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Institut für Angewandte Analysis und Numerische Simulation (IANS)
Fakultät Mathematik und Physik
Fachbereich Mathematik
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: ians-preprints@mathematik.uni-stuttgart.de
WWW: http://preprints.ians.uni-stuttgart.de

ISSN **1611-4176**

ON A TIME-DEPENDENT FLUID-SOLID COUPLING IN 3D WITH NONSTANDARD BOUNDARY CONDITIONS

CHRISTINA SURULESCU

Abstract. We study the motion of a Stokes fluid through an elastic half cylinder with thickness. The fluid is driven by a small time-dependent pressure drop between the outflow and the inflow ends of the tube. We consider small displacements of the elastic structure, thus the domains involved are not moving in time. We prove existence and uniqueness of a weak solution for this three dimensional fluid-elastic structure interaction problem. Under supplementary regularity assumptions on the data we obtain strong energy estimates and show th existence of a pressure with improved properties.

1. Introduction

We consider a time-dependent fluid-structure interaction problem in 3D: a viscous incompressible fluid flows through an elastic tube with thickness. The flow is driven by the difference of the pressures at the ends of the tube (as in Jäger & Mikelić [18], Conca, Murat & Pironneau [8] or Čanić & Mikelić [5]). We suppose that the pressure drop between the inflow and the outflow is small and that the viscous effects of the fluid are strongly predominant when compared to the inertial ones. The displacements of the elastic wall are assumed to be small, so that we can consider the fluid-structure interface (and thus the involved domains) as being fixed. Furthermore, all deformations are supposed to happen only in the normal direction.

Thus, we model the fluid by the Stokes equations, the behavior of the elastic structure is described by the Lamé equations for linearized elasticity and we deal with cylindrical domains. Concerning the boundary conditions, the coupling is expressed through the equilibrium of surface forces and the continuity of velocities at the interface. The elastic wall is considered to be clamped on its entire boundary, excepting the interface between the two media and boundary conditions involving the pressure are taken for the fluid at the ends of the tube. These are nonstandard boundary conditions for a fluid flow; for other references on this type of conditions, though in different contexts, see e.g., [1], [4], [20]. We show the existence of a unique weak solution to the coupled problem described above. Moreover, for smooth enough data improved regularity is available for the solution and the existence of a pressure satisfying the fluid equations in the sense of distributions is proved.

In the remaining of this section we give a short overview of some related works on mathematical analysis of fluid-structure interaction problems. Starting with *the stationary case*, we refer to [16] and [2] for a 2D fluid interacting with a 1D elastic structure and to [17], [23], [24] and [22] for 3D models. In *the time dependent case* we distinguish between models dealing with cylindrical domains and models where the domains are moving in time. For the former ones see for instance [5] (handling a problem similar to the one considered here, however in a 2D/1D setting). Concerning the latter case, in [6] is studied the interaction between a Navier-Stokes fluid contained in a cavity with an elastic plate as cover and having the rest of the boundary fixed and rigid (3D/2D problem). In [22] similar problems are treated, namely a Navier-Stokes fluid flowing through a box with an elastic cover and having inflow and outflow sections (with boundary conditions involving the pressure), respectively a Navier-Stokes fluid moving in a cylinder bounded by a thin

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Date: June 6th 2007.

1991 *Mathematics Subject Classification.* 74F10, 35Q30, 76D03.

Key words and phrases. Fluid-structure interaction, Stokes equations, boundary conditions involving the pressure, existence and uniqueness of solutions.

elastic shell and with prescribed velocities at the tube's ends (both in the 3D/2D setting). Other time-dependent fluid-structure interaction problems with time moving domains were considered for instance in [13] (1D fluid, 1D structure), [19],[21], [3] (2D fluid, 1D structure) or in [11], [10], [25] for higher dimensions.

2. Problem setting

Let $\Omega_f := S_f \times (0,L)$ be the fluid domain, where $S_f := \{(a\cos\theta, a\sin\theta) : a \in (0,r),\ \theta \in (0,\pi)\}$ is the half disk centered at 0 and having radius r and L denotes the length of the half cylinder. The domain occupied by the elastic structure is $\Omega_s := S_s \times (0,L)$, with $R > r$ and $S_s := \{(a\cos\theta, a\sin\theta) : a \in (r,R),\ \theta \in (0,\pi)\}$. These subdomains form together the domain $\Omega := \Omega_f \cup \Omega_s$. Let Γ_{fs} denote the fluid-structure interface, $\Gamma_{f,ends,k}$ $(k = 1,2)$ be the fluid boundaries at the ends of the tube, Γ_{ext} be the exterior lateral boundary of the elastic cylinder, $\Gamma_{s,ends} = \Gamma_{s,ends,1} \cup \Gamma_{s,ends,2}$ be its boundaries at the tube's ends, and $\Gamma_{bot} = \Gamma_{f,bot} \cup \Gamma_{s,bot}$ be the bottom part of the boundary; $\Gamma_{f,bot}$ and $\Gamma_{s,bot}$ are the corresponding boundaries for the fluid, respectively the elastic structure parts.

Recall that we characterise the elastic structure with the aid of the Lamé system for linearized elasticity:

$$\partial_{tt}\mathbf{u} - \text{div}\ (\lambda \text{trace}\ \mathbf{e}(\mathbf{u})\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u})) = \mathbf{g} \text{ in } (0,T) \times \Omega_s.$$

This can also be written in the equivalent form:

$$\partial_{tt}\mathbf{u} - \lambda\nabla(\text{div}\ \mathbf{u}) - 2\mu\nabla \cdot \mathbf{e}(\mathbf{u}) = \mathbf{g} \text{ in } (0,T) \times \Omega_s. \tag{1}$$

Here $\lambda,\ \mu > 0$ are the Lamé constants for the considered St. Venant-Kirchhoff elastic material, $\mathbf{e}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$ is Green's linear strain tensor and $\mathbf{g}$ is the given loading force.

The elastic structure is supposed to be clamped on its entire boundary, excepting the interface with the fluid, thus we supplement (1) with the boundary conditions:

$$\mathbf{u} = 0 \text{ on } (0,T) \times (\Gamma_{ext} \cup \Gamma_{s,ends} \cup \Gamma_{s,bot}) \tag{2}$$

and we also have to take some initial conditions for the displacement $\mathbf{u}$ and its velocity:

$$\mathbf{u}(0) = 0, \quad \partial_t\mathbf{u}(0) = \mathbf{u}_{01} \text{ in } \Omega_s \tag{3}$$

Here we assume for simplicity of further writing that there is no initial displacement, however this can be easily extended to the case with $\mathbf{u}(0) \neq 0$.

For the Stokes flow we consider the system:

$$\begin{aligned} \partial_t\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p &= \mathbf{f} \text{ in } (0,T) \times \Omega_f \\ \text{div}\ \mathbf{v} &= 0 \text{ in } (0,T) \times \Omega_f \\ \mathbf{v} \times \mathbf{n} &= 0 \text{ on } (0,T) \times \Gamma_{f,ends} \\ \mathbf{v} &= 0 \text{ on } (0,T) \times \Gamma_{f,bot} \\ p &= 0 \text{ on } (0,T) \times \Gamma_{f,ends,2} \\ p &= P(t) \text{ on } (0,T) \times \Gamma_{f,ends,1} \\ \mathbf{v}(0) &= \mathbf{v}_0 \text{ in } \Omega_f, \end{aligned} \tag{4}$$

where $\mathbf{v}$ stands for the velocity of the fluid, p for the pressure, $\mathbf{f}$ is a given body force and $P(t)$ is the time dependent pressure drop between the inflow and outflow sections.

The coupling conditions ensuring the equilibrium of surface forces and the continuity of velocities at the interface are:

$$\begin{aligned} (\lambda\text{trace}\ \mathbf{e}(\mathbf{u})\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u})) \cdot \mathbf{n}_s &= p \cdot \mathbf{n}_f - \nu(\nabla \times \mathbf{v}) \times \mathbf{n}_f \text{ on } (0,T) \times \Gamma_{fs} \\ \partial_t\mathbf{u} &= \mathbf{v} \text{ on } (0,T) \times \Gamma_{fs}. \end{aligned} \tag{5}$$

Remark 2.1. Here we consider the case of a fixed fluid-structure interface. This can be done when assuming that the displacements of the structure (and thus of the interface) are small enough; this is not the case for large displacements. However, for the viscous fluid sticking to the interface we could not consider a homogeneous Dirichlet condition, since even if the displacements are small, there is no guarantee that their velocity is small, too.

Now, having set the equations, we can state our

Problem 1. *Determine a solution* $(\mathbf{u},\mathbf{v})$ *in* $(0,T)\times\Omega$ *of the system (1)-(3) and (4), together with the coupling conditions (5).*

3. Weak formulation and main result

In this section we give the weak formulation of the coupled problem and state the main result. We consider the following function spaces:

$$\begin{aligned}\mathcal{V} &:= \{\boldsymbol{\varphi}\in\mathcal{D}(\bar{\Omega}) \ : \ \text{div}\ \boldsymbol{\varphi}=0 \text{ in } \Omega_f,\\ &\qquad \boldsymbol{\varphi}\times\mathbf{n}=0 \text{ on } \Gamma_{f,ends},\ \boldsymbol{\varphi}=0 \text{ on } \Gamma_{ext}\cup\Gamma_{s,ends}\cup\Gamma_{bot}\}\\ \mathbf{H}(\Omega) &= \overline{\mathcal{V}}^{(\mathbf{L}^2(\Omega),(\cdot,\cdot)_{f,s})},\ \mathbf{V}(\Omega)=\overline{\mathcal{V}}^{\mathbf{H}^1(\Omega)},\\ \mathbf{V}_f &:= \{\mathbf{v}\in\mathbf{H}^1(\Omega_f) \ : \ \text{div}\ \mathbf{v}=0 \text{ in } \Omega_f,\\ &\qquad \mathbf{v}\times\mathbf{n}=0 \text{ on } \Gamma_{f,ends},\ \mathbf{v}=0 \text{ on } \Gamma_{f,bot}\}.\end{aligned}$$

We shall use throughout the paper the notation

$$\mathbf{H}^{s,1}_{0,\Gamma} := \mathbf{H}^1_{0,\Gamma_{ext}\cup\Gamma_{s,ends}\cup\Gamma_{s,bot}}(\Omega_s).$$

We denote by $(\boldsymbol{\xi},\boldsymbol{\varphi})_{f,s}$ the $\mathbf{L}^2$-inner product

$$(\boldsymbol{\xi},\boldsymbol{\varphi})_{f,s} := (\boldsymbol{\xi},\boldsymbol{\varphi})_{\Omega_f} + (\boldsymbol{\xi},\boldsymbol{\varphi})_{\Omega_s},\ \forall\boldsymbol{\xi},\boldsymbol{\varphi}\in\mathbf{L}^2(\Omega).$$

The norm in $\mathbf{L}^2(\Omega)$ is equivalent to the norm generated by this inner product.

Assume now that

$$\begin{gathered}\mathbf{g}\in\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_s)),\ \mathbf{f}\in\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_f)) \quad \text{and} \quad P\in L^2(0,T;L^2(\Gamma_{f,ends,1})),\\ \mathbf{v}_0\in\mathbf{V}_f,\ \mathbf{u}_{01}\in\mathbf{H}^{s,1}_{0,\Gamma} \quad \text{with} \quad \mathbf{v}_0=\mathbf{u}_{01} \text{ on } \Gamma_{fs}\end{gathered} \tag{6}$$

and let the compatibility condition

$$\int_{\Gamma_{f,ends,2}} v_3 - \int_{\Gamma_{f,ends,1}} v_3 + \int_{\Gamma_{fs}} \mathbf{v}\cdot\mathbf{n} = 0$$

be satisfied, too.

One can prove (like in e.g., [8], see also [15] ch.I, S.3) that the following coercivity condition involving the curl of the fluid's velocity is satisfied:

$$\exists C_{curl}>0 \quad : \quad \forall\mathbf{v}\in\mathbf{L}^2(0,T;\mathbf{V}_f),\ \|\nabla\times\mathbf{v}\|^2_{\mathbf{L}^2(\Omega_f)} \geq C_{curl}\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega_f)}. \tag{7}$$

Also assume (like in [8]) the existence of a function $U_0\in\mathbf{L}^2(0,T;\mathbf{V}_f)$ such that

$$\text{div}\ \mathbf{U}_0 = 0 \text{ in } (0,T)\times\Omega_f \tag{8}$$

$$\mathbf{U}_0 = 0 \text{ on } (0,T)\times\Gamma_{f,bot} \tag{9}$$

$$\mathbf{U}_0\times\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma_{f,ends}. \tag{10}$$

Remark 3.1. The existence of such a function is not obvious, however it can be justified e.g., upon considering it to be the solution of the Stokes problem

$$\begin{aligned}\partial_t\mathbf{U}_0-\nu\Delta\mathbf{U}_0+\nabla p &= 0 \text{ in } (0,T)\times\Omega_f\\ \text{div}\ \mathbf{U}_0 &= 0 \text{ in } (0,T)\times\Omega_f\\ \mathbf{U}_0\times\mathbf{n} &= 0 \text{ on } (0,T)\times\Gamma_{f,ends}\\ \mathbf{U}_0 &= 0 \text{ on } (0,T)\times\Gamma_{f,bot}\\ \mathbf{U}_0 &= \mathbf{h} \text{ on } (0,T)\times\Gamma_{fs}\\ p &= p_0 \text{ on } (0,T)\times\Gamma_{f,ends},\end{aligned} \tag{11}$$

with both $p_0\in L^2(0,T;H^{1/2}(\Omega_f))$ and $\mathbf{h}\in\mathbf{L}^2(0,T;\mathbf{H}^{1/2}(\Gamma_{fs}))$ given.

The proof in [4] can be easily adapted to show the existence of a solution to (11).

The weak formulation of problem (1)-(5) is obtained by testing the equations for the fluid and those for the structure by $\boldsymbol{\varphi} \in \mathbf{V}(\Omega)$. The resulting weak problem reads:

Problem 2. *Find* $(\mathbf{u}, \mathbf{v}) \in \mathbf{L}^2(0,T;\mathbf{H}^{s,1}_{0,\Gamma}) \times \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))$ *such that*

$$\mathbf{v} - \mathbf{U}_0 \in \mathbf{L}^2(0,T;\mathbf{V}_f), \tag{12}$$

$$\frac{d}{dt}((\partial_t \mathbf{u}, \boldsymbol{\varphi})_{\Omega_s} + (\mathbf{v}, \boldsymbol{\varphi})_{\Omega_f}) + a(\mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \times \mathbf{v}, \nabla \times \boldsymbol{\varphi})_{\Omega_f} \tag{13}$$
$$= (\mathbf{g}, \boldsymbol{\varphi})_{\Omega_s} + (\mathbf{f}, \boldsymbol{\varphi})_{\Omega_f} + \int_{\Gamma_{f,ends,1}} P(t)\varphi_3, \ \forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega),$$

where $a(\mathbf{u}, \boldsymbol{\varphi})$ *is the continuous, bilinear form (see* [7]*):*

$$a(\mathbf{u}, \boldsymbol{\varphi}) := \lambda(div\ \mathbf{u}, div\ \boldsymbol{\varphi})_{\Omega_s} + 2\mu(\mathbf{e}(\mathbf{u}), \mathbf{e}(\boldsymbol{\varphi}))_{\Omega_s}$$

$$\mathbf{u}(0) = 0,\ \partial_t \mathbf{u}(0) = \mathbf{u}_{01},\ \mathbf{v}(0) = \mathbf{v}_0 \ and \int_0^t \mathbf{v}(s)ds = \mathbf{u}(t) \ a.e.\ t\ on\ \Gamma_{fs}.$$

Definition 3.1. $(\mathbf{u}, \mathbf{v}) \in \mathbf{L}^2(0,T;\mathbf{H}^{s,1}_{0,\Gamma}) \times \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))$ *is called a weak solution of Problem 1 if for all* $\boldsymbol{\varphi} \in \mathbf{V}(\Omega)$ *it satisfies the variational formulation in (13).*

Next, as in [11] or [12] we define a global velocity, together with its corresponding initial condition and a global exterior force. This will allow us to treat the problem as a whole, unlikely in e.g., [3], [17] or [23], [24], where it was splitted in two subproblems (one for the fluid and one for the elastic structure), each of them being handled separately and eventually ensuring the coupling by a fixed-point procedure.

With the following notations (χ_{Ω_s}, respectively χ_{Ω_f} stand for the characteristic functions of Ω_s, respectively Ω_f):

$$\boldsymbol{\omega} := \partial_t \mathbf{u}\chi_{\Omega_s} + \mathbf{v}\chi_{\Omega_f}, \quad \boldsymbol{\omega}_0 := \mathbf{u}_{01}\chi_{\Omega_s} + \mathbf{v}_0\chi_{\Omega_f} \text{ and } \mathbf{G} := \mathbf{g}\chi_{\Omega_s} + \mathbf{f}\chi_{\Omega_f},$$

we obtain the problem (equivalent to Problem 2):

Problem 3. *Find* $\boldsymbol{\omega}$ *such that* $\mathbf{v} - \mathbf{U}_0 \in \mathbf{L}^2(0,T;\mathbf{V}_f)$ *and*

$$< \partial_t \boldsymbol{\omega}, \boldsymbol{\varphi} >_{f,s} + a(\int_0^t \boldsymbol{\omega}(s)ds, \boldsymbol{\varphi}) + \nu(\nabla \times \boldsymbol{\omega}, \nabla \times \boldsymbol{\varphi})_{\Omega_f} \tag{14}$$
$$= (\mathbf{G}(t), \boldsymbol{\varphi})_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)\varphi_3, \ \forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega) \ a.e.\ t \in [0,T],$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 \ in\ \mathbf{V}'(\Omega) \tag{15}$$

and

$$\int_0^t \boldsymbol{\omega}(s)\chi_{\Omega_s} ds = \int_0^t \boldsymbol{\omega}(s)\chi_{\Omega_f} ds \ on\ \Gamma_{fs},\ a.e.\ t, \tag{16}$$

where $\mathbf{V}'(\Omega)$ *is the dual space of* $\mathbf{V}(\Omega)$.

We have denoted by $< .,. >_{f,s}$ the duality pairing between $\mathbf{V}'(\Omega)$ and $\mathbf{V}(\Omega)$, that is generated from the inner product $(.,.)_{f,s}$. Having in mind the assumptions made on the data of the problem, $\boldsymbol{\omega}_0$ as defined above satisfies $\boldsymbol{\omega}_0 \in \mathbf{V}(\Omega)$ and the initial condition on $\boldsymbol{\omega}$ is equivalent to

$$< \boldsymbol{\omega}(0), \boldsymbol{\varphi} >_{f,s} = (\boldsymbol{\omega}_0, \boldsymbol{\varphi})_{f,s},\ \forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega).$$

We can now state the main result:

Theorem 3.2. *Under the assumptions (6) there exists a unique weak solution of Problem 1.*

4. Proof of the existence

4.1. **Galerkin approximations.**

For the proof we use the Galerkin method (see e.g., Evans [14]), thus we build a weak solution of the problem by first constructing solutions of certain finite dimensional approximations and then passing to limits. We therefore take the functions $\mathbf{w}_k = \mathbf{w}_k(\mathbf{x})$ $(k = 1, 2, ...)$ such that

$$\{\mathbf{w}_k\}_k \text{ is a basis of } \mathbf{V}(\Omega). \tag{17}$$

We take $\{\mathbf{w}_k\}_k$ to be the complete set of eigenfunctions of the eigenvalue problem

$$\mathbf{w} \in \mathbf{V}(\Omega) \ : \ (\nabla \mathbf{w}, \nabla \boldsymbol{\varphi})_{f,s} = \alpha(\mathbf{w}, \boldsymbol{\varphi})_{f,s}, \ \forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega),$$

and assume that $\{\mathbf{w}_i\}_{i=1,2...}$ is orthonormalized with the $\mathbf{H}^1_{0,\Gamma_{ext}\cup\Gamma_{s,ends}\cup\Gamma_{bot}}(\Omega)$ -inner product $(\nabla ., \nabla .)_{f,s}$. Moreover, observe that $\{\mathbf{w}_k\}_k$ is orthogonal w.r.t. the L^2-inner product $(.,.)_{f,s}$.

Now fix a positive integer m and write

$$\boldsymbol{\omega}_m(t) := \sum_{k=1}^{m} c_{km}(t)\mathbf{w}_k, \tag{18}$$

where the coefficients $c_{km}(t)$ $(0 \le t \le T,\ k = 1, \dots, m)$ are taken such that

$$(\boldsymbol{\omega}_m(0), \mathbf{w}_k)_{f,s} = (\boldsymbol{\omega}_0, \mathbf{w}_k)_{f,s} \tag{19}$$

be satisfied.

The Galerkin approximation corresponding to (14) writes $(0 \le t \le T,\ k = 1, ..., m)$:

$$\begin{aligned}(\partial_t \boldsymbol{\omega}_m(t), \mathbf{w}_k)_{f,s} \quad &+ \quad a(\int_0^t \boldsymbol{\omega}_m(s)ds, \mathbf{w}_k) + \nu(\nabla \times \boldsymbol{\omega}_m(t), \nabla \times \mathbf{w}_k)_{\Omega_f} \\ &= \quad (\mathbf{G}(t), \mathbf{w}_k)_{f,s} + \int_{\Gamma_{f,ends,1}} P(t) w_{k,3}.\end{aligned} \tag{20}$$

The coupling condition (16) is clearly satisfied for the Galerkin approximation defined in (18), i.e. we have

$$\int_0^t \boldsymbol{\omega}_m(s)\chi_{\Omega_s} ds = \int_0^t \boldsymbol{\omega}_m(s)\chi_{\Omega_f} ds \text{ on } \Gamma_{fs}, \text{ a.e. } t. \tag{21}$$

Now denoting the right hand side in (20) by $F_k(t)$ observe that the system (19), (20) can be written in the form of a linear ODE system of first order for the Galerkin coefficients $c_{km}(t)$ and for $d_{km}(t) := \int_0^t c_{km}(s)ds$:

$$\begin{aligned}&\sum_{l=1}^{m}(\mathbf{w}_l, \mathbf{w}_k)_{f,s} c'_{lm}(t) + \nu \sum_{l=1}^{m}(\nabla \times \mathbf{w}_l, \nabla \times \mathbf{w}_k)_{\Omega_f} c_{lm}(t) \\ &\qquad + \sum_{l=1}^{m} a(\mathbf{w}_l, \mathbf{w}_k) d_{lm}(t) = F_k(t),\end{aligned} \tag{22}$$

with

$$d'_{lm}(t) = c_{lm}(t), \ l = 1, \dots, m$$

and with the initial conditions

$$\begin{aligned}\sum_{l=1}^{m}(\mathbf{w}_l, \mathbf{w}_k)_{f,s} c_{lm}(0) \quad &= \quad (\boldsymbol{\omega}_0, \mathbf{w}_k)_{f,s}, \\ d_{lm}(0) \quad &= \quad 0, \ l = 1, \dots, m.\end{aligned}$$

By the classical theory of these systems and using the properties of $\{\mathbf{w}_k\}_k$ it follows that there exists a unique solution $(c_{1m}, \dots, c_{mm},\ d_{1m}, \dots, d_{mm}) \in C^1(0, T)$ of (22) with the conditions above. This leads to the existence of a unique solution $\boldsymbol{\omega}_m$ for the system (20) together with the compatibility condition (21).

4.2. **Energy estimates.**

We intend to pass to the limit with $m \to \infty$ in (20) and for this we need some estimates that should be uniform in m. These are given by the following

Proposition 4.1. *There exists a constant $C > 0$ such that*

$$\sup_{0\le t\le T}\Big(\|\boldsymbol{\omega}_m(t)\|^2_{f,s} + \|\int_0^t \boldsymbol{\omega}_m(s)ds\|^2_{\mathbf{H}^1(\Omega_s)}\Big) + \|\boldsymbol{\omega}_m\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))} + \|\boldsymbol{\omega}'_m\|^2_{\mathbf{L}^2(0,T;\mathbf{V}'(\Omega))} \le C(\|G\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|P\|^2_{L^2(0,T;L^2(\Gamma_{f,ends,1}))} + \|\mathbf{u}_{01}\|^2_{\mathbf{H}^1(\Omega_s))} + \|\mathbf{v}_0\|^2_{\mathbf{H}^1(\Omega_f)}). \tag{23}$$

The constant C depends on the fixed $T > 0$, on C_{curl}, ν and on the constants C_{trace} and C_{Korn} in the inequalities (with ζ in the corresponding spaces):

$$\|\zeta\|_{L^2(\Gamma_{f,ends,1})} \le C_{trace}\|\zeta\|_{H^1(\Omega_f)} \text{ (Sobolev embeddings)} \tag{24}$$

$$\|\zeta\|^2_{H^1(\Omega_s)} \le C_{Korn}a(\zeta,\zeta) \text{ (with Korn's inequality, see [7]).} \tag{25}$$

Proof. Multiply (20) by $c_{km}(t)$. Upon summing up after $k = 1, \dots, m$ and taking into account (18), we get:

$$(\partial_t\boldsymbol{\omega}_m(t), \boldsymbol{\omega}_m(t))_{f,s} + a(\int_0^t \boldsymbol{\omega}_m(s)ds, \boldsymbol{\omega}_m(t)) + \nu(\nabla\times\boldsymbol{\omega}_m(t), \nabla\times\boldsymbol{\omega}_m(t))_{\Omega_f} = (\mathbf{G}(t), \boldsymbol{\omega}_m(t))_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)\omega_{m,3}(t),$$

from which we deduce

$$\frac{1}{2}\frac{d}{dt}\Big[\|\boldsymbol{\omega}_m(t)\|^2_{f,s} + a(\int_0^t \boldsymbol{\omega}_m(s)ds, \int_0^t \boldsymbol{\omega}_m(s)ds)\Big] + \nu\|\nabla\times\boldsymbol{\omega}_m(t)\|^2_{\mathbf{L}^2(\Omega_f)} \le \frac{1}{2}\|\mathbf{G}(t)\|^2_{f,s} + \frac{1}{2}\|\boldsymbol{\omega}_m(t)\|^2_{f,s} + \frac{\delta}{2}\|P(t)\|^2_{L^2(\Gamma_{f,ends,1})} + \frac{C^2_{trace}}{2\delta}\|\boldsymbol{\omega}_m(t)\|^2_{\mathbf{H}^1(\Omega_f)}, \tag{26}$$

with the constant δ chosen such that $\delta \ge \frac{C^2_{trace}}{2\nu C_{curl}}$.

By using (7) and applying the differential form of Gronwall's inequality it follows that:

$$\|\boldsymbol{\omega}_m(t)\|^2_{f,s} + a(\int_0^t \boldsymbol{\omega}_m(s)ds, \int_0^t \boldsymbol{\omega}_m(s)ds) \le e^T(\|\boldsymbol{\omega}_m(0)\|^2_{f,s} + \|\mathbf{G}\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} + \delta\|P\|^2_{L^2(0,T;L^2(\Gamma_{f,ends,1}))}).$$

Now with (25) we obtain:

$$\|\boldsymbol{\omega}_m(t)\|^2_{f,s} + C_{Korn}\|\int_0^t \boldsymbol{\omega}_m(s)ds\|^2_{\mathbf{H}^1(\Omega_s)} \le e^T(\|\boldsymbol{\omega}_m(0)\|^2_{f,s} + \|\mathbf{G}\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} + \delta\|P\|^2_{L^2(0,T;L^2(\Gamma_{f,ends,1}))}). \tag{27}$$

Integrate in time in (26) and use again (7) and (27) to deduce that

$$\|\boldsymbol{\omega}_m\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))} \le const(T, C_{curl}, C_{Korn}, C_{trace}, \nu)\cdot \cdot(\|\mathbf{v}_0\|^2_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{u}_{01}\|^2_{\mathbf{L}^2(\Omega_s)} + \|\mathbf{G}\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|P\|^2_{L^2(0,T;L^\infty(\Gamma_{f,ends,1}))}). \tag{28}$$

In order to obtain (23), we still need some estimate for the time derivative of $\boldsymbol{\omega}_m$. For this let $\boldsymbol{\zeta} \in \mathbf{V}(\Omega)$ with $||\boldsymbol{\zeta}||_{\mathbf{H}^1} \leq 1$ and write $\boldsymbol{\zeta} = \mathbf{P}_m\boldsymbol{\zeta} + (\mathbf{I}-\mathbf{P}_m)\boldsymbol{\zeta}$, where $\mathbf{P}_m$ is the projection from $\mathbf{L}^2(\bar{\Omega})$ onto span $\{\mathbf{w}_k\}_{k=1,\dots,m}$, i.e. $\forall\ \boldsymbol{\zeta} \in \mathbf{L}^2(\bar{\Omega})$ it is $(\mathbf{P}_m\boldsymbol{\zeta}, \mathbf{w})_{f,s} = (\boldsymbol{\zeta}, \mathbf{w})_{f,s}$, $\forall \mathbf{w} \in$ span $\{\mathbf{w}_k\}_{k=1,\dots,m}$.

Since $\boldsymbol{\omega}'_m(t) \in$ span $\{\mathbf{w}_k\}_{k=1,\dots,m}$, we have

$$\begin{aligned} < \boldsymbol{\omega}'_m(t), \boldsymbol{\zeta} >_{f,s} &= (\boldsymbol{\omega}'_m(t), \mathbf{P}_m\boldsymbol{\zeta})_{f,s} + (\boldsymbol{\omega}'_m(t), (\mathbf{I}-\mathbf{P}_m)\boldsymbol{\zeta})_{f,s} \\ &= (\boldsymbol{\omega}'_m(t), \mathbf{P}_m\boldsymbol{\zeta})_{f,s}. \end{aligned} \tag{29}$$

Then we can write:

$$< \boldsymbol{\omega}'_m(t), \boldsymbol{\zeta} >_{f,s} = (\mathbf{G}(t), \mathbf{P}_m\boldsymbol{\zeta})_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)(\mathbf{P}_m\boldsymbol{\zeta})_3$$

$$-a(\int_0^t \boldsymbol{\omega}'_m(s)ds, \mathbf{P}_m\boldsymbol{\zeta}) - \nu(\nabla \times \boldsymbol{\omega}_m(t), \nabla \times \mathbf{P}_m\boldsymbol{\zeta})_{\Omega_f},$$

from which it follows that

$$\|\boldsymbol{\omega}'_m\|^2_{\mathbf{L}^2(0,T;\mathbf{V}'(\Omega))} \leq C(T, \nu, C_{curl}, C_{Korn}, C_{trace}) \cdot \tag{30}$$

$$\cdot(\|\mathbf{v}_0\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{u}_{01}\|^2_{\mathbf{H}^1(\Omega_s)} + \|\mathbf{G}\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|P\|^2_{L^2(0,T;L^2(\Gamma_{f,ends,1}))}),$$

upon using (7), (26), (28) and the fact that $\|\mathbf{P}_m\boldsymbol{\zeta}\|_{\mathbf{H}^1} \leq \|\boldsymbol{\zeta}\|_{\mathbf{H}^1} \leq 1$.

Now it is clear that we obtain (23) from (27), (28) and (30). □

4.3. **Existence of a weak solution.**

We now pass to limits (for $m \to \infty$) in the Galerkin approximation.

The estimate (23) implies that:

$$(\boldsymbol{\omega}_m)_m \text{ is bounded in } \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)) \tag{31}$$

$$(\int_0^t \boldsymbol{\omega}_m(s)ds)_m \text{ is bounded in } \mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega_s)) \tag{32}$$

$$(\boldsymbol{\omega}_m)_m \text{ is bounded in } \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f)) \tag{33}$$

and

$$(\boldsymbol{\omega}'_m)_m \text{ is bounded in } \mathbf{L}^2(0,T;\mathbf{V}'(\Omega)). \tag{34}$$

Consequently, there exists a subsequence $(\boldsymbol{\omega}_{m_k})_k \subset (\boldsymbol{\omega}_m)_m$ and a function $\boldsymbol{\omega} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))$ with $\int_0^t \boldsymbol{\omega}(s)\chi_{\Omega_s}ds \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_s))$, $\boldsymbol{\omega}\chi_{\Omega_f} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))$ and $\boldsymbol{\omega}' \in \mathbf{L}^2(0,T;\mathbf{V}'(\Omega))$ such that

$$\boldsymbol{\omega}_{m_k} \overset{k\to\infty}{\rightharpoonup} \boldsymbol{\omega} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)) \tag{35}$$

$$\int_0^t \boldsymbol{\omega}_{m_k}(s)\chi_{\Omega_s}ds \overset{k\to\infty}{\rightharpoonup} \int_0^t \boldsymbol{\omega}(s)\chi_{\Omega_s}ds \text{ in } \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_s)) \tag{36}$$

$$\boldsymbol{\omega}_{m_k}\chi_{\Omega_f} \overset{k\to\infty}{\rightharpoonup} \boldsymbol{\omega}\chi_{\Omega_f} \text{ in } \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f)) \tag{37}$$

and

$$\boldsymbol{\omega}'_{m_k} \overset{k\to\infty}{\rightharpoonup} \boldsymbol{\omega}' \text{ in } \mathbf{L}^2(0,T;\mathbf{V}'(\Omega)). \tag{38}$$

We now fix an integer N and choose a function $\boldsymbol{\varphi} \in \mathbf{C}^1(0,T;\mathbf{V}(\Omega))$ of the form

$$\boldsymbol{\varphi}(t) := \sum_{k=1}^{N} \alpha_k(t)\mathbf{w}_k, \tag{39}$$

where $\{\alpha_k\}_{k=\overline{1,N}}$ are smooth functions. We choose N such that $N \le m$, multiply (20) by $\alpha_k(t)$, sum after $k = 1, ..., N$ and integrate with respect to time to obtain:

$$\int_0^T \Big[<\boldsymbol{\omega}'_m(t), \boldsymbol{\varphi}(t)>_{f,s} + a(\int_0^t \boldsymbol{\omega}_m(s)ds, \boldsymbol{\varphi}(t)) + \nu(\nabla \times \boldsymbol{\omega}_m(t), \nabla \times \boldsymbol{\varphi}(t))_{\Omega_f} \Big]$$

$$= \int_0^T \Big[(\mathbf{G}(t), \boldsymbol{\varphi}(t))_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)\varphi_3(t) \Big] dt. \tag{40}$$

Now we may pass to the limit in the above identity, in virtue of (35)-(38) (set $m = m_k$); we obtain:

$$\int_0^T \Big[<\boldsymbol{\omega}'(t), \boldsymbol{\varphi}(t)>_{f,s} + a(\int_0^t \boldsymbol{\omega}(s)ds, \boldsymbol{\varphi}(t)) + \nu(\nabla \times \boldsymbol{\omega}(t), \nabla \times \boldsymbol{\varphi}(t))_{\Omega_f} \Big] dt$$

$$= \int_0^T \Big[(\mathbf{G}(t), \boldsymbol{\varphi}(t))_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)\varphi_3(t) \Big] dt. \tag{41}$$

Observe that (41) holds for all functions $\boldsymbol{\varphi} \in \mathbf{L}^2(0, T; \mathbf{V}(\Omega))$, since functions of the form (39) are dense in this space. It also follows from (41) that

$$<\boldsymbol{\omega}'(t), \boldsymbol{\varphi}(t)>_{f,s} + a(\int_0^t \boldsymbol{\omega}(s)ds, \boldsymbol{\varphi}(t)) + \nu(\nabla \times \boldsymbol{\omega}(t), \nabla \times \boldsymbol{\varphi}(t))_{\Omega_f}$$

$$= (\mathbf{G}(t), \boldsymbol{\varphi}(t))_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)\varphi_3(t),$$

for all $\boldsymbol{\varphi} \in \mathbf{V}(\Omega)$ and a.e. $0 \le t \le T$. Also notice that $\boldsymbol{\omega} \in \mathbf{C}(0, T; \mathbf{H}^{-1}(\Omega))$.

The compatibility condition (16) follows by passing to the limit in (21) and using (32), (33), as well as the convergences (in $\mathbf{L}^2(0, T; \mathbf{H}^{1/2}(\Gamma_{fs}))$) for the respective traces on the interface Γ_{fs} .

The existence result is proved if we verify that

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 \text{ in } \Omega. \tag{42}$$

We therefore choose any function $\boldsymbol{\varphi} \in \mathbf{C}^1(0, T; \mathbf{V}(\Omega))$ with $\boldsymbol{\varphi}(T) = 0$ and integrate by parts in time in (41) to obtain

$$\int_0^T \Big[-(\boldsymbol{\omega}(t), \boldsymbol{\varphi}'(t))_{f,s} + a(\int_0^t \boldsymbol{\omega}(s)ds, \boldsymbol{\varphi}(t)) + \nu(\nabla \times \boldsymbol{\omega}(t), \nabla \times \boldsymbol{\varphi}(t))_{\Omega_f} \Big] dt$$

$$= \int_0^T \Big[(\mathbf{G}(t), \boldsymbol{\varphi}(t))_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)\varphi_3(t) \Big] dt - (\boldsymbol{\omega}(0), \boldsymbol{\varphi}(0))_{f,s}. \tag{43}$$

From (40) we deduce in an analogous way that

$$\int_0^T \Big[-(\boldsymbol{\omega}_m(t), \boldsymbol{\varphi}'(t))_{f,s} + a(\int_0^t \boldsymbol{\omega}_m(s)ds, \boldsymbol{\varphi}(t)) + \nu(\nabla \times \boldsymbol{\omega}_m(t), \nabla \times \boldsymbol{\varphi}(t))_{\Omega_f} \Big]$$

$$= \int_0^T \Big[(\mathbf{G}(t), \boldsymbol{\varphi}(t))_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)\varphi_3(t) \Big] dt - (\boldsymbol{\omega}_m(0), \boldsymbol{\varphi}(0))_{f,s}. \tag{44}$$

We set again $m = m_k$ and deduce from (19) and (35)-(38) (after passing to the limit) that

$$\int_0^T \Big[-(\boldsymbol{\omega}(t), \boldsymbol{\varphi}'(t))_{f,s} + a(\int_0^t \boldsymbol{\omega}(s)ds, \boldsymbol{\varphi}(t)) + \nu(\nabla \times \boldsymbol{\omega}(t), \nabla \times \boldsymbol{\varphi}(t))_{\Omega_f} \Big]$$

$$= \int_0^T \Big[(\mathbf{G}(t), \boldsymbol{\varphi}(t))_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)\varphi_3(t) \Big] dt - (\boldsymbol{\omega}_0, \boldsymbol{\varphi}(0))_{f,s}. \tag{45}$$

Compare now the identities (43) and (45) to deduce (42), since $\boldsymbol{\varphi}(0)$ was arbitrary.

5. Proof of the uniqueness

In this section we prove the uniqueness of the weak solution found in Section 4. In order to do that, it suffices to show that the only weak solution of Problem 3 with $P(t) \equiv 0$ and $\mathbf{G}(t) \equiv \mathbf{0}$ for all $0 \le t \le T$ is

$$\boldsymbol{\omega} \equiv \mathbf{0}. \tag{46}$$

Thus, we know that $\boldsymbol{\omega} \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$, $\boldsymbol{\omega}\chi_{\Omega_f} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))$, $\boldsymbol{\omega}' \in \mathbf{L}^2(0,T;\mathbf{V}'(\Omega))$, $\int_0^t \boldsymbol{\omega}(s)\chi_{\Omega_s} ds \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_s))$ and

$$< \boldsymbol{\omega}', \boldsymbol{\varphi} >_{f,s} + a(\int_0^t \boldsymbol{\omega}(s)ds, \boldsymbol{\varphi}) + \nu(\nabla \times \boldsymbol{\omega}, \nabla \times \boldsymbol{\varphi})_{\Omega_f} = 0, \tag{47}$$

for all $\boldsymbol{\varphi} \in \mathbf{V}(\Omega)$ and with the initial condition

$$\boldsymbol{\omega}(0) = \mathbf{0}. \tag{48}$$

Let us denote $\boldsymbol{\psi}(t) := \int_0^t \boldsymbol{\omega}(s)ds$. Then notice that

$$< \boldsymbol{\psi}'', \boldsymbol{\varphi} >_{f,s} + a(\boldsymbol{\psi}(t), \boldsymbol{\varphi}) + \nu(\nabla \times \boldsymbol{\psi}'(t), \nabla \times \boldsymbol{\varphi})_{\Omega_f} = 0, \tag{49}$$

for all $\boldsymbol{\varphi} \in \mathbf{V}(\Omega)$,

$$\boldsymbol{\psi}(0) = \mathbf{0}, \quad \boldsymbol{\psi}'(0) = \mathbf{0} \tag{50}$$

and

$$\boldsymbol{\psi} \in \mathbf{L}^2(0,T;\mathbf{V}(\Omega)), \ \boldsymbol{\psi}' \in \mathbf{L}^2(0,T;\mathbf{H}(\Omega)), \ \boldsymbol{\psi}'' \in \mathbf{L}^2(0,T;\mathbf{V}'(\Omega)).$$

Fix $0 \le s \le T$ and take

$$\boldsymbol{\zeta}(t) := \begin{cases} \int_t^s \boldsymbol{\psi}(\tau)d\tau & \text{if} \quad 0 \le t \le s \\ 0 & \text{if} \quad s \le t \le T. \end{cases} \tag{51}$$

Then (49) can be written:

$$\int_0^s [(\boldsymbol{\psi}''(t), \boldsymbol{\zeta}(t))_{f,s} + a(\boldsymbol{\psi}(t), \boldsymbol{\zeta}(t)) + \nu(\nabla \times \boldsymbol{\psi}'(t), \nabla \times \boldsymbol{\zeta}(t))_{\Omega_f}]dt = 0,$$

since $\boldsymbol{\zeta}(t) \in \mathbf{V}(\Omega)$, $\forall t \in (0,T)$, by the regularity of $\boldsymbol{\psi}$.

Upon integrating by parts with respect to time, it follows that (observe that $\boldsymbol{\zeta}'(t) = -\boldsymbol{\psi}(t)$ for $0 \le t \le s$):

$$-\int_0^s [(\boldsymbol{\psi}'(t), \boldsymbol{\zeta}'(t))_{f,s} + \int_0^s a(\boldsymbol{\zeta}'(t), \boldsymbol{\zeta}(t)) + \nu \int_0^s (\nabla \times \boldsymbol{\psi}(t), \nabla \times \boldsymbol{\zeta}'(t))_{\Omega_f}]dt = 0,$$

thus

$$\int_0^s (\boldsymbol{\psi}'(t), \boldsymbol{\psi}(t))_{f,s} dt - \int_0^s a(\boldsymbol{\zeta}'(t), \boldsymbol{\zeta}(t)) dt + \nu \int_0^s |\nabla \times \boldsymbol{\psi}(t)|^2_{\Omega_f} dt = 0.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\int_0^s [\|\boldsymbol{\psi}(t)\|^2_{f,s} - a(\boldsymbol{\zeta}(t), \boldsymbol{\zeta}(t))] dt = -\nu \int_0^s |\nabla \times \boldsymbol{\psi}(t)|^2_{\Omega_f} dt \leq 0,$$

thus

$$\frac{1}{2}[\|\boldsymbol{\psi}(t)\|^2_{f,s} + a(\boldsymbol{\zeta}(0), \boldsymbol{\zeta}(0))] \leq 0$$

and after applying (25) it follows that $\boldsymbol{\psi}(s) = \mathbf{0}$. Now, since $s \in (0,T)$ was arbitrary, this implies that $\boldsymbol{\psi} \equiv \mathbf{0}$ in $(0,T) \times \Omega$ and the conclusion follows. ■

6. Improved regularity

Assuming more regularity on the data, stronger energy estimates are available for the weak solutions $\boldsymbol{\omega}_m$ obtained from the Galerkin approximations, thus enhancing the regularity properties of $\mathbf{v}$ and $\mathbf{u}$.

6.1. **Strong energy estimates.**

Theorem 6.1. *Let $\mathbf{g}$, $\mathbf{f}$, P, $\mathbf{v}_0$ and $\mathbf{u}_{01}$ satisfy (6). Additionally, let*

$$\mathbf{g}' \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_s)),\ \mathbf{f}' \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_f)),\ P' \in L^2(0,T;L^2(\Gamma_{f,ends,1})), \tag{52}$$

$$\mathbf{v}_0 \in \mathbf{H}^2(\Omega_f) \text{ with } \mathbf{v}_0 \times \mathbf{n} = 0 \text{ on } \Gamma_{f,ends}, \tag{53}$$

$$\mathbf{u}_1 \in \mathbf{H}^2(\Omega_s),\ P(0) \in L^2(\Gamma_{f,ends,1}) \tag{54}$$

and assume the existence of an initial pressure $p_0 \in H^1(\Omega_f)$ such that

$$p_0 \cdot \mathbf{n}_f - \nu(\nabla \times \mathbf{v}_0) \times \mathbf{n}_f = 0 \text{ on } (0,T) \times \Gamma_{fs}. \tag{55}$$

Then for all $m \in \mathbb{N}$ the solution $\boldsymbol{\omega}_m$ of (19),(20) satisfies the estimate

$$\|\boldsymbol{\omega}'_m(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\omega}'_m\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))} + \|\int_0^t \boldsymbol{\omega}'_m(s) ds\|^2_{\mathbf{H}^1(\Omega_s)} \tag{56}$$

$$\leq Ce^{CT}\Big(\|\mathbf{G}\|^2_{\mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))} + \|P'\|^2_{L^2(0,T;L^2(\Gamma_{f,ends,1}))} + \|\mathbf{v}_0\|^2_{\mathbf{H}^2(\Omega_f)} + \|p_0\|^2_{H^1(\Omega_f)} + \|\mathbf{u}_{01}\|^2_{\mathbf{H}^1(\Omega_s)}\Big)$$

for all $t \in [0,T]$.

Remark 6.2. Compare (55) with (5) and remember that we considered $\mathbf{u}_0 = 0$.

Proof. Denote $\boldsymbol{\zeta}_m := \partial_t \boldsymbol{\omega}_m$. Differentiating in (20) w.r.t. t and taking $\mathbf{w}_k := \boldsymbol{\zeta}_m(t,\cdot)$ it follows for each $t \in [0,T]$:

$$\frac{1}{2}\frac{d}{dt}\Big[\|\boldsymbol{\zeta}_m(t)\|^2_{f,s} + a(\int_0^t \boldsymbol{\zeta}_m(s) ds, \int_0^t \boldsymbol{\zeta}_m(s) ds)\Big] + \nu\|\nabla \times \boldsymbol{\zeta}_m\|^2_{\Omega_f}$$

$$= (\mathbf{G}'(t), \boldsymbol{\zeta}_m(t))_{f,s} + \int_{\Gamma_{f,ends,1}} P'(t)\zeta_{m,3}(t) d\sigma + a(\boldsymbol{\omega}_m(0), \boldsymbol{\zeta}_m(t)),$$

hence

$$\frac{1}{2}\frac{d}{dt}\Big[\|\boldsymbol{\zeta}_m(t)\|^2_{f,s} + a(\int_0^t \boldsymbol{\zeta}_m(s) ds, \int_0^t \boldsymbol{\zeta}_m(s) ds)\Big] + \nu\|\nabla \times \boldsymbol{\zeta}_m\|^2_{\Omega_f}$$

$$\leq C\Big(\|\boldsymbol{\zeta}_m(t)\|^2_{f,s} + \frac{\alpha}{2}\|\boldsymbol{\zeta}_m(t)\|^2_{\mathbf{H}^1(\Omega_f)} + a(\boldsymbol{\omega}_m(0), \boldsymbol{\zeta}_m(t))$$

$$+\|\mathbf{G}'(t)\|^2_{f,s} + \frac{1}{2\alpha}\|P'(t)\|^2_{L^2(\Gamma_{f,ends,1})}\Big),$$

with the constant C depending on C_{trace} in (24).

Further, integrate the above inequality between 0 and t ($t \in (0,T)$ fixed) and apply inequalities (7) and (25) to obtain

$$\frac{1}{2}\|\boldsymbol{\zeta}_m(t)\|^2_{f,s} + \frac{1}{2}(\frac{1}{C_{Korn}} - \beta C)\|\int_0^t \boldsymbol{\zeta}_m(s)ds\|^2_{\mathbf{H}^1(\Omega_s)} + (\nu C_{curl} - \frac{\alpha}{2})\int_0^t \|\boldsymbol{\zeta}_m(s)\|^2_{\mathbf{H}^1(\Omega_f)}ds$$

$$\leq C\Big(\int_0^t \|\boldsymbol{\zeta}_m(s)\|^2_{f,s} + \|\mathbf{G}'(t)\|^2_{f,s} + \frac{1}{2\alpha}\|P'(t)\|^2_{L^2(\Gamma_{f,ends,1})} + \frac{1}{2\beta}\|\boldsymbol{\omega}_m(0)\|^2_{\mathbf{H}^1(\Omega_s)} + \|\boldsymbol{\zeta}_m(0)\|^2_{f,s}\Big), \tag{57}$$

where the constants α and β are chosen s.t. $\frac{1}{C\cdot C_{Korn}} < \beta$ and $2\nu C_{curl} < \alpha$, C being a generic constant depending on C_{trace}.

Applying Gronwall's integral inequality and the estimation (see Subsection 4.2)

$$\|\boldsymbol{\omega}_m(0)\|^2_{\mathbf{H}^1(\Omega_s)} \leq \|\mathbf{v}_0\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{u}_{01}\|^2_{\mathbf{H}^1(\Omega_s)}$$

it follows

$$\|\boldsymbol{\zeta}_m(t)\|^2_{f,s} \leq Ce^{CT}\Big(\|\mathbf{G}'(t)\|^2_{f,s} + \frac{1}{2\alpha}\|P'(t)\|^2_{L^2(\Gamma_{f,ends,1})} + \|\mathbf{v}_0\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{u}_{01}\|^2_{\mathbf{H}^1(\Omega_s)} + \|\boldsymbol{\zeta}_m(0)\|^2_{f,s}\Big). \tag{58}$$

Now we need an estimate for $\|\boldsymbol{\zeta}_m(0)\|^2_{f,s} = \|\boldsymbol{\omega}'_m(0)\|^2_{f,s}$ and we use for this the idea in [12], adding to (20) the term $\nu(\nabla\times(\boldsymbol{\omega}_m(0) - \mathbf{v}_0), \nabla\times\mathbf{w}_k)_{\Omega_f}$, evaluating the new equation in $t=0$ and setting $\mathbf{w}_k := \boldsymbol{\omega}'_m(0)$. Thus we get

$$\|\boldsymbol{\omega}'_m(0)\|^2_{f,s} = (\mathbf{G}(0), \boldsymbol{\omega}'_m(0))_{f,s} + \int_{\Gamma_{f,ends,1}} P(0)\omega'_{m,3}(0)d\sigma - \nu(\nabla\times\mathbf{v}_0, \nabla\times\boldsymbol{\omega}'_m(0))_{\Omega_f}.$$

Remark 6.3. Adding the above term to the weak formulation (20) has only the purpose of allowing to estimate $\|\boldsymbol{\zeta}_m(0)\|^2_{f,s}$. It does not affect the estimates of the solution and the existence proof, since

$$\|\boldsymbol{\omega}_m(0) - \mathbf{v}_0\|_{\mathbf{H}^1(\Omega_f)} = \|\mathbf{P}_m\boldsymbol{\omega}_0 - \boldsymbol{\omega}_0\|_{\mathbf{H^1(\Omega_f)}} \leq \|\mathbf{P}_m\boldsymbol{\omega}_0 - \boldsymbol{\omega}_0\|_{\mathbf{H^1(\Omega)}} \overset{m\to\infty}{\to} 0$$

(the proof of this goes analogously to the one in [27], Ch.21).

Now observe that $\boldsymbol{\omega}'_m(0) \in \text{span }\{\mathbf{w}_k\}_k$, thus div $\boldsymbol{\omega}'_m(0) = 0$, hence we deduce

$$\|\boldsymbol{\omega}'_m(0)\|^2_{f,s} = (\mathbf{G}(0), \boldsymbol{\omega}'_m(0))_{f,s} + \int_{\Gamma_{f,ends,1}} P(0)\omega'_{m,3}(0)d\sigma - \nu(\nabla\times\mathbf{v}_0, \nabla\times\boldsymbol{\omega}'_m(0))_{\Omega_f}$$
$$-a(\mathbf{u}_0, \boldsymbol{\omega}'_m(0)) + (p_0, \text{div }\boldsymbol{\omega}'_m(0))_{\Omega_f},$$

hence with $\mathbf{u}_0 = 0$

$$\begin{aligned}\|\boldsymbol{\omega}'_m(0)\|^2_{f,s} &= (\mathbf{G}(0), \boldsymbol{\omega}'_m(0))_{f,s} + (\Delta\mathbf{v}_0, \boldsymbol{\omega}'_m(0))_{\Omega_f} \\ &+ (p_0, \text{div}\boldsymbol{\omega}'_m(0))_{\Omega_f} + \int_{\Gamma_{fs}} (p_0\cdot\mathbf{n}_f - \nu(\nabla\times\mathbf{v}_0)\times\mathbf{n}_f)\cdot\boldsymbol{\omega}'_m(0)d\sigma\end{aligned} \tag{59}$$

With hypothesis (55), (59) becomes

$$\begin{aligned}\|\boldsymbol{\omega}'_m(0)\|^2_{f,s} &= (\mathbf{G}(0), \boldsymbol{\omega}'_m(0))_{f,s} + (\boldsymbol{\Delta}\mathbf{v}_0 - \nabla p_0, \boldsymbol{\omega}'_m(0))_{\Omega_f} \\ &\leq C(\|\mathbf{G}(0)\|^2_{f,s} + \|\mathbf{v}_0\|^2_{\mathbf{H}^2(\Omega_f)} + \|p_0\|^2_{H^1(\Omega_f)}) + \gamma\|\boldsymbol{\omega}'_m(0)\|^2_{f,s} \quad (\gamma < 1),\end{aligned}$$

hence with $\|\mathbf{G}(\mathbf{0})\|^2_{f,s} \leq C(\|\mathbf{G}\|^2_{f,s} + \|\mathbf{G}'\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)})$ it follows

$$\|\boldsymbol{\omega}'_m(0)\|^2_{f,s} \leq C(\|\mathbf{G}\|^2_{\mathbf{H}^1(0,T;\mathbf{L}^2(\Omega)} + \|\mathbf{v}_0\|^2_{\mathbf{H}^2(\Omega_f)} + \|p_0\|^2_{H^1(\Omega_f)}). \tag{60}$$

Now plug (60) into (58) to get the estimate (56) in the conclusion of the theorem. □

Upon passing to the limit for $m \to \infty$ Theorem 6.1 implies some strong estimates for the solution of (13):

Theorem 6.4. *In the hypotheses of Theorem 6.1 the solution* $(\mathbf{u}, \mathbf{v})$ *of (13) satisfies the estimate*

$$\|\mathbf{v}'(t)\|^2_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{u}''(t)\|_{\mathbf{L}^2(\Omega_s)} + \|\mathbf{v}'\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1_{0,\Gamma_{f,bot}}(\Omega_f))} + \|\mathbf{u}'(t)\|^2_{\mathbf{H}^1(\Omega_s)}$$
$$\leq Ce^{CT}\Big(\|\mathbf{G}\|^2_{\mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))} + \|P'\|^2_{L^2(0,T;L^2(\Gamma_{f,ends,1}))} + \|\mathbf{v}_0\|^2_{\mathbf{H}^2(\Omega_f)}$$
$$+\|p_0\|^2_{H^1(\Omega_f)} + \|\mathbf{u}_{01}\|^2_{\mathbf{H}^1(\Omega_s)}\Big). \tag{61}$$

Moreover, for $\mathbf{U}_0 \in \mathbf{L}^2(0,T;\mathbf{V}_f)$ *as in Section 3,* $\mathbf{v}$ *satisfies*

$$div\ \mathbf{v} = 0$$

in the sense of distributions in Ω *and the boundary conditions on* $\mathbf{v}$ *in (4) in the sense of traces of functions of* $\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))$.

Proof. The latter statement of the theorem follows from $\mathbf{v} - \mathbf{U}_0 \in \mathbf{L}^2(0,T;\mathbf{V}_f)$ and the properties of $\mathbf{U}_0$ in Section 3. □

6.2. **Existence of a pressure.**

Theorem 6.5. *Assume* $\nabla \cdot \mathbf{f} \in \mathbf{L}^2(\Omega_f)$. *Then in the hypotheses of Theorem 6.1 there exists a class of functions* $p \in H(\Delta, \Omega_f)/\mathbb{R}$ *such that*

$$\partial_t \mathbf{v} - \nu \boldsymbol{\Delta} \mathbf{v} + \nabla p = \mathbf{f} \tag{62}$$

is satisfied in the sense of distributions in Ω_f.

Moreover, for $\nabla \times \mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_f))$ *we have* $\nabla \times \mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}(\Delta,\Omega_f))$.

Remark 6.6. Here

$$H(\Delta,\Omega_f) := \{q \in L^2(\Omega_f)\ :\ \Delta q \in L^2(\Omega_f)\}$$

and $p \in H(\Delta,\Omega_f)/\mathbb{R}$ means that $p \in H(\Delta,\Omega_f)$ and it is unique up to an additive constant.

Proof. Take a divergence-free function $\boldsymbol{\varphi}$ of $\mathcal{D}(\Omega_f)$ (prolongated by zero into Ω_s) as the test function in (13). Then using the definition of the distribution derivative we have

$$(\partial_t \mathbf{v}, \boldsymbol{\varphi})_{\Omega_f} + < \nu \nabla \times (\nabla \times \mathbf{v}), \boldsymbol{\varphi} >= (\mathbf{f}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \mathcal{D}(\Omega_f)\ :\ \text{div}\ \boldsymbol{\varphi} = 0 \text{ in } \Omega_f,$$

where $< \cdot, \cdot >$ denotes the duality between $\mathcal{D}'(\Omega_f)$ and $\mathcal{D}(\Omega_f)$. Since Ω_f has a locally Lipschitz boundary, Theorem I.2.3 in Girault & Raviart [15] can be applied to ensure the existence of $p \in L^2(\Omega_f)/\mathbb{R}$ such that

$$\partial_t \mathbf{v} + \nu \nabla \times (\nabla \times \mathbf{v}) + \nabla p = \mathbf{f}$$

in the sense of distributions in Ω_f.

Applying the divergence operator to this equation it follows that

$$\Delta p = \nabla \cdot f - \partial_t \nabla \cdot \mathbf{v},$$

which together with Theorem 6.4 proves the result.

Further, applying the curl operator on (62) leads to

$$-\nu \boldsymbol{\Delta}(\nabla \times \mathbf{v}) = \nabla \times \mathbf{f} - \nabla \times \partial_t \mathbf{v}$$

from which with Theorem 6.4 and the hypothesis on $\nabla \times \mathbf{f}$ the last statement in this theorem follows. □

7. Conclusion

In this paper we have considered a 3D/3D fluid-elastic structure interaction problem. The viscous, incompressible fluid was moving through a half tube having a rigid, fixed bottom and flexible and thick walls. We considered very small displacements, in order to be able to assume that the domains involved were cylindrical. For the coupled problem with the fluid behavior described by the Stokes equations with boundary conditions involving the pressure (at the in- and outflow) and with the Lamé equations for linearized elasticity characterising the behavior of the deformable structure, we have shown the existence of a unique solution (velocity and displacement) and derived enhanced regularity properties upon assuming smooth enough data. The existence of a pressure defined up to an addditive constant has also been proved. The method seems not to be directly adaptable to the case of a Navier-Stokes fluid, because of the nonlinearity of the convective term. Furthermore, allowing for larger displacements of the fluid-structure interface would normally lead to dropping the assumption of cylindrical domains. By our knowledge, this kind of time-dependent problems has not been treated yet in the 3D/3D case, but only for 3D fluid/2D structure interactions ([6], [22]) or in lower dimensions.

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Institut für Angewandte Analyis und Numerische Simulation, Pfaffenwaldring 57, 70569 Stuttgart, Germany
E-mail address: christina.surulescu@mathematik.uni-stuttgart.de

Christina Surulescu
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: Christina.Surulescu@mathematik.uni-stuttgart.de

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