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Abstract

We consider a boundary-value problem for the Poisson equation in a thick junction Ω_ε which is the union of a domain Ω_0 and a large number of $\varepsilon-$periodically situated thin curvilinear cylinders. The following nonlinear Robin boundary condition $\partial_\nu u_\varepsilon + \varepsilon\kappa(u_\varepsilon) = 0$ is given on the lateral surfaces of the thin cylinders. The asymptotic analysis of this problem is made as $\varepsilon \to 0$, i.e., when the number of the thin cylinders infinitely increases and their thickness tends to zero. We prove the convergence theorem and show that the nonlinear Robin boundary condition is transformed (as $\varepsilon \to 0$) in the blow up term of the corresponding ordinary differential equation in the region that is filled up by the thin cylinders in the limit passage. The convergence of the energy integral is proved as well. Using the method of matched asymptotic expansions, the approximation for the solution is constructed and the corresponding asymptotic error estimate in the Sobolev space $H^1(\Omega_\varepsilon)$ is proved.

KEY WORDS: homogenization; thick junction; nonlinear boundary conditions; approximate model

MOS subject classification: 35B27, 35J65, 35J60, 74K30

1 Introduction

The subject of this research is related to the asymptotic analysis of boundary-value problems in perturbed domains. There are many kinds of the domain perturbations (perforated domains, partly perforated domains, lattice frames, thin domains, junctions of domains with different limit dimensions, etc.) and we need different asymptotic methods to study boundary-value problems in such domains (see, e.g., [2, 4, 10, 11, 18, 19, 22, 31, 32, 35] and references therein).

In this paper we consider new kind of perturbed domains, namely thick junctions. Boundary-value problems in thick junctions are now very extensively investigated (see [3, 5]-[9], [12]-[16], [23]-[29]). Sometime such domains are called domains perforated by narrow parallel channels or sheets [22, 3, 35], or domains with highly oscillating boundary [5, 6, 7, 9].

A thick junction (or thick multi-structure) of type $k : p : d$ is the union of some domain in $\mathbb{R}^n$, which is called the junction's body, and a large number of ε-periodically situated thin domains along some manifolds on the boundary of the junction's body (see Fig. 1). This manifold is called the joint zone. Here ε is a small parameter, which characterizes the distance between neighboring thin domains and their thickness. The type $k : p : d$ of a thick junction refers respectively to the limiting dimensions of the body, the joint zone, and each of the attached thin domains.

As it was shown in [3, 35], such problems lose the coercitivity as $\varepsilon \to 0$ and this creates special difficulties in the asymptotic investigation. The classification of such thick junctions was given in [23]-[28] and basic results were obtained for boundary-value problems in thick junctions of different types. It was pointed out that qualitative properties of solutions essentially depend on the junction type and on the conditions given on the boundaries of the attached thin domains.

Boundary-value problems in thick junctions are mathematical models of widely used engineering and industrial constructions as well as many other physical and biological systems with very distinct characteristic scales (see, e.g., [1]). In last time such thick multi-structures are in common practice as thick absorbers. In paper [21] the following experimental data were obtained: the electron microphotographs of the surface of a thick absorber have shown that these structures have the chemical activity without interference of an external fields in reactions of degradation of organic solutes in water; in addition the analysis of the absorption spectra has shown that mainly oxidative degradation of organic molecules takes place. The point is that very small chemical activity holds always between the surface of a thick absorber and water (the corresponding linear mathematical model reads as the following Robin boundary condition $\partial_\nu u + \varepsilon k_0 u = 0$, where u is the chemical concentration, εk_0 is the small absorbent coefficient), but large surface concentration in the thick absorber supplies the sufficiently large total chemical activity. Mathematical justification of this fact for a linear model in thick junction of type $3:2:2$ is presented in [12].

The recent development of reaction diffusion systems in biology, ecology and biochemistry, and the traditional importance of these systems in physics, heat-mass transfer and engineering lead to extensive study in various aspects of nonlinear boundary-value problems in thick multi-structures.

The aim of this paper is to develop rigorous asymptotic methods (convergence and approximation) for boundary-value problems in thick junctions with nonlinear Robin boundary conditions. The convergence theorems for monotone differential equations in thick junctions with the uniform Neumann conditions were proved in [5, 6, 7].

1.1 Statement of the problem

Let $a,\ h$ be positive numbers and N be a large positive integer. Define a small parameter $\varepsilon = \frac{a}{N}$. A model thick junction Ω_ε of type $3:2:1$ (see Fig. 1) consist of the "body"

$$\Omega_0 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 :\ x' = (x_1, x_2) \in \Xi_0 = (0,a) \times (0,a), \quad -\gamma(x') < x_3 < 0\}$$

and a large number of thin curvilinear cylinders $G_\varepsilon = \bigcup_{i,j=0}^{N-1} G_\varepsilon(i,j)$,

$$G_\varepsilon(i,j) = \left\{ x \in \mathbb{R}^3 :\ 0 < x_3 < h, \quad \left(\frac{x_1}{\varepsilon} - \frac{1}{2} - i\right)^2 + \left(\frac{x_2}{\varepsilon} - \frac{1}{2} - j\right)^2 < \varrho^2(x_3) \right\},$$

where the given functions γ and ϱ are smooth and positive on $[0,a]\times[0,a]$ and $[0,h]$ respectively; in addition $\varrho < \frac{1}{2}$. Obviously, the thin curvilinear cylinders fill out the parallelepiped $\Omega^+ = \Xi_0 \times (0,h)$ in the limit passage as $N \to +\infty$ ($\varepsilon \to 0$).

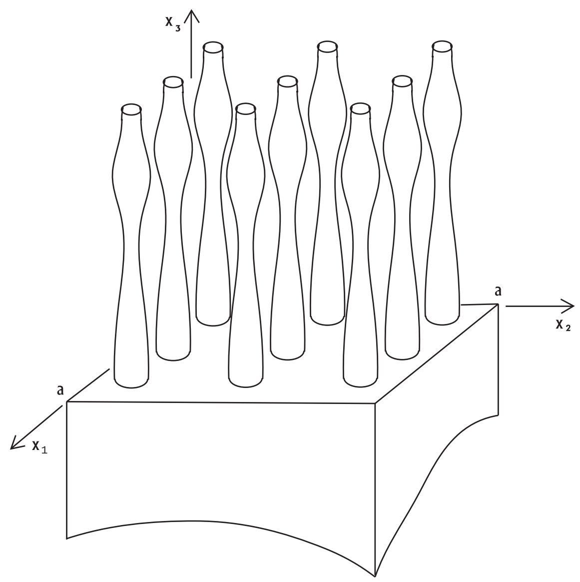

Figure 1: Thick multi-structure Ω_ε of type $3:2:1$.

In Ω_ε we consider the following boundary-value problem

$$
\begin{array}{rcll}
-\Delta_x u_\varepsilon(x) & = & f_0(x), & x \in \Omega_0, \\
-\Delta_x u_\varepsilon(x) & = & 0, & x \in G_\varepsilon, \\
-\partial_\nu u_\varepsilon(x) & = & \varepsilon\kappa\big(u_\varepsilon(x)\big), & x \in \Gamma_\varepsilon, \\
\partial_\nu u_\varepsilon(x) & = & 0, & x \in \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \\
[u_\varepsilon]_{|x_3=0} = [\partial_{x_3} u_\varepsilon]_{|x_3=0} & = & 0, & x' \in \Xi_0,
\end{array}
\tag{1}
$$

where $\partial_\nu = \partial/\partial_\nu$ is the outward normal derivative, Γ_ε is the union of the lateral surfaces of the thin cylinders, the brackets denote the jump of the enclosed quantities, $f_0 \in L^2(\Omega_0)$ and its support is compactly embedded in Ω_0, the function $\kappa : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies the following condition:

$$
\exists\, c_1 > 0 \quad \exists\, c_2 > 0 \quad \forall\, t \in \mathbb{R}: \quad c_1 \le \kappa'(t) \le c_2. \tag{2}
$$

Recall that a function $u_\varepsilon \in H^1(\Omega_\varepsilon)$ is called a weak solution to problem (1) if it satisfies the following integral identity

$$
\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla\varphi \, dx + \varepsilon \int_{\Gamma_\varepsilon} \kappa\big(u_\varepsilon\big)\, \varphi \, d\sigma_x = \int_{\Omega_0} f_0\, \varphi \, dx \quad \forall \varphi \in H^1(\Omega_\varepsilon). \tag{3}
$$

Our goal is to study the asymptotic behaviour of the solution to problem (1) as $\varepsilon \to 0$, i.e., when the number of the thin cylinders infinitely increases and the thickness tends to zero.

1.1.1 Comments to the statement

In a typical interpretation the solution to problem (1) denotes the density of some quantity (chemical concentration, temperature, electronic potential) at equilibrium within the thick junction Ω_ε. Usually the source of the quantity is located in the junction's body. Therefore the right-hand side f_0 is defined in Ω_0.

The nonlinear Robin condition means that there is very small flux of this quantity through the lateral sides of the thin curvilinear cylinders. At first sight it seems that there is not any difference between this Robin condition and the homogeneous Neumann condition since the function κ is multiplied by ε. But as we will see this is quite false; this condition is transformed (as $\varepsilon \to 0$) in the zeroth-order nonlinear term of the corresponding ordinary differential equation in the region that is filled up by the thin cylinders in the limit passage (see (28)). Just the appearance of this blow up term mathematically justifies the experimental data mentioned above.

To prove the convergence theorem we can consider more weak conditions for κ as in [33]: $\kappa(x,t)$ is continuous, monotone non-decreasing in t and satisfies the uniform Lipschitz condition $|\kappa(x,t_1) - \kappa(x,t_2)| \le C|t_1 - t_2|$ (for $x \in \Omega^+$, $\forall\, t_1, t_2 \in \mathbb{R}$). But this is inessential generalization. In addition, to construct the asymptotics for the solution we need the smoothness of κ.

The case of general form of the thin curvilinear cylinders

$$
G_\varepsilon(i,j) = \big\{x \in \mathbb{R}^3 : \ 0 < x_3 < h, \quad \big(\varepsilon^{-1}x_1 - i, \varepsilon^{-1}x_2 - j\big) \in \omega(x_3)\big\} \tag{4}
$$

is of the same type of analysis and in special situations it requires additional cumbersome calculations. Therefore, we prefer the simpler one to make the formulas and calculations clearer and shorter. In (4), $\omega(x_3)$ is a plane domain, which is strictly situated in the square $\{\xi' = (\xi_1, \xi_2) : 0 < \xi_1 < 1, \ 0 < \xi_2 < 1\}$ for any $x_3 \in [0,h]$, and the surface $\{(\xi', x_3) : \ \xi' \in \partial\omega, \ x_3 \in [0,h]\}$ is smooth.

1.2 Features and techniques of the investigation

1. To homogenize boundary value problems in thick multi-structures with the linear non-homogeneous Neumann or Fourier conditions on the boundaries of the thin attached domains, the method of special integral identities was suggested in [28, 29]. In Sec. 2 we prove the corresponding integral

identity for our problem. With the help of it we deduce certain uniformly estimates with respect to ε and the uniformly estimate for the solution to problem (1) (see (13)). Also this identity is essential in proving the convergence theorem, in finding the boundary residuals and in proving the asymptotic estimates.

2. In fact the existence of the solution to problem (1) at a fixed value of ε it follows from [33], where the method of sub/super-solutions was used. However, in § 2.1 we shortly demonstrate the direct method in the calculus of variations to prove the existence. This approach is easier and suitable for such kind of problems.

3. As was showed in [23]-[28] thick junctions are not strongly connected domains, i.e., there is no any sequence of extension operators $\{ \mathbf{P}_\varepsilon : H^1(\Omega_\varepsilon) \mapsto H^1(\mathbb{R}^n) \}_{\varepsilon>0}$ whose norms are uniformly bounded in ε. This fact creates one of the main difficulties in the proofs of convergence theorems for solutions to boundary-value problems in thick junctions. At the same time the availability of an uniformly bounded family of extension operators is typical supposition in overwhelming majority of the existing homogenization schemes.

The convergence theorems for solutions to boundary-value problems in thick junctions of different types were proved in [23]-[28], [12]-[14] with the help of special extension operators which preserve the space class of the solutions and whose H^1-norms are uniformly bounded in ε only for the solutions.

In [9], where a homogeneous Neumann boundary-value problem was studied in a thick junction, it was shown that if the boundaries of thin rods are rectilinear, then the solution can be extended by zero to prove the convergence theorems. This is explained by the fact that this extension preserves the weak derivative with respect to x_2 due to the rectilinearity of the boundaries of the rods along the Ox_2-axis. This approach was used in [5, 6, 7, 8]. Also in [9], the homogeneous Neumann problem was considered in a bounded plane domain whose boundary is waved by the function $x_2 = h(x_1/\varepsilon)$, where h must be a continuously differentiable periodic function and the reciprocal functions of h on some intervals have to be existed to construct special extension operator. But this extension does not preserve the space class of the solution and it was constructed under the assumption that the right-hand side $f \in H^1$.

In Sec. 3 we present a new approach to prove the convergence theorem using the extension by zero even if the thin domains of a thick junction are not rectilinear (this became a reality due to the integral identity (5)) and in a case of non-homogeneous boundary conditions.

4. There are two ways to study boundary-value problems in perturbed domains. The first one consists in the proof of a convergence theorem. The second one is related to obtaining the formal asymptotics for a solution and its justification (establish the asymptotic error estimate). It should be noted that an important problem for any asymptotic method is its accuracy. Therefore, the proof of the error estimate of discrepancy between the constructed approximation and the exact solution is general principle that has been applied to the analysis of the efficiency of an asymptotic method.

To construct the asymptotic approximation for the solution to problem (1), the approach within the conceptual framework of [24, 25, 28] is used. At the end of Sec.4 we prove the corresponding asymptotic error estimates in Theorem 3 and Corollary 1.

2 Auxiliary uniform estimates

In what follows we will often use the following identity (see [28] for the rectilinear cylinders)

$$\varepsilon \int\limits_{\Gamma_\varepsilon} \frac{\varphi(x)\, d\sigma_x}{\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}} = \int\limits_{G_\varepsilon} g(x_3)\, \varphi\, dx + \varepsilon \int\limits_{G_\varepsilon} \nabla_{\xi'} Y(\xi', x_3)|_{\xi'=\frac{x'}{\varepsilon}} \cdot \nabla_{x'} \varphi\, dx \quad \forall\, \varphi \in H^1(G_\varepsilon). \tag{5}$$

Here $g(x_3) = \frac{l_\omega(x_3)}{|\omega(x_3)|}$, $\omega(x_3) = \left\{ \xi' \in \mathbb{R}^2 : \ (\xi_1 - \frac{1}{2})^2 + (\xi_2 - \frac{1}{2})^2 < \varrho^2(x_3) \right\}$, $|\omega(x_3)|$ is the area of the circle $\omega(x_3)$, $l_\omega(x_3)$ is the length of $\partial\omega(x_3)$ for any fixed $x_3 \in [0, h]$. The function Y is a unique solution of the following problem

$$\Delta_{\xi'} Y = g(x_3) \;\text{ in }\; \omega(x_3), \quad \partial_{\nu'(\xi')} Y = 1 \;\text{ on }\; \partial\omega(x_3), \quad \int_{\omega(x_3)} Y(\xi', x_3)\, d\xi' = 0, \tag{6}$$

and then it is $1-$periodically continued with respect to ξ_1 and ξ_2; $\xi' = x'/\varepsilon, \nu'(\xi') = \big(\nu_1(\xi'), \nu_2(\xi')\big)$ is the outward normal to ω. The variable $x_3 \in [0, h]$ is regarded as a parameter in problem (6).

To obtain (5) we have to integrate by parts the last integral in (5) and take into account the boundary condition for Y and coordinates of the outward normal to the lateral surfaces of each cylinder $G_\varepsilon(i,j),\ i,j = 0, \ldots, N-1:$

$$\overline{\nu} = \frac{1}{\sqrt{1 + \varepsilon^2 |\varrho'(x_3)|^2}} \,\big(\nu_1(x'/\varepsilon), \nu_2(x'/\varepsilon), -\varepsilon\varrho'(x_3)\big). \tag{7}$$

Remark 1. *We don't simplify the view of the function g in (5) to take into account the general form of the thin curvilinear cylinders (4).*

It is easy to verify that $Y(\xi', x_3) = \varrho(x_3)\, W(\eta'),\ \eta' = \big(\xi_1 - \frac{1}{2},\, \xi_2 - \frac{1}{2}\big)/\, \varrho(x_3)$, where W is the unique solution to the problem

$$\Delta_{\eta'} W = 2 \;\text{ in }\; \{|\eta'| < 1\}, \quad \partial_{\nu'(\eta')} W = 1 \;\text{ on }\; \{|\eta'| = 1\}, \quad \int_{|\eta'|<1} W(\eta')\, d\eta' = 0.$$

Due to the regularity properties of solutions to elliptic problems we have

$$\sup_{x \in G_\varepsilon} \big|\nabla_{\xi'} Y(\xi', x_3)|_{\xi' = \frac{x'}{\varepsilon}}\big| = \sup_{\xi' \in \omega(x_3),\ x_3 \in (0,h)} \big|\nabla_{\xi'} Y(\xi', x_3)\big| = \sup_{|\eta'|<1} \big|\nabla_{\eta'} W(\eta')\big| \le C_0. \tag{8}$$

Using Cauchy's inequality with δ ($ab \le \delta a^2 + \frac{b^2}{4\delta},\ \ a, b, \delta > 0$) and (8), we deduce from (5) the following estimates

$$\varepsilon \int_{\Gamma_\varepsilon} \varphi^2 \, d\sigma_x \le C_1 \left(\varepsilon^2 \int_{G_\varepsilon} |\nabla_{x'} \varphi|^2 \, dx + \int_{G_\varepsilon} \varphi^2 \, dx \right), \tag{9}$$

$$\int_{G_\varepsilon} \varphi^2 \, dx \le C_2 \left(\varepsilon^2 \int_{G_\varepsilon} |\nabla_{x'} \varphi|^2 \, dx + \varepsilon \int_{\Gamma_\varepsilon} \varphi^2 \, d\sigma_x \right) \qquad \forall\, \varphi \in H^1(G_\varepsilon), \tag{10}$$

where the constant C_1 and C_2 are independent of ε.

Remark 2. *In what follows all constants $\{C_i\}$ and $\{c_i\}$ in inequalities are independent of the parameter ε.*

2.1 Existence and uniqueness of the solution to problem (1)

Associated with (1), we consider the energy functional

$$I_\varepsilon[u] := \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 \, dx + \varepsilon \int_{\Gamma_\varepsilon} B(u)\, d\sigma_x - \int_{\Omega_0} u(x) f_0(x)\, dx \quad \text{on } H^1(\Omega_\varepsilon), \tag{11}$$

where

$$B(z) = \int_0^z \kappa(t)\, dt \quad \forall\, z \in \mathbb{R}. \tag{12}$$

By the same way as in [17, §8.2.3], it is easy to prove that if u_ε is a minimizer of I_ε at a fixed value of ε, then u_ε is a weak solution to problem (1).

Theorem 1. *At each fixed value of ε problem (1) has exactly one solution $u_\varepsilon \in H^1(\Omega_\varepsilon)$ for which the following estimate*

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C(1 + \|f_0\|_{L^2(\Omega_0)}) \tag{13}$$

holds, where the constant C is independent of ε, f_0 and u_ε.

Proof. 1. Integrating inequalities in (2), we obtain the following inequalities

$$c_1 t^2 + \kappa(0)t \le \kappa(t)\, t \le c_2 t^2 + \kappa(0)t \quad \forall\, t \in \mathbb{R}, \tag{14}$$

and

$$\frac{c_1}{2}\, z^2 + \kappa(0)z \le B(z) \le \frac{c_2}{2}\, z^2 + \kappa(0)z \quad \forall\, z \in \mathbb{R}. \tag{15}$$

2. Now let us prove a coercitivity condition on I_ε. Due to (15) we have

$$I_\varepsilon[u] \ge \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2\, dx + \varepsilon \frac{c_1}{2}\int_{\Gamma_\varepsilon} u^2\, d\sigma_x + \varepsilon\, \kappa(0)\int_{\Gamma_\varepsilon} u\, d\sigma_x - \int_{\Omega_0} u\, f_0\, dx. \tag{16}$$

Applying Cauchy's inequality with δ_1 and δ_2 to the last two integrals in (16) respectively, we deduce

$$I_\varepsilon[u] \ge \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2\, dx + \varepsilon\, \frac{c_1}{2}\int_{\Gamma_\varepsilon} u^2\, d\sigma_x - \\ - \varepsilon\, |\kappa(0)|\delta_1 \int_{\Gamma_\varepsilon} u^2\, d\sigma_x - \varepsilon\, |\kappa(0)| \frac{1}{4\,\delta_1}\int_{\Gamma_\varepsilon} d\sigma_x - \delta_2 \int_{\Omega_0} u^2\, dx - \frac{1}{4\delta_2}\int_{\Omega_0} f_0^2\, dx.$$

Set $\delta_1 = \frac{c_1}{4|\kappa(0)|}$ and take $C_1 = \min\{\frac{1}{2}, \frac{c_1}{4}\}$. Thus,

$$I_\varepsilon[u] \ge C_1\Big(\int_{\Omega_\varepsilon} |\nabla u|^2\, dx + \varepsilon \int_{\Gamma_\varepsilon} u^2\, d\sigma_x\Big) - C_2 - \delta_2 \int_{\Omega_0} u^2\, dx - \frac{1}{4\delta_2}\|f_0\|^2_{L^2(\Omega_0)}, \tag{17}$$

where the constant C_2 is independent of ε. It should be stressed here that measure of Γ_ε is greater than some positive constant multiplied by ε^{-1}.

Using (9) and (10), by the same arguments as in [29, Lemma 1] we can prove that the usual norm $\|\cdot\|_{H^1(\Omega_\varepsilon)}$ is uniformly in ε equivalent to a new norm

$$\|u\|_\varepsilon := \Big(\int_{\Omega_\varepsilon} |\nabla u|^2\, dx + \varepsilon \int_{\Gamma_\varepsilon} u^2\, d\sigma_x\Big)^{1/2}$$

in the space $H^1(\Omega_\varepsilon)$, i.e., there exist constants $C_3 > 0$, $C_4 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $u \in H^1(\Omega_\varepsilon)$ the following relations hold

$$C_3\|u\|_{H^1(\Omega_\varepsilon)} \le \|u\|_\varepsilon \le C_4\|u\|_{H^1(\Omega_\varepsilon)}. \tag{18}$$

Thanks to (18) we can choose δ_2 independent of u and ε such that the following inequality

$$I_\varepsilon[u] \ge C_5\|u\|^2_{H^1(\Omega_\varepsilon)} - C_6 \tag{19}$$

holds for any function $u \in H^1(\Omega_\varepsilon)$.

3. The existence of a minimizer follows from (19), the fact that the function $\frac{1}{2}|p|^2$, $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, is convex, and general approach of the direct variational methods (see [17, Chapter 8.2]). We need only note that if $\{u_{k_j}\}_{j=1}^\infty \in H^1(\Omega_\varepsilon)$ is a minimizing subsequence with $u_{k_j} \to u$ weakly in $H^1(\Omega_\varepsilon)$, then by compactness we have $u_{k_j} \to u$ strongly in $L^2(\Omega_0)$ and the sequence of the traces $u_{k_j}|_{\Gamma_\varepsilon} \to u|_{\Gamma_\varepsilon}$ strongly in $L^2(\Gamma_\varepsilon)$ as $j \to +\infty$.

4. We now prove uniqueness. Assume u_1 and u_2 are two weak solution to problem (1). Then its difference satisfies the following identity

$$\int_{\Omega_\varepsilon} \nabla(u_1 - u_2)\cdot\nabla\varphi\, dx + \varepsilon\int_{\Gamma_\varepsilon} \big(\kappa(u_1) - \kappa(u_2)\big)\, \varphi\, d\sigma_x = 0 \quad \forall\varphi \in H^1(\Omega_\varepsilon). \tag{20}$$

Set $\varphi = u_1 - u_2$ in (20). Then

$$\int_{\Omega_\varepsilon} |\nabla(u_1 - u_2)|^2\, dx + \varepsilon\int_{\Gamma_\varepsilon} \kappa'\big(u_1 + \theta(u_2 - u_1)\big)\, \big(u_1 - u_2\big)^2\, d\sigma_x = 0 \tag{21}$$

for some $\theta \in [0,1]$. In view of (2), it follows from (21) that $u_1 = u_2$ a.e. in Ω_ε.

5. (*Uniform estimate*) Denote by u_ε the solution to problem (1). Setting $\varphi = u_\varepsilon$ in (3) and taking into account the left inequality in (14), we get

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \varepsilon\,c_1 \int_{\Gamma_\varepsilon} u_\varepsilon^2\,d\sigma_x + \varepsilon\,\kappa(0)\int_{\Gamma_\varepsilon} u_\varepsilon\,d\sigma_x \le \int_{\Omega_0} f_0 u_\varepsilon\,dx$$

from which

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \varepsilon\,c_1 \int_{\Gamma_\varepsilon} u_\varepsilon^2\,d\sigma_x \le C_1\Big(\varepsilon\int_{\Gamma_\varepsilon} u_\varepsilon^2\,d\sigma_x\Big)^{\frac{1}{2}} + \|f_0\|_{L^2(\Omega_0)}\|u_\varepsilon\|_{L^2(\Omega_0)}. \tag{22}$$

Then using (18) and (9), we derive the estimate (13) from (22). □

3 Convergence theorem

In the sequel, $\widetilde{y}$ denotes the zero-extension to the parallelepiped $\Omega^+ = \Xi_0 \times (0,h)$ which is filled up by the thin curvilinear cylinders in the limit passage as $\varepsilon \to 0$, namely,

$$\widetilde{y(x)} = \begin{cases} y(x), & x \in G_\varepsilon, \\ 0, & x \in \Omega^+ \setminus G_\varepsilon. \end{cases}$$

Also we introduce the following characteristic function

$$\chi_{G_\varepsilon}(x) = \begin{cases} 1, & x \in G_\varepsilon, \\ 0, & x \in \Omega^+ \setminus G_\varepsilon. \end{cases}$$

Similarly as in [13, Sec. 4], we can prove that

$$\chi_{G_\varepsilon} \to |\omega| \quad \text{weakly in} \quad L^2(\Omega^+) \quad \text{as} \quad \varepsilon \to 0. \tag{23}$$

Lemma 1. *Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a sequence in $H^1(G_\varepsilon)$ uniformly bounded in ε and such that*

$$\widetilde{\kappa(v_\varepsilon)} \to \zeta \quad \textit{weakly in} \quad L^2(\Omega^+) \quad \textit{as} \quad \varepsilon \to 0.$$

Then

$$\varepsilon\int_{\Gamma_\varepsilon} \kappa(v_\varepsilon)\,\varphi\,d\sigma_x \to \int_{\Omega^+} g(x_3)\,\zeta(x)\,\varphi(x)\,dx \quad \textit{as} \quad \varepsilon \to 0 \quad \forall \varphi \in C^1\big(\overline{\Omega^+}\big). \tag{24}$$

In addition,

$$\varepsilon\int_{\Gamma_\varepsilon} \varphi\,d\sigma_x \to \int_{\Omega^+} l_\omega(x_3)\,\varphi(x)\,dx \quad \textit{as} \quad \varepsilon \to 0 \quad \forall \varphi \in C^1\big(\overline{\Omega^+}\big). \tag{25}$$

Proof. By virtue of (5) we have

$$\varepsilon\int_{\Gamma_\varepsilon} \kappa(v_\varepsilon)\,\varphi\,d\sigma_x = \int_{\Omega^+} g(x_3)\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,\widetilde{\kappa(v_\varepsilon)}\,\varphi\,dx +$$
$$+\,\varepsilon\int_{G_\varepsilon} \sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,\nabla_{\xi'}Y(\xi',x_3)|_{\xi'=\frac{x'}{\varepsilon}} \cdot \nabla_{x'}\big(v_\varepsilon\,\varphi\big)\,dx \qquad \forall \varphi \in C^1\big(\overline{\Omega^+}\big).$$

Thanks to the Lemma's condition and (8), the first summand tends to $\int_{\Omega^+} g(x_3)\,\zeta(x)\,\varphi\,dx$ and the second one vanishes as $\varepsilon \to 0$.

To prove (25), we have to pass to the limit in the following identity

$$\varepsilon\int_{\Gamma_\varepsilon}\varphi\,d\sigma_x=\int_{\Omega^+}g(x_3)\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,\chi_{G_\varepsilon}\,\varphi\,dx+ \\ +\varepsilon\int_{G_\varepsilon}\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,\nabla_{\xi'}Y(\xi',x_3)|_{\xi'=\frac{x'}{\varepsilon}}\cdot\nabla_{x'}\varphi\,dx\quad\forall\varphi\in C^1\big(\overline{\Omega^+}\big)$$

and take into account (23). □

Remark 3. *From Lemma 1 and (2) it follows that for any sequence* $\{v_\varepsilon\}_{\varepsilon>0}\in H^1(G_\varepsilon)$, *which is uniformly bounded with respect to* ε *in* $H^1(G_\varepsilon)$, *there exists a subsequence* $\{\varepsilon'\}\subset\{\varepsilon\}$ *(again denoted by* ε*) and a function* $\zeta\in L^2(\Omega^+)$ *such that the convergence (24) holds.*

Theorem 2. *The solution* u_ε *to problem (1) satisfies the following relations*

$$\left.\begin{array}{rcll} u_\varepsilon & \xrightarrow{w} & v_0^- & \text{in } H^1(\Omega_0),\\ \widetilde{u_\varepsilon} & \xrightarrow{w} & |\omega(x_3)|\,v_0^+ & \text{in } L^2(\Omega^+),\\ \widetilde{\partial_{x_3}u_\varepsilon} & \xrightarrow{w} & |\omega(x_3)|\,\partial_{x_3}v_0^+ & \text{in } L^2(\Omega^+),\\ \widetilde{\partial_{x_i}u_\varepsilon} & \xrightarrow{w} & 0 & \text{in } L^2(\Omega^+),\ (i=1,2) \end{array}\right\}\quad as\quad \varepsilon\to 0, \tag{26}$$

where the function

$$v_0(x)=\begin{cases}v_0^-(x), & x\in\Omega_0,\\ v_0^+(x), & x\in\Omega^+,\end{cases} \tag{27}$$

is a unique weak solution to the following problem

$$\begin{array}{rcll} -\Delta_x\ v_0^-(x) &=& f_0(x), & x\in\Omega_0\\ \partial_\nu v_0^-(x) &=& 0, & x\in\partial\Omega_0\setminus\Xi_0,\\ -\partial_{x_3}\big(|\omega(x_3)|\,\partial_{x_3}v_0^+(x)\big) &=& -l_\omega(x_3)\,\kappa\big(v_0^+(x)\big), & x\in\Omega^+,\\ v_0^-(x',0) &=& v_0^+(x',0), & (x',0)\in\Xi_0,\\ \partial_{x_3}v_0^-(x',0) &=& |\omega(0)|\,\partial_{x_3}v_0^+(x',0), & (x',0)\in\Xi_0,\\ \partial_{x_3}v_0^+(x',h) &=& 0, & (x',h)\in\Xi_h, \end{array} \tag{28}$$

which is called homogenized problem for (1). Here $\Xi_h=\{x:\ x_3=h,\ x'\in\Xi_0\}$.

Furthermore, the following energy convergence holds as $\varepsilon\to 0:$

$$E_\varepsilon(u_\varepsilon)=\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,dx+\varepsilon\int_{\Gamma_\varepsilon}\kappa(u_\varepsilon)\,u_\varepsilon\,d\sigma_x\ \to \\ \int_{\Omega_0}|\nabla v_0^-|^2\,dx+\int_{\Omega^+}|\omega(x_3)|\,|\partial_{x_3}v_0^+|^2\,dx+\int_{\Omega^+}l_\omega(x_3)\,\kappa\big(v_0^+\big)\,v_0^+\,dx=E_0(v_0). \tag{29}$$

Proof. **1.** It follows from (13) and (2) that the values

$$\|u_\varepsilon\|_{H^1(\Omega_0)},\quad \|\widetilde{u_\varepsilon}\|_{L^2(\Omega^+)},\quad \|\widetilde{\partial_{x_i}u_\varepsilon}\|_{L^2(\Omega^+)}\quad(i=1,2,3),\quad \|\widetilde{\kappa(u_\varepsilon)}\|_{L^2(\Omega^+)}$$

are uniformly bounded with respect to ε. Hence, there exists a subsequence $\{\varepsilon'\}\subset\{\varepsilon\}$, again denoted by ε, such that

$$\left.\begin{array}{rcll} u_\varepsilon & \xrightarrow{w} & v_0^- & \text{in } H^1(\Omega_0),\\ \widetilde{u_\varepsilon} & \xrightarrow{w} & |\omega(x_3)|\big(|\omega(x_3)|^{-1}\,v\big)=:|\omega|\,v_0^+ & \text{in } L^2(\Omega^+),\\ \widetilde{\partial_{x_i}u_\varepsilon} & \xrightarrow{w} & \eta_i & \text{in } L^2(\Omega^+),\\ \widetilde{\kappa(u_\varepsilon)} & \xrightarrow{w} & \zeta & \text{in } L^2(\Omega^+), \end{array}\right\}\quad \text{as } \varepsilon\to 0, \tag{30}$$

where v_0^-, v_0^+, $\eta_i\ (i=1,2,3)$, ζ are some functions which will be determined in what follows.

2. At first we determine η_3. Take any function $\phi \in C_0^\infty(\Omega^+)$ and with the help of (5) perform the following calculations:

$$\int_{\Omega^+} \widetilde{\partial_{x_3} u_\varepsilon}\, \phi\, dx = \int_{G_\varepsilon} \partial_{x_3} u_\varepsilon\, \phi\, dx = -\int_{G_\varepsilon} u_\varepsilon\, \partial_{x_3}\phi\, dx - \varepsilon \int_{\Gamma_\varepsilon} \frac{\varrho'(x_3)\, u_\varepsilon \phi}{\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}}\, d\sigma_x =$$

$$= -\int_{\Omega^+} \widetilde{u_\varepsilon}\, \partial_{x_3}\phi\, dx -$$

$$-\int_{\Omega^+} \varrho'(x_3)\, g(x_3)\, \widetilde{u_\varepsilon}\, \phi\, dx - \varepsilon \int\limits_{G_\varepsilon} \varrho'(x_3)\, \nabla_{\xi'} Y(\xi', x_3)|_{\xi'=\frac{x'}{\varepsilon}} \cdot \nabla_{x'}\big(u_\varepsilon \phi\big)\, dx. \quad (31)$$

Taking into account (8) and (13), and passing to the limit in (31) as $\varepsilon \to 0$, we obtain

$$\int_{\Omega^+} \eta_3\, \phi\, dx = -\int_{\Omega^+} \big(|\omega(x_3)|\, v_0^+\, \partial_{x_3}\phi \,+\, |\omega(x_3)|'\, v_0^+\, \phi\big)\, dx \quad \forall \phi \in C_0^\infty(D_1),$$

whence it follows that there exists a weak derivative $\partial_{x_3} v_0^+$ and $\eta_3(x) = |\omega(x_3)|\, \partial_{x_3} v_0^+(x)$ for a.e. $x \in \Omega^+$.

Now let us find $\eta_i,\ i=1,2$. Consider the functions

$$Y_i(\xi_i) = -\xi_i + \frac{1}{2} + [\xi_i], \quad i=1,2, \quad (32)$$

where $[t]$ is the integer part of t. With the help of these functions we determine the following test functions

$$\Phi_i(x) = \begin{cases} 0, & x \in \Omega_0, \\ \varepsilon Y_i\left(\frac{x_i}{\varepsilon}\right)\, \psi(x), & x \in G_\varepsilon, \quad i=1,2, \end{cases} \qquad \forall\, \psi \in C_0^\infty(\Omega^+).$$

It is easy to see that $\Phi_i \in H^1(\Omega_\varepsilon)$ and

$$\begin{array}{l} \nabla\Phi_1 = \left(-\psi + \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1}\psi,\ \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right) \partial_{x_2}\psi,\ \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right) \partial_{x_3}\psi\right), \\ \nabla\Phi_2 = \left(\varepsilon Y_2\left(\frac{x_2}{\varepsilon}\right) \partial_{x_1}\psi,\ -\psi + \varepsilon Y_2\left(\frac{x_2}{\varepsilon}\right) \partial_{x_2}\psi,\ \varepsilon Y_2\left(\frac{x_2}{\varepsilon}\right) \partial_{x_3}\psi\right), \end{array} \quad x \in G_\varepsilon.$$

Substituting the functions Φ_1 and Φ_2 into the integral identity (3), we get

$$\int_{G_\varepsilon} \left(-\frac{\partial u_\varepsilon}{\partial x_i}\, \psi + \varepsilon Y_i \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial \psi}{\partial x_1} + \varepsilon Y_i \frac{\partial u_\varepsilon}{\partial x_2}\frac{\partial \psi}{\partial x_2} + \varepsilon Y_i \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \psi}{\partial x_3}\right) dx =$$

$$= -\varepsilon^2 \int_{\Gamma_\varepsilon} \kappa\big(u_\varepsilon\big)\, Y_i \psi\, d\sigma_x, \quad i=1,2,$$

from which with the help of (13), (9) and (2) we deduce that

$$\left|\int_{\Omega^+} \widetilde{\partial_{x_i} u_\varepsilon}\, \psi\, dx\right| \le \varepsilon c_1 \|\psi\|_{H^1(\Omega^+)}, \quad i=1,2. \quad (33)$$

Passing to the limit as $\varepsilon \to 0$ in (33) gives $\int_{\Omega^+} \eta_i\, \psi\, dx = 0$ for any $\psi \in C_0^\infty(\Omega^+)$. This means that $\eta_i = 0$ a.e. in $\Omega^+,\ i=1,2$.

3. Let us show that the traces $v_0^+|_{\Xi_0}$ and $v_0^-|_{\Xi_0}$ are equal. By virtue of the compactness of the trace operator and the first relation in (30), we have

$$u_\varepsilon(x', 0) \longrightarrow v_0^-(x', 0) \quad \text{in} \quad L^2(\Xi_0) \quad \text{as} \quad \varepsilon \to 0. \quad (34)$$

Consider the following equality

$$\widetilde{u_\varepsilon}(x', 0) = \chi_{\omega_0}\left(\frac{x'}{\varepsilon}\right) u_\varepsilon(x', 0), \qquad x' \in \Xi_0, \quad (35)$$

where $\chi_{\omega_0}(\xi')$, $\xi' \in \mathbb{R}^2$, is the 1-periodic function defined on the square $[0,1]\times[0,1]$ as follows:

$$\chi_{\omega_0}(\xi') = \begin{cases} 1, & \xi' \in \overline{\omega(0)}, \\ 0, & [0,1]\times[0,1] \setminus \overline{\omega(0)}. \end{cases}$$

Obviously, $\chi_{\omega_0}(\frac{x'}{\varepsilon}) \stackrel{w}{\longrightarrow} |\omega(0)|$ in $L^2(\Xi_0)$ as $\varepsilon \to 0$. Due to (34) the right-hand side of equality (35) converges weakly to $|\omega(0)|\, v_0^-(\cdot, 0)$ in $L^2(\Xi_0)$ as $\varepsilon \to 0$.

On the other hand, with the help of (5) we have

$$\int_{\Xi_0} \widetilde{u_\varepsilon}(x',0)\,\psi(x')\,dx' = \frac{1}{h}\Big(\int_{\Omega^+} \widetilde{u_\varepsilon}(x)\,\psi(x')\,dx + \int_{\Omega^+} (x_3-h)\,\widetilde{\partial_{x_3} u_\varepsilon}(x)\,\psi(x')\,dx +$$
$$\int_{\Omega^+} g(x_3)\,\varrho'(x_3)\,(x_3-h)\widetilde{u_\varepsilon}\,\psi\,dx \,+\, \varepsilon\int_{G_\varepsilon} \varrho'(x_3)\,(x_3-h)\nabla_{\xi'}Y(\xi',x_3)|_{\xi'=\frac{x'}{\varepsilon}} \cdot \nabla_{x'}\big(u_\varepsilon\psi\big)\,dx\Big) \quad (36)$$

for any function $\psi \in C_0^\infty(\Xi_0)$. Taking into account convergence results obtained above and passing to the limit in (36) as $\varepsilon \to 0$, we get the following identity

$$\int_{\Xi_0} |\omega(0)|\, v_0^-(x',0)\,\psi(x')\,dx' = \frac{1}{h}\Big(\int_{\Omega^+} |\omega(x_3)|\, v_0^+(x)\,\psi(x')\,dx +$$
$$\int_{\Omega^+} (x_3-h)\,|\omega(x_3)|\,\partial_{x_3} v_0^+(x)\,\psi\,dx + \int_{\Omega^+} (x_3-h)\,g(x_3)\,\varrho'(x_3)\,|\omega(x_3)|\,v_0^+(x)\,\psi(x')\,dx\Big) =$$
$$= \frac{1}{h}\int_{\Omega^+} \Big(|\omega(x_3)| v_0^+(x)\,\psi(x') + (x_3-h)\,\psi(x')\,\partial_{x_3}\big(|\omega(x_3)| v_0^+(x)\big)\Big)\,dx =$$
$$= \int_{\Xi_0} |\omega(0)|\, v_0^+(x',0)\,\psi(x')\,dx' \quad \forall\, \psi \in C_0^\infty(\Xi_0),$$

which implies that

$$v_0^+(x',0) = v_0^-(x',0) \quad \text{for a.e. } x' \in \Xi_0. \qquad (37)$$

4. Using the extension by zero and the identity (5), we rewrite the integral identity (3) in the following way

$$\int_{\Omega_0} f_0\,\varphi\,dx = \int_{\Omega_0} \nabla u_\varepsilon \cdot \nabla\varphi\,dx + \int_{\Omega^+} \Big(\widetilde{\partial_{x_1} u_\varepsilon}\,\partial_{x_1}\varphi + \widetilde{\partial_{x_2} u_\varepsilon}\,\partial_{x_2}\varphi + \widetilde{\partial_{x_3} u_\varepsilon}\,\partial_{x_3}\varphi\Big)dx +$$
$$+\, \varepsilon\int_{G_\varepsilon} \sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,\nabla_{\xi'}Y(\xi',x_3)|_{\xi'=\frac{x'}{\varepsilon}} \cdot \nabla_{x'}\big(\kappa\big(u_\varepsilon\big)\,\varphi\big)\,dx +$$
$$+ \int_{\Omega^+} g(x_3)\,\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,\widetilde{\kappa\big(u_\varepsilon\big)}\,\varphi\,dx \qquad \forall\, \varphi \in C^1(\overline{\Omega}), \quad (38)$$

where $\Omega = \Omega_0 \cup \Xi_0 \cup \Omega^+$. It is easy to see that the third summand vanishes due to (2), (8) and (13). Taking into account (30) and results obtained in the second item, we pass to the limit in (38) as $\varepsilon \to 0$. As a result we get the identity

$$\int\limits_{\Omega_0} f_0\,\varphi\,dx = \int\limits_{\Omega_0} \nabla v_0^- \cdot \nabla\varphi\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,\partial_{x_3} v_0^+\,\partial_{x_3}\varphi\,dx + \int\limits_{\Omega^+} g(x_3)\,\zeta(x)\,\varphi\,dx \qquad (39)$$

for any $\varphi \in C^1(\overline{\Omega})$. Since the space $C^1(\overline{\Omega})$ is dense in the anisotropic Sobolev space

$$\mathcal{H} = \{v \in L^2(\Omega) : \quad \partial_{x_3} v \in L^2(\Omega), \quad v|_{\Omega_0} \in H^1(\Omega_0)\},$$

identity (39) is valid for any function $\varphi \in \mathcal{H}$. It should be stressed here that there are traces $v|_{\Xi_0+0}$ and $v|_{\Xi_0-0}$ for any function $v \in \mathcal{H}$ and they are equal (see [36]).

With the help of (3) and (39) we can find that

$$\lim_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \varepsilon\int_{\Gamma_\varepsilon}\kappa\big(u_\varepsilon\big)\,u_\varepsilon\,d\sigma_x\Big) = \lim_{\varepsilon\to 0}\int_{\Omega_0} f_0\,u_\varepsilon\,dx = \int_{\Omega_0} f_0\,v_0^-\,dx =$$
$$= \int_{\Omega_0}|\nabla v_0^-|^2\,dx + \int_{\Omega^+}|\omega(x_3)|\,|\partial_{x_3}v_0^+|^2\,dx + \int_{\Omega^+} g(x_3)\,\zeta(x)\,v_0^+\,dx. \quad (40)$$

5. Now it remains to determine the function ζ. For this we will use the method of Browder and Minty, a remarkable technique which somehow applies to the corresponding inequality of monotonicity to justify passing to a weak limit within a nonlinearity. Thanks to (2), the inequality of monotonicity in our case reads as follows

$$\int_{\Omega_0}|\nabla u_\varepsilon - \nabla\varphi|^2\,dx + \int_{G_\varepsilon}|\partial_{x_3}u_\varepsilon - \partial_{x_3}\varphi|^2\,dx +$$
$$+\,\varepsilon\int_{\Gamma_\varepsilon}\big(\kappa(u_\varepsilon)-\kappa(\varphi)\big)\,\big(u_\varepsilon-\varphi\big)\,d\sigma_x \ge 0 \quad \forall\,\varphi\in C^1(\overline{\Omega}), \quad (41)$$

which leads to

$$\int_{\Omega_0}|\nabla u_\varepsilon|^2\,dx + \int_{G_\varepsilon}|\partial_{x_3}u_\varepsilon|^2\,dx + \varepsilon\int_{\Gamma_\varepsilon}\kappa\big(u_\varepsilon\big)\,u_\varepsilon\,d\sigma_x -$$
$$-\,2\int_{\Omega_0}\nabla u_\varepsilon\cdot\nabla\varphi\,dx - 2\int_{\Omega^+}\widetilde{\partial_{x_3}u_\varepsilon}\cdot\partial_{x_3}\varphi\,dx + \int_{\Omega_0}|\nabla\varphi|^2\,dx + \int_{\Omega^+}\chi_{G_\varepsilon}\,|\partial_{x_3}\varphi|^2\,dx -$$
$$-\,\varepsilon\int_{\Gamma_\varepsilon}\kappa\big(u_\varepsilon\big)\,\varphi\,d\sigma_x - \varepsilon\int_{\Gamma_\varepsilon}\kappa\big(\varphi\big)\,u_\varepsilon\,d\sigma_x + \varepsilon\int_{\Gamma_\varepsilon}\kappa\big(\varphi\big)\,\varphi\,d\sigma_x \ge 0. \quad (42)$$

Taking into account (33), the limit of the first line in (42) is given by (40). With regard to the results of the second item and (23) we know the limit of the second line in (42) And with the help of Lemma 1 we can find the limits of the other three integrals. As a result we get

$$\int_{\Omega_0}|\nabla v_0^- - \nabla\varphi|^2\,dx + \int_{\Omega^+}|\omega(x_3)|\,\big(\partial_{x_3}v_0^+ - \partial_{x_3}\varphi\big)^2\,dx +$$
$$+\int_{\Omega^+} g(x_3)\big(\zeta(x) - |\omega(x_3)|\,\kappa(\varphi)\big)\,(v_0^+ - \varphi)\,dx \ge 0 \quad \forall\,\varphi\in C^1(\overline{\Omega}). \quad (43)$$

Since $C^1(\overline{\Omega})$ is dense in $\mathcal{H}$, inequality (43) is valid for any function $\varphi\in\mathcal{H}$.

Fix any $\psi\in C^1(\overline{\Omega})$ and set $\varphi := v_0 - \lambda\psi$ ($\lambda > 0$) in (43), where

$$v_0(x) = \begin{cases} v_0^-(x), & x\in\Omega_0,\\ v_0^+(x), & x\in\Omega^+.\end{cases}$$

We get then

$$\lambda\int_{\Omega_0}|\nabla\psi|^2\,dx + \lambda\int_{\Omega^+}|\omega(x_3)|\,|\partial_{x_3}\psi|^2\,dx +$$
$$+\int_{\Omega^+} g(x_3)\big(\zeta(x) - |\omega(x_3)|\,\kappa(v_0^+ - \lambda\psi)\big)\,\psi\,dx \ge 0.$$

In the limit (as $\lambda\to 0$) we obtain

$$\int_{\Omega^+} g(x_3)\big(\zeta(x) - |\omega(x_3)|\,\kappa(v_0^+)\big)\,\psi\,dx \ge 0 \quad \forall\,\psi\in C^1(\overline{\Omega}).$$

Replacing ψ by $-\psi$, we deduce that in fact quality holds above. Thus

$$\zeta(x) = |\omega(x_3)|\,\kappa\big(v_0^+(x)\big) \quad \text{for a.e. } x\in\Omega^+. \quad (44)$$

6. Returning to (39), we see that the function v_0 belongs to $\mathcal{H}$ due to (37) and it satisfies the following integral identity

$$\int\limits_{\Omega_0} \nabla v_0^- \cdot \nabla\varphi\, dx + \int\limits_{\Omega^+} |\omega(x_3)|\, \partial_{x_3} v_0^+\, \partial_{x_3}\varphi\, dx + \int\limits_{\Omega^+} l_\omega(x_3)\, \kappa\big(v_0^+\big)\, \varphi\, dx = \int\limits_{\Omega_0} f_0\, \varphi\, dx \quad \forall\, \varphi \in \mathcal{H}. \quad (45)$$

Hence v_0 is a weak solution to the limit problem (28).

Assume that $v_0^{(1)}$ and $v_0^{(2)}$ are two weak solutions to problem (28). Consequently

$$\int_{\Omega_0} \nabla\big(v_0^{(1,-)} - v_0^{(2,-)}\big) \cdot \nabla\varphi\, dx + \int_{\Omega^+} |\omega(x_3)|\, \partial_{x_3}\big(v_0^{(1,-)} - v_0^{(2,-)}\big)\, \partial_{x_3}\varphi\, dx + \\ + \int_{\Omega^+} l_\omega(x_3) \Big(\kappa\big(v_0^{(1,+)}\big) - \kappa\big(v_0^{(2,+)}\big)\Big)\, \varphi\, dx = 0 \quad \forall\, \varphi \in \mathcal{H}. \quad (46)$$

We set $\varphi = v_0^{(1)} - v_0^{(2)}$ in (46), and use (2) to deduce that $v_0^{(1)} = v_0^{(2)}$ a.e. in Ω.

Due to the uniqueness of the solution to problem (28), the above argumentations hold for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. By replacing (44) in (40), one obtains the convergence of energies (29). □

4 Asymptotic representations for the solution

Here the leading terms of asymptotic expansions for the solution u_ε are constructed both in the junction body and in each thin cylinder as well as in a neighborhood of the joint zone. Then, using the method of matched asymptotic expansions, we construct an asymptotic approximation and prove the corresponding estimates. In this section we assume that function ϱ is constant in a small enough neighborhood of zero.

4.1 Outer expansions

We seek the main terms of the asymptotics for the solution u_ε, restricted to Ω_0, in the form

$$u_\varepsilon \approx v_0^-(x) + \sum_{n=1}^{\infty} \varepsilon^n v_n^-(x), \quad (47)$$

and, restricted to each thin curvilinear cylinder $G_\varepsilon(i,j)$, in the form

$$u_\varepsilon \approx v_0^+(x) + \sum_{n=1}^{\infty} \varepsilon^n v_n^+(x, \xi_1 - i, \xi_2 - j), \qquad \xi_1 = \varepsilon^{-1} x_1, \quad \xi_2 = \varepsilon^{-1} x_2. \quad (48)$$

The expansions (47) and (48) are usually called *outer expansions*.

Substituting the series (47) in (1) and collecting the terms of order ε^0 in the equation and in the boundary conditions on $\Omega_0 \setminus \Xi_0$, we get the first two relations for v_0^- in (28).

Decomposing formally the function v_n^+ in the Taylor series with respect to the variables x_1, x_2 in a neighborhood of the point $x_1^{(i)} = \varepsilon(i+1/2)$, $x_1^{(j)} = \varepsilon(j+1/2)$, we rewrite (48) in the following form :

$$u_\varepsilon = v_0^+(i,j,x_3) + \sum_{n=1}^{2} \varepsilon^n V_n^{i,j}(x_3, \xi_1, \xi_2) + O(\varepsilon^3), \quad x \in G_\varepsilon(i,j), \quad (49)$$

$$V_n^{i,j} = \sum_{m=0}^{n} \frac{1}{m!} \left((\xi_1 - i - 1/2) \frac{\partial}{\partial x_1} + (\xi_2 - j - 1/2) \frac{\partial}{\partial x_2} \right)^m v_{n-m}^+(i,j,x_3,\xi_1,\xi_2), \quad (50)$$

where $v_n^+(i,j,\ldots) = v_n^+(x_1^{(i)}, x_2^{(j)}, \ldots)$.

Substituting (49) in (1) and collecting corresponding terms lead us to the following problems for functions $V_1^{i,j}$ and $V_2^{i,j}$:

$$\Delta_{\xi'} V_1^{i,j}(x_3,\xi') = 0 \ \text{ in } \ \omega(x_3), \quad \partial_{\nu(\xi')} V_1^{i,j}(x_3,\xi') = 0 \ \text{ on } \ \partial\omega(x_3), \tag{51}$$

$$\begin{aligned} -\Delta_{\xi'} V_2^{i,j}(x_3,\xi') &= \partial^2_{x_3x_3} v_0^+(i,j,x_3), & \xi' \in \omega(x_3) \\ -\partial_{\nu(\xi')} V_2^{i,j}(x_3,\xi') &= -\varrho'(x_3)\,\partial_{x_3} v_0^+ + \kappa\big(v_0^+(i,j,x_3)\big), & \xi' \in \partial\omega(x_3). \end{aligned} \tag{52}$$

Here $\xi' = (\xi_1,\xi_2)$, the variable $x_3 \in (0,h)$ is regarded as a parameter in these problems. To obtain the second relation in (52), we have to decompose function κ in the Taylor series as well.

From (51) it follows that the function $V_1^{i,j}$ doesn't depend on ξ'. We restrict ourselves to the leading term of the asymptotics, and thus set $V_1^{i,j} = 0$. Then, by virtue of (50), we have

$$v_1^+(i,j,x_3,\xi') = -\partial_{x_1} v_0^+(i,j,x_3)\,(\xi_1 - i - 1/2) - \partial_{x_2} v_0^+(i,j,x_3)\,(\xi_2 - j - 1/2)\,. \tag{53}$$

The solvability condition for problem (52) is given by the ordinary differential equation with respect to x_3 :

$$\partial_{x_3}\big(|\omega(x_3)|\,\partial_{x_3} v_0^+(i,j,x_3)\big) \;=\; l_\omega(x_3)\,\kappa\big(v_0^+(i,j,x_3)\big).$$

Since the intervals $\{x:\ x_1^{(i)} = \varepsilon(i+1/2),\ \ x_1^{(j)} = \varepsilon(j+1/2),\ \ 0 < x_3 < h\}$, $i,j = 1,\ldots,N-1$, fill out the domain Ω^+ in the limit passage as $\varepsilon \to 0$, we extend the last equation over Ω^+ (for comparison see the third equation in (28)). Obviously, it must be required that v_0^+ satisfies the last equality in (28) as well.

So, it remains to find conditions for the functions v_0^- and v_0^+ in the joint zone Ξ_0. For this we should match the outer asymptotic expansions (47), (48) with an *inner* expansion

$$u_\varepsilon \approx v_0^-(x',0) + \varepsilon \sum_{i=1}^{3} Z_i(x/\varepsilon)\partial_{x_i} v_0^-(x',0) + \ldots \tag{54}$$

in a neighborhood of Ξ_0, namely, the asymptotic behaviour of the first terms of the outer expansion as $x_3 \to \pm 0$ must coincide with the asymptotic behaviour of the first terms of the inner expansion as $\xi_3 \to \pm\infty$.

In (54), $\{Z_i\}$ are functions of the junction-layer type defined in the union Π of the infinite semi-cylinders

$$\Pi^+ = \omega(0) \times [0,+\infty), \quad \text{and} \quad \Pi^- = (0,1)\times(0,1)\times(-\infty,0).$$

It is clear that, since the thin cylinders are located periodically, the functions Z_i, $i = 1,2,3$, must be 1−periodic in ξ_1 and ξ_2. The other relations for these functions can be obtained by the substitution of (54) in problem (1). As a result, we get

$$\begin{aligned} -\Delta_{\xi\xi} Z_i(\xi) &= 0, & \xi \in \Pi^\pm, \\ \partial_{\nu_{\xi'}} Z_i(\xi) &= -\delta_{1,i}\nu_1(\xi') - \delta_{2,i}\nu_2(\xi'), & \xi \in \partial\Pi^+ \setminus \omega(0), \\ \partial_{\xi_3} Z_i(\xi',0) &= 0, & (\xi',0) \in \partial\Pi^- \setminus \omega(0), \\ \partial^k_{\xi_j} Z_i|_{\xi_j=0} &= \partial^k_{\xi_j} Z_i|_{\xi_j=1},\ k=0,1,\ j=1,2, & \xi \in \partial\Pi^- \cup \{\xi_3<0\}, \\ [Z_i]_{|\xi_3=0} &= [\partial_{\xi_3} Z_i]_{|\xi_3=0} \;=\; 0, & \xi' \in \omega(0), \end{aligned} \tag{55}$$

where $\delta_{i,j}$ is the Kronecker symbol.

The solvability and properties of solutions of these problems were investigated in [25]. It follows from the results of [25] that there exist unique solutions $Z_i \in H^1_{\sharp,loc}(\Pi)$, $i = 1,2,3$, of these problems with the following differentiable asymptotics:

$$Z_i(\xi) = \begin{cases} -\xi_i + \frac{1}{2} + \mathcal{O}(\exp(-\delta_i\xi_3), & \xi_3 \to +\infty, \\ \mathcal{O}(\exp(\delta_i\xi_3)), & \xi_3 \to -\infty, \end{cases} \quad i = 1,2, \tag{56}$$

$$Z_3(\xi) = \begin{cases} \frac{1}{|\omega(0)|}\,\xi_3 + \mathcal{O}(\exp(-\delta_3\xi_3)), & \xi_3 \to +\infty, \\ C_3 + \xi_3 + \mathcal{O}(\exp(\delta_0\xi_3)), & \xi_3 \to -\infty, \end{cases} \tag{57}$$

where $\delta_i,\ i = 1, 2, 3$, are certain positive constants, $H^1_{\sharp,loc}(\Pi) = \{u : \Pi \to \mathbb{R} \mid u$ is periodic in ξ_1 and ξ_2 for any $\xi_3 < 0,\ u \in H^1(\Pi_R)$ for any $R > 0\},\ \Pi_R = \Pi \cap \{\xi :\ -R < \xi_3 < R\}$. Furthermore, these functions possess the following symmetry properties with respect to the axis $(1/2, 1/2, \xi_3)$: Z_1 is odd with respect to ξ_1 and even with respect to ξ_2, Z_2 is even with respect to ξ_1 and odd with respect to ξ_2, and Z_3 is even with respect to ξ_1 and ξ_2.

It should be noted that general results about the existence and the main asymptotic relations for solutions to elliptic problems in domains with different exits to infinity can be found in [30].

Expanding the first terms $v_0^\pm$ of the outer asymptotic series (47) and (48) in the Taylor series with respect to x_3 in a neighborhood of a point $(x', 0) \in \Xi_0$ and passing to the variable ξ_3, we obtain

$$v_0^\pm(x, t) = v_0^\pm(x', 0) + \varepsilon\xi_3\partial_{x_3}v_0^\pm(x', 0) + \mathcal{O}(\varepsilon^2\xi_3^2).$$

Equating the coefficients of these expansions and the coefficients of the asymptotics (as $\xi_3 \to \pm\infty$) for the first terms of inner series (54) and taking (56) and (57) into account, we establish that

$$v_0^+(x', 0) = v_0^-(x', 0) \quad \text{and} \quad |\omega(0)|\,\partial_{x_3}v_0^+(x', 0) = \partial_{x_3}v_0^-(x', 0), \quad x' \in \Xi_0.$$

Thus, the first terms $v_0^\pm$ of the asymptotic expansions (47) and (48) satisfy equalities of the limit problem (28).

4.2 Asymptotic approximation

An approximating function R_ε is constructed as the sum of the first terms of outer expansions (47), (48) and inner expansion (54) with the subtraction of the identical terms of their asymptotics (as $x_3 \to \pm 0$ and $\xi_3 \to \pm\infty$, respectively) because they are summed twice. As a result, we get

$$R_\varepsilon = v_0^+(x) + \varepsilon\Big(\sum_{i=1}^{2} Y_i(\xi_i)\partial_{x_i}v_0^+(x) + \chi_0(x_3)\,\mathcal{N}^+(\xi, x')\Big)\big|_{\xi=\frac{x}{\varepsilon}}, \quad x \in G_\varepsilon. \tag{58}$$

$$R_\varepsilon(x) = v_0^-(x) + \varepsilon\chi_0(x_3)\,\mathcal{N}^-(\xi, x')|_{\xi=\frac{x}{\varepsilon}}, \quad x \in \Omega_0, \tag{59}$$

Here $Y_i,\ i = 1, 2$, are defined in (32); $\chi_0 \in C_0^\infty(\mathbb{R})$ is a cut-off function such that

$$\chi_0(\xi_3) = \begin{cases} 1, & |\xi_3| \le \theta_0/2, \\ 0, & |\xi_3| \ge \theta_0, \end{cases}$$

where θ_0 is a small enough fixed positive number such that the function ϱ is constant on the segment $[0, \theta_0]$,

$$\mathcal{N}^+(\xi, x') = \sum_{i=1}^{3}\left(Z_i(\xi)\ -\ \delta_{i,1}\,Y_1(\xi_1)\ -\ \delta_{i,2}\,Y_2(\xi_2)\ -\ \delta_{i,3}\,\frac{1}{|\omega(0)|}\,\xi_3\right)\partial_{x_i}v_0^-(x', 0);$$

$$\mathcal{N}^-(\xi, x') = \sum_{i=1}^{3}\left(Z_i(\xi) - \delta_{i,3}\ \xi_3\right)\partial_{x_i}v_0^-(x', 0).$$

Theorem 3. *For any $\delta \in (0, \frac{1}{2})$ there exist positive constants C_0, ε_0 such that for all values $\varepsilon \in (0, \varepsilon_0)$ the difference between the solution u_ε to problem (1) and the approximation function R_ε defined by (58) and (59) satisfies the following estimate*

$$\|u_\varepsilon - R_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C_0\,\varepsilon^{1-\delta}. \tag{60}$$

Proof. **Discrepancies in the domain** Ω_0. Taking into account the properties of the functions $\{Z_i\}$ and function v_0^-, we conclude that R_ε satisfies the Neumann boundary conditions on $\partial\Omega_0 \cap \partial\Omega_\varepsilon$ for problem (1).

Putting R_ε in the corresponding differential equation of problem (1) gives

$$\begin{aligned} -\Delta_x R_\varepsilon(x) - f_0(x) = -\chi_0'(x_3)\big(\partial_{\xi_3}\mathcal{N}^-(\xi,x')\big)|_{\xi=x/\varepsilon} - \\ -\chi_0(x_3)\textstyle\sum_{i=1}^2\big(\partial^2_{x_i\xi_i}\mathcal{N}^-(\xi,x')\big)|_{\xi=x/\varepsilon} - \varepsilon\partial_{x_3}\big(\chi_0'(x_3)\mathcal{N}^-(x/\varepsilon,x')\big) - \\ -\varepsilon\chi_0(x_3)\textstyle\sum_{i=1}^2\partial_{x_i}\big(\big(\partial_{x_i}\mathcal{N}^-(\xi,x')\big)|_{\xi=x/\varepsilon}\big), \quad x\in\Omega_0. \end{aligned} \tag{61}$$

Further, the arguments of functions involved in calculations are indicated only if their absence may cause confusion. We multiply the identity (61) by a test function $\varphi\in H^1(\Omega_\varepsilon)$ and integrate by parts in Ω_0 :

$$-\int_{\Xi_0(\varepsilon)}\partial_{x_3}R_\varepsilon(x',0-0)\,\varphi\,dx' + \int_{\Omega_0}\nabla_x R_\varepsilon\cdot\nabla_x\varphi\,dx - \int_{\Omega_0} f_0\varphi\,dx = I_1^- + \ldots + I_4^-, \tag{62}$$

where $\Xi_0(\varepsilon)=\Xi_0\cap\overline{G_\varepsilon}$,

$$I_1^-(\varepsilon,\varphi) = -\int_{\Omega_0}\chi_0'(x_3)\big(\partial_{\xi_3}\mathcal{N}^-(\xi,x')\big)|_{\xi=x/\varepsilon}\,\varphi\,dx,$$

$$I_2^-(\varepsilon,\varphi) = -\sum_{i=1}^2\int_{\Omega_0}\chi_0(x_3)\big(\partial^2_{x_i\xi_i}\mathcal{N}^-(\xi,x')\big)|_{\xi=x/\varepsilon}\,\varphi\,dx,$$

$$I_3^-(\varepsilon,\varphi) = \varepsilon\int_{\Omega_0}\chi_0'(x_3)\mathcal{N}^-(x/\varepsilon,x')\,\partial_{x_3}\varphi\,dx,$$

$$I_4^-(\varepsilon,\varphi) = \varepsilon\sum_{i=1}^2\int_{\Omega_0}\chi_0(x_3)\big(\partial_{x_i}\mathcal{N}^-(\xi,x')\big)|_{\xi=x/\varepsilon}\big)\,\partial_{x_i}\varphi\,dx.$$

Discrepancies in the thin curvilinear cylinders. It is easy to calculate that $\partial_{x_3}R_\varepsilon(x',h)=0$,

$$\partial_{x_3}R_\varepsilon(x',0+0) = \varepsilon\sum_{i=1}^2 Y_i\big(\frac{x_i}{\varepsilon}\big)\partial^2_{x_3x_i}v_0^-(x',0) + \partial_{x_3}R_\varepsilon(x',0-0), \quad x'\in\Xi_0(\varepsilon), \tag{63}$$

and

$$\begin{aligned}\partial_\nu R_\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2|\varrho'|^2}}\Bigg\{\varepsilon\left[\sum_{i=1}^2 Y_i\big(\frac{x_i}{\varepsilon}\big)\,\partial_{\nu'}\big(\partial_{x_i}v_0^+\big) + \chi_0\sum_{k=1}^2\big(\partial_{x_k}\mathcal{N}^+(\xi,x')\big)|_{\xi=\frac{x}{\varepsilon}}\big)\,\nu_k\big(\frac{x'}{\varepsilon}\big)\right] - \\ -\varepsilon\varrho'(x_3)\,\partial_{x_3}\Big(v_0^+(x)+\varepsilon\sum_{i=1}^2 Y_i\big(\frac{x_i}{\varepsilon}\big)\partial_{x_i}v_0^+(x)\Big)\Bigg\}, \quad x\in\Gamma_\varepsilon,\end{aligned} \tag{64}$$

where $\nu'=\big(\nu_1(x'/\varepsilon),\nu_2(x'/\varepsilon)\big)$.

Putting R_ε in the corresponding differential equation of problem (1) gives

$$\begin{aligned} -\Delta_x R_\varepsilon = -\chi_0'(x_3)\big(\partial_{\xi_3}\mathcal{N}^+(\xi,x')\big)|_{\xi=x/\varepsilon} - \\ -\chi_0(x_3)\textstyle\sum_{i=1}^2\big(\partial^2_{x_i\xi_i}\mathcal{N}^+(\xi,x')\big)|_{\xi=x/\varepsilon} - \varepsilon\partial_{x_3}\big(\chi_0'(x_3)\mathcal{N}^+(x/\varepsilon,x')\big) - \\ -\varepsilon\chi_0(x_3)\textstyle\sum_{i=1}^2\partial_{x_i}\big(\big(\partial_{x_i}\mathcal{N}^+(\xi,x')\big)|_{\xi=x/\varepsilon}\big) - \\ -\varepsilon\textstyle\sum_{k=1}^3\sum_{i=1}^2\partial_{x_k}\Big(Y_i\big(\frac{x_i}{\varepsilon}\big)\partial^2_{x_ix_k}v_0^+(x)\Big) + \\ +\partial_{x_3}\big(\ln|\omega(x_3)|\big)\,\partial_{x_3}v_0^+(x) - g(x_3)\,\kappa\big(v_0^+\big), \quad x\in G_\varepsilon. \end{aligned} \tag{65}$$

Using (5) and taking into account the boundary values of $\partial_\nu R_\varepsilon$ (see (63), (64)), we multiply (65) by a test function $\varphi \in H^1(\Omega_\varepsilon)$ and integrate by parts in G_ε. This yields

$$\int_{\Xi_0(\varepsilon)} \partial_{x_3} R_\varepsilon(x', 0+0)\,\varphi\,dx' + \int_{G_\varepsilon} \nabla_x R_\varepsilon \cdot \nabla_x \varphi\,dx + \varepsilon \int_{\Gamma_\varepsilon} \kappa\big(R_\varepsilon\big)\,\varphi\,d\sigma_x =$$

$$= I_1^+(\varepsilon,\varphi) + \ldots + I_7^+(\varepsilon,\varphi), \qquad (66)$$

where

$$I_1^+(\varepsilon,\varphi) = -\int_{G_\varepsilon} \chi_0'(x_3)\big(\partial_{\xi_3}\mathcal{N}^+(\xi,x')\big)|_{\xi=x/\varepsilon}\,\varphi\,dx,$$

$$I_2^+(\varepsilon,\varphi) = -\int_{G_\varepsilon} \chi_0(x_3)\sum_{i=1}^{2}\big(\partial^2_{x_i\xi_i}\mathcal{N}^+(\xi,x')\big)|_{\xi=x/\varepsilon}\,\varphi\,dx,$$

$$I_3^+(\varepsilon,\varphi) = \varepsilon\int_{G_\varepsilon} \chi_0'(x_3)\mathcal{N}^+(x/\varepsilon,x')\,\partial_{x_3}\varphi\,dx,$$

$$I_4^+(\varepsilon,\varphi) = \varepsilon\int_{G_\varepsilon} \chi_0(x_3)\sum_{i=1}^{2}\big(\partial_{x_i}\mathcal{N}^+(\xi,x')\big)|_{\xi=x/\varepsilon}\,\partial_{x_i}\varphi\,dx,$$

$$I_5^+(\varepsilon,\varphi) = \varepsilon\int_{G_\varepsilon} \sum_{i=1}^{2} Y_i\big(\frac{x_i}{\varepsilon}\big)\nabla_x\big(\partial_{x_i} v_0^+\big)\cdot\nabla_x\varphi\,dx$$

$$I_6^+(\varepsilon,\varphi) = -\varepsilon\int_{G_\varepsilon} \varrho'(x_3)\nabla_{\xi'}Y(\xi',x_3)|_{\xi'=\frac{x'}{\varepsilon}}\cdot\nabla_{x'}\big(\partial_{x_3}v_0^+\,\varphi\big)\,dx$$

$$I_7^+(\varepsilon,\varphi) = \varepsilon\int_{\Gamma_\varepsilon}\Big(\kappa\big(R_\varepsilon\big) - \frac{\kappa\big(v_0^+\big)}{\sqrt{1+\varepsilon^2|\varrho'|^2}}\Big)\,\varphi\,d\sigma_x + \varepsilon\int_{G_\varepsilon}\big(\nabla_{\xi'}Y\big)\cdot\nabla_{x'}\big(\kappa(v_0^+)\,\varphi\big)\,dx.$$

Asymptotic estimates. Summing (62) and (66), we see that function R_ε constructed by formulas (58) and (59) satisfies the following integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla_x R_\varepsilon\cdot\nabla_x\varphi\,dx + \varepsilon\int\limits_{\Gamma_\varepsilon}\kappa\big(R_\varepsilon\big)\,\varphi\,d\sigma_x - \int\limits_{\Omega_0} f_0\varphi\,dx = F_\varepsilon(\varphi) \quad \forall\,\varphi\in H^1(\Omega_\varepsilon), \qquad (67)$$

where $F_\varepsilon(\varphi) = I_1^\pm(\varepsilon,\varphi) + \ldots + I_4^\pm(\varepsilon,\varphi) + I_5^+(\varepsilon,\varphi) + \ldots + I_7^+(\varepsilon,\varphi)$, $I_j^\pm(\varepsilon,\varphi) = I_j^+(\varepsilon,\varphi) + I_j^-(\varepsilon,\varphi)$, $j = 1,\ldots,4$.

Subtracting the integral identity (3) from (67), we get

$$\int\limits_{\Omega_\varepsilon} \nabla_x\big(R_\varepsilon - u_\varepsilon\big)\cdot\nabla_x\varphi\,dx + \varepsilon\int\limits_{\Gamma_\varepsilon}\Big(\kappa\big(R_\varepsilon\big) - \kappa\big(u_\varepsilon\big)\Big)\,\varphi\,d\sigma_x = F_\varepsilon(\varphi) \quad \forall\,\varphi\in H^1(\Omega_\varepsilon). \qquad (68)$$

Now we are going to estimate the value $F_\varepsilon(\varphi)$. Integrals in $I_1^\pm(\varepsilon,\varphi)$ and $I_3^\pm(\varepsilon,\varphi)$ are, in fact, over $\operatorname{supp}\big(\chi_0'(x_3)\big) \cap \Omega_\varepsilon = \{x : \theta_0/2 < |x_3| < \theta_0\} \cap \Omega_\varepsilon$ respectively. In these sets, by virtue of (56) and (57), the functions $\big(\partial_{\xi_3}\mathcal{N}^+(\xi,x')\big)|_{\xi=x/\varepsilon}$ and $\big(\partial_{\xi_3}\mathcal{N}^-(\xi,x')\big)|_{\xi=x/\varepsilon}$ are exponentially small and the functions $\mathcal{N}^+$ and $\mathcal{N}^-$ are uniformly bounded with respect to ε. Therefore, $|I_1^\pm(\varepsilon,\varphi)| + |I_3^\pm(\varepsilon,\varphi)| \le \varepsilon\,C_1\|\varphi\|_{H^1(\Omega_\varepsilon)}$.

In order to estimate the term $I_2^\pm(\varepsilon,\varphi)$ we use the following statement.

Proposition 1.([24])*Assume that a function $\mathcal{N}$ is 1-periodic in ξ_1 and ξ_2, it belongs to the space $L_2(\Pi)$ and exponentially decreasing at infinity as $\xi_3\to\infty$, i.e., there exist positive constants c, R, σ such that for any $|\xi_3| \ge R$*

$$\big|\mathcal{N}(\xi)\big| \le c\,\exp(-\sigma|\xi_3|).$$

Then for any $\delta > 0$ there exist positive constants c_1, ε_0 such that for all values $\varepsilon \in (0, \varepsilon_0)$ the following inequality is valid

$$\Big| \int_{\Omega_\varepsilon} \mathcal{N}\big(\frac{x}{\varepsilon}\big)\, \psi(x)\, dx \Big| \le c_1 \varepsilon^{1-\delta} \|\psi\|_{H^1(\Omega_\varepsilon)}, \quad \forall\, \psi \in H^1(\Omega_\varepsilon).$$

Since all functions of ξ entering in $\partial^2_{x_i \xi_i} \mathcal{N}^{\pm}(\xi, x'),\ i = 1, 2,$ exponentially decrease as $|\xi_3| \to +\infty$, on the basis of Proposition 1 we get $|I_2^{\pm}(\varepsilon, \varphi)| \le \varepsilon^{1-\delta}\, C_2 \|\varphi\|_{H^1(\Omega_\varepsilon)}$, where δ is arbitrary fixed positive number.

Integrals in $I_4^{\pm}(\varepsilon, \psi)$ are over $\{x : \ |x_3| < \theta_0\} \cap \Omega_\varepsilon$ and they can be estimated by the following way. We extract if necessary the exponentially decreasing part in the corresponding integrand and then with the help of the Cauchy-Bunyakovsky inequality we estimate the integral. As a result, we have $|I_4^{\pm}(\varepsilon, \varphi)| \le \varepsilon^{3/2}\, C_4 \|\varphi\|_{H^1(\Omega_\varepsilon)}$.

Remark 4. *The constants C_2 and C_4 depend on the quantities*

$$\sup_{x' \in \Xi_0} \Big| \mathcal{D}^{\alpha}_x \big(v_0^-(x', 0)\big) \Big|, \quad \alpha_1 + \alpha_2 + \alpha_3 \le 2. \tag{69}$$

Applying the even extension to the limit problem (28) with respect to the planes $x_i = 0, x_i = a,\ i = 1, 2,$ and taking into account that the support of f_0 is located in Ω_0, we establish that function v_0^- and its derivatives have no singularities in a neighborhood of Ξ_0. Then, by virtue of classical results on the smoothness of solutions to boundary-value problems, the quantities (69) are bounded. Also according to results on the smoothness of solutions to quasi-linear boundary-value problems ([20]) and to the continuous dependence of solutions to ordinary differential equations on parameters (see [34]), we conclude that $\partial_{x_i} v_0^+ \in H^1(\Omega^+),\ i = 1, 2.$

Due to the properties of the functions $Y_i,\ i = 1, 2,$ and Y (see (32), (8)), the terms I_5^+ and I_6^+ admit the following bounds $|I_5^+(\varepsilon, \varphi)| + |I_6^+(\varepsilon, \varphi)| \le C_5 \varepsilon \|\varphi\|_{H^1(\Omega_\varepsilon)}$.

It remains to estimate I_7^+. Since $\big(1 + \varepsilon^2 |\varrho'|^2\big)^{-1/2} = 1 + \mathcal{O}(\varepsilon^2)$ and

$$\kappa\big(R_\varepsilon\big) = \kappa\big(v_0^+\big) + \varepsilon \kappa'(\cdot)\Big(\sum_{i=1}^{2} Y_i\, \partial_{x_i} v_0^+(x) + \chi_0(x_3)\, \mathcal{N}^+ \Big),$$

with the help of (2), (9) and (8) we deduce that I_7^+ is not greater than $C\varepsilon \|\varphi\|_{H^1(\Omega_\varepsilon)}$.

Thus, with regard to the inequalities obtained above,

$$|F_\varepsilon(\varphi)| \le c(\delta)\, \varepsilon^{1-\delta} \|\varphi\|_{H^1(\Omega_\varepsilon)}. \tag{70}$$

Set now $\varphi = R_\varepsilon - u_\varepsilon$ in (68). Then, taking into account (70) and (2), we have

$$\int_{\Omega_\varepsilon} |\nabla_x \big(R_\varepsilon - u_\varepsilon\big)|^2\, dx + \varepsilon\, c_1 \int_{\Gamma_\varepsilon} \big(R_\varepsilon - u_\varepsilon\big)^2\, d\sigma_x \le c(\delta)\, \varepsilon^{1-\delta} \|\varphi\|_{H^1(\Omega_\varepsilon)},$$

whence thanks to (18) it follows (60). □

Corollary 1. *From (60) it follows that $\|u_\varepsilon - v_0\|_{L^2(\Omega_\varepsilon)} \le c_2\, \varepsilon^{1-\delta}$.*

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