0 \ \text{ in } D_0. \tag{6.10}$$

Using similar arguments as above, we can prove for each $\ell\in\mathbb{N}$ that

$$\mathbf{v}^e_{(2\ell+1)}=\mathbf{0}, \quad p^e_{(2\ell+1)}=0 \ \text{ in } D_e,$$

$$\mathbf{v}^i_{(2\ell+1)}=\mathbf{0}, \quad p^i_{(2\ell+1)}=0 \ \text{ in } D_0.$$

Therefore, in view of (6.1), (6.4), (6.6) and (6.8), the expansion of the inner velocity field $\mathbf{v}^i$ with respect to χ_0, up to the order $\mathcal{O}(\chi_0^4)$, has the form

$$v^i_k(\mathbf{x})=U^\infty_k-\chi_0^2\frac{U^\infty_j}{8\pi}\int_{D_0}\mathcal{G}_{kj}(\mathbf{x}-\mathbf{y})d\mathbf{y}+\mathcal{O}(\chi_0^4), \quad \mathbf{x}\in D_0,\ k=1,2,3. \tag{6.11}$$

This asymptotic formula has been previously obtained in [21] and [12], by using different arguments.

On the other hand, taking into account the boundary condition (2.6), and the continuity and Brinkman equations (2.3) and (2.4), we obtain the non-dimensional hydrodynamic force $\mathbf{F}$, exerted by the exterior Stokes flow on the porous particle, in the form

$$\begin{aligned} F_k &= \int_S t_k(\mathbf{v}^e)(\mathbf{x})dS(\mathbf{x})=\int_S t_k(\mathbf{v}^i)(\mathbf{x})dS(\mathbf{x})\\ &= \int_{D_0}\Big\{-\frac{\partial p^i(\mathbf{x})}{\partial x_k}+\nabla^2 v^i_k(\mathbf{x})\Big\}d\mathbf{x}=\chi_0^2\int_{D_0}v^i_k(\mathbf{x})d\mathbf{x}, \end{aligned}$$

i.e.,

$$F_k = \chi_0^2 \int_{D_0} v_k^i(\mathbf{x}) d\mathbf{x} \quad \text{for} \quad k = 1, 2, 3. \tag{6.12}$$

Further, substituting the asymptotic formula (6.11) into the formula (6.12), we obtain the following asymptotic result for the non-dimensional hydrodynamic force:

$$F_k = \chi_0^2 U_k^\infty |D_0| - \frac{\chi_0^4}{8\pi} U_j^\infty \int_{D_0} \int_{D_0} \mathcal{G}_{kj}(\mathbf{x}-\mathbf{y}) d\mathbf{x} d\mathbf{y} + \mathcal{O}(\chi_0^6),$$

which, in view of the divergence formula, becomes (see also [21, 12])

$$F_k = \chi_0^2 U_k^\infty |D_0| - \frac{\chi_0^4 U_j^\infty}{8\pi} \int_S \int_S (-\delta_{kj} n_l(\mathbf{x}) n_l(\mathbf{y}) + n_k(\mathbf{x}) n_j(\mathbf{y})) r dS(\mathbf{x}) dS(\mathbf{y}) + \mathcal{O}(\chi_0^6). \tag{6.13}$$

Now, denoting by $\overline{\mathbf{F}}$ the hydrodynamic force on the porous particle, and reverting to dimensional variables, the formula (6.13) takes the form

$$\overline{F}_k = \frac{\mu}{k_0} u_k^\infty |D_0| - \frac{\mu}{8\pi k_0^2} \int_S \int_S (-\delta_{kj} n_l(\mathbf{x}) n_l(\mathbf{y}) + n_k(\mathbf{x}) n_j(\mathbf{y})) r dS(\mathbf{x}) dS(\mathbf{y}) + \mathcal{O}(k_0^{-3}), \tag{6.14}$$

where μ is the fluid viscosity.

In the case of a porous sphere of radius a, which is immersed in a uniform flow with velocity field $\mathbf{u}^\infty$, we obtain from the formula (6.14) the following asymptotic result(see also [21, 12, 28]):

$$\overline{F}_k = \frac{4}{3}\pi\mu a u_k^\infty \left\{ \frac{a^2}{k_0} - \frac{4}{15}\frac{a^4}{k_0^2} + \mathcal{O}\left(\frac{a^6}{k_0^3}\right) \right\}, \quad k = 1, 2, 3.$$

6.2 Flow past a void in a porous medium with large permeability

Let us now consider the case of Brinkman flow in a porous medium with large permeability and in the presence of a void, i.e., $\chi_1 \ll 1$ and $\chi_0 = 0$ (see also [20]).

Using similar arguments as in the previous case ($\chi_0 \ll 1$, $\chi_1 = 0$), one obtains the following asymptotic formula for the exterior velocity field:

$$v_k^e(\mathbf{x}) = U_k^\infty + \chi_1^2 \frac{U_j^\infty}{8\pi} \int_{D_0} \mathcal{G}_{kj}(\mathbf{x}-\mathbf{y}) d\mathbf{y} + \mathcal{O}(\chi_1^4), \quad \mathbf{x} \in D_e,\ k = 1, 2, 3.$$

In addition, from the property (6.12) we deduce that the non-dimensional hydrodynamic force exerted on the void is equal to zero, i.e., $\mathbf{F} = \mathbf{0}$.

7 Appendix

In this section we will show the compactness of the linear integral operators given in Theorem 4.2 *a*), based on the assumption that S is a strong Lipschitz boundary. In order to obtain these results, we need the following imbedding property.

Lemma 7.1 *Let S be a strong Lipschitz boundary and let $q \in (2, 4)$. Then the imbedding $H^{\frac{1}{2}}(S) \hookrightarrow L^q(S)$ is compact.*

Proof. For given $q \in (2,4)$ define $s = 1 - \frac{2}{q}$. Then $s \in (0, \frac{1}{2})$. Since S is a compact Lipschitz manifold and the Sobolev spaces $H^\ell = W^{\ell,2}$ are invariant under Lipschitz coordinate transformations for $|\ell| \le 1$, the norms $\|v\|_{H^{\frac{1}{2}}(S)}$ and $\|v\|_{L^q(S)}$ are equivalent to the norms

$$\sum_{j=1}^N \|\phi_j v\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \quad \text{and} \quad \sum_{j=1}^N \|\phi_j v\|_{L^q(\mathbb{R}^2)},$$

where ϕ_j, $j = 1, \cdots, N$, define a partition of the unity on S.

Now, for each term we have the imbeddings

$$\phi_j v \in H^{\frac{1}{2}}(\mathbb{R}^2) \overset{\text{comp}}{\hookrightarrow} H^s(\mathbb{R}^2) \overset{\text{cont}}{\longrightarrow} L^q(\mathbb{R}^2)$$

due to the imbedding theorem 7.58 in [1] (where $0 < s < \frac{1}{2}$, $n = k = 2$, $p = 2$ and $\chi = 0 = s - 1 + \frac{2}{q}$). $\square$

Further, using the representations of the kernels to the single- and double-layer integral operators for the Brinkman and Stokes systems (see Section 3.1), as well as the power series expansions of the functions A_j and D_ℓ ($j = 1, 2$, $\ell = 1, 2, 3$), one obtains (see [21]):

$$\begin{aligned} \mathcal{G}_{ij}^{\chi^2,0}(\hat{\mathbf{x}}) &= (\mathcal{G}_{ij}^{\chi^2} - \mathcal{G}_{ij})(\hat{\mathbf{x}}) \\ &= \frac{1}{|\hat{\mathbf{x}}|}(A_1(\chi|\hat{\mathbf{x}}|) - 1)\delta_{ij} + \frac{\hat{x}_i \hat{x}_j}{|\hat{\mathbf{x}}|^3}(A_2(\chi|\hat{\mathbf{x}}|) - 1) \\ &= -\chi \frac{4}{3}\delta_{ij} - \chi^2 \left\{ \frac{1}{4}\frac{\hat{x}_i \hat{x}_j}{|\hat{\mathbf{x}}|} - \frac{3}{4}|\hat{\mathbf{x}}|\delta_{ij} \right\} + \mathcal{O}\big(\chi^3 |\hat{\mathbf{x}}|^2\big), \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} K_{ij}^{\chi^2,0}(\hat{\mathbf{x}}) &= (K_{ij}^{\chi^2} - K_{ij})(\hat{\mathbf{x}}) \\ &= -2\left\{ \delta_{ik}\frac{\hat{x}_j}{|\hat{\mathbf{x}}|^3} D_1(\chi|\hat{\mathbf{x}}|) + \left(\delta_{kj}\frac{\hat{x}_i}{|\hat{\mathbf{x}}|^3} + \delta_{ij}\frac{\hat{x}_k}{|\hat{\mathbf{x}}|^3} \right) D_2(\chi|\hat{\mathbf{x}}|) \right. \\ &\quad \left. + \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{|\hat{\mathbf{x}}|^5}(D_3(\chi|\hat{\mathbf{x}}|) - 3) \right\} n_k(\mathbf{y}) \\ &= -\frac{\chi^2}{2|\hat{\mathbf{x}}|}\left\{ \delta_{ik}\hat{x}_j + \delta_{kj}\hat{x}_i + \delta_{ij}\hat{x}_k - \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{|\hat{\mathbf{x}}|^2} \right\} n_k(\mathbf{y}) + \mathcal{O}(\chi^3 |\hat{\mathbf{x}}|), \end{aligned} \tag{7.2}$$

where $\widehat{\mathbf{x}} = \mathbf{y} - \mathbf{x}$, $\widehat{x}_\ell = y_\ell - x_\ell$, and

$$A_1(z) = 1 + 2\sum_{k=1}^\infty \frac{(k+1)^2}{(k+2)!}(-z)^k, \quad A_2(z) = 1 - 2\sum_{k=1}^\infty \frac{k^2-1}{(k+2)!}(-z)^k,$$

$$D_1(z) = 2\sum_{k=2}^\infty \frac{k^2-1}{(k+2)!}(-z)^k, \quad D_2(z) = -\frac{1}{4}z^2 + \frac{1}{5}z^3 - \sum_{k=4}^\infty \frac{k(k^2-1)}{(k+2)!}(-z)^k,$$

$$D_3(z) = 3 - \frac{1}{4}z^2 + 2\sum_{k=4}^\infty \frac{(k-3)(k^2-1)}{(k+2)!}(-z)^k.$$

These expressions provide the kernel's asymptotic behaviour for $z = \chi|\mathbf{y} - \mathbf{x}| \to 0$.

In order to obtain an estimate for the difference of the hypersingular integral operators, we have to estimate the difference of their hypersingular kernels. For this aim, we need the difference of the pressure terms (see [21])

$$\Lambda_{ik}^{\chi^2,0}(\hat{\mathbf{x}}) = \big(\Lambda_{ik}^{\chi^2} - \Lambda_{ik}\big)(\hat{\mathbf{x}}) = 2\chi^2 \frac{1}{|\hat{\mathbf{x}}|}\delta_{ik}, \tag{7.3}$$

as well as the sum of derivatives of the double-layer kernels

$$\begin{aligned}\frac{\partial K_{ij}^{\chi^2,0}(\hat{\mathbf{x}})}{\partial x_\ell} + \frac{\partial K_{i\ell}^{\chi^2,0}(\hat{\mathbf{x}})}{\partial x_j} =& \chi^2 \Bigg\{ \frac{1}{|\hat{\mathbf{x}}|}\big(\delta_{ij}\delta_{k\ell} + \delta_{kj}\delta_{i\ell} + \delta_{ik}\delta_{j\ell}\big) \\ & - \frac{1}{|\hat{\mathbf{x}}|^3}\Big[\delta_{ik}\hat{x}_j\hat{x}_\ell + \delta_{kj}\hat{x}_i\hat{x}_\ell + \delta_{ij}\hat{x}_k\hat{x}_\ell + \delta_{k\ell}\hat{x}_i\hat{x}_j + \delta_{j\ell}\hat{x}_i\hat{x}_k + \delta_{i\ell}\hat{x}_j\hat{x}_k\Big] \\ & + \frac{3}{|\hat{\mathbf{x}}|^5}\hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_\ell\Bigg\} n_k(\mathbf{y}) + \mathcal{O}(\chi^3). \end{aligned} \tag{7.4}$$

The complementary double-layer integral operator

The complementary double-layer integral operator is given by

$$\mathbf{K}_{\chi^2,0} : H^{1/2}(S) \to H^{1/2}(S), \quad \mathbf{K}_{\chi^2,0} = \mathbf{K}_{\chi^2} - \mathbf{K}.$$

Referring to a chart of some local, regular parametric representation $\mathbf{x}(s)$ of S, $s = (s_1, s_2)$, and using the formula (7.2), it follows that there exist constants $c_1 > 0$ and $c_2 > 0$ such that for each $\Phi \in L^q(S)$ and $\mathbf{x} \in S$ we have

$$\begin{aligned}\left|\left(\frac{\partial}{\partial s_\nu}\mathbf{K}_{\chi^2,0}\Phi\right)(\mathbf{x})\right| &\le c_1 \int_S \frac{1}{|\hat{\mathbf{x}}|}|\Phi(\mathbf{y})|dS(\mathbf{y}) \\ &\le c_2 \left[\int_S \frac{1}{|\widehat{x}|^{q'}}dS(\mathbf{y})\right]^{1/q'} \|\Phi\|_{L^q(S)}, \end{aligned} \tag{7.5}$$

where q and q' are positive numbers satisfying $2 < q < 4$ and

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Therefore, $q' < 2$ and, hence, there exists a constant $c > 0$ such that

$$\int_S \frac{1}{|\hat{\mathbf{x}}|^{q'}}dS(\mathbf{y}) < c. \tag{7.6}$$

From (7.5) and (7.6) we obtain the estimate

$$\left|\left(\frac{\partial}{\partial s_\nu}\mathbf{K}_{\chi^2,0}\Phi\right)(\mathbf{x})\right| \le C_1\|\Phi\|_{L^q(S)}, \tag{7.7}$$

where the constant $C_1 > 0$ is independent of $s = (s_1, s_2)$ and $\Phi \in L^q(S)$.

Further, in view of (7.7), we obtain the estimate

$$\left\|\frac{\partial}{\partial s_\nu}\mathbf{K}_{\chi^2,0}\Phi\right\|_{L^2(S)} \le C\|\Phi\|_{L^q(S)} \quad \text{for all} \quad \Phi \in L^q(S),$$

where the constant $C > 0$ does not depend on $s = (s_1, s_2)$ or $\Phi \in L^q(S)$.

Therefore, the operators

$$\frac{\partial}{\partial s_\nu}\mathbf{K}_{\chi^2,0} : L^q(S) \to L^2(S), \quad \nu = 1, 2 \tag{7.8}$$

are continuous and, in view of the compact imbedding

$$L^2(S) \overset{\text{comp}}{\hookrightarrow} H^{-1/2}(S) \tag{7.9}$$

and Lemma 7.1, we deduce that the following linear integral operators:

$$\frac{\partial}{\partial s_\nu}\mathbf{K}_{\chi^2,0} : H^{1/2}(S) \overset{\text{comp}}{\longrightarrow} L^2(S) \overset{\text{comp}}{\hookrightarrow} H^{-1/2}(S), \quad \nu = 1, 2$$

are compact.

In addition, from (7.8) and Lemma 7.1 it follows the compactness property of the operator

$$\mathbf{K}_{\chi^2,0} : H^{1/2}(S) \to H^1(S)$$

and, since the inclusion

$$H^1(S) \overset{\text{comp}}{\hookrightarrow} H^{1/2}(S)$$

is compact, we find that the complementary double-layer integral operator

$$\mathbf{K}_{\chi^2,0} : H^{1/2}(S) \overset{\text{comp}}{\longrightarrow} H^1(S) \overset{\text{comp}}{\hookrightarrow} H^{1/2}(S)$$

is also compact as a mapping into $H^{1/2}(S)$, as asserted.

The complementary hypersingular boundary integral operator

Next, we show that the integral operator

$$\mathbf{D}_{\chi^2,0} : H^{1/2}(S) \to H^{-1/2}(S), \quad \mathbf{D}_{\chi^2,0} = \mathbf{D}_{\chi^2} - \mathbf{D}$$

given by the formula (4.21), is compact. Note that $\mathbf{D}_0$ corresponds to the Stoke system (i.e., to the case $\chi = 0$), and

$$\begin{aligned}
(\mathbf{D}_{\chi^2,0}\Phi)_j(\mathbf{x}) &= \Big(\mathbf{T}(\mathbf{W}_{\chi^2,0}(\Phi))\Big)_j(\mathbf{x}) \\
&= n_\ell(\mathbf{x})\int_S \left\{ -\Lambda_{ik}^{\chi^2,0}(\widehat{\mathbf{x}})n_k(\mathbf{y})\delta_{jl} + \frac{\partial K_{i\ell}^{\chi^2,0}(\widehat{\mathbf{x}})}{\partial x_j} + \frac{\partial K_{ij}^{\chi^2,0}(\widehat{\mathbf{x}})}{\partial x_\ell} \right\} \phi_i(\mathbf{y})dS(\mathbf{y}) \\
&= -\int_S \Lambda_{ik}^{\chi^2,0}(\widehat{\mathbf{x}})n_k(\mathbf{y})n_j(\mathbf{x})\phi_i(\mathbf{y})dS(\mathbf{y}) + \int_S \frac{\partial K_{i\ell}^{\chi^2,0}(\widehat{\mathbf{x}})}{\partial x_j} n_\ell(\mathbf{x})\phi_i(y)dS(\mathbf{y}) \\
&\quad + \int_S \frac{\partial K_{ij}^{\chi^2,0}(\widehat{\mathbf{x}})}{\partial x_\ell}\phi_i(\mathbf{y})n_\ell(\mathbf{x})dS(\mathbf{y}) \quad \text{for all } \Phi = (\phi_1, \phi_2, \phi_3) \in H^{1/2}(S).
\end{aligned}$$

Now, let $q \in (2, 4)$. According to the estimates (7.3) and (7.4), we deduce that there exist constants $c_3 > 0$ and $c_4 > 0$ such that the following estimates hold:

$$|(\mathbf{D}_{\chi^2,0}\Phi)(\mathbf{x})| \le c_3 \int_S \frac{1}{|\widehat{\mathbf{x}}|}|\Phi(\mathbf{y})|dS(\mathbf{y}) \le c_4\|\Phi\|_{L^q(S)},$$

for all $\Phi \in L^q(S)$. Hence we have the estimate

$$\|\mathbf{D}_{\chi^2,0}\Phi\|_{L^2(S)} \leq c\|\Phi\|_{L^q(S)} \quad \text{for all} \quad \Phi \in L^q(S),$$

where the constant $c > 0$ does not depend on $\Phi \in L^q(S)$. Therefore, the linear integral operator

$$\mathbf{D}_{\chi^2,0} : L^q(S) \to L^2(S)$$

is continuous. Using again Lemma 7.1 it follows that

$$\mathbf{D}_{\chi^2,0} : H^{1/2}(S) \to L^2(S)$$

is compact and, in view of the compact imbedding property (7.9), we find that the boundary integral operator

$$\mathbf{D}_{\chi^2,0} : H^{1/2}(S) \stackrel{\text{comp}}{\longrightarrow} L^2(S) \stackrel{\text{comp}}{\hookrightarrow} H^{-1/2}(S)$$

is indeed compact as a mapping into $H^{-1/2}(S)$.

The complementary single-layer integral operator

Let us now prove that the complementary single-layer operator

$$\mathcal{V}_{\chi^2,0} : H^{-1/2}(S) \to H^{1/2}(S), \quad \mathcal{V}_{\chi^2,0} = \mathcal{V}_{\chi^2} - \mathcal{V},$$

given by the formula

$$(\mathcal{V}_{\chi^2,0}\Psi)_j(\mathbf{x}) = \int_S \mathcal{G}_{ji}^{\chi^2,0}(\widehat{\mathbf{x}})\psi_i(\mathbf{y})dS(\mathbf{y}) \quad \text{for all} \quad \Psi = (\psi_1, \psi_2, \psi_2) \in H^{-1/2}(S),\ \mathbf{x} \in S$$

is compact.

For this purpose, we use the fact that for each $\Psi \in H^{-1/2}(S)$ the Neumann boundary value problem

$$(-\Delta + 1)\mathbf{w} = \mathbf{0} \text{ in } D_0, \quad \frac{\partial \mathbf{w}}{\partial n}\Big|_S = \Psi \tag{7.10}$$

has a unique solution $\mathbf{w} \in H^1(D_0)$ and $\mathbf{w}|_S \in H^{1/2}(S)$ (see [3]).

Therefore, considering the complementary single-layer potential $\mathbf{V}_{\chi^2,0}(\cdot, \Psi)$ with the density Ψ, i.e.,

$$\mathbf{V}_{\chi^2,0}(\cdot, \Psi) = \mathbf{V}_{\chi^2}(\cdot, \Psi) - \mathbf{V}_0(\cdot, \Psi),$$

where $\mathbf{V}_0(\cdot, \Psi)$ is the single-layer potential corresponding to the Stokes system, and applying the second Green's formula for the operator $-\Delta + 1$, we obtain

$$\begin{aligned}(\mathbf{V}_{\chi^2,0})_j(\mathbf{x}, \Psi) &= \int_S \mathcal{G}_{ji}^{\chi^2,0}(\widehat{\mathbf{x}})\frac{\partial w_i(\mathbf{y})}{\partial n(\mathbf{y})}dS(\mathbf{y}) \\ &= \int_S w_i\big|_S(\mathbf{y})\frac{\partial \mathcal{G}_{ji}^{\chi^2,0}(\widehat{\mathbf{x}})}{\partial n(\mathbf{y})}dS(\mathbf{y}) + \int_{D_0} [(-\Delta + 1)\mathcal{G}_{ji}^{\chi^2,0}(\widehat{\mathbf{x}})]w_i(\mathbf{y})d\mathbf{y},\ \mathbf{x} \in D_0.\end{aligned} \tag{7.11}$$

Let us now consider the decomposition

$$\int_{D_0} [(-\Delta + 1)\mathcal{G}_{ji}^{\chi^2,0}(\widehat{\mathbf{x}})]w_i(\mathbf{y})d\mathbf{y} = (\mathbf{K}_1\mathbf{w})_j(\mathbf{x}) + (\mathbf{K}_2\mathbf{w})_j(\mathbf{x}), \quad \mathbf{x} \in D_0, \tag{7.12}$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are the volume integral operators defined on $H^1(D_0)$ by the relations

$$(\mathbf{K}_1\mathbf{w})_j(\mathbf{x}) = -\int_{D_0} (\Delta_{\mathbf{y}}\mathcal{G}_{ji}^{\chi^2,0}(\widehat{\mathbf{x}}))w_i(\mathbf{y})d\mathbf{y}, \quad \mathbf{w} \in H^1(D_0)$$

$$(\mathbf{K}_2\mathbf{w})_j(\mathbf{x}) = \int_{D_0} \mathcal{G}_{ji}^{\chi^2,0}(\widehat{\mathbf{x}})w_i(\mathbf{y})d\mathbf{y}, \quad \mathbf{w} \in H^1(D_0).$$

The formula (7.1) shows that $\mathbf{K}_1$ admits a pseudohomogeneous kernel expansion and, hence, it is a pseudodifferential operator in $\mathbb{R}^3$ of order -2. Therefore, the mapping

$$\mathbf{K}_1 : H^0_{\text{comp}}(\mathbb{R}^3) \to H^2_{\text{loc}}(\mathbb{R}^3)$$

is continuous (see [13]).

Now, considering the extension $\widetilde{\mathbf{w}}$ of $\mathbf{w} \in H^1(D_0) \overset{\text{comp}}{\hookrightarrow} H^0(D_0) = L^2(D_0)$, given by

$$\widetilde{\mathbf{w}} := \begin{cases} \mathbf{w} & \text{in } \overline{D}_0 \\ \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{D}_0, \end{cases} \tag{7.13}$$

we deduce that $\widetilde{\mathbf{w}} \in H^0_{\text{comp}}(\mathbb{R}^3) = \{\mathbf{v} \in H^0(\mathbb{R}^3) \mid \mathbf{v} \text{ has compact support in } \mathbb{R}^3\}$. Therefore, we have the property

$$\mathbf{K}_1\widetilde{\mathbf{w}} \in H^2_{\text{loc}}(\mathbb{R}^3) \hookrightarrow H^{3/2}_{\text{loc}}(\mathbb{R}^3) \overset{\gamma_0}{\longmapsto} H^1(S),$$

where the imbedding $H^2_{\text{loc}}(\mathbb{R}^3) \hookrightarrow H^{3/2}_{\text{loc}}(\mathbb{R}^3)$ is compact and the Gagliardo trace mapping $\gamma_0 : H^{3/2}_{\text{loc}}(\mathbb{R}^3) \to H^1(S)$ is continuous (see Lemma 3.1).

Consequently, $\gamma_0\mathbf{K}_1\widetilde{\mathbf{w}} \in H^1(S)$. In addition, the mapping $\gamma_0\mathbf{K}_1 : H^0_{\text{comp}}(\mathbb{R}^3) \to H^1(S)$ is compact, as well as the mapping

$$\gamma_0\mathbf{K}_1 : H^1(D_0) \to H^1(S).$$

Since $\mathbf{K}_2$ is a pseudodifferential operator in $\mathbb{R}^3$ of order -4, the mapping

$$\mathbf{K}_2 : H^0_{\text{comp}}(\mathbb{R}^3) \to H^4_{\text{loc}}(\mathbb{R}^3)$$

is continuous, and, hence,

$$\mathbf{K}_2\widetilde{\mathbf{w}} \in H^4_{\text{loc}}(\mathbb{R}^3) \overset{\text{comp}}{\hookrightarrow} H^{3/2}(\mathbb{R}^3) \overset{\gamma_0}{\longmapsto} H^1(S).$$

Thus, $\gamma_0\mathbf{K}_2\widetilde{\mathbf{w}} \in H^1(S)$. In addition, the mapping $\gamma_0\mathbf{K}_2 : H^0_{\text{comp}}(\mathbb{R}^3) \to H^1(S)$ is compact, as well as the mapping

$$\gamma_0\mathbf{K}_2 : H^1(D_0) \to H^1(S).$$

Let us now consider the first of the integral operators on the right hand side of (7.11), which has a bounded kernel, namely

$$\mathbf{A}_{\chi^2,0} : H^{1/2}(S) \to L^2(D_0), \tag{7.14}$$

given by the formula

$$(\mathbf{A}_{\chi^2,0}\mathbf{h})(\mathbf{x}) = \int_S \frac{\partial\mathcal{G}^{\chi^2,0}(\widehat{\mathbf{x}})}{\partial n(\mathbf{y})} \cdot \mathbf{h}(\mathbf{y})dS(\mathbf{y}) \quad \text{for all } \mathbf{h} \in H^{1/2}(S), \quad \mathbf{x} \in D_0. \tag{7.15}$$

According to the fact that the kernel of the linear operator $\gamma_0 \mathbf{A}_{\chi^2,0}$ has a similar expansion to that in (7.2), the derivatives $\frac{\partial}{\partial s_\nu}(\gamma_0 \mathbf{A}_{\chi^2,0})$, $\nu = 1, 2$, define operators with weakly singular kernels. Then $\frac{\partial}{\partial s_\nu}(\gamma_0 \mathbf{A}_{\chi^2,0}) : L^q(S) \to L^2(S)$, $\nu = 1, 2$, are bounded. In addition, the imbedding $H^{\frac{1}{2}}(S) \hookrightarrow L^q(S)$ for $2 < q < 4$ is compact. Therefore, $\gamma_0 \mathbf{A}_{\chi^2,0} : H^{1/2}(S) \to H^1(S)$ is a compact operator.

Now, let us consider the isomorphism

$$\mathcal{F} : H^{-1/2}(S) \to H^1(D_0) \tag{7.16}$$

mapping an arbitrary element $\Psi \in H^{-1/2}(S)$ onto an element $\mathbf{w} \in H^1(D_0)$, which is the unique solution of the boundary value problem (7.10). Consequently, the mapping

$$(\gamma_0 \mathbf{A}_{\chi^2,0})(\gamma_0 \mathcal{F}) : H^{-1/2}(S) \to H^1(S) \tag{7.17}$$

is compact.

Repeating the previous arguments, we deduce that the following mapping:

$$\big(\gamma_0(\mathbf{K}_1 + \mathbf{K}_2)\big)\mathcal{F} : H^{-1/2}(S) \to H^1(S) \tag{7.18}$$

is compact.

Now, taking into account the relations (7.11) and (7.12), we deduce that the complementary single-layer integral operator

$$\mathcal{V}_{\chi^2,0} = (\gamma_0 \mathbf{A}_{\chi^2,0})(\gamma_0 \mathcal{F}) + \big(\gamma_0(\mathbf{K}_1 + \mathbf{K}_2)\big)\mathcal{F} : H^{-1/2}(S) \stackrel{\text{comp}}{\longrightarrow} H^1(S) \stackrel{\text{comp}}{\hookrightarrow} H^{1/2}(S)$$

is compact as a mapping into $H^{1/2}(S)$, as asserted.

The adjoint of the complementary double-layer integral operator

Let us now refer to the adjoint of the complementary double-layer integral operator, i.e., to the integral operator

$$\mathbf{K}'_{\chi^2,0} : H^{-1/2}(S) \to L^2(S),$$

given by (4.20), and show that it is compact.

Note that for each $\Psi = (\psi_1, \psi_2, \psi_3) \in H^{-1/2}(S)$ we have

$$\begin{aligned}(\mathbf{K}'_{\chi^2,0}\Psi)_j(\mathbf{x}_0) &= \int_S K_{ji}^{\chi^2,0}(\mathbf{x}_0, \mathbf{y})\psi_i(\mathbf{y})dS(\mathbf{y}) \\ &= \int_S S_{jik}^{\chi^2,0}(\mathbf{x}_0 - \mathbf{y})n_k(\mathbf{x}_0)\psi_i(\mathbf{y})dS(\mathbf{y}), \quad \mathbf{x}_0 \in S,\end{aligned}$$

and

$$\begin{aligned}\int_S S_{jik}^{\chi^2,0}(\mathbf{x} - \mathbf{y})\psi_i(\mathbf{y})dS(\mathbf{y}) &= \int_S \left\{ \frac{\partial \mathcal{G}_{ji}^{\chi^2,0}(\mathbf{x} - \mathbf{y})}{\partial x_k} + \frac{\partial \mathcal{G}_{ki}^{\chi^2,0}(\mathbf{x} - \mathbf{y})}{\partial x_j} \right\} \psi_i(\mathbf{y})dS(\mathbf{y}) \\ &= \frac{\partial}{\partial x_k}(\mathbf{V}_{\chi^2,0}(\mathbf{x}, \Psi))_j + \frac{\partial}{\partial x_j}(\mathbf{V}_{\chi^2,0}(\mathbf{x}, \Psi))_k, \quad \mathbf{x} \in D_0.\end{aligned}$$

Therefore,

$$\mathbf{K}'_{\chi^2,0}\Psi = \Big\{ \gamma_0 \Big(\nabla_{\mathbf{x}} \mathbf{V}_{\chi^2,0}(\cdot, \Psi) + \big(\nabla_{\mathbf{x}} \mathbf{V}_{\chi^2,0}(\cdot, \Psi)\big)^T \Big) \Big\} \cdot \mathbf{n} \ \text{ for all } \ \Psi \in H^{-1/2}(S), \tag{7.19}$$

where the superscript T means the transpose of a matrix.

On the other hand, according to the formula (7.11), we find that

$$\nabla_{\mathbf{x}} \mathbf{V}_{\chi^2,0}(\mathbf{x}, \Psi) = \big(\mathbf{K}_3 \mathbf{w}\big|_S\big)(\mathbf{x}) + \Big((\mathbf{K}_4 + \mathbf{K}_5)\mathbf{w}\Big)(\mathbf{x}), \quad \mathbf{x} \in D_0, \tag{7.20}$$

where $\mathbf{w} \in H^1(D_0)$ is the unique solution of the boundary value problem (7.10) with given boundary data $\Psi \in H^{-1/2}(S)$. In addition, $\mathbf{K}_j$, $j = 3, 4, 5$, are the integral operators given for each $\mathbf{x} \in D_0$ by

$$\big(\mathbf{K}_3 \mathbf{w}\big|_S\big)(\mathbf{x}) = \int_S \mathbf{w}\big|_S(\mathbf{y}) \cdot \nabla_{\mathbf{x}} \frac{\partial \mathcal{G}^{\chi^2,0}(\widehat{\mathbf{x}})}{\partial n(\mathbf{y})} dS(\mathbf{y}),$$

$$\big(\mathbf{K}_4 \mathbf{w}\big)(\mathbf{x}) = -\int_{D_0} \big[\Delta_{\mathbf{y}} \nabla_{\mathbf{x}} \mathcal{G}^{\chi^2,0}(\widehat{\mathbf{x}})\big] \cdot \mathbf{w}(\mathbf{y}) d\mathbf{y},$$

$$\big(\mathbf{K}_5 \mathbf{w}\big)(\mathbf{x}) = \int_{D_0} \big[\nabla_{\mathbf{x}} \mathcal{G}^{\chi^2,0}(\widehat{\mathbf{x}})\big] \cdot \mathbf{w}(\mathbf{y}) d\mathbf{y}.$$

Further, we take into account that the kernel of $\gamma_0 \mathbf{K}_3$ is weakly singular. Hence, for any $q \in (2, 4)$ the linear mapping $\gamma_0 \mathbf{K}_3 : L^q(S) \to L^2(S)$ is continuous, and, in view of Lemma 7.1, $\gamma_0 \mathbf{K}_3 : H^{1/2}(S) \to L^2(S)$ is a compact operator. So, with $\mathbf{w}\big|_S = \gamma_0 \mathcal{F} \Psi$, we find that

$$(\gamma_0 \mathbf{K}_3)(\gamma_0 \mathcal{F}) : H^{-1/2}(S) \overset{\text{comp}}{\longrightarrow} L^2(S) \overset{\text{comp}}{\hookrightarrow} H^{-1/2}(S),$$

where $\mathcal{F}$ is the mapping given in (7.16).

Also, since $\mathbf{K}_4$ is a pseudodifferential operator of order -1 in $\mathbb{R}^3$ we have with $\widetilde{\mathbf{w}}$ given in (7.13) that $\mathbf{K}_4 \widetilde{\mathbf{w}} \in H^1_{\text{loc}}(\mathbb{R}^3)$ and with Gagliardo's trace lemma the property

$$\gamma_0 \mathbf{K}_4 : H^1(D_0) \overset{\text{cont}}{\longrightarrow} H^{1/2}(S) \overset{\text{comp}}{\hookrightarrow} L^2(S),$$

i.e.,

$$\big(\gamma_0 \mathbf{K}_4)\mathcal{F} : H^{-1/2}(S) \overset{\text{cont}}{\longrightarrow} H^{1/2}(S) \overset{\text{comp}}{\hookrightarrow} L^2(S).$$

Similarly, since $\mathbf{K}_5$ is a pseudodifferential operator of order -4 in $\mathbb{R}^3$, one obtains that

$$\gamma_0 \mathbf{K}_5 : H^1(D_0) \overset{\text{comp}}{\longrightarrow} H^1(S) \overset{\text{comp}}{\hookrightarrow} L^2(S),$$

and hence we deduce the property

$$\big(\gamma_0 \mathbf{K}_5)\mathcal{F} : H^{-1/2}(S) \overset{\text{comp}}{\longrightarrow} H^1(S) \overset{\text{comp}}{\hookrightarrow} L^2(S).$$

Now, using the decomposition formula (7.20), as well as the compactness properties of the linear operators $(\gamma_0 \mathbf{K}_3)(\gamma_0 \mathcal{F})$, $(\gamma_0 \mathbf{K}_4)\mathcal{F}$ and $(\gamma_0 \mathbf{K}_5)\mathcal{F}$ into the space $L^2(S)$, we deduce that the linear operator

$$\mathbf{B}_{\chi^2,0} : H^{-1/2}(S) \to L^2(S),$$

given by

$$\begin{aligned} \mathbf{B}_{\chi^2,0} \Psi &:= \Big(\gamma_0 \big(\nabla \mathbf{V}_{\chi^2,0}(\cdot, \Psi)\big)\Big) \cdot \mathbf{n} \\ &= \Big\{\Big((\gamma_0 \mathbf{K}_3)(\gamma_0 \mathcal{F}) + (\gamma_0 \mathbf{K}_4)\mathcal{F} + (\gamma_0 \mathbf{K}_5)\mathcal{F}\Big)\Psi\Big\} \cdot \mathbf{n} \quad \text{for all} \quad \Psi \in H^{-1/2}(S), \end{aligned} \tag{7.21}$$

is compact. Similarly, the operator $\mathbf{C}_{\chi^2,0} : H^{-1/2}(S) \to L^2(S)$, given by

$$\mathbf{C}_{\chi^2,0}\Psi := \Big(\gamma_0\big(\nabla \mathbf{V}_{\chi^2,0}(\cdot,\Psi)\big)^T\Big)\cdot \mathbf{n} \quad \text{for all} \quad \Psi \in H^{-1/2}(S),$$

is also compact.

Finally, taking into account the relation (7.19), we conclude that the linear operator

$$\mathbf{K}'_{\chi^2,0} = \mathbf{B}_{\chi^2,0} + \mathbf{C}_{\chi^2,0} : H^{-1/2}(S) \stackrel{\text{comp}}{\longrightarrow} L^2(S) \stackrel{\text{comp}}{\hookrightarrow} H^{-1/2}(S)$$

is compact as a mapping into $H^{-1/2}(S)$.

This completes the proof of compactness results in Theorem 4.2 a).

Since the proof of the corresponding compactness results in the case of a Lyapunov boundary can be obtained with similar arguments as above (see also [21]), we omit it.

The proof of the last part of Theorem 4.2 can be consulted in [41] and [13].

References

[1] R.A. Adams, Sobolev Spaces (Academic Press, New York, 1975).

[2] A. H-D Cheng and D. Ouazar, Groundwater, Chapter 8 in: Boundary Element Techniques in Geomechanics, edited by G. D. Manolis and T. G. Davies (CMP/Elsevier, 1993), pp. 243-294.

[3] M. Costabel, Boundary integral operators on Lipschitz domains: Elementary results, SIAM J. Math. Anal. **19**, 613-626 (1988).

[4] M. Costabel and E. P. Stephan, Integral equations for transmission problem in linear elasticity, J. Integral Equations Appl. **2**, 211–223 (1990).

[5] R. H. Davis and H. A. Stone, Flow through beds of porous particles, Chem. Engng. Sci. **48**, 3393–4005 (1993).

[6] E. B. Fabes, C. E. Kenig and G. C. Verchota, The Dirichlet problem for the Stokes system on Lipschitz domains Duke Math. J. **57**, 769–793 (1988).

[7] E. B. Fabes, C. E. Kenig and G. C. Verchota, Boundary value problems for the systems of elastostatics in Lipschitz domains, Duke Math. J. **57**, 795–818 (1988).

[8] T. M. Fischer, An integral equation procedure for the exterior three-dimensional slow viscous flow, Integral Equations Operator Theory. **5**, 490–505, (1982).

[9] T. M. Fischer, G. C. Hsiao and W. L. Wendland, Singular perturbations for the exterior three–dimensional slow viscous flow problem, J. Math. Anal. Appl. **110**, 583–603 (1984).

[10] E. Gagliardo, Proprietà di alcune classi di funzioni in piu variabili, Ricerche Mat. **7**, 102-137 (1958)

[11] P. Grisvard, Boundary Value Problems in Non-Smooth Domains (Pitman, London, 1985).

[12] J. J. L. Higdon and M. Kojima, On the calculation of Stokes flow past porous particles, Int. J. Multiphase Flow, **7**, 719–727 (1981).

[13] G. C. Hsiao and W. L. Wendland, Boundary Integral Equations: Variational Methods (Springer–Verlag: Heidelberg, submitted).

[14] G. W. Jackson and D. F. James, The permeability of fibrous porous media, Can. J. Chem. Engng. **64**, 364–374 (1986).

[15] M. Kohr, A mixed boundary value problem for the unsteady Stokes system in a bounded domain in $\mathbb{R}^n$, Eng. Anal. Bound. Elem. **29**, 936–943 (2005).

[16] M. Kohr, The Dirichlet problems for the Stokes resolvent equations in bounded and exterior domains in $\mathbb{R}^n$, Math. Nachr. **280**, 534–559 (2007).

[17] M. Kohr, The interior Neumann problem for the Stokes resolvent system in a bounded domain in $\mathbb{R}^n$, Arch. Mech. **3** (2007). To appear.

[18] M. Kohr and I. Pop, Viscous Incompressible Flow for Low Reynolds Numbers (WIT Press, Southampton (UK), 2004).

[19] M. Kohr and G. P. Raja Sekhar, Existence and uniqueness result for two-dimensional porous media flows with porous inclusions based on Brinkman equation, Eng. Anal. Bound. Elem. To appear.

[20] M. Kohr and G. P. Raja Sekhar, Existence and uniqueness result for the problem of viscous flow in a granular material with a void. Submitted.

[21] M. Kohr, G. P. Raja Sekhar and W. L. Wendland, Boundary integral method for Stokes flow past a porous body. Submitted.

[22] M. Kohr and W. L. Wendland, Variational boundary integral equations for the Stokes system, Appl. Anal. **85**, 1343–1372 (2006).

[23] R. Kress, Linear Integral Equations (Springer, Berlin, 1989).

[24] J. A. Liggette and P. L. F. Liu, The Boundary Integral Equation Method for Porous Media Flow (George Allen and Unwinn, London, 1983).

[25] B. S. Padmavathi, T. Amaranath and S. D. Nigam, Stokes flow past a porous sphere using Brinkman's model, Z. Angew. Math. Phys. **44**, 929–939 (1993).

[26] D. Palanippan, Arbitrary Stokes flow past a porous sphere, Mech. Res. Comm. **20**, 309–317 (1993).

[27] Yu Qin and P. N. Kaloni, Creeping flow past a porous spherical shell, Z. Angew. Math. Mech. **73**, 77–84 (1983).

[28] Yu Qin and P. N. Kaloni, A Cartesian-tensor solution of the Brinkman equation, J. Engng. Math. **22**, 177–188 (1988).

[29] G. P. Raja Sekhar, B. S. Padmavathi and T. Amaranath, Complete general solution of the Brinkman Equations, Z. Angew. Math. Mech. **77**, 555–556 (1997).

[30] G. P. Raja Sekhar and T. Amaranath, A Stokes flow inside a porous spherical shell, Z. Angew. Math. Phys. **51**, 481–490 (2000).

[31] H. Power and L. C. Wrobel, Boundary Integral Methods in Fluid Mechanics (WIT Press: Computational Mechanics Publications, Southampton, 1995).

[32] C. Pozrikidis, Boundary Integral and Singularity Methods for Linearized Viscous Flow (Cambridge University Press, Cambridge, 1992).

[33] G. P. Raja Sekhar, M. K. Partha and P. V. S. N. Murthy, Viscous flow past a spherical void in porous media - Effect of stress jump boundary condition, J. Porous Media, **9**, 745–767 (2006).

[34] G. P. Raja Sekhar and O. Sano, Viscous flow past a circular /spherical void in porous medium-an application to measurement of the velocity of ground water by single boring method, J. Phys. Soc. Jpn. **69**, 37–42 (2000).

[35] G. P. Raja Sekhar and O. Sano, Two - dimensional viscous flow past a slightly deformed circular cavity in a porous medium, Fluid. Dyn. Res. **28**, 281–293 (2001).

[36] G. P. Raja Sekhar and O. Sano, Two-dimensional viscous flow in a granular material with a void of arbitrary shape, Phys. Fluids, **15**, 554–567 (2003).

[37] O. Sano, Viscous flow past a cylidrical hole bored inside porous media - with application to measurement of the velocity of subterranean water by the single boring method, Nagare. **2**, 252–259 (1983).

[38] L. Spielman and S. L. Goren, Model for predicting pressure drop and filtration efficiency in fibrous media, Environ. Sci. Technol. **2**, 279–287 (1968).

[39] M. E. Taylor, Pseudodifferential Operators (Princeton University Press, Princeton, 1981).

[40] R. Y. Tsay and S. Weinbaum, Viscous flow in a channel with periodic cross-bridging fibres: exact solutions and Brinkman approximation, J. Fluid Mech. **226**, 125–148 (1991).

[41] W. Varnhorn, An explicit potential theory for the Stokes resolvent boundary value problems in three dimensions, Manuscr. Math. **70**, 339–361 (1991).

[42] W. Varnhorn, The Stokes Equations (Akademie Verlag, Berlin, 1994).

[43] S. Weinbaum, R. Tsay and F. E. Curry, A three-dimensional junction-pore-matrix for capillary permeability, Microvasc. Res. **44**, 85 (1992).

[44] T. Zlatanovski, Axisymmetric creeping flow past a porous prolate spheroidal particle using the Brinkman model, Q. J. Mech. Appl. Math. **52**, 111–126 (1999).

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