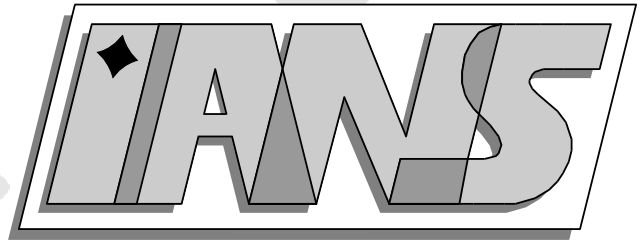


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Stuttgart**



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**Sparse Grid Interpolation Toolbox User's Guide**

V5.0, December 23, 2007

Andreas Klimke

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**Berichte aus dem Institut für  
Angewandte Analysis und Numerische Simulation**

Documentation 2007/017



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# 1 Getting Started

## 1.1 What is the Sparse Grid Interpolation Toolbox?

### Introduction

The interpolation problem considered with sparse grid interpolation is an optimal recovery problem (i.e. the selection of points such that a smooth multivariate function can be approximated with a suitable interpolation formula). Depending on the characteristics of the function to interpolate (degree of smoothness, periodicity), various interpolation techniques based on sparse grids exist. All of them employ Smolyak's construction, which forms the basis of all sparse grid methods. With Smolyak's famous method, well-known univariate interpolation formulas are extended to the multivariate case by using tensor products in a special way. As a result, one obtains a powerful interpolation method that requires significantly fewer support nodes than conventional interpolation on a full grid. The points comprising the multidimensional sparse grid are selected in a predefined fashion. The difference in the number of required points can be several orders of magnitude with increasing problem dimension. The most important property of the method constitutes the fact that the asymptotic error decay of full grid interpolation with increasing grid resolution is preserved up to a logarithmic factor. An additional benefit of the method is its hierarchical structure, which can be used to obtain an estimate of the current approximation error. Thus, one can easily develop an interpolation algorithm that aborts automatically when a desired accuracy is reached.

### Major features

This Matlab toolbox includes hierarchical sparse grid interpolation algorithms based on both piecewise multilinear and polynomial basis functions. Special emphasis is placed on an efficient implementation that performs well even for very large dimensions  $d > 10$ .

There are many ways to customize the behavior of the interpolation routines. Furthermore, additional tasks involving the interpolants can be performed, such as computing derivatives or performing an optimization or integration. The following list gives an overview of the options that are available:

- Enable vectorized processing. Speed up the construction of the interpolant for functions that allow for vectorized evaluation.
- Create multiple interpolants at once for functions with multiple output arguments.
- Choose from three different grid types for piecewise linear interpolation. Depending on your objective function, a certain grid type may perform better than the others.

- If very high accuracies are required, you may use the Chebyshev-Gauss-Lobatto sparse grid, which employs polynomial basis functions.
- Compute gradients along with the interpolated values at just a small additional cost.
- Integrate the interpolant.
- Perform a search for minima and maxima using several efficient algorithms.
- Use the dimension-adaptive algorithm to automatically detect separability, and to take the importance of the dimensions into account when constructing the interpolant. This is especially useful in case of very high-dimensional problems when a regular sparse grid refinement leads to too many support nodes.
- Specify the minimum or maximum sparse grid depth to compute, or specify the minimum and maximum number of function evaluations to use (for the dimension-adaptive sparse grid).
- Last but not least, the Sparse Grid Interpolation toolbox is designed to easily integrate with your models in Matlab as well as external models.

For additional information on the theoretical and algorithmic aspects of sparse grid interpolation, please refer to the references provided at the end of this chapter.

### 1.2 Initialization of the toolbox

To initialize the toolbox, it must be added to the Matlab search path. You can do this by calling the function `spinit`, i.e. go to the directory containing the sparse grid interpolation toolbox and enter `spinit` at the Matlab prompt.

If you would like to use the Sparse Grid Interpolation Toolbox from within another Matlab application, you may call the `spinit` function as well. It will automatically add the correct paths to the Matlab path.

### 1.3 A first example

Let us interpolate a simple two-variate function

$$f(x,y) = \sin(x) + \cos(x)$$

with the default settings of the sparse grid interpolation package. Here, we interpolate the function for the domain  $[0, \pi] \times [0, \pi]$ .



## Constructing the interpolant

First, we compute the hierarchical surpluses (i.e. the coefficients) of the interpolant.

```
f = @(x,y) sin(x) + cos(y);
z = spvals(f,2,[0 pi; 0 pi])

z =
    vals: {[65x1 double]}
  gridType: 'Clenshaw-Curtis'
        d: 2
   range: [2x2 double]
  maxLevel: 4
 estRelError: 0.0063
 estAbsError: 0.0188
 fevalRange: [-1 2]
 minGridVal: [0 1]
 maxGridVal: [0.5000 0]
   nPoints: 65
   fevalTime: 0.0502
 surplusCompTime: 0.0024
   indices: [1x1 struct]
```

The function `spvals` returns these hierarchical surpluses, and also includes some additional information collected during the construction process of the interpolant. For instance, We obtain information on the estimated relative and absolute error. The number of sparse grid support nodes is provided, as well as the computing time for evaluating the function and computing the hierarchical surpluses. The surpluses themselves are stored under the field `vals`.

## Computing interpolated values

To compute interpolated values, we can now use the `spinterp` function. To increase efficiency, multiple interpolated values can be computed at once. Below, we compute the interpolated values for five randomly chosen points and compare them to the exact function value by computing the maximum absolute error.

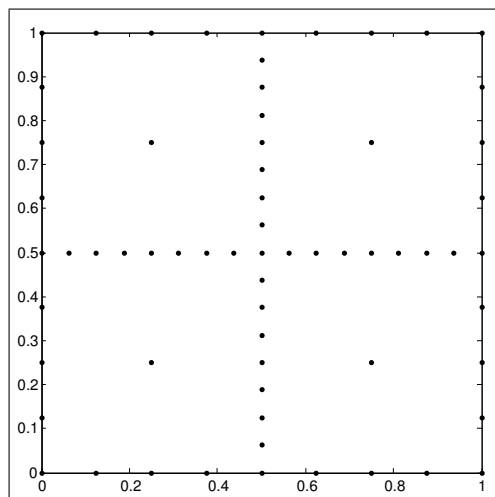
```
x1 = pi*rand(1,5); x2 = pi*rand(1,5);
y = spinterp(z,x1,x2)

error = max(abs(y - f(x1,x2)))
y =
    1.7173    0.7210    0.2675    0.7701    0.5510
error =
    0.0076
```

## Visualizing the sparse grid

Let us now visualize the sparse grid. From the information returned by `spvals`, we see that the used sparse grid is of the type Clenshaw-Curtis, and the maximum level was 4. In two and three dimensions, we can easily plot the sparse grid with the `plotgrid` function. It takes the level and the dimension as input arguments. Optional is an options structure containing the sparse grid type, created with `spset`. The default grid type is the Clenshaw-Curtis grid, we thus do not have to specify the grid type here.

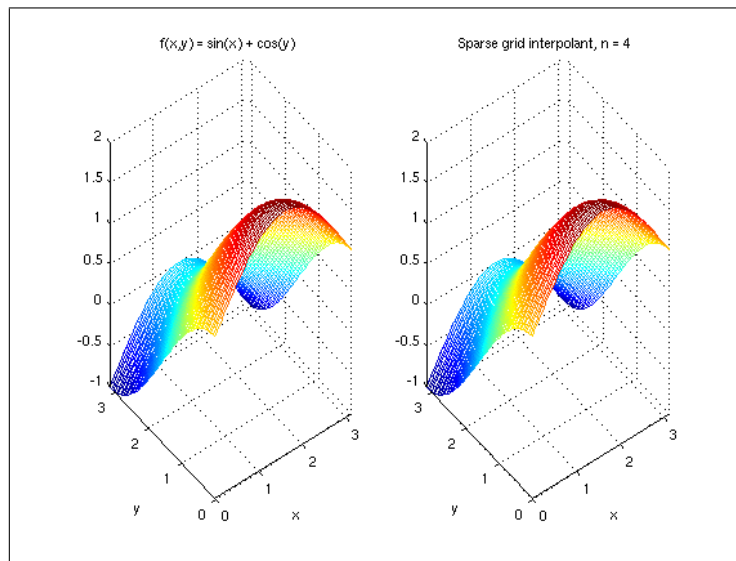
```
plotgrid(4,2)
```



## Visualizing the interpolant

To visualize the original function and compare it to the interpolant, we can plot both functions, for instance, by using `ezmesh`.

```
subplot(1,2,1);  
ezmesh(f, [0 pi]);  
title('f(x,y) = sin(x) + cos(y)');  
subplot(1,2,2);  
ezmesh(@(x,y) spinterp(z,x,y), [0 pi]);  
title('Sparse grid interpolant, n=4');
```



## 1.4 Piecewise linear basis functions

Piecewise linear basis functions provide a good compromise between accuracy and computational cost due to their bounded support. The Sparse Grid Interpolation package includes three different grid types that work with piecewise multilinear basis functions:

- The Clenshaw-Curtis grid type “ClenshawCurtis” (CC)
- the "classical" maximum-norm-based grid type “Maximum” (M), as described e.g. by Bungartz/Griebel in [1],
- The maximum-norm-based grid without points on the boundary “NoBoundary” (NB), with basis functions that extrapolate towards the boundary (it is not assumed that the objective function must be zero at the boundary).

For a detailed description of the piecewise multilinear basis functions implemented here, please see [2] or [3, ch. 3], and the references stated therein.

### Accuracy of piecewise multilinear interpolation

We now take a brief look at the approximation quality. An a priori error estimate can be obtained for a  $d$ -variate function  $f$  if continuous mixed derivatives

$$D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}},$$

with  $\beta \in \mathbb{N}_0^d$ ,  $|\beta| = \sum_{i=1}^d \beta_i$ , and  $\beta_1, \dots, \beta_d \leq 2$ , exist. According to [4] or [5], the order of the interpolation error in the maximum norm is then given by

$$\|f - A_{q,d}(f)\|_\infty = \mathcal{O}(N^{-2} \cdot |\log_2 N|^{3 \cdot (d-1)}),$$

where  $A_{q,d}(f)$  denotes the sparse grid interpolant of  $f$ , and  $N$  denotes the number of grid points of the sparse grids of type CC or M (the NB grid type has not yet been analyzed, but shows the same order of convergence in numerical tests). Note that the number of grid points  $N$  of  $A_{q,d}(f)$  can be computed by  $\text{spdim}(q-d, d)$ . Piecewise multilinear approximation on a full grid with  $N^*$  grid points is much less efficient, i.e.  $\mathcal{O}(N^{*-2/d})$ .

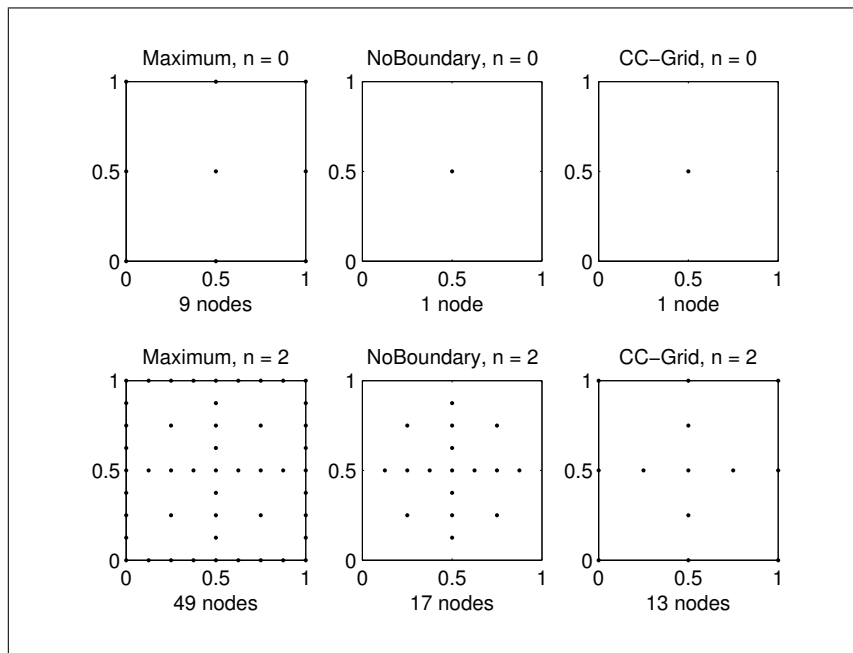
## Number of grid points

Table 1.1 shows the number of grid points of the non-adaptive sparse grid interpolant depending on the interpolation depth  $n$ .

Table 1.1: Comparison: Number of grid points for level  $n = q - d$ .

| $n$ | $d = 2$ |      |     | $d = 4$ |       |      | $d = 8$ |        |       |
|-----|---------|------|-----|---------|-------|------|---------|--------|-------|
|     | M       | NB   | CC  | M       | NB    | CC   | M       | NB     | CC    |
| 0   | 9       | 1    | 1   | 81      | 1     | 1    | 6561    | 1      | 1     |
| 1   | 21      | 5    | 5   | 297     | 9     | 9    | 41553   | 17     | 17    |
| 2   | 49      | 17   | 13  | 945     | 49    | 41   | 1.9e5   | 161    | 145   |
| 3   | 113     | 49   | 29  | 2769    | 209   | 137  | 7.7e5   | 1121   | 849   |
| 4   | 257     | 129  | 65  | 7681    | 769   | 401  | 2.8e6   | 6401   | 3937  |
| 5   | 577     | 321  | 145 | 20481   | 2561  | 1105 | 9.3e6   | 31745  | 15713 |
| 6   | 1281    | 769  | 321 | 52993   | 7937  | 2929 | 3.0e7   | 141569 | 56737 |
| 7   | 2817    | 1793 | 705 | 1.3e5   | 23297 | 7537 | 9.1e7   | 5.8e5  | 1.9e5 |

The following graph illustrates the sparse grids of level 0 and level 2 of the three respective grids in two dimensions.



### Which piecewise linear interpolation scheme works best?

Although the performance of the three grid types is rather similar for lower-dimensional problems, there are two important points to be mentioned:

- The ClenshawCurtis grid and the NoBoundary grid have just a single node at the lowest interpolation level  $n=0$  (this means that an interpolant of level 0 of these grid types is just a constant function). The Maximum grid has  $3^d$  nodes at the lowest level. Therefore, the Maximum is not well-suited for higher-dimensional problems. For instance, for  $d=10$ , already 59049 support nodes would be required to obtain an initial interpolant.
- Since the CC-grid is the most versatile grid working well in both lower and higher dimensions, at this point, the dimension-adaptive algorithms are implemented for this grid type only.

Therefore, for most practical applications, we recommend using the Clenshaw-Curtis grid. Occasionally, the other grid types may perform better by a small factor, as numerical experiments show (try running the demo `spcompare` from the command line or from the Sparse Grid Interpolation demo page).

## 1.5 Polynomial basis functions

The piecewise multilinear approach can be significantly improved by using higher-order basis functions, such as the Lagrangian characteristic polynomials. The approximation properties of sparse grid interpolation techniques using polynomial basis functions have been studied extensively in [4], where error bounds depending on the smoothness of the function were derived.

From the one-dimensional case, we know that one should not use equidistant nodes for higher-order polynomial interpolation. This directly suggests using Chebyshev-based node distributions. Since an additional requirement of an efficient sparse grid algorithm is the nesting of the sets of nodes, the Chebyshev Gauss-Lobatto nodes are clearly the best choice, and are therefore also suggested in [4]. In this toolbox, this grid type (CGL) is selected by the value "Chebyshev" for the GridType property configurable with the `spset` function.

Since version 5.0, an additional polynomial sparse grid is available, the Gauss-Patterson sparse grid. This grid is based on the abscissae of Gauss-Patterson integration. The Gauss-Patterson formula is a nested quadrature rule that achieves a higher degree of exactness than integration at the Chebyshev Gauss-Lobatto nodes. See [6, 7] for additional details.

For a detailed description of the polynomial basis functions implemented here, please see [3, ch. 3], and the references stated therein. Since version v3.2, the toolbox uses an improved construction algorithm employing the fast discrete cosine transform, see [8].

## Accuracy of polynomial interpolation

From the error bounds of the univariate case, the following general error bounds depending on the smoothness of the objective function  $f$  are derived in [4]. For  $f \in F_d^k$ ,

$$F_d^k = \{f : [-1, 1]^d \rightarrow \mathbb{R} \mid D^\beta f \text{ continuous if } \beta_i \leq k \forall i\},$$

with  $\beta \in \mathbb{N}_0^d$ ,

$$D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}},$$

the order of the interpolation error in the maximum norm is given by

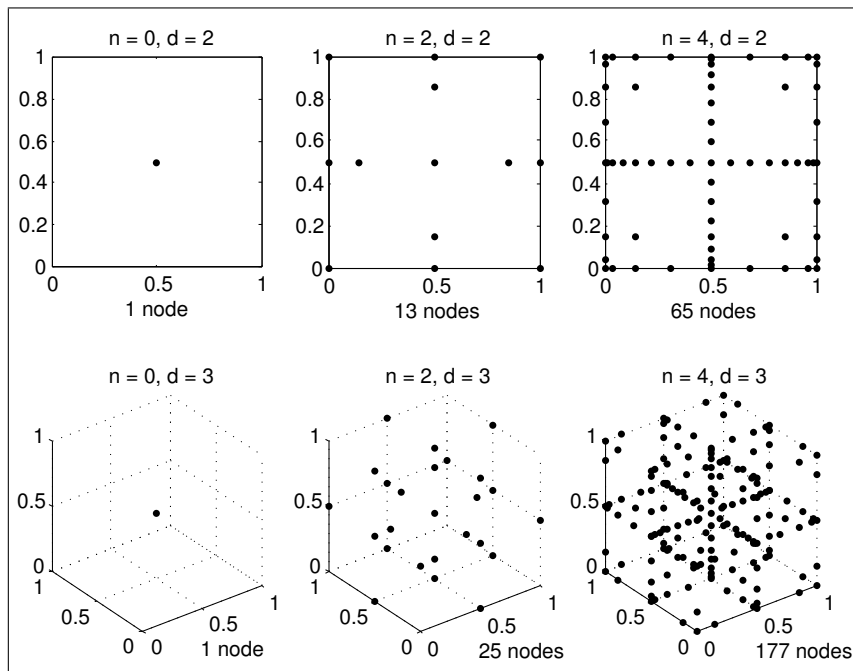
$$\|f - A_{q,d}(f)\|_\infty = \mathcal{O}(N^{-k} \cdot |\log N|^{(k+2)(d+1)+1}),$$

where  $A_{q,d}(f)$  denotes the sparse grid interpolant of  $f$ , and  $N$  denotes the number of grid points of the sparse grids of type CGL. Note that the number of grid points  $N$  of  $A_{q,d}(f)$  can be computed by `spdim(q-d, d)`.

## Number of grid points

The number of grid points of the CGL-grid is identical to the one of the Clenshaw-Curtis (CC) grid. The number of grid points of the Gauss-Patterson grid is identical to the one of the NB grid.

The following graph illustrates the sparse grids of level 0 and level 2 of the CGL-grid in two and three dimensions.



### When should I use polynomial rather than linear basis functions?

There is obviously some trade-off between the accuracy gain and the computing time required to construct as well as interpolate the interpolant. Since the higher-order accuracy only becomes effective with increasing number of nodes, we recommend to use the polynomial approach only if the following two conditions are met:

- The objective function to be recovered is known to be very smooth.
- High relative accuracies smaller than  $10^{-2}$  are required.

## 1.6 Dimensional adaptivity

With the standard sparse grid approach, all dimensions are treated equally, i.e. in each coordinate direction, the number of grid points is equal. The question arises as to whether one can further reduce the computational effort for objective functions where not all input variables carry equal weight. This is especially important in the case of higher-dimensional problems, where this property is often observed. Unfortunately, it is usually not known a priori which variables (or, in this context, dimensions) are important. Therefore, an efficient approach should be able to automatically detect which dimensions are the more or less important ones, without wasting any function evaluations. Hegland [9] and Gerstner and Griebel [10] show that this is indeed possible. They propose an approach to generalize sparse grids such that the number of nodes in each dimension is adaptively adjusted to the problem at hand. Here, the adaptive refinement is not performed in a spatial manner, as it is commonly done in two- and three-dimensional

problems (e.g. [5]), where more support nodes are placed in areas of increased nonlinearity or non-smoothness (this can become impractical in higher dimensions due to the required complex data structure and refinement criteria).

Besides being able to balancing the number of nodes in each coordinate direction, dimension-adaptive sparse grids are capable of automatically detecting separability (or partial separability) encountered in problems with additive (or nearly additive) structure.

The Sparse Grid Interpolation Toolbox includes a powerful dimension-adaptive algorithm based on the approach by Gerstner and Griebel [10], but also includes the significant performance enhancements described in [3, ch. 3]. Applying the dimension-adaptive algorithm is very easy - it can be switched "on" or "off" with a single parameter of the sparse grid options structure that can be set with the `spset` function. Furthermore, the degree of dimensional adaptivity can be chosen as a scalar from the interval [0,1] where 1 stands for greedy (= purely adaptive) and 0 stands for conservative (= standard sparse grid) refinement.

## Example

Consider the following quadratic test function:

$$f(x) = \left[ \sum_{i=1}^d (x_i - 1)^2 \right] - \left[ \sum_{i=2}^d x_i x_{i-1} \right],$$

implemented in Matlab by the following code:

```
function y = trid(x)
% TRID Quadratic function with a tridiagonal Hessian.
% Y = TRID(X) returns the function value Y for a D-
% dimensional input vector X.
%
% f(x)=[sum_{i=1}^d (x_i-1)^2] - [sum_{i=2}^d x_i x_{i-1}]
%
% The test function is due to Arnold Neumaier, listed
% on the global optimization Web page at
% http://www.mat.univie.ac.at/~neum/glopt/

d = length(x);
y = sum((x-1).^2) - sum(x(2:d).*x(1:d-1));
```

The function clearly exhibits additive structure, however, the function is not fully separable due to the second term coupling the variables. Consider the high-dimensional case  $d = 100$ . A traditional tensor-product approach would completely fail in interpolating a high-dimensional function of this type, since at least  $2^{100}$  nodes are required if an interpolation formula with two nodes is extended to the multivariate case via conventional tensor products. With the dimension-adaptive sparse grid algorithm, however, the structure is automatically detected, and the function is successfully recovered using just  $\mathcal{O}(d^2)$  points. For the interpolation domain, we have used  $[-d^2, d^2]$  in each dimension.

Using piecewise multilinear basis functions and the Clenshaw-Curtis-Grid,  $f$  can be recovered with an estimated relative error of below 0.1 percent (the relative error is given with respect



to the estimated range of the function) with about 27000 function evaluations, as the following code shows.

```
d = 100;
range = repmat([-d^2 d^2],d,1);
options = spset('DimensionAdaptive', 'on', ...
               'DimadaptDegree', 1, ...
               'FunctionArgType', 'vector', ...
               'RelTol', 1e-3, ...
               'MaxPoints', 40000);
z1 = spvals(@trid, d, range, options)
```

```
z1 =
      vals: {[26993x1 double]}
      gridType: 'Clenshaw-Curtis'
           d: 100
      range: [100x2 double]
  estRelError: 3.2552e-04
  estAbsError: 9.7656e+04
  fevalRange: [100 300000100]
  minGridVal: [1x100 double]
  maxGridVal: [1x100 double]
      nPoints: 26993
      fevalTime: 5.0122
  surplusCompTime: 0.4784
      indices: [1x1 struct]
      maxLevel: [1x100 double]
  activeIndices: [5149x1 uint32]
  activeIndices2: [5749x1 uint32]
           E: [1x5749 double]
           G: [5749x1 double]
           G2: [5749x1 double]
  maxSetPoints: 6
      dimAdapt: 1
```

Since the objective function is quadratic, we can even approximate the function up to floating point accuracy with the polynomial basis functions and the Chebyshev-Gauss-Lobatto grid:

```
options = spset(options, 'GridType', 'Chebyshev');
z2 = spvals(@trid, d, range, options)
```

```
z2 =
      vals: {[20201x1 double]}
      gridType: 'Chebyshev'
           d: 100
      range: [100x2 double]
  estRelError: 2.4835e-17
  estAbsError: 7.4506e-09
  fevalRange: [100 300000100]
```

```
minGridVal: [1x100 double]
maxGridVal: [1x100 double]
nPoints: 20201
fevalTime: 3.8847
surplusCompTime: 3.5568
indices: [1x1 struct]
maxLevel: [1x100 double]
activeIndices: [4951x1 uint32]
activeIndices2: [5151x1 uint32]
E: [1x5151 double]
G: [5151x1 double]
G2: [5151x1 double]
maxSetPoints: 2
dimAdapt: 1
```

We can verify the quality of the interpolants by computing the maximum relative error for 100 randomly sampled points (the relative error is computed with respect to the range of the function values that occurred during the sparse grid construction). In this case, the estimated error was too optimistic in the piecewise linear case- however, the relative error for the sampled points is still below 1 percent.

```
% Compute 100 randomly sampled points
p = 100;
rand('state', 0);
x = -d^2 + 2*d^2*rand(p,d);

% Compute exact function values
y = zeros(p,1);
for k = 1:p
    y(k) = trid(x(k,:));
end

% Compute interpolated function values
xcell = num2cell(x,1);
ip1 = spinterp(z1, xcell{:});
ip2 = spinterp(z2, xcell{:});

% Compute relative errors
err_CC = max(abs(y-ip1))/(z1.fevalRange(2)-z1.fevalRange(1))
err_CGL = max(abs(y-ip2))/(z2.fevalRange(2)-z2.fevalRange(1))

err_CC =
    0.0061
err_CGL =
    1.2716e-14
```

## 2 Advanced Topics

### 2.1 Multiple output variables

Real-world problems usually have more than a single output argument. Sparse grids are very well-suited to deal with such kind of problems, since the regular structure allows to construct good approximations for multiple output variables at once.

The sparse grid interpolation package is designed to make dealing with multiple output arguments easy, as the following example demonstrates.

#### Example

Consider the following simple test function with multiple output arguments:

```
function [out1, out2, out3, out4] = multiout(x,y)
% MULTIOUT Test function with four output arguments
out1 = (x+y).^2;
out2 = 1./exp(1+(x-0.5).^2+(y-0.3).^2);
out3 = sin(pi*(2-x))+cos(pi*(1-y));
out4 = sinh(4.*(x-0.5));
```

`spvals` will automatically compute interpolants with respect to all four output variables if the number of output variables is specified in the sparse grid `OPTIONS` structure:

```
nout = 4;
options = spset('NumberOfOutputs', nout, 'Vectorized', 'on');
z = spvals(@multiout, 2, [], options)
```

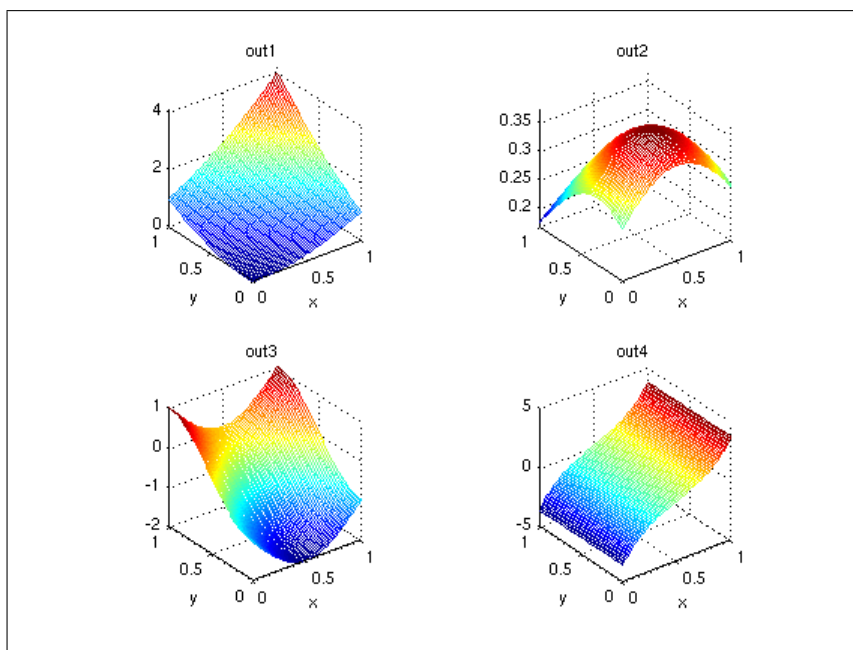
```
z =
      vals: {4x1 cell}
    gridType: 'Clenshaw-Curtis'
         d: 2
    range: []
  maxLevel: 5
estRelError: 0.0034
estAbsError: 0.0249
 fevalRange: [4x2 double]
 minGridVal: [4x2 double]
 maxGridVal: [4x2 double]
   nPoints: 145
   fevalTime: 0.0392
surplusCompTime: 0.0129
```

indices: [1x1 struct]

Note that the output parameters of the objective function must all be scalar. The number of outputs `nout` specified in the options structure may be smaller than the actual number of outputs. In this case, interpolants are constructed only with respect to the first `nout` arguments.

To compute interpolated values, the desired output argument must now be specified. This is done by adding an additional field `selectOutput` to the structure `z` prior to the call to the `spinterp` function. The following code plots the four computed interpolants:

```
for k = 1:nout
    z.selectOutput = k;
    subplot(2,2,k);
    ezmesh(@(x,y) spinterp(z,x,y), [0 1]);
    axis square;
    title(['out' num2str(k)]);
end
```



An additional example of using multiple output arguments with `spvals` is given by the demo `spdemovarout` available at the command line or from the demos page.

## 2.2 Derivatives

One of the primary purposes of sparse grid interpolation is the construction of surrogate functions for local or global optimization. While some optimization methods work well using function values only, many efficient algorithms exist that require computation of the gradient. With the Sparse Grid Interpolation Toolbox, it is possible to obtain gradients of the interpolants

directly – up to floating point accuracy – as opposed to approximating them numerically, such as by finite differences. This is demonstrated in the following.

## How to obtain the derivatives?

Computing derivatives is extremely simple. One just calls the method `spinterp`, but instead of a single left-hand argument, one uses the syntax `[ip, ipgrad] = spinterp(z, x1, ..., xn)`. This returns not only the function value at the point(s)  $(x_1, \dots, x_n)$ , but also the complete gradient vector(s). Please see the reference page of `spinterp` for further details (or the examples provided below).

It is important to note that the entire procedure of computing the hierarchical surpluses of the sparse grid interpolant with `spvals` remains the same, regardless of whether derivatives are desired or not. Also, purging of the interpolant (see `sppurge`) can be performed in the usual manner if desired. This makes using derivative information very flexible, and it can be decided ad-hoc, well after interpolant construction (for example, when different optimization algorithms are applied), whether to use derivatives or not.

Furthermore, the derivatives computed by `spinterp` are **not** additional approximations of the derivatives of the original function, but rather, the **exact** derivatives of the interpolant (up to floating point accuracy). The advantage of this approach is that **no additional memory is required** to store derivative information. The derivatives are computed on-the-fly by efficient algorithms.

## Derivatives of piecewise multilinear interpolants

We start with derivatives of piecewise multilinear Clenshaw-Curtis Sparse Grid Interpolants. Deriving a piecewise linear function leads to piecewise constant derivatives with respect to the variable that the function is differentiated for. Since the interpolant is non-differentiable at the kinks, the left-sided (or right-sided) derivative is computed at these points only. The following example in two dimensions illustrates the nature of the derivatives.

First, we define the example objective function. We also define its derivatives (this is only for comparison to give an idea of the quality of the computed derivatives).

```
% Define function and its derivatives
f = inline('1./(cos(2*x).^2+sin(y).^2+1)+0.2*y');
fdx = inline('4*cos(2*x).*sin(2*x)./(cos(2*x).^2+sin(y).^2+1).^2');
fdy = inline('-2*cos(y).*sin(y)./(cos(2*x).^2+sin(y).^2+1).^2+0.2');
```

Next, we compute the interpolant. In this case, using the regular Clenshaw- Curtis sparse grid. We limit the sparse grid depth to 4 here (i.e.,  $A_{q,d}(f) = A_{4+2,2}(f)$  is computed).

```
d = 2;
maxDepth = 4;
options = spset('Vectorized', 'on', 'SparseIndices', 'off', ...
    'MaxDepth', maxDepth);
z = spvals(f, d, [], options);
```

## 2 Advanced Topics

---

```
Warning: MaxDepth = 4 reached before accuracies RelTol = 0.01 or
AbsTol = 1e-06 were achieved.
```

```
The current estimated relative accuracy is 0.02877.
```

Our aim here is to plot the derivatives. Therefore, we define a suitable grid, and set up an array for the vectorized evaluation of the interpolant and its derivatives. We set up the grid such that the jumps of the derivative will be clearly visible.

```
np = 2^double(z.maxLevel)+1;
x = linspace(0,1,np);
xstep = zeros(1,(np-1)*2);
xstep(1:2:end-1) = x(1:end-1);
xstep(2:2:end) = x(2:end) - eps;
[x,y] = ndgrid(xstep);
```

Next, we evaluate the interpolant at the grid points. As result, `ipgrad` will contain an array of the shape of the input arrays `x` and `y`, with the difference that it is a cell array instead of a double array, where each cell contains the entire gradient at the point of the corresponding array entry of `x` and `y`.

```
[ip,ipgrad] = spinterp(z,x,y);
```

For plotting, it is convenient to convert the data returned as a cell array back to a double array. This is achieved by the following statements. We extract the derivatives with respect to `y` here.

```
% Convert returned cell array to matrix
ipgradmat = cell2mat(ipgrad);
% Get the derivatives with respect to y
ipdy = ipgradmat(2:d:end,:);
```

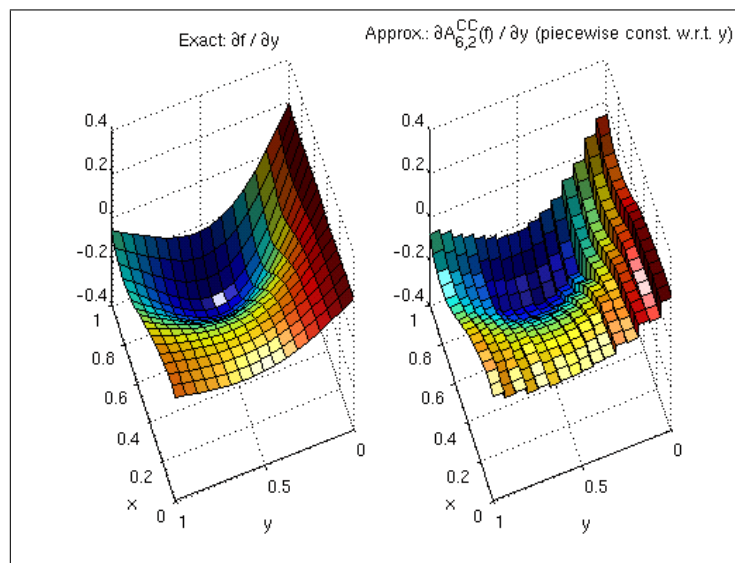
Similarly, we can get all derivatives with respect to `x` with the command

```
ipdx = ipgradmat(1:d:end,:);
```

This approach of transforming the cell array to a double array can be easily extended to other dimensions.

Finally, we plot the obtained derivatives next to the exact derivatives.

```
subplot(1,2,1,'align');
surf(x,y,fdy(x,y));
view(250,50); xlabel('x'); ylabel('y'); light;
title('Exact:  $\frac{\partial f}{\partial y}$ ');
subplot(1,2,2,'align');
surf(x,y,ipdy);
view(250,50); xlabel('x'); ylabel('y'); light;
title('Approx.:  $A^{CC}_{6,2}(f)_{y_{piecewise\ const. w.r.t. y}}$ ');
```



### Augmented derivatives to achieve continuity

Obtaining the derivatives of the interpolant is usually not the primary goal, but rather, serves a secondary purpose. For instance, in an optimization procedure, one is not interested in the derivatives per se. Instead, the gradient vector enters an iterative procedure to achieve the primary goal, which is to numerically compute a local optimizer. Unfortunately, the discontinuous derivatives of a piecewise multilinear interpolant have a serious drawback: the first order optimality condition  $\text{grad } f = 0$  cannot be fulfilled. Instead, the sign of the gradient components will oscillate about the optimizer of the continuous interpolant, resulting in slow convergence.

To overcome this limitation, the Sparse Grid Interpolation Toolbox offers a powerful alternative to computing the exact derivatives of a piecewise multilinear interpolant. By setting a simple flag, augmented derivatives can be computed that artificially enforce continuity. This is achieved by linear interpolation with respect to the derived variable.

Let us consider an example (we use the same test function and interpolant from above).

First, we re-define the evaluation grid (there are no more jumps to emphasize).

```
np = 2^double(z.maxLevel+1)+1;
x = linspace(0,1,np);
[x,y] = ndgrid(x);
```

Prior to evaluating the interpolant, we set the flag `continuousDerivatives = 'on'`.

```
z.continuousDerivatives = 'on';
```

Computing interpolated values and gradients of the sparse grid interpolant is done as before with the command

```
[ip,ipgrad] = spinterp(z,x,y);
```

Finally, we generate the plot. Compare the plot to the previous one. Note that the derivatives are now continuous.

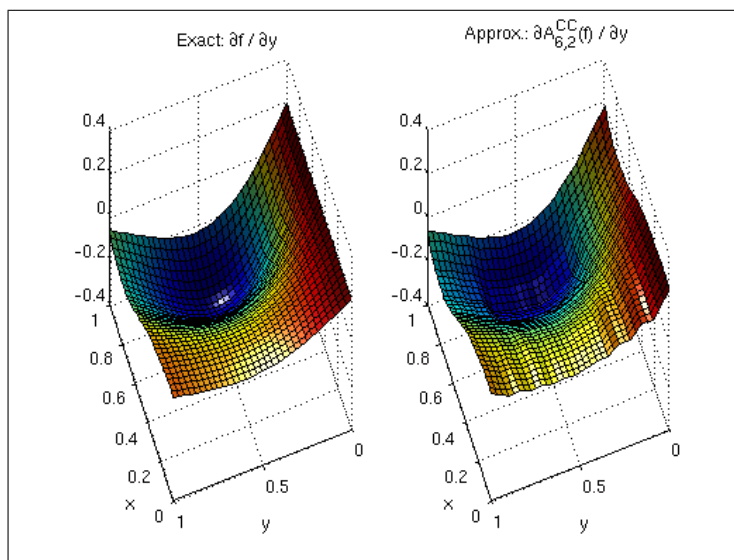
```

% Convert returned cell array to matrix
ipgradmat = cell2mat(ipgrad);

% Get the derivatives with respect to y
ipdy = ipgradmat(2:d:end,:);

% Plot exact derivatives and derivatives of interpolant
subplot(1,2,1,'align');
surf(x,y,fdy(x,y));
view(250,50); xlabel('x'); ylabel('y'); light;
title('Exact:  $\frac{\partial f}{\partial y}$ ');
subplot(1,2,2,'align');
surf(x,y,ipdy);
view(250,50); xlabel('x'); ylabel('y'); light;
title('Approx.:  $\frac{\partial A_{6,2}^{CC}(f)}{\partial y}$ ');

```



To conclude this section: If piecewise multilinear sparse grid interpolants (the Clenshaw-Curtis grid) are used, augmented derivatives can help improving efficiency when solving optimization problems with methods that require computation of the gradient.

### Derivatives of polynomial interpolants

The Chebyshev-Gauss-Lobatto (CGL) sparse grid uses globally defined polynomial basis functions. These basis functions are infinitely smooth, and thus, the derivatives are infinitely smooth, too. The Sparse Grid Interpolation Toolbox offers efficient algorithms involving barycentric interpolation and the discrete cosine transform to compute gradients, with excellent numerical stability.



Consider the following example. Using the test function from above, we compute a CGL-type sparse grid interpolant (again with `maxDepth = 4`).

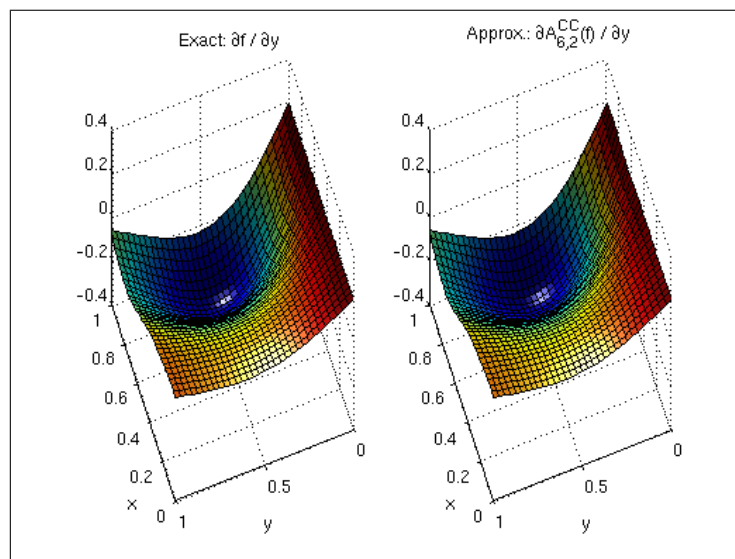
```
maxDepth = 4;
options = spset('Vectorized','on','SparseIndices','off', ...
    'MaxDepth', maxDepth, 'GridType', 'Chebyshev');
z = spvals(f,d,[],options);
```

Warning: MaxDepth = 4 reached before accuracies RelTol = 0.01 or AbsTol = 1e-06 were achieved.

The current estimated relative accuracy is 0.020306.

The remaining code evaluates the interpolant and its derivatives at the full grid and creates the plot, just as above.

```
[ip,ipgrad] = spinterp(z,x,y);
ipgradmat = cell2mat(ipgrad);
ipdy = ipgradmat(2:d:end,:);
subplot(1,2,1,'align');
surf(x,y,fdy(x,y));
view(250,50); xlabel('x'); ylabel('y'); light;
title('Exact:  $\frac{\partial f}{\partial y}$ ');
subplot(1,2,2,'align');
surf(x,y,ipdy);
view(250,50); xlabel('x'); ylabel('y'); light;
title('Approx.:  $\frac{\partial A_{6,2}^{CC}(f)}{\partial y}$ ');
```



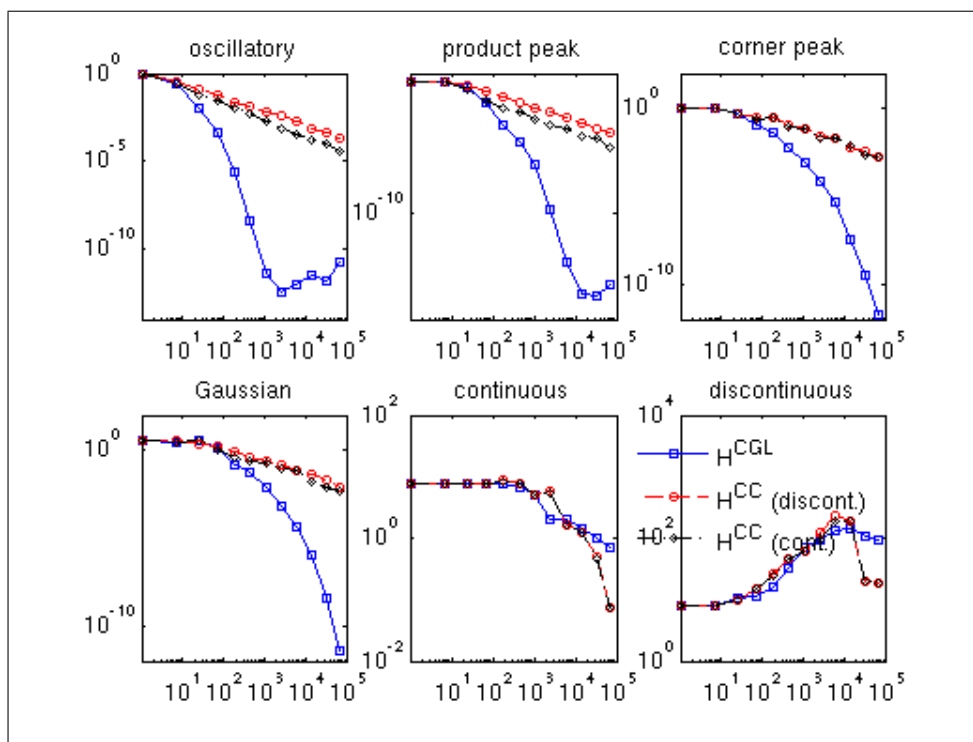
### Approximation quality

Although we see the main application of computing derivatives of sparse grid interpolants in obtaining gradients during an optimization algorithm, it is interesting to investigate the approx-

imation quality with respect to the derivatives of the original function.

This is illustrated by the example `spcomparederiv.m` that plots an approximation of the error in the maximum norm by computing the maximum absolute error of the derivatives for the six test functions of Genz (see `testderivatives.m`) at 100 randomly sampled points. The plot presented below is for dimension  $d = 3$ .

```
% Reset random number generator (to generate reproducible results)
rand('state',0);
% Run the demo.
spcomparederiv;
```



The legend indicates the three types of derivatives: discontinuous ( $H^{CC}$  grid), augmented continuous ( $H^{CC}$  grid), and smooth ( $H^{CGL}$  grid).

**Remark:** For the original functions labeled 'continuous' and 'discontinuous', single-sided derivatives are computed at the points where the function is not continuously differentiable. Note that the approximations of the derivatives of both of these two functions do not converge, for the following reasons:

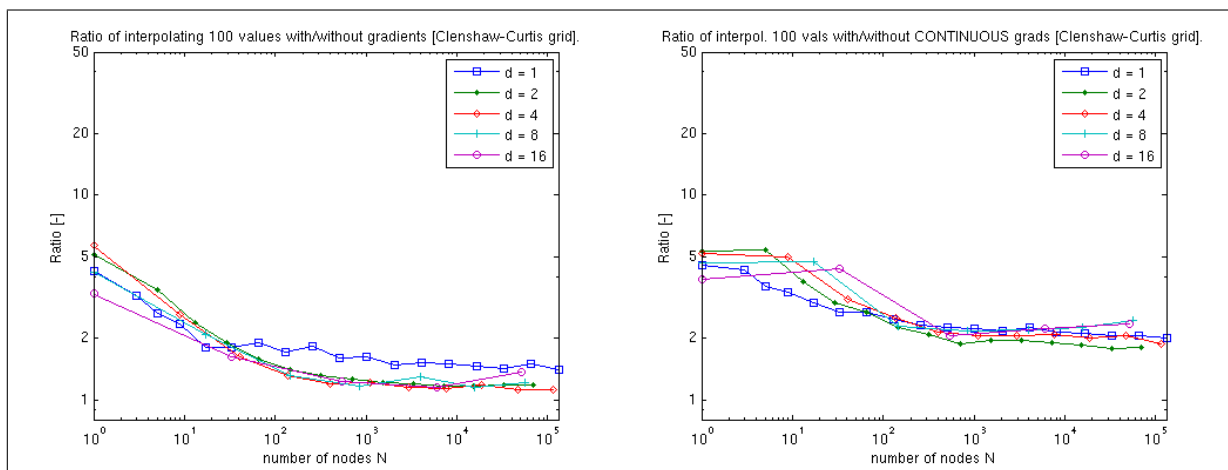
- Since the **discontinuous** function itself cannot be approximated by a continuous sparse grid interpolant in the first place, the approximations of the derivatives will not converge, either.

- What is less obvious is that the derivatives of the **continuous** function **cannot** be successfully approximated for the whole considered box, either. Although the plot labeled 'continuous' appears to suggest slow convergence, **convergence in the maximum norm is not achieved** in regions close to the kink(s). However, this/these region(s) becomes smaller with increasing number of support nodes. The decreasing size of the non-converging region(s) close to the kink(s) explain(s) the decay of the absolute error in the plot: It becomes less likely that any of the 100 randomly sampled points are placed here.

## Computational cost

### Cost for the piecewise multilinear Clenshaw-Curtis sparse grid

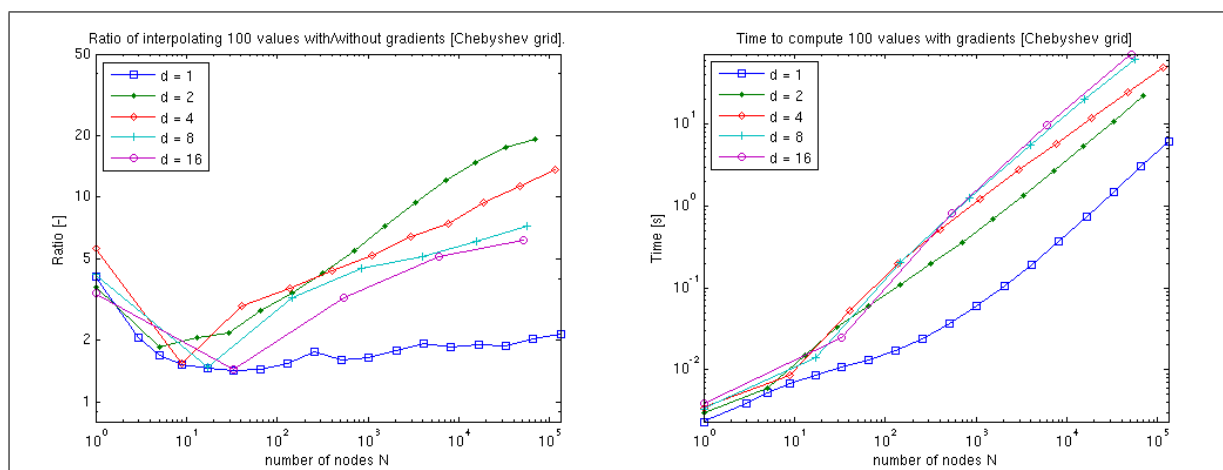
The gradient vector of the piecewise multilinear Clenshaw-Curtis sparse grid interpolant can be obtained at a very small additional cost. The following two plots illustrate that this additional cost for computing either the exact derivatives or the augmented continuous derivatives amounts to just a small factor that is almost independent of the problem dimension.



### Cost for the polynomial Chebyshev-Gauss-Lobatto sparse grid

Due to the more sophisticated algorithms required in the polynomial case, the additional cost of computing the gradients is considerably higher compared to a mere interpolation of function values. However, as the dimension increases, the additional cost decreases, as fewer subgrids will require a derivative computation (many subgrids are lower-dimensional than the final interpolant, and thus, must not be differentiated with respect to the dimensions that are omitted).

Thus, the performance is very competitive especially in higher dimensions. For instance, consider applying numerical differentiation instead (which you can easily do alternatively). This would require  $d + 1$  interpolant evaluations to compute the gradient if single-sided differences are used, or even  $2d + 1$  if the more accurate centered difference formula is used.



**Remark:** We produced the previous graphs with the example function `timespderiv.m`, which is included in the toolbox (see demos). The test was performed using Matlab R14 SP3 running on a Linux 2.6.15 machine equipped with one AMD Athlon 1.5 GHz CPU and 512 MB of memory.

## 2.3 Numerical Integration (Quadrature)

Once you have computed a sparse grid interpolant of an objective function, you can compute the integral value of it for the given range. You can do this for any grid type, and for both regular and dimension-adaptive sparse grid interpolants by simply calling the `spquad` function. A couple of examples are provided below.

### Integration of regular sparse grid interpolants

Consider the task of integrating the test function

$$f(\mathbf{x}) = (1 + 1/d)^d \prod_{i=1}^d (x_i)^{1/d}$$

for the domain  $[0, 1]^d$ , and  $d = 5$ . The exact value of the integral is 1. We reproduce the results of Table 1 for the columns labelled "Trapez", "Clenshaw", and "Gauss-Patterson" from [7]. Note that the grid type Clenshaw-Curtis and Chebyshev of the Sparse Grid Interpolation Toolbox correspond to the sparse grids based on the Trapez rule and the Clenshaw Curtis rule in the paper, respectively.

Define the objective function, dimension, and maximum depth:

```
d = 5;
maxDepth = 6;
f = @(x) (1+1/d)^d * prod(x)^(1/d);
```

Compute integral for increasing sparse grid depth, and generate the results table:

```
warning('off', 'MATLAB:spinterp:insufficientDepth');
z = cell(3,1);
quadr = zeros(3,1);
disp('└─Clenshaw-Curtis─┘└─Chebyshev─┘└─Gauss-Patterson─┘');
disp('└─points─┘└─error─┘└─points─┘└─error─┘└─points─┘└─error─┘');
for k = 0:maxDepth
    options = spset('MinDepth', k, 'MaxDepth', k, ...
        'FunctionArgType', 'vector');

    % Compute integral values with Clenshaw-Curtis grid
    options = spset(options, 'GridType', 'Clenshaw-Curtis', ...
        'PrevResults', z{1});
    z{1} = spvals(f,d,[],options);
    quadr(1) = spquad(z{1});

    % Compute integral values with Chebyshev grid
    options = spset(options, 'GridType', 'Chebyshev', ...
        'PrevResults', z{2});
    z{2} = spvals(f,d,[],options);
    quadr(2) = spquad(z{2});

    % Compute integral values with Gauss-Patterson grid
    options = spset(options, 'GridType', 'Gauss-Patterson', ...
        'PrevResults', z{3});
    z{3} = spvals(f,d,[],options);
    quadr(3) = spquad(z{3});

    % Results output
    disp(sprintf('%5d└─%8.3e┘└─%5d└─%8.3e┘└─%5d└─%8.3e┘', ...
        z{1}.nPoints, abs(1-quadr(1)), ...
        z{2}.nPoints, abs(1-quadr(2)), ...
        z{3}.nPoints, abs(1-quadr(3))));
end
```

| Clenshaw-Curtis | Chebyshev | Gauss-Patterson |           |        |           |
|-----------------|-----------|-----------------|-----------|--------|-----------|
| points          | error     | points          | error     | points | error     |
| 1               | 2.442e-01 | 1               | 2.442e-01 | 1      | 2.442e-01 |
| 11              | 1.080e+00 | 11              | 6.385e-01 | 11     | 8.936e-03 |
| 61              | 7.578e-02 | 61              | 1.441e-01 | 71     | 8.073e-04 |
| 241             | 2.864e-01 | 241             | 1.237e-01 | 351    | 2.070e-04 |
| 801             | 1.079e-01 | 801             | 6.650e-03 | 1471   | 2.256e-05 |
| 2433            | 8.001e-02 | 2433            | 1.060e-02 | 5503   | 1.420e-06 |
| 6993            | 5.030e-02 | 6993            | 1.743e-03 | 18943  | 3.437e-09 |

## Integration of dimension-adaptive sparse grid interpolants

To illustrate a higher-dimensional, dimension-adaptive case, we consider the absorption problem from W. Morokoff, R. Caflisch, "Quasi-monte carlo integration", J. Comp. Phys. 122, pp. 218-230, 1995, given by the integral equation

$$y(x) = \int_x^1 \gamma y(x') dx' + x.$$

The exact solution of this equation is given by

$$y(x) = \frac{1}{\gamma} (1 - (1 - \gamma) \exp(\gamma(1 - x))).$$

Two alternate representations are given in the paper, the first being an infinite integral with an integrand with a jump, and the second one with a smooth integrand.

The sparse grid method does not work well for the first representation, since it is a non-smooth function where the discontinuities are not parallel to the coordinate directions (see S. Dirnsdorfer, "Numerical Quadrature on Sparse Grids", Diploma Thesis, TU Munich, 2000). However, in case of the second representation, very accurate results can be computed using the dimension-adaptive approach, as shown below.

We define the integrand of the absorption problem as follows. The optional parameter named SMOOTH indicates which of the two representation should be used.

```
function y = absorb(varargin)
% ABSORB(R1,...,RD,GAMMA,X,SMOOTH)  Finite sum integrand of
% the integrand representation of the absorption problem
% paper W. Morokoff, R. Caflisch, "Quasi-monte carlo
% integration", J. Comp. Phys. 122, pp. 218-230, 1995.

d = length(varargin) - 3;
gamma = varargin{end-2};
x = varargin{end-1};
smooth = varargin{end};

% Compute F as in paper
if ~smooth
% First representation with jump
phi = @(z) (1 .* (z >= 0)); % Heaviside function
d = d - 1;
sumR = cell(d+1,1);
sumR{1} = varargin{1};
for k = 2:d+1
sumR{k} = sumR{k-1} + varargin{k};
end
F = @(n) (gamma^n * phi(1-x-sumR{n}) ...
.* phi(sumR{n+1}-(1-x)));
else
```

```

% Second, smooth representation
prodR1 = cell(d,1);
prodR1{1} = ones(size(varargin{1}));
for k = 2:d
    prodR1{k} = prodR1{k-1};
    for l = 1:k-1
        prodR1{k} = prodR1{k} .* varargin{l};
    end
end
prodR2 = cell(d,1);
prodR2{1} = varargin{1};
for k = 2:d
    prodR2{k} = prodR2{k-1} .* varargin{k};
end
F = @(n) (gamma^n * (1 - x)^n * ...
    prodR1{n} .* (1 - (1 - x) * prodR2{n}));
end

% Compute integrand value(s) (finite sum)
y = zeros(size(varargin{1}));
for n = 1:d
    y = y + F(n);
end

```

The following loop computes increasingly accurate approximations to the solution of the absorption problem with  $d = 20$ ,  $\gamma = 0.5$ , and  $x = 0$  by computing a dimension-adaptive polynomial interpolant of the smooth integrand which is then integrated using the `spquad` function. For comparison, we also compute the integral using crude Monte Carlo (MC) with the same number of points.

```

gamma = 0.5; x = 0; d = 20;

% Show exact solution
I_exact = 1/gamma - (1-gamma)/gamma*exp(gamma*(1-x))

options = spset('DimensionAdaptive','on', 'DimadaptDegree', 1, ...
    'GridType', 'Chebyshev', 'Vectorized', 'on');
Nmax = 50000;
N = 2*d;
z = [];

warning('off', 'MATLAB:spinterp:maxPointsReached');
while N <= Nmax
    % Compute integral via sparse grid
    spoptions = spset(options, 'MinPoints', N, ...
        'MaxPoints', N, 'PrevResults', z);
    z = spvals(@absorb, d, [], spoptions, gamma, x, true);
end

```

```
e1 = abs(I_exact - spquad(z));

% Compute integral via MC (error average for 10 runs)
e2 = 0;
for k = 1:10
    p = rand(z.nPoints,d);
    p = num2cell(p,1);
    I = sum(absorb(p{:}, gamma, x, true)) / double(z.nPoints);
    e2 = e2 + abs(I_exact - I);
end
e2 = e2 / 10;

disp(['_points:_ ' sprintf('%5d', z.nPoints) ...
      '_|_error_(CGL):_' sprintf('%9.3e',e1) ...
      '_|_error_(MC):_' sprintf('%9.3e',e2)]);
N = round(z.nPoints .* 2);
end
warning('on', 'MATLAB:spinterp:maxPointsReached');
```

```
I_exact =
    0.3513
```

```
points:    41 | error (CGL): 4.622e-04 | error (MC): 1.298e-02
points:    87 | error (CGL): 5.606e-06 | error (MC): 6.008e-03
points:   177 | error (CGL): 6.010e-07 | error (MC): 7.791e-03
points:   367 | error (CGL): 1.566e-07 | error (MC): 3.442e-03
points:   739 | error (CGL): 3.893e-08 | error (MC): 4.761e-03
points:  1531 | error (CGL): 2.461e-08 | error (MC): 1.895e-03
points:  3085 | error (CGL): 1.061e-09 | error (MC): 1.036e-03
points:  6181 | error (CGL): 2.750e-09 | error (MC): 6.147e-04
points: 12393 | error (CGL): 1.335e-09 | error (MC): 6.609e-04
points: 24795 | error (CGL): 3.006e-10 | error (MC): 5.420e-04
points: 49739 | error (CGL): 1.791e-10 | error (MC): 3.305e-04
```

## 2.4 Optimization

Once a sparse grid interpolant providing a surrogate function or meta-model of an expensive to evaluate model has been obtained, a very common task to be performed is often a search for local/global minimizers or maximizers. Since version 4.0 of the toolbox, several efficient optimization methods are available to perform this task. Furthermore, it is easy to use third-party optimization codes on sparse grid interpolants.

### Available algorithms

The following optimization algorithms are included with the toolbox:



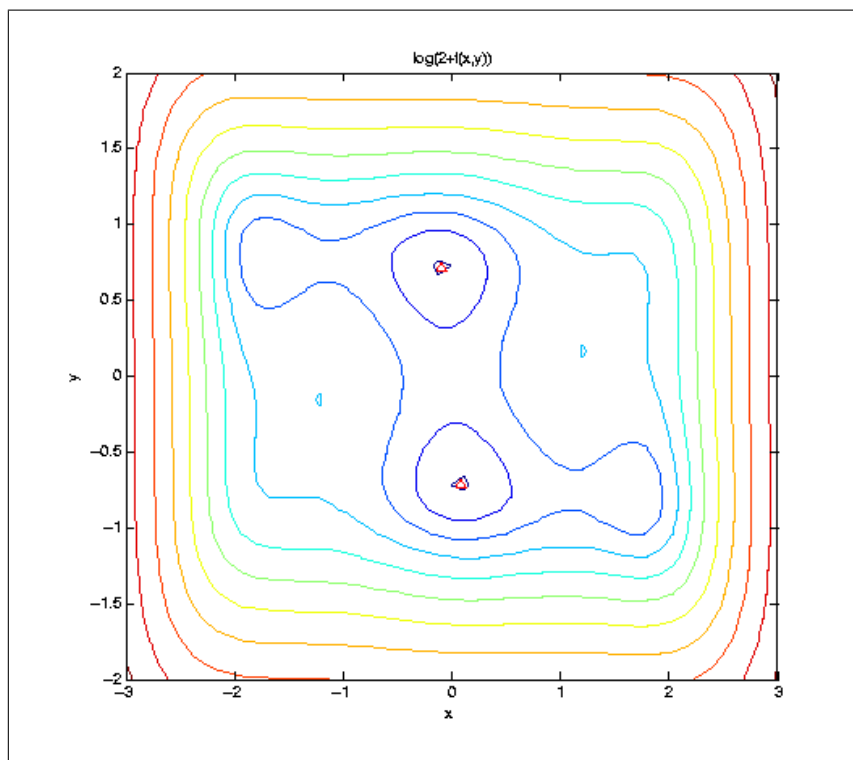
- `spcgsearch` - suitable for optimizing polynomial sparse grid interpolants.
- `spcompssearch` - suitable for optimizing piecewise linear sparse grid interpolants.
- `spfminsearch` - works for all types of interpolants, but usually less efficient than `spcompssearch` or `spcgsearch`.
- `spmultistart` - a multiple random start search method that uses any of the above methods for the local searches.

Many parameters of these algorithms can be configured with an options structure created with the `spoptimset` function.

### Optimizing piecewise linear interpolants

We consider a simple algebraic test function  $f$ , the well-known six-hump camel back function. Here, we visualize  $f$  slightly shifted and in logarithmic scaling to clearly show the 6 minima. Two minima are global, indicated by the red triangle.

```
f = @(x,y) (4-2.1.*x.^2+x.^4./3).*x.^2+x.*y+(-4+4.*y.^2).*y.^2;
ezcontour(@(x,y) log(2+f(x,y)), [-3 3 -2 2], 51);
hold on;
plot([ 0.08984201 -0.08984201], [-0.71265640 0.71265640], 'r^');
```



We construct a sequence of piecewise linear interpolants, and optimize them by the `spcompsearch` function in each step. Here, we use a maximum of  $N = 705$  points, as used by the Clenshaw-Curtis grid of level  $n_{\max} = 5$ , to approximate the (global) minimum.

```
nmax = 7;
z = [];
range = [-3 3; -2 2];
f_exact = -1.0316284535

warning('off', 'MATLAB:spinterp:insufficientDepth');
tic;
for n = 1:nmax
    spoptions = spset('MinDepth', n, 'MaxDepth', n, 'PrevResults', z, ...
        'KeepFunctionValues', 'on');
    z = spvals(f,2,range,spoptions);
    [xopt, fval, exitflag, output] = spcompsearch(z,range);
    disp(['_grid_pnts:_ ' sprintf('%3d', z.nPoints) ...
        '_|_optim_fevals:_ ' sprintf('%3d', output.nFEvals) ...
        '_|_fval:_ ' sprintf('%+5f', fval) ...
        '_|_abs._error:_ ' num2str(abs(f_exact-fval))]);
end
toc;
warning('on', 'MATLAB:spinterp:insufficientDepth');
```

```
f_exact =
    -1.0316
```

```
grid pnts:  5 | optim fevals:  8 | fval: +0.000000 | abs. error: 1.0316
grid pnts: 13 | optim fevals:  8 | fval: +0.000000 | abs. error: 1.0316
grid pnts: 29 | optim fevals: 12 | fval: -0.750000 | abs. error: 0.2816
grid pnts: 65 | optim fevals: 12 | fval: -0.984375 | abs. error: 0.0472
grid pnts: 145 | optim fevals: 20 | fval: -0.986956 | abs. error: 0.0446
grid pnts: 321 | optim fevals: 20 | fval: -1.026468 | abs. error: 0.0051
grid pnts: 705 | optim fevals: 24 | fval: -1.031286 | abs. error: 0.0003
Elapsed time is 0.641286 seconds.
```

## Optimizing polynomial interpolants

If the objective function is smooth, polynomial interpolants are a good choice. In the example below, by using the Chebyshev-Gauss-Lobatto sparse grid, we achieve an exponential convergence rate for the considered analytic function. To further reduce the number of sparse grid points, we use a dimension-adaptive interpolant. We start with  $N = 5$  nodes, and increase the number of nodes by about a factor of 1.5 in each step of the loop, up to about 100 points.

```
Nmax = 100;
N = 5;
z = [];
```

```
warning('off', 'MATLAB:spinterp:maxPointsReached');
tic;
while N <= Nmax
    spoptions = spset('MinPoints', N, 'MaxPoints', N, 'PrevResults', z, ...
        'GridType', 'Chebyshev', 'DimensionAdaptive', 'on', ...
        'KeepFunctionValues', 'on');
    z = spvals(f,2,range,spoptions);
    N = round(z.nPoints .* 1.5);
    [xopt, fval, exitflag, output] = spcgsearch(z,range);
    disp(['_grid_pnts:_ ' sprintf('%3d', z.nPoints) ...
        '_optim_fevals:_ ' sprintf('%3d', output.nFEvals) ...
        '_fval:_ ' sprintf('%+5f', fval) ...
        '_abs_error:_ ' num2str(abs(f_exact-fval))]);
end
toc;
warning('on', 'MATLAB:spinterp:maxPointsReached');
```

```
grid pnts:   5 | optim fevals:   1 | fval: +0.000000 | abs. error: 1.0316
grid pnts:  11 | optim fevals:   9 | fval: +0.000000 | abs. error: 1.0316
grid pnts:  17 | optim fevals:  20 | fval: -0.537875 | abs. error: 0.49375
grid pnts:  29 | optim fevals:  29 | fval: -1.031628 | abs. error: 1.9e-11
grid pnts:  53 | optim fevals:  30 | fval: -1.031628 | abs. error: 1.9e-11
grid pnts:  85 | optim fevals:  30 | fval: -1.031628 | abs. error: 1.9e-11
Elapsed time is 1.146093 seconds.
```

## A high-dimensional example

Let us look at the optimization of a higher-dimensional function. We consider again the function `trid.m` that was already used to illustrate the dimension-adaptive algorithm:

```
type('trid.m');
function y = trid(x)
% TRID Quadratic function with a tridiagonal Hessian.
% Y = TRID(X) returns the function value Y for a D-
% dimensional input vector X.
%
% The test function is due to Arnold Neumaier, listed
% on the global optimization Web page at
% http://www.mat.univie.ac.at/~neum/glopt/

d = length(x);
y = sum((x-1).^2) - sum(x(2:d).*x(1:d-1));
```

We let  $d = 100$ , and compute the known exact minimal value for comparison:

```
d = 100;
range = repmat([-d^2 d^2], d, 1);
f_exact = -d*(d+4)*(d-1)/6
```

```
f_exact =
    -171600
```

For high-dimensional problems, it is important to use dimensional adaptivity. Note that here, as well as in the examples above, we use the `KeepFunctionValues` property to indicate that the function values obtained during the sparse grid construction should be retained in order to save time when selecting a good start point for the search.

```
options = spset('DimensionAdaptive', 'on', ...
               'DimadaptDegree', 1, ...
               'FunctionArgType', 'vector', ...
               'RelTol', 1e-3, ...
               'GridType', 'Chebyshev', ...
               'KeepFunctionValues', 'on');

Nmax = 40000;
N = 2*d;
z = [];
warning('off', 'MATLAB:spinterp:maxPointsReached');
tic;
xopt = [];
fval = [];
while N <= Nmax
    spoptions = spset(options, 'MinPoints', N, ...
                      'MaxPoints', N, 'PrevResults', z);
    z = spvals(@trid,d,range,spoptions);
    z = sppurge(z);
    spoptions = spoptimset('TolFun',1e-6);
    [xopt,fval,exitflag,output] = spcgsearch(z,range,spoptions);
    N = round(z.nPoints .* 2);
    disp(['_grid_pnts:_ ' sprintf('%5d', z.nPoints) ...
         '_|_optim_fevals:_ ' sprintf('%4d', output.nFEvals) ...
         '_|_fval:_ ' sprintf('%+9.1f', fval) ...
         '_|_abs._error:_ ' num2str(abs(f_exact-fval))]);
end
toc;
warning('on', 'MATLAB:spinterp:maxPointsReached');
```

```
grid pnts:  201 | optim fevals:  11 | fval:    -0 | abs. error: 171600
grid pnts:  443 | optim fevals:  29 | fval:   -18 | abs. error: 171582
grid pnts:  923 | optim fevals:  48 | fval:  -132 | abs. error: 171468
grid pnts: 1883 | optim fevals:  93 | fval:  -537 | abs. error: 171063
grid pnts: 3899 | optim fevals: 117 | fval: -1202 | abs. error: 170398
grid pnts: 7889 | optim fevals: 188 | fval: -3293 | abs. error: 168307
grid pnts:16043 | optim fevals: 248 | fval: -10725 | abs. error: 160874
grid pnts:32477 | optim fevals: 305 | fval:-171600 | abs. error: 5.6e-08
Elapsed time is 188.371965 seconds.
```

## Using third-party optimization algorithms

Instead of using the optimization algorithms provided with the Sparse Grid Interpolation Toolbox, you can also use third-party optimization methods. In the following example, we use `fmincon` from The Mathwork's Optimization Toolbox on `spsurf` to optimize the sparse grid interpolant obtained in the last step of the loop from the example above.

```
optimsetoptions = optimset('GradObj','on', ...
    'LargeScale','off');
[xopt,fval,exitflag,output] = fmincon(@(x) spsurf(x,z), ...
    range(:,1)+range(:,2))/2, [], [], [], [], range(:,1),range(:,2), ...
    [], optimsetoptions);
disp(['_grid_pnts:_ ' sprintf('%5d', z.nPoints) ...
    '_optim_fevals:_ ' sprintf('%4d', output.funcCount) ...
    '_fval:_ ' sprintf('%+9.1f', fval) ...
    '_abs_error:_ ' num2str(abs(f_exact-fval))]);
```

```
Optimization terminated: magnitude of directional derivative in search
direction less than 2*options.TolFun and maximum constraint violation
is less than options.TolCon.
```

```
No active inequalities
```

```
grid pnts: 32477 | optim fevals: 5711 | fval:-171600 | abs. error: 1.1e-07
```

## 2.5 Improving performance

The aim of this section is to provide an overview on how to optimize the performance of the Sparse Grid Interpolation Toolbox.

### Vectorizing the objective function

Vectorizing the objective function is most beneficial if the function evaluations are very cheap, in the order of less than 1/100 s. In this case, providing a vectorized function can improve the performance of the `spvals` function. Consider the following function

$$f(x_1, x_2) = (x_1 + x_2)^2$$

and the following two m-files implementing it:

```
function y = fun(x1, x2)
y = x1 * x2;
y = y^2;
```

```
function y = fun_vec(x1, x2)
y = x1 .* x2;    % Use '.' before any '^', '*' or '/' to enable
y = y.^2;        % vectorized evaluation of expressions
```

The first m-file allows for evaluation at a single real-valued point only, the second one permits vectorized evaluation. Since in case of cheap functions, the function calls in Matlab represent a significant overhead, the function evaluation part of the `spvals` algorithm is much slower if the non-vectorized form is used. This is demonstrated by the following code.

```
tic, z1 = spvals('fun',2); toc;
tic, z2 = spvals('fun_vec',2,[],spset('Vectorized','on')); toc;
z1.fevalTime
z2.fevalTime
```

```
Elapsed time is 0.112452 seconds.
```

```
Elapsed time is 0.069006 seconds.
```

```
ans =
```

```
0.1021
```

```
ans =
```

```
0.0480
```

### Reusing previous results

An important feature of the toolbox is that you do not have to discard previously computed results. A "best practice" is, therefore, to embed the interpolant construction in a loop. Proceeding in this way has two advantages: First, it gives the user a maximum of control in monitoring the decay of the estimated interpolation error. Second, it makes it possible to start with a low number of required points, and to increase this number slowly if the targeted accuracy is not yet achieved. There are several examples on how to implement such a loop in the provided demos. See, for instance, `spadaptanim.m` or `spcompare.m` in the `examples` directory. A small example on implementing dimension-adaptive interpolant construction in a loop is provided below.

```
np = 2;
z = [];
options = spset('Vectorized', 'on', 'DimensionAdaptive', 'on', ...
    'RelTol', inf);
while np < 4000
    options = spset(options, 'PrevResults', z, 'MinPoints', np, ...
        'MaxPoints', np);
    z = spvals('fun_vec',2,[],options);
    np = z.nPoints;
    disp(['np=_' num2str(np) ', e_rel=_' num2str(z.estRelError)]);
    np = np * 2;
end
```

```
np = 5, e_rel = 0.75
```

```
np = 13, e_rel = 0.5625
```

```
np = 29, e_rel = 0.046875
```

```
np = 73, e_rel = 0.011719
```

```
np = 177, e_rel = 0.0029297
```

```

np = 417, e_rel = 0.00073242
np = 897, e_rel = 6.1035e-05
np = 1921, e_rel = 4.5776e-05
np = 4097, e_rel = 3.8147e-06

```

## Purging interpolant data

Since version v3.2 of the toolbox, a new function called `sppurge` is available. This function serves to "purge" or "clean up" the interpolant data from sub-grids that do not contribute significantly to the result. This is done by introducing a drop tolerance that is applied to the hierarchical surpluses. Sub-grids where the absolute value of all hierarchical surpluses fall below this drop tolerance are marked and neglected during the interpolation process. By default, very conservative purging parameters are used, guaranteeing that the accuracy of the interpolation will not be affected up to about the 12th significant digit. However, if the accuracy requirements are lower, the user may use higher drop tolerances, and thus, trade improved interpolation speed against lower accuracy. This is illustrated by the following example. We assume that an interpolant was computed for the function `fun_vec` by the code above with 4097 points, using piecewise multilinear basis functions. The following code generates a plot that shows the time required to compute 1000 randomly sampled points for different drop tolerances. The maximum absolute error is shown for comparison. This example only uses absolute drop tolerances (the relative drop tolerance is set to zero).

```

% Define drop tolerances
dropTols = [1e-5, 1e-4, 1e-3];

% Generate 1000 random points
rand('state',0);
x = rand(1000,1); y = rand(1000,1);

% Compute exact function values
f_exact = fun_vec(x,y);

e = zeros(3,1); t = zeros(3,1);
for k = 1:3
    % Purge interpolant with drop tolerance
    z = sppurge(z,spset('DropTol', [dropTols(k), 0]));
    % Interpolate and measure time
    tic, ip = spinterp(z, x, y); t(k) = toc;
    % Compute maximum error
    e(k) = max(abs(f_exact - ip));
end

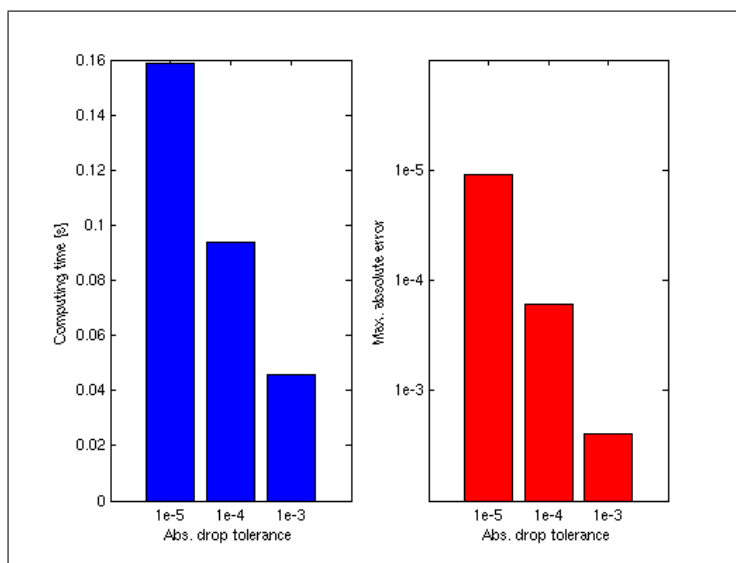
% Plot results
subplot(1,2,1);
bar(t, 'b');
set(gca,'XTickLabel', {'1e-5','1e-4','1e-3'})

```

```

xlabel('Abs. drop tolerance');
ylabel('Computing time [s]');
subplot(1,2,2);
bar(log10(e), 'r');
set(gca,'XTickLabel', {'1e-5','1e-4','1e-3'})
set(gca,'YDir','reverse');
set(gca,'YLim', [-6 -2]);
set(gca,'YTick', [-5 -4 -3]);
set(gca,'YTickLabel', {'1e-5','1e-4','1e-3'})
xlabel('Abs. drop tolerance');
ylabel('Max. absolute error');

```



For another example using the default relative drop tolerance, see `sppurge` in the function reference.

## Vectorized interpolant evaluation

The `spinterp` function is designed for vectorized evaluation. Since the sparse grid algorithm involves more computational overhead than other, simpler interpolation methods, and due to the fact that MATLAB is relatively slow if many function calls are performed (since it is an interpreted language), it is recommended to evaluate as many interpolation points at a time as possible. The following code illustrates non-vectorized vs. vectorized evaluation at 1000 points for the interpolant from above.

```

% Non-vectorized interpolation
tic
for k = 1:1000
    ip = spinterp(z,x(k),y(k));

```



```

end
toc

% Vectorized interpolation
tic, ip = spinterp(z,x,y); toc

```

Elapsed time is 2.526127 seconds.

Elapsed time is 0.045744 seconds.

## 2.6 Interfacing concepts

Applying the `spvals` method to construct interpolants sometimes requires a small interface function. In this section, we show the most important categories of Matlab function headers and (if necessary) how to design an appropriate interface function for them. Tables 2.1, 2.2 show the basic function header types discussed here. Combinations of those are of course also possible and can be derived from the treated cases. In the tables, the objective interpolation variables (all must be real-valued scalars) are denoted by  $x_1, \dots, x_n$ . Examples of the presented cases are provided below.

Table 2.1: Interface function NOT required.

| # | header  | variable types  |
|---|---|---|
| 1 | <code>out = fun(x<sub>1</sub>,x<sub>2</sub>,...,x<sub>n</sub>)</code>   | $x_1, \dots, x_n$ are real scalars  |
| 2 | <code>out = fun(x<sub>1</sub>,x<sub>2</sub>,...,x<sub>n</sub>,p<sub>1</sub>,p<sub>2</sub>,...,p<sub>m</sub>)</code>   | $x_1, \dots, x_n$ are real scalars, $p_1, \dots, p_m$ are parameters of arbitrary type (double array, cell array, structure, etc.)  |
| 3 | <code>out = fun(x<sub>1</sub>,...,x<sub>i<sub>1</sub></sub>,p<sub>1</sub>,...,p<sub>j<sub>1</sub></sub>,x<sub>i<sub>1</sub>+1</sub>,...,x<sub>i<sub>2</sub></sub>,p<sub>j<sub>1</sub>+1</sub>,...,p<sub>j<sub>2</sub></sub>,...)</code> | $x_1, \dots, x_n$ are real scalars, $p_1, \dots, p_m$ are parameters of arbitrary type (double array, cell array, structure, etc.)  |
| 4 | <code>out = fun(v)</code>   | $v$ is a row or column vector with the entries $x_1, \dots, x_n$  |
| 5 | <code>out = fun(v,p<sub>1</sub>,p<sub>2</sub>,...,p<sub>m</sub>)</code>   | $v$ is a row or column vector with the real scalar entries $x_1, \dots, x_n$ , and $p_1, \dots, p_m$ are parameters of arbitrary type (double array, cell array, structure, etc.) |
| 6 | <code>[out<sub>1</sub>,out<sub>2</sub>,...,out<sub>n</sub>] = fun(...)</code>   | $out_1, \dots, out_n$ are real scalar output parameters, input parameters the same as one of the above  |
| 7 | <code>varargout = fun(...)</code>   | <code>varargout</code> is a cell array of real scalar output parameters $out_1, \dots, out_n$ , input parameters the same as one of the above                                     |

Table 2.2: Interface function REQUIRED (only some exemplary cases).

| # | header  | variable types   |
|---|---|--|
| 8 | <code>out = fun(A,p<sub>1</sub>,p<sub>2</sub>,...,p<sub>m</sub>)</code>           | $A$ is a matrix where some of its entries are the objective interpolation parameters $x_1, \dots, x_n$ , and $p_1, \dots, p_m$ are parameters of arbitrary type as above |
| 9 | <code>v<sub>out</sub> = fun(x<sub>1</sub>,x<sub>2</sub>,...,x<sub>n</sub>)</code> | $v_{out}$ is a row or column vector with real scalar outputs   |

### Examples

**Type 1:**  $out = fun(x_1, x_2, \dots, x_n)$

Objective function:

```
function y = fun1(x1, x2)
y = x1 .* x2;    % Use '.' before any '^', '*' or '/' to enable
y = y.^2;       % vectorized evaluation of expressions
```

Example for call to spvals:

```
options = spset('Vectorized', 'on');
range   = [0,2; 0,2];
z = spvals(@fun1, 2, range, options);
```

**Type 2:**  $out = fun(x_1, x_2, \dots, x_n, p_1, p_2, \dots, p_m)$

Objective function:

```
function y = fun2(x1, x2, c, params)
y = c .* (params.p1 .* x1 + length(params.p2) .* x2);
```

Example for call to spvals:

```
options = spset('Vectorized', 'on');
range = []; % use default range [0,1]^d
c = 2;
params = struct('p1', 3, 'p2', 'hello');
z = spvals(@fun2, 2, range, options, c, params);
```

**Type 3:**  $out = fun(x_1, \dots, x_{i_1}, p_1, \dots, p_{j_1}, x_{i_1+1}, \dots, x_{i_2}, p_{j_1+1}, \dots, p_{j_2}, \dots)$

Objective function:

```
function y = fun3(p1, x1, p2, x2)
y = p1 .* x1 + p2 .* x2;
```

Example for call to spvals:

```
options = spset('VariablePositions', [2 4], 'Vectorized', 'on');
range = [0,1; -1,2];
p1 = 2; p2 = 3;
z = spvals(@fun3, 2, range, options, p1, p2);
```

**Type 4:**  $out = fun(v)$

Objective function:

```
function y = fun4(x)
y = prod(x);
```

Example for call to `spvals`:

```
options = spset('FunctionArgType', 'vector');
range = [0 1; 1 2; 2 3; 3 4; 4 5];
z = spvals(@fun4, 5, range, options);
```

**Type 5:**  $out = \text{fun}(v, p_1, p_2, \dots, p_m)$

Objective function:

```
function y = fun5(x,p);
y = x(:)'*p; % Compute dot product
```

Example for call to `spvals`:

```
options = spset('FunctionArgType', 'vector');
range = []; % use default range [0,1]^d
p = rand(3,1);
z = spvals(@fun5, 3, range, options, p);
```

**Type 6:**  $[out_1, out_2, \dots, out_n] = \text{fun}(\dots)$

Objective function:

```
function [y1, y2] = fun6(x1, x2);
y1 = 2*x1 + 3*x2;
y2 = 4*x1 - 1*x2;
```

Example for call to `spvals`:

```
options = spset('NumberOfOutputs', 2);
range = []; % use default range [0,1]^d
z = spvals(@fun6, 2, range, options);
```

To compute interpolated values of functions with multiple output parameters, see Section 2.1.

**Type 7:**  $\text{varargout} = \text{fun}(\dots)$

Objective function:

```
function varargout = fun7(x1,x2,nout);
for k = 1:nout
    varargout{k} = x1.^k + k.*x2;
end
```

Example for call to `spvals`:

```
nout = 4;
options = spset('NumberOfOutputs', nout, 'Vectorized', 'on');
range = []; % use default range [0,1]^d
z = spvals(@fun7, 2, range, options, nout);
```

To compute interpolated values of functions with multiple output parameters, see Section 2.1.

**Type 8:**  $out = fun(A, p_1, p_2, \dots, p_m)$

Objective function:

```
function y = fun8(A, f);
y = A\f;
```

Assume that the diagonal entries of  $A$ , i.e.  $a_{11}, a_{22}, \dots, a_{nn}$  vary in some given range. An interpolant of `fun8` is sought for these varying diagonal entries of  $A$ :

```
d = 3;
A = magic(d); f = ones(d,1);
range = [diag(A)-0.5 diag(A)+0.5];
nout = d;
options = spset('NumberOfOutputs', nout, 'FunctionArgType', 'vector');
z = spvals(@interface_fun8, d, range, options, A, f);
```

The interface function `interface_fun8` looks like this:

```
function varargout = interface_fun8(a, A, f);
% Interface function to fun8

% Write the modifiable entries into A
for k = 1:length(a);
    A(k,k) = a(k);
end

% Call objective function fun8
y = fun8(A,f);

% Put the results in cell array (outputs must be cell row vector
% of scalars to be treated by spvals)
varargout = num2cell(y)';
```

Note that the original output, a column vector from the solution of the linear equation system is transformed into a cell array with a single row to match one of the admissible output variants. The original input is also modified to contain the interpolation parameters as a vector, which is permitted by `spvals`. The original Matrix as well as the right-hand side  $f$  are passed as additional parameters. To compute interpolated values of functions with multiple output parameters, see Section 2.1.

**Type 9:**  $v_{out} = fun(x_1, x_2, \dots, x_n)$

Objective function:

```
function y = fun9(x1, x2)
y = [x2 .* cos(x1); ...
     x2 .* sin(x1); ...
     x2];
```

Assume that the output of `fun9` is not a list of real scalars or a `varargout` cell array. In this case, a conversion of the output is required. The interface function uses Matlab's `num2cell` function to achieve this.

```
function varargout = interface_fun9(x1, x2);
y = fun9(x1, x2);
varargout = num2cell(y)';
```

Example for call to `spvals`:

```
nout = 3;
options = spset('NumberOfOutputs', nout);
z = spvals(@interface_fun9, 2, [], options);
```

To compute interpolated values of functions with multiple output parameters, see Section 2.1.

## 2.7 Approximating ODEs

As an example for a more complex function with multiple input- and output arguments, we show how to handle an ordinary differential equation. The model considered is a second order differential equation

$$Q''(t) + aQ'(t) + b = 50 \cos(t)$$

from [11, pp. 145–162] simulating an electrical circuit.

### The ODE model in Matlab

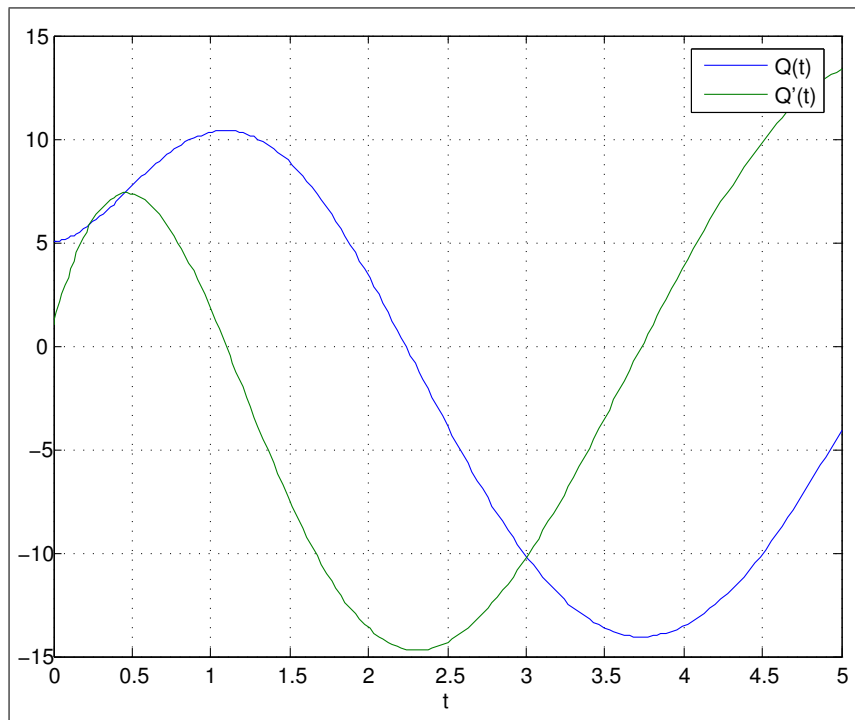
Rewriting this second-order equation as a system of first order equations, we can define the ODE file in Matlab format as follows:

```
function [out1, out2, out3] = circuit(t, u, flag, a, b);
% CIRCUIT definition of the electrical circuit ODE.

switch flag
case ''
    out1 = [u(2); 50*cos(t) - a*u(2) - b*u(1)];
case 'init'
    out1 = [0; 5];           % tspan
    out2 = [5; 1];          % initial conditions
    out3 = odeset('RelTol', 1e-6);
end
```

We can solve this ODE for  $a = 2$ ,  $b = 4$ , and the default initial conditions and time span as defined in the ODE file using the MATLAB solver `ode45`.

```
[t,Q] = ode45('circuit', [], [], [], 2, 4);
plot(t,Q)
xlabel('t');
grid on;
legend('Q(t)', 'Q''(t)');
```



## Interpolation problem statement and possible applications

We now consider the initial conditions and the parameters  $a, b$  to vary in some range, that is we assume intervals for  $Q(0)$ ,  $Q'(0)$ ,  $a$ , and  $b$ , and compute an error-controlled sparse grid interpolant for the ODE model at each time step. The interpolant can then be used to do several useful analyses, for instance, perform a Monte Carlo simulation with random variables, optimize the model for the given range of parameters and initial conditions, e.g. minimize or maximize the amplitude, or compute an envelope of the result using fuzzy calculus or interval analysis. In many cases, this can be done considerably faster than by using the original ODE directly, since the construction and evaluation of the interpolant is very fast.

## The interface function

We proceed as follows. First of all, we write an interface function of the ODE model to enable its evaluation by the `spvals` function.

```
function varargout = interface_circuit(Q0, Q0prime, a, b, tspan, nsteps)
% Definition of the complete model as a function of the uncertain
% input parameters.

% The time steps must be at fixed steps such that the number of
% outputs and time steps stay the same for each parameter
% variation.
t = linspace(tspan(1), tspan(2), nsteps);
```

```

% Call the ODE solver
[t, Q] = ode45('circuit', t, [Q0 Q0prime], [], a, b);

% Convert result vector to parameter list. This conversion is
% necessary, since the output arguments of the objective function
% to SPVALS must all be scalar. In this case, we assume that only
% the first column (i.e. Q, not Q') is of interest and thus
% returned.
varargout = num2cell(Q(:,1)');

```

### Interpolant construction

Next, we construct the interpolant, simultaneously for all time steps. Here, we use the intervals  $[Q(0)] = [4,6]$ ,  $[Q'(0)] = [0,2]$ ,  $[a] = [1,3]$ , and  $[b] = [3,5]$ .

```

% Problem dimension
d = 4;

% Define the time span considered
tspan = [0 5];

% Define the number of steps to consider
nsteps = 101;

% Define the objective range of the initial conditions and the
% parameters
range = [4 6; % [Q(0)]
         0 2; % [Q'(0)]
         1 3; % [a]
         3 5]; % [b]

% Maximum number of sparse grid levels to compute
nmax = 3;

% Initialize z
z = [];

% Turn insufficient depth warning off, since it is anticipated.
warning('off', 'MATLAB:spinterp:insufficientDepth');

% Compute increasingly accurate interpolants; use previous results;
% display estimated maximum relative error over all time steps at
% each iteration.
for n = 1:nmax
    options = spset('Vectorized', 'off', 'MinDepth', n, 'MaxDepth', ...

```

```

        n, 'NumberOfOutputs', nsteps, 'PrevResults', z);
z = spvals('interface_circuit', d, range, options, tspan, nsteps);
disp(['Current (estimated) maximum relative error over all time' ...
      'steps: ', num2str(z.estRelError)]);
end

% Turn insufficient depth warning back on
warning('on', 'MATLAB:spinterp:insufficientDepth');

```

```

Current (estimated) maximum relative error over all timesteps: 0.64844
Current (estimated) maximum relative error over all timesteps: 0.34119
Current (estimated) maximum relative error over all timesteps: 0.057381

```

### Computing interpolated values

We can now compute interpolated values at each time step, for any combination of parameters within the range that the interpolant was computed for. The structure `z` contains all the required information. We only need to select the desired output parameter (i.e. the time step in this example). To compute 10 randomly distributed values at time  $t = 5$  (which is step #101 with the chosen discretization) within the box  $[Q(0)] \times [Q'(0)] \times [a] \times [b]$ , we would simply use the following commands:

```

% Compute 10 randomly distributed points in [0,1] and re-scale them to
% the objective range
x = cell(1,4);
for k = 1:d
    x{k} = range(k,1) + rand(1,10) .* (range(k,2) - range(k,1));
end

% Select output parameter #101
z.selectOutput = 101;
% Compute and display interpolated values
y = spinterp(z, x{:})

```

```

y =
Columns 1 through 7
-2.7824 -1.3367 -1.8990 -3.4497 -1.9133 -4.1978 -0.1284
Columns 8 through 10
-4.9939 -8.1073 -3.3928

```

## 2.8 External models

Through system calls available in Matlab, one can easily execute external programs computing external models. The results from the external program can either be passed as an output stream



(requires subsequent parsing of the stream to retrieve the results in usable format), or by saving the results to a file and reading the results from Matlab.

By embedding the system calls, reading/parsing of the result, etc., in Matlab functions, one can obtain wrapper functions that are treatable like regular Matlab functions, and thus, easily accessible to the spvals algorithm. In the following, we present Matlab pseudo-code for a possible approach.

```
function [varargout] = external_model(external_config, x1, ..., xd)

try
    store permutation (x1,...xd) to external_config.inputfile

    % Start external program, pass input file name to program, pass
    % output file name to program.
    system([external_config.program '□-i□' external_config.inputfile ...
           '□-o□' external_config.outputfile]);

    read result from external_config.outputfile into varargout
catch
    Do some error handling
end
```

In the presented case, the call to spvals would look like this:

```
external_config.program = 'myprog.exe';
external_config.inputfile = 'in.txt';
external_config.outputfile = 'out.txt';
options = spset('VariablePositions', [1 + 1:d], 'NumberOfOutputs', nout);
z = spvals(@external_model, d, range, options, external_config);
```



## 3 Functions – Alphabetical List

### **cmpgrids**

Compare the available sparse grid types.

#### **Syntax**

```
cmpgrids  
cmpgrids(N)  
cmpgrids(N,D)
```

#### **Description**

`cmpgrids` Compares the maximum-norm-based grid, the no-boundary-nodes grid, the Clenshaw-Curtis grid, the Chebyshev-Gauss-Lobatto grid, and the Gauss-Patterson grid in dimension  $D = 2$  and level  $N = 3$ .

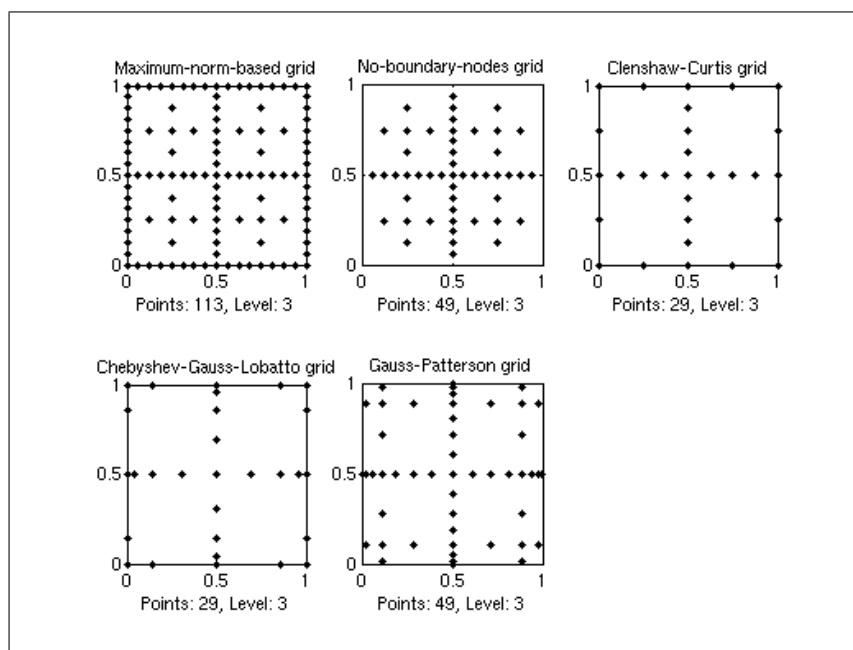
`cmpgrids(N)` Compares the grids for level  $N$ .

`cmpgrids(N,D)` Compares the grids in dimension  $D$ . Permitted are only the values  $D = 2$  or  $D = 3$ .

#### **Examples**

The following statement plots the four available sparse grids with level  $N = 3$  in two dimensions, producing the following graph.

```
cmpgrids(3,2);
```



**See Also** `plotgrid`, `plotindices`, `spgrid`.

---

## plotgrid

Plots a sparse grid.

### Syntax

```
plotgrid(N,D)
plotgrid(N,D,OPTIONS)
H = plotgrid(...)
```

### Description

`plotgrid(N,D)` Plots the sparse grid of level  $N$  and dimension  $D$ . By default, the Clenshaw-Curtis sparse grid type is selected.

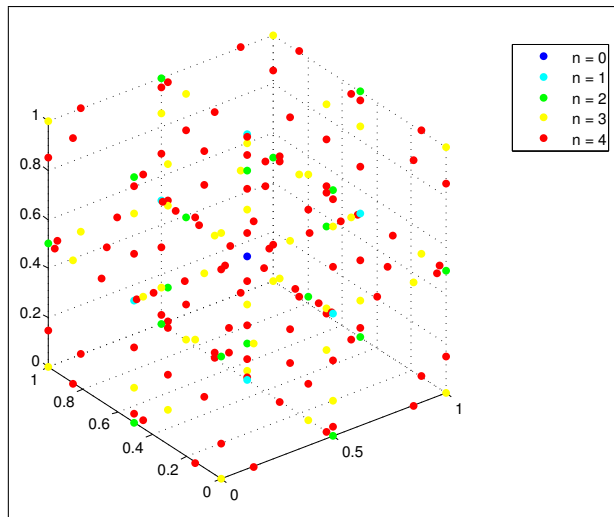
`plotgrid(N,D,OPTIONS)` Plots the sparse grid, but with the grid type as specified in `OPTIONS`. `OPTIONS` must be a structure created with the `spset` function. `H = PLOTGRID(...)` Returns a vector of handles to the grid points (useful for changing the look of the plotted grid).

---

## Examples

The following statements can be used to plot the Chebyshev-Gauss-Lobatto sparse grid of level  $N = 4$  in three dimensions, highlighting the grid points of the levels in different colors:

```
options = spset('GridType', 'Chebyshev');
n = 4;
h = plotgrid(n,3,options);
cols = brighten(jet(n+1),-1);
legendstr = cell(1,n+1);
for k = 0:n
    set(h(k+1), 'Color', cols(k+1,:), 'MarkerSize', 20);
    legendstr{k+1} = ['n_□=□' num2str(k)];
end
grid on;
legend(legendstr);
```



**See Also** `cmpgrids`, `plotindices`, `spgrid`.

---

## plotindices

Visualizes the index sets of a two-dimensional dimension-adaptive sparse grid.

### Syntax

`plotindices(Z)`

## Description

`plotindices(Z)` Plots the set of multi-indices  $S_k$  of a two-dimensional dimension-adaptive sparse grid interpolant  $A_{S_k}(f)$ . `Z` must be the sparse grid data as returned by `spvals`. `spvals` must be called with the option 'DimensionAdaptive' switched 'on' (this can be done using `spset`).

## Examples

The following code constructs a dimension-adaptive sparse grid interpolant of the function

$$f(x,y) = \sin(10x^2) + y^2$$

using greedy grid refinement (the degree of dimensional adaptivity is set to 1). The default interpolation box is `range = [0,1]^2`.

```
f = inline('sin(10.*x)+y.^2');
options = spset('DimensionAdaptive', 'on', 'DimAdaptDegree', 1);
z = spvals(f, 2, [], options)
```

`z =`

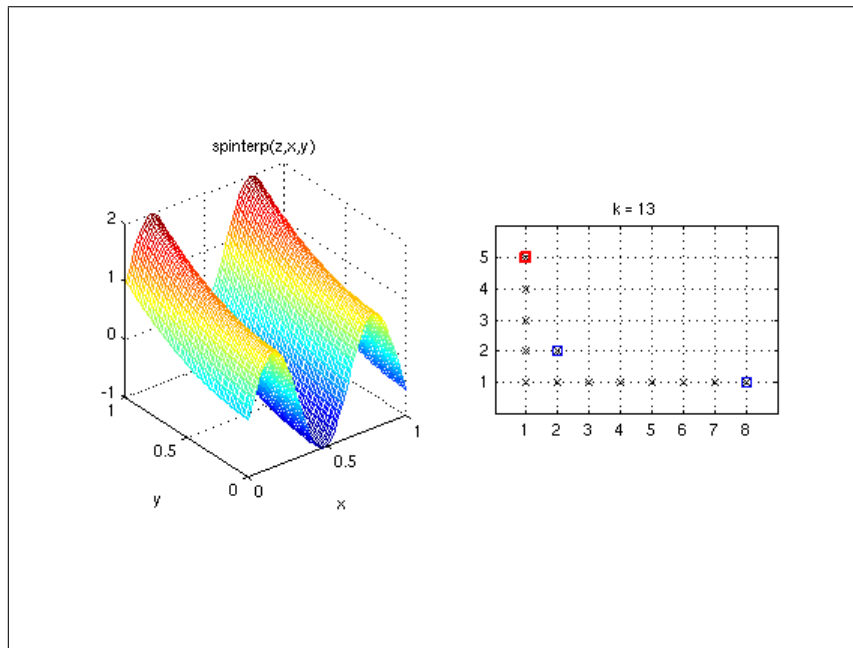
```
      vals: {[149x1 double]}
      gridType: 'Clenshaw-Curtis'
          d: 2
      range: []
  estRelError: 0.0018
  estAbsError: 0.0039
   fevalRange: [-0.9589 1.2500]
   minGridVal: [0.5000 0]
   maxGridVal: [0.1562 0.5000]
     nPoints: 149
     fevalTime: 0.3286
surplusCompTime: 0.0089
     indices: [1x1 struct]
     maxLevel: [7 4]
   activeIndices: [3x1 uint32]
  activeIndices2: [13x1 uint32]
             E: [1x13 double]
             G: [13x1 double]
             G2: [13x1 double]
   maxSetPoints: 7
     dimAdapt: 1
```

The resulting interpolant is plotted using Matlab's `ezmesh` command and an anonymous function containing the call to `spinterp`. Plotting the multi-index sets used by the interpolant reveals that the refinement is more dense in the `x`-direction, since more points are required to resolve the oscillation of the sine curve. Due to the greedy refinement, only a single index (2,2)

---

is computed in joint dimensions, since the error indicator of the multi-index (2,2) is equal to zero ( $f$  is a separable function).

```
subplot(1,2,1);  
ezmesh(@(x,y) spinterp(z,x,y), [0 1]);  
axis square;  
subplot(1,2,2);  
plotindices(z);
```



**See Also** `plotgrid`, `plotindices`, `spgrid`.

---

## spcgsearch

Optimizes the sparse grid interpolant using the CG method. Recommended for optimizing polynomial sparse grids (Chebyshev grid). It is discouraged to apply this method to piecewise linear sparse grids since they are not smooth enough for the algorithm to perform well (use `spcomsearch` instead for these grid types).

### Syntax

```
X = spcgsearch(Z)  
X = spcgsearch(Z,XBOX)  
X = spcgsearch(Z,XBOX,OPTIONS)
```

```
[X,FVAL] = spcgsearch(...)  
[X,FVAL,EXITFLAG] = spcgsearch(...)  
[X,FVAL,EXITFLAG,OUTPUT] = spcgsearch(...)
```

#### Description

`X = spcgsearch(Z)` Starts the search at the best available sparse grid point and attempts to find a local minimizer of the sparse grid interpolant `Z`. The entire range of the sparse grid interpolant is searched.

`X = spcgsearch(Z,XBOX)` Uses the search box `XBOX = [a1, b1; a2, b2; ...]`. The size of search box `XBOX` must be smaller than or equal to the range of the interpolant.

`X = spcgsearch(Z,XBOX,OPTIONS)` Minimizes with the default optimization parameters replaced by values in the structure `OPTIONS`, created with the `spsoptimset` function. See `spsoptimset` for details.

`[X,FVAL] = spcgsearch(...)` Returns the value of the sparse grid interpolant at `X`.

`[X,FVAL,EXITFLAG] = spcgsearch(...)` Returns an `EXITFLAG` that describes the exit condition of `spcgsearch`. Possible values of `EXITFLAG` and the corresponding exit conditions are

- 1 – `spcgsearch` converged to a solution `X`.
- 0 – Maximum number of function evaluations or iterations reached.

`[X,FVAL,EXITFLAG,OUTPUT] = spcgsearch(...)` Returns a structure `OUTPUT` with the number of function evaluations in `OUTPUT.nFEvals`, the number of gradients in `.nGradEvals`, and the computing time in `.time`.

#### Examples

Usually, the objective function will be expensive to evaluate. Here, we just consider the well-known the six-hump camel-back for function simplicity.

```
f = @(x,y) (4-2.1.*x.^2+x.^4./3).*x.^2+x.*y+(-4+4.*y.^2).*y.^2;
```

Before applying the `spcgsearch` algorithm, we need to create a sparse grid interpolant of the objective function. This is done as usual using the `spvals` algorithm.

In preparation to calling `spvals`, we first set up the interpolant construction with adequate parameters. A conjugate gradient (CG) line search algorithm uses derivatives to determine the search direction, it best to use the smooth Chebyshev grid in order to obtain an interpolant with accurate, smooth derivatives. Furthermore, it is useful to keep the function values as they can be used by the optimization algorithm to select good starting values for the optimization.

```
options = spset('keepFunctionValues','on', 'GridType', 'Chebyshev', ...  
              'DimensionAdaptive', 'on', 'DimAdaptDegree', 1, 'MinPoints', 10);
```

We construct the interpolant for the range that we are interested in optimizing the objective function for.



---

```
range = [-3 3; -2 2];
```

Now, we are ready to construct the sparse grid interpolant.

```
z = spvals(f, 2, range, options)
```

```
z =
    vals: {[37x1 double]}
    gridType: 'Chebyshev'
    d: 2
    range: [2x2 double]
    estRelError: 6.7208e-16
    estAbsError: 1.1013e-13
    fevalRange: [-0.9706 162.9000]
    minGridVal: [0.5000 0.6913]
    maxGridVal: [0 0]
    nPoints: 37
    fevalTime: 0.0690
    surplusCompTime: 0.3137
    indices: [1x1 struct]
    maxLevel: [4 3]
    activeIndices: [4x1 uint32]
    activeIndices2: [11x1 uint32]
    E: [Inf 108.9000 48 48.6000 10.7392 6 16.0000 7.1054e-15
1.1013e-13 7.1054e-15 1.4211e-14]
    G: [11x1 double]
    G2: [11x1 double]
    maxSetPoints: 4
    dimAdapt: 1
    fvals: {[37x1 double]}
```

Having obtained the interpolant, we can now search for the minimizer using `spcgsearch`. This is achieved by simply calling

```
[xopt, fval] = spcgsearch(z)
```

```
xopt =
    -0.0898
     0.7127
fval =
    -1.0316
```

There are multiple ways of configuring the search using an options structure defined with `sptoptimset`. For instance, you can display information at each iteration. Additional information on the optimization can be obtained by specifying optional left-hand parameters:

```
optoptions = sptoptimset('Display', 'iter');
[xopt, fval, exitflag, output] = spcgsearch(z, [], optoptions)
```

| Iteration | Func-count | Grad-count | f(x)      | Procedure   |
|-----------|------------|------------|-----------|-------------|
| 0         | 1          | 1          | -0.970563 | start point |
| 1         | 10         | 1          | -1.024    | line search |
| 2         | 17         | 2          | -1.03161  | line search |
| 3         | 24         | 3          | -1.03163  | line search |
| 4         | 29         | 4          | -1.03163  | line search |

xopt =  
-0.0898  
0.7127

fval =  
-1.0316

exitflag =  
1

output =  
nFEvals: 29  
nGradEvals: 4  
time: 0.2737

**See Also** spoptimset.

---

## spcompsearch

Optimizes the sparse grid interpolant using the compass (coordinate) search method. Best-suited for piecewise multilinear sparse grids.

### Syntax

```
X = spcompsearch(Z)
X = spcompsearch(Z,XBOX)
X = spcompsearch(Z,XBOX,OPTIONS)
[X,FVAL] = spcompsearch(...)
[X,FVAL,EXITFLAG] = spcompsearch(...)
[X,FVAL,EXITFLAG,OUTPUT] = spcompsearch(...)
```

### Description

`X = spcompsearch(Z)` Starts the search at the best available sparse grid point and attempts to find a local minimizer of the sparse grid interpolant `Z`. The entire range of the sparse grid interpolant is searched.

`X = spcompsearch(Z,XBOX)` Uses the search box `XBOX = [a1, b1; a2, b2; ...]`. The size of search box `XBOX` must be smaller than or equal to the range of the interpolant.

`X = spcompsearch(Z,XBOX,OPTIONS)` Minimizes with the default optimization parameters

---

replaced by values in the structure `OPTIONS`, created with the `spoptimset` function. See `spoptimset` for details.

`[X,FVAL] = spcompsearch(...)` Returns the value of the sparse grid interpolant at `X`.

`[X,FVAL,EXITFLAG] = spcompsearch(...)` Returns an `EXITFLAG` that describes the exit condition of `spcompsearch`. Possible values of `EXITFLAG` and the corresponding exit conditions are

- 1 – `spcompsearch` converged to a solution `X`.
- 0 – Maximum number of function evaluations or iterations reached.

`[X,FVAL,EXITFLAG,OUTPUT] = spcompsearch(...)` Returns a structure `OUTPUT` with the number of function evaluations in `OUTPUT.nFEvals` and the computing time in `.time`.

## Examples

A compass search algorithm is a direct search method that does not need derivatives, so it is well-suited to optimize a piecewise multilinear sparse grid interpolant computed for the grid types `Maximum`, `NoBoundary`, or `Clenshaw-Curtis`.

As with the example presented for `spcgsearch`, we consider the six-hump camel-back function.

```
f = @(x,y) (4-2.1.*x.^2+x.^4./3).*x.^2+x.*y+(-4+4.*y.^2).*y.^2;
```

We create the sparse grid interpolant using `spvals` as follows. Note that it is useful to keep the function values as they can be used by the optimization algorithm to select good starting values for the optimization without having to evaluate the interpolant.

```
options = spset('keepFunctionValues','on', 'GridType', 'Clenshaw-Curtis', ...  
    'DimensionAdaptive', 'on', 'DimAdaptDegree', 1, 'MinPoints', 10);
```

The next steps are setting the interpolation range (the optimization range will be the same by default), constructing the interpolant, providing additional (optional) optimization parameters, and finally, the call to the `spcompsearch` algorithm.

```
range = [-3 3; -2 2];  
z = spvals(f, 2, range, options)  
optoptions = spoptimset('Display', 'iter');  
[xopt, fval] = spcompsearch(z, [], optoptions)
```

```
z =  
    vals: {[205x1 double]}  
    gridType: 'Clenshaw-Curtis'  
    d: 2  
    range: [2x2 double]  
    estRelError: 0.0037  
    estAbsError: 0.6031  
    fevalRange: [-0.9970 162.9000]
```

```

    minGridVal: [0.5000 0.3281]
    maxGridVal: [0 0]
    nPoints: 205
    fevalTime: 0.1031
    surplusCompTime: 0.0077
    indices: [1x1 struct]
    maxLevel: [7 6]
    activeIndices: [4x1 uint32]
    activeIndices2: [17x1 uint32]
    E: [Inf 108.9000 48 52.2844 45.8547 6 24 12.7500 22.3788
4.3594 7.6817 0 0 1.2568 2.2379 0.3364 0.6031]
    G: [17x1 double]
    G2: [17x1 double]
    maxSetPoints: 7
    dimAdapt: 1
    fvals: {[205x1 double]}
Iteration   Func-count  Grad-count    f(x)           Procedure
    0         0         0      -0.997009     start point
    1         4         0      -0.997009     contract step
    2         8         0      -0.997009     contract step
    3        12         0      -0.997009     contract step
    4        16         0      -1.02647     coordinate step
    5        20         0      -1.02647     contract step
    6        24         0      -1.02647     contract step
xopt =
    0.0938
   -0.6875
fval =
   -1.0265

```

**See Also** spoptimset.

---

## spdim

Computes the number of sparse grid points.

### Syntax

```

P = spdim(N,D)
P = spdim(N,D,OPTIONS)

```

---

## Description

`P = spdim(N,D)` Computes the number of points of the sparse grid of dimension  $D$  and level  $N$ .

`P = spdim(N,D,OPTIONS)` Computes the number of points as above, but with the default grid type replaced by the grid type specified in `OPTIONS`, an argument created with `spset`. See `spset` for details.

## Examples

Compute the number of support nodes of the 10-dimensional sparse grid of level 7 for the Clenshaw-Curtis (default) grid with the following command:

```
spdim(7,10)
```

```
ans =  
    652065
```

For comparison, compute the number of nodes of the maximum-norm-based sparse grid:

```
options = spset('GridType','Maximum');  
spdim(7,10,options)
```

```
ans =  
    1.8317e+09
```

---

## spfminsearch

Optimizes the sparse grid interpolant using MATLAB's `fminsearch` method.

### Syntax

```
X = spfminsearch(Z)  
X = spfminsearch(Z,XBOX)  
X = spfminsearch(Z,XBOX,OPTIONS)  
[X,FVAL] = spfminsearch(...)  
[X,FVAL,EXITFLAG] = spfminsearch(...)  
[X,FVAL,EXITFLAG,OUTPUT] = spfminsearch(...)
```

### Description

`X = spfminsearch(Z)` Starts the search at the best available sparse grid point and attempts to find a local minimizer of the sparse grid interpolant  $Z$ . The entire range of the sparse grid interpolant is searched.

`X = spfminsearch(Z,XBOX)` Uses the search box `XBOX = [a1, b1; a2, b2; ...]`. The size of search box `XBOX` must be smaller than or equal to the range of the interpolant.

`X = spfminsearch(Z,XBOX,OPTIONS)` Minimizes with the default optimization parameters replaced by values in the structure `OPTIONS`, created with the `spoptimset` function. See `spoptimset` for details.

`[X,FVAL] = spfminsearch(...)` Returns the value of the sparse grid interpolant at `X`.

`[X,FVAL,EXITFLAG] = spfminsearch(...)` Returns an `EXITFLAG` that describes the exit condition of `spfminsearch`. Possible values of `EXITFLAG` and the corresponding exit conditions are

- 1 – `spfminsearch` converged to a solution `X`.
- 0 – Maximum number of function evaluations or iterations reached.

`[X,FVAL,EXITFLAG,OUTPUT] = spfminsearch(...)` Returns a structure `OUTPUT` with the number of function evaluations in `OUTPUT.nFEvals` and the computing time in `.time`. The `OUTPUT` result from the `fminsearch` call is returned as `OUTPUT.fminsearchOutput`.

## Examples

`spfminsearch` internally calls MATLAB's `fminsearch` function to perform the search. The sparse grid interpolant is modified by a penalty function such that the search is restricted to the provided search box.

`spfminsearch` is a derivative-free method that is suitable for all sparse grid types. However, it is usually outperformed by `spcompsearch` for the grid types `Maximum`, `NoBoundary`, or `Clenshaw-Curtis`, and by `spcgsearch` for the grid type `Chebyshev`.

As with the example presented for `spcgsearch`, we consider the six-hump camel-back function (see that example for further details).

```
f = @(x,y) (4-2.1.*x.^2+x.^4./3).*x.^2+x.*y+(-4+4.*y.^2).*y.^2;
```

Interpolant creation and setting optional parameters:

```
options = spset('keepFunctionValues','on', 'GridType', 'Chebyshev', ...
    'DimensionAdaptive', 'on', 'DimAdaptDegree', 1, 'MinPoints', 10);
range = [-3 3; -2 2];
z = spvals(f, 2, range, options);
optoptions = spoptimset('Display', 'iter');
```

Performing the optimization:

```
[xopt, fval] = spfminsearch(z, [], optoptions)
```

| Iteration | Func-count | min f(x)  | Procedure       |
|-----------|------------|-----------|-----------------|
| 0         | 1          | -0.970563 |                 |
| 1         | 3          | -0.970563 | initial simplex |
| 2         | 5          | -0.997137 | expand          |
| 3         | 7          | -0.99731  | reflect         |

---

|    |    |           |                 |
|----|----|-----------|-----------------|
| 4  | 9  | -0.99731  | contract inside |
| 5  | 11 | -0.999861 | contract inside |
| 6  | 13 | -1.00004  | reflect         |
| 7  | 15 | -1.00004  | contract inside |
| 8  | 17 | -1.00004  | contract inside |
| 9  | 19 | -1.00004  | contract inside |
| 10 | 21 | -1.0002   | expand          |
| 11 | 23 | -1.00055  | expand          |
| 12 | 25 | -1.00087  | expand          |
| 13 | 27 | -1.00192  | expand          |
| 14 | 29 | -1.00227  | expand          |
| 15 | 31 | -1.00483  | expand          |
| 16 | 32 | -1.00483  | reflect         |
| 17 | 34 | -1.00771  | expand          |
| 18 | 36 | -1.01172  | expand          |
| 19 | 38 | -1.01615  | expand          |
| 20 | 40 | -1.02567  | expand          |
| 21 | 41 | -1.02567  | reflect         |
| 22 | 43 | -1.03063  | reflect         |
| 23 | 44 | -1.03063  | reflect         |
| 24 | 46 | -1.03083  | reflect         |
| 25 | 48 | -1.03119  | contract inside |
| 26 | 50 | -1.03155  | contract inside |
| 27 | 52 | -1.03155  | contract inside |
| 28 | 54 | -1.03155  | contract inside |
| 29 | 56 | -1.03162  | contract inside |
| 30 | 58 | -1.03162  | contract inside |
| 31 | 60 | -1.03162  | contract inside |
| 32 | 62 | -1.03162  | reflect         |
| 33 | 64 | -1.03163  | contract inside |
| 34 | 66 | -1.03163  | contract inside |
| 35 | 68 | -1.03163  | contract inside |
| 36 | 70 | -1.03163  | contract inside |
| 37 | 72 | -1.03163  | contract inside |
| 38 | 74 | -1.03163  | contract inside |
| 39 | 76 | -1.03163  | contract inside |
| 40 | 78 | -1.03163  | contract inside |
| 41 | 80 | -1.03163  | contract inside |
| 42 | 82 | -1.03163  | reflect         |
| 43 | 84 | -1.03163  | contract inside |

Optimization terminated:

the current x satisfies the termination criteria using OPTIONS.TolX of  
1.000000e-04  
and F(X) satisfies the convergence criteria using OPTIONS.TolFun of  
1.000000e-04

```
xopt =  
    -0.0899  
    0.7127  
fval =  
    -1.0316
```

**See Also** spoptimset.

---

## spget

Get sparse grid interpolation OPTIONS parameters.

### Syntax

```
VAL = spget(OPTIONS, 'NAME')  
VAL = spget(OPTIONS, 'NAME', DEFAULT)
```

### Description

VAL = spget(OPTIONS, 'NAME') Extracts the value of the property NAME from the sparse grid options structure OPTIONS, returning an empty matrix if the property value is not specified in OPTIONS. It is sufficient to type only the leading characters that uniquely identify the property. Case is ignored for property names. [] is a valid OPTIONS argument.

VAL = spget(OPTIONS, 'NAME', DEFAULT) Extracts the named property as above, but returns VAL = DEFAULT if the named property is not specified in OPTIONS.

### Examples

Assume that an options structure has been created using the spset command:

```
options = spset('GridType', 'Maximum', 'MaxDepth', 4)
```

```
options =  
    GridType: 'Maximum'  
    RelTol: []  
    AbsTol: []  
    Vectorized: []  
    MinDepth: []  
    MaxDepth: 4  
    VariablePositions: []  
    NumberOfOutputs: []  
    PrevResults: []
```



---

```
FunctionArgType: []
KeepFunctionValues: []
    KeepGrid: []
DimensionAdaptive: []
    MinPoints: []
    MaxPoints: []
DimadaptDegree: []
    SparseIndices: []
```

Using `spget`, we can extract the contents of the structure:

```
gridType = spget(options, 'GridType')
```

```
gridType =
    Maximum
```

By using the third argument, we can set a default in case the according property of the structure is empty:

```
minDepth = spget(options, 'MinDepth')
minDepth = spget(options, 'MinDepth', 2)
```

```
minDepth =
    []
```

```
minDepth =
    2
```

Note that the default argument has no effect if the accessed property contains a value.

```
gridType = spget(options, 'GridType', 'Clenshaw-Curtis')
```

```
gridType =
    Maximum
```

**See Also** `spset`.

---

## spgrid

Compute the sparse grid point coordinates.

### Syntax

```
X = spgrid(N,D)
X = spgrid(N,D,OPTIONS)
```

#### Description

$X = \text{spgrid}(N, D)$  Computes the sparse grid points of level  $N$  and problem dimension  $D$ . The coordinate value of dimension  $i = 1 \dots d$  is stored in column  $i$  of the matrix  $X$ . One row of the matrix  $X$  represents one grid point.

$X = \text{spgrid}(N, D, \text{OPTIONS})$  computes the sparse grid points as above, but with default grid type replaced by the grid type specified in `OPTIONS`, an argument created with the `spset` function.

**Remark** Note that `spgrid` only computes the grid points that are added to the interpolant at level  $N$ .

#### Examples

Compute the grid points of the Clenshaw-Curtis (default) grid for the first 3 levels ( $n = 0 \dots 2$ ), dimension  $d = 2$ , and display them:

```
d = 2;
for n = 0:2
    x = spgrid(n,d)
end
```

```
x =
    0.5000    0.5000
```

```
x =
     0    0.5000
    1.0000    0.5000
    0.5000     0
    0.5000    1.0000
```

```
x =
    0.2500    0.5000
    0.7500    0.5000
     0         0
    1.0000     0
     0         1.0000
    1.0000    1.0000
    0.5000    0.2500
    0.5000    0.7500
```

**See Also** `cmpgrids`, `plotgrid`, `spset`.

---

---

## spinit

Initialize the Sparse Grid Interpolation toolbox.

### Syntax

```
spinit
```

### Description

`spinit` Simply adds the sparse grid toolbox directories to the Matlab path. This ensures that the sparse grid routines are found when working in other directories.

**Remark** To run the sparse grid interpolation demos from the help browser, `spinit` must be started first.

---

## spinterp

Evaluation of the sparse grid interpolant.

### Syntax

```
IP = spinterp(Z, Y1, ..., YD)  
[IP, IPGRAD] = spinterp(Z, Y1, ..., YD)
```

### Description

`IP = spinterp(Z, Y1, ..., YD)` Computes the interpolated values `IP` at the point(s)  $(Y_1, \dots, Y_D)$  over the sparse grid. The input parameters  $Y_i$  may be double arrays of equal size for vectorized processing. The sparse grid data must be given as a structure `Z` containing the hierarchical surpluses (computed with `spvals`).

`[IP, IPGRAD] = spinterp(Z, Y1, ..., YD)` Computes interpolated values `IP` and derivatives `IPGRAD` at the specified points. The derivatives are returned as  $D \times 1$  gradient vectors inside of a cell array that has equal size as the double array `IP`. See Section 2.2 for additional information.

### Examples

Assume a sparse grid interpolant of the Matlab peaks function has been computed for the domain  $[0, 2]^2$  using the following command:

```
z = spvals(@(x,y) peaks(x,y), 2, [0,2; 0,2]);
```

Then, we can evaluate `z` at a single point, e.g. the point  $(0.5, 0.5)$ , simply like this:

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---

```
ip = spinterp(z, 0.5, 0.5);
```

```
ip =  
    0.3754
```

If multiple evaluations of the interpolant are required, it is best to use a vectorized call to `spinterp` for fast processing. For example, to evaluate the interpolant at the full grid  $[0, 2] \times [0, 2]$  at  $50 \times 50$  points (equidistant spacing), we can proceed as follows:

```
x = linspace(0,2,50); y = linspace(0,2,50);  
[xmat,yamat] = meshgrid(x,y);  
tic; ip = spinterp(z, xmat, ymat); toc
```

Elapsed time is 0.202514 seconds.

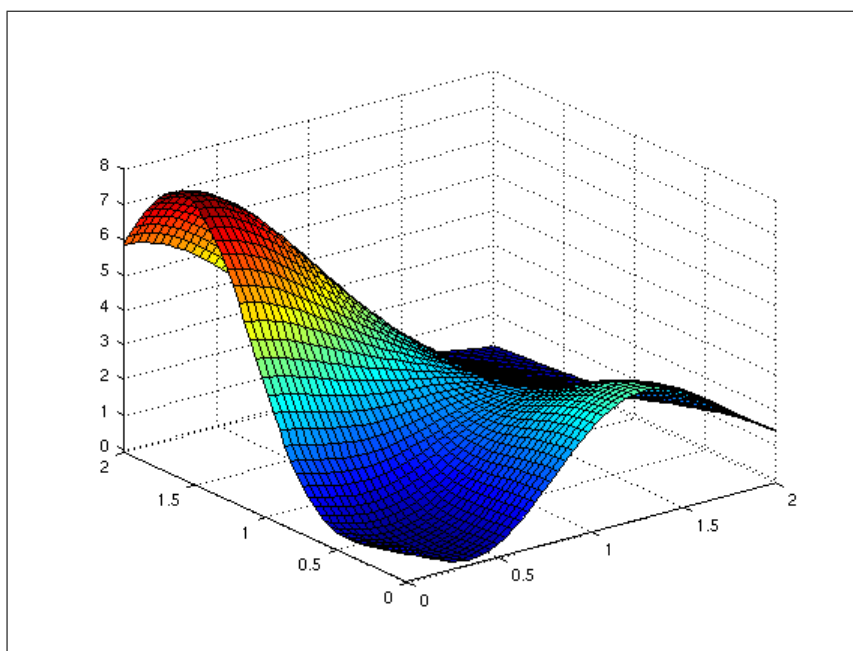
Note that the output size of `ip` matches the size of the input matrices:

```
size(xmat)  
size(ip)
```

```
ans =  
    50    50  
ans =  
    50    50
```

We could visualize the result using the `surf` command:

```
surf(xmat, ymat, ip);  
axis tight;
```



---

**See Also** `spvals`.

---

## **spmultistart**

Attempts to find a global optimizer of the sparse grid interpolant by performing multiple local searches starting from random start points.

### **Syntax**

```
X = spmultistart(Z)
X = spmultistart(Z,XBOX)
X = spmultistart(Z,XBOX,OPTIONS)
[X,FVAL] = spmultistart(...)
[X,FVAL,EXITFLAG] = spmultistart(...)
[X,FVAL,EXITFLAG,OUTPUT] = spmultistart(...)
```

### **Description**

`X = spmultistart(Z)` Attempts to find a global optimizer `X` of the sparse grid interpolant `Z` by performing multiple local searches starting from random start points. The entire range of the sparse grid interpolant is searched. Using the default settings, the first start point is not random but the best available sparse grid point. By default, `spcompsearch` is used for the local searches if the grid type is not of type Chebyshev. If it is, `spcgsearch` is used.

`X = spmultistart(Z,XBOX)` Uses the search box `XBOX = [a1, b1; a2, b2; ...]`. The size of search box `XBOX` must be smaller than or equal to the range of the interpolant.

`X = spmultistart(Z,XBOX,OPTIONS)` Minimizes with the default optimization parameters replaced by values in the structure `OPTIONS`, created with the `spoptimset` function. For instance, the local optimization method can be selected. See `spoptimset` for details.

`[X,FVAL] = spmultistart(...)` Returns the value of the sparse grid interpolant at `X`.

`[X,FVAL,EXITFLAG] = spmultistart(...)` Returns an `EXITFLAG` that describes the exit condition of `spmultistart`. Possible values of `EXITFLAG` and the corresponding exit conditions are

- 1 – `spmultistart` converged to at least one solution `X`.
- 0 – Maximum number of function evaluations or iterations reached for all local searches.

`[X,FVAL,EXITFLAG,OUTPUT] = spmultistart(...)` Returns a structure `OUTPUT` with the total computing time in `.time`, and a cell array of all local search results `.allResults`.

## Examples

The following example demonstrates the usage of `spsmultistart` with its default options. Many parameters can be modified using `spoptimset`. Here, we optimize Branin's function, a function with three global optimizers.

```
f = inline(['(5/pi*x-5.1/(4*pi^2)*x.^2+y-6).^2_+ ...  
          '10*(1-1/(8*pi))*cos(x)+10']);
```

Interpolant creation:

```
range = [-5 10; 0 15];  
options = spset('keepFunctionValues','on', 'GridType', 'Chebyshev', ...  
              'DimensionAdaptive','on', 'DimAdaptDegree', 1, ...  
              'MinPoints', 10);  
z = spvals(f, 2, range, options);
```

Performing the optimization:

```
[xopt, fval, exitflag, output] = spmultistart(z)
```

```
xopt =  
    -3.1416  
    12.2751  
fval =  
    0.3978  
exitflag =  
     1  
output =  
    time: 4.0003  
    allResults: [1x1 struct]
```

All local optimization results are available from the output structure:

```
output.allResults.x  
output.allResults.fval  
  
ans =  
    3.1416    9.4248    3.1416    9.4248    3.1416    9.4248   -3.1416  
    9.4248   -3.1416    9.4248  
    2.2750    2.4750    2.2750    2.4750    2.2750    2.4750   12.2751  
    2.4750   12.2751    2.4750  
ans =  
    0.3980    0.3979    0.3980    0.3979    0.3980    0.3979    0.3978  
    0.3979    0.3978    0.3979
```

**See Also** `spoptimset`.

---

---

## spoptimget

Get sparse grid optimization OPTIONS parameters.

### Syntax

```
VAL = spoptimget(OPTIONS, 'NAME')
VAL = spoptimget(OPTIONS, 'NAME', DEFAULT)
```

### Description

VAL = spoptimget(OPTIONS, 'NAME') Extracts the value of the property NAME from the sparse grid optimization options structure OPTIONS, returning an empty matrix if the property value is not specified in OPTIONS. It is sufficient to type only the leading characters that uniquely identify the property. Case is ignored for property names. [] is a valid OPTIONS argument.

VAL = spoptimget(OPTIONS, 'NAME', DEFAULT) Extracts the named property as above, but returns VAL = DEFAULT if the named property is not specified in OPTIONS.

### Examples

Assume that an options structure has been created using the spoptimset command:

```
options = spoptimset('TolFun', 1e-4, 'MaxIter', 200)
```

```
options =
    Minimize: []
    Maximize: []
    TolFun: 1.0000e-04
    TolX: []
    MaxIter: 200
    StartPoint: []
    TestCorners: []
    PrevResult: []
    Method: []
    NumStarts: []
    OptimsetOptions: []
    Display: []
```

Using spoptimget, we can extract the contents of the structure:

```
tolFun = spoptimget(options, 'TolFun')
```

```
tolFun =
    1.0000e-04
```

By using the third argument, we can set a default in case the according property of the structure is empty:

```
startPoint = spoptimget(options, 'StartPoint')
startPoint = spoptimget(options, 'StartPoint', 'best')
```

```
startPoint =
    []
startPoint =
    best
```

Note that the default argument has no effect if the accessed property contains a value.

```
maxIter = spoptimget(options, 'MaxIter', 500)
```

```
maxIter =
    200
```

**See Also** `spoptimset`.

---

## spoptimset

Create/alter a sparse grid optimization `OPTIONS` structure.

### Syntax

```
OPTIONS = spoptimset('NAME1',VALUE1,'NAME2',VALUE2,...)
OPTIONS = spoptimset(OLDOPTS,'NAME1',VALUE1,...)
OPTIONS = spoptimset(OLDOPTS,NEWOPTS)
```

### Description

`spoptimset` with no input arguments displays all property names and their possible values.

`OPTIONS = spoptimset('NAME1',VALUE1,'NAME2',VALUE2,...)` creates options structure `OPTIONS` in which the named properties have the specified values. Any unspecified properties have default values. It is sufficient to type only the leading characters that uniquely identify the property. Case is ignored for property names.

`OPTIONS = spoptimset(OLDOPTS,'NAME1',VALUE1,...)` alters an existing options structure `OLDOPTS`.

`OPTIONS = spoptimset(OLDOPTS,NEWOPTS)` combines an existing options structure `OLDOPTS` with a new options structure `NEWOPTS`. Any new properties overwrite corresponding old properties.

### Properties

The properties configurable with `spoptimset` are listed in Table 3.1.



Table 3.1: Properties configurable with `spoptimset`.

| <i>Property</i>                               | <i>Value {default}</i>  | <i>Description</i>  |
|---|---|---|
| <code>Minimize</code>                         | <code>{on}   off</code>   | If set to on, the optimization algorithm will search for a minimizer.   |
| <code>Maximize</code>                         | <code>on   {off}</code>   | If set to on, the optimization algorithm will search for a maximizer (searching for the minimizer and the maximizer at the same time is allowed). Note that searching for a maximizer is currently not supported by <code>spcgsearch</code> .   |
| <code>TolX</code>                             | positive scalar<br><code>{1e-4}</code>  | Termination tolerance on $X$ . Note that the tolerance on $X$ is taken with respect to the problem being re-scaled to the unit interval in each coordinate direction. That is, for instance, a sparse grid interpolant defined for the box $[0, 1e6] \times [0, 1e-6]$ with <code>TolX</code> = 0.1 would mean a break tolerance of $1e5$ in $x_1$ and a tolerance of $1e-7$ in $x_2$ -direction. This parameter does not apply to <code>spcgsearch</code> .  |
| <code>TolFun</code>                           | positive scalar<br><code>{1e-6}</code>  | The search is terminated when the change of the function value from one iteration to the next is smaller than <code>TolFun</code> .   |
| <code>MaxIter</code>                          | integer <code>{100}</code>  | Maximum number of allowed iterations.   |
| <code>StartPoint</code>                       | <code>{best}   random</code><br>  $D \times 1$ vector                                   | Start search from best available, random, or specified start point.   |
| <code>TestCorners</code>                      | <code>on   {off}</code>   | Specifically includes the $2^D$ corner points of the search box as potential start points of the search.  |
| <code>PrevResult</code>                       | $(d+1) \times \{1 2\}$<br>double array  | Specifies a possible best start point, such as from a previous search over a subdomain of the current search box. Format: <code>[xoptmin; ymin xoptmax; ymax]</code> , where <code>xoptmin</code> and <code>xoptmax</code> are column vectors. Depending on the contents of the <code>Minimize</code> and <code>Maximize</code> fields, minima and/or maxima information should be provided. <code>PrevResult</code> is only considered as a start point if <code>StartPoint</code> is set to <code>best</code> . |
| <code>Method</code>                           | <code>{spcgsearch}  </code><br><code>{spcompsearch}</code><br>  <code>spfmsearch</code> | Specifies the method used by the multiple random start search <code>spmultistart</code> . <code>spcgsearch</code> is the default for the Chebyshev grid, otherwise, it is <code>spcompsearch</code> .   |
| <code>NumStarts</code>                        | integer <code>{10}</code>   | Number of local searches to perform for the multiple random start method <code>spmultistart</code> . The following points are considered: (best) + (NumStarts-1 random points).   |
| <code>Optimset</code><br><code>Options</code> | struct <code>{[]}</code>  | This feature is useful if additional configuration of the <code>fminsearch</code> algorithm used by <code>spfmsearch</code> is required beyond the parameters available through <code>spoptimset</code> . Example: <code>options = spoptimset('Optimset', optimset('FunValCheck', 'on'));</code>  |
| <code>Display</code>                          | <code>{off}   iter</code>   | Optionally, displays information at each iteration.   |

## Examples

As a preliminary to the following example, we construct a sparse grid interpolant of a test function (Branin's function) as follows.

```
f = inline(['(5/pi*x-5.1/(4*pi^2)*x.^2+y-6).^2+ ...  
          '10*(1-1/(8*pi))*cos(x)+10']);  
range = [-5 10; 0 15];  
options = spset('keepFunctionValues','on', 'GridType', 'Chebyshev', ...  
              'DimensionAdaptive', 'on', 'DimAdaptDegree', 1, 'MinPoints', 10, ...  
              'RelTol', 1e-6);  
z = spvals(f, 2, range, options);
```

A typical case of a modification of the sparse grid optimization options structure is given by the need to specify a more stringent error tolerance on the function value to be used by the `spcgsearch` algorithm.

```
optoptions = spoptimset('TolFun', 1e-10);  
format long e;  
[xopt, fval] = spcgsearch(z, [], optoptions)  
format short e;
```

```
xopt =  
    3.141592655097273e+00  
    2.274999997132531e+00  
fval =  
    3.978873577297303e-01
```

**See Also** `spoptimget`.

---

## sppurge

Purge sparse grid data.

### Syntax

```
Z = sppurge(Z)  
Z = sppurge(Z, OPTIONS)
```

### Description

`Z = sppurge(Z)` Marks indices that have corresponding hierarchical surplus values larger than the default drop tolerance `[0, 100*eps]`. The `sppurge` function returns the same sparse grid interpolant data `Z`, but enhanced by a field `purgeData` that is used by `spinterp` to only consider the marked indices in the interpolation process, thus saving computing time.

---

`Z = sppurge(Z,OPTIONS)` The parameter `OPTIONS` must be an options structure generated with `spset`. Only the value of the `DropTol` property is used, which enables the user to set any absolute and relative drop tolerance to be used by the purging algorithm.

## Examples

We consider the quadratic test function

$$f(x) = \left[ \sum_{i=1}^d (x_i - 1)^2 \right] - \left[ \sum_{i=2}^d x_i x_{i-1} \right],$$

implemented in Matlab by the following code:

```
function y = trid(x)
% TRID Quadratic function with a tridiagonal Hessian.
% Y = TRID(X) returns the function value Y for a D-
% dimensional input vector X.
%
% The test function is due to Arnold Neumaier, listed
% on the global optimization Web page at
% http://www.mat.univie.ac.at/~neum/glopt/

d = length(x);
y = sum((x-1).^2) - sum(x(2:d).*x(1:d-1));
```

During the construction of the interpolant, many sub-grids are encountered that do not contribute to the interpolant, i.e., they have hierarchical surpluses that are all zero (up to floating point accuracy). An adaptive algorithm cannot know these non-contributing sub-grids in advance. However, using the `DropTol` feature, we can tell the interpolation function `spinterp` to neglect the sub-grids that do not contribute, and thus, save a significant amount of computing time. We consider the high-dimensional case  $d = 100$ . With the dimension-adaptive algorithm, the problem structure is automatically detected, and the function is successfully recovered using just  $\mathcal{O}(d^2)$  function evaluations. For the interpolation domain, we use  $[-d^2, d^2]$  in each dimension.

```
d = 100;
range = repmat([-d^2 d^2],d,1);
options = spset('DimensionAdaptive', 'on', ...
               'DimadaptDegree', 1, ...
               'FunctionArgType', 'vector', ...
               'RelTol', 1e-3, ...
               'MaxPoints', 40000);
z = spvals('trid',d,range,options);
```

We now evaluate the obtained interpolant, first without, and thereafter, with the `DropTol` feature set to the default value of  $[0, 100 * \text{eps}]$  (absolute drop tolerance is zero, relative drop tolerance is  $100 * \text{eps}$ ). We evaluate the interpolant at 100 random points, measure the time, the absolute error, and compare the timing results in a plot.

```
% Compute 100 randomly sampled points
p = 100;
rand('state', 0);
x = -d^2 + 2*d^2*rand(p,d);

% Compute exact function values
y = zeros(p,1);
for k = 1:p
    y(k) = trid(x(k,:));
end

xcell = num2cell(x,1);
tic;
% Compute interpolated function values, no dropped indices
ip1 = spinterp(z, xcell{:});
t1 = toc

% Perform purging of interpolant data
tic;
z = sppurge(z);
t2 = toc
tic;
% Compute interpolated function values
% Some indices dropped according to drop tolerance
ip2 = spinterp(z, xcell{:});
t3 = toc

% Compute relative errors
err_ndt = max(abs(y-ip1))/(z.fevalRange(2)-z.fevalRange(1))
err_wdt = max(abs(y-ip2))/(z.fevalRange(2)-z.fevalRange(1))
```

```
t1 =
    2.1393
```

```
t2 =
    0.0128
```

```
t3 =
    0.2933
```

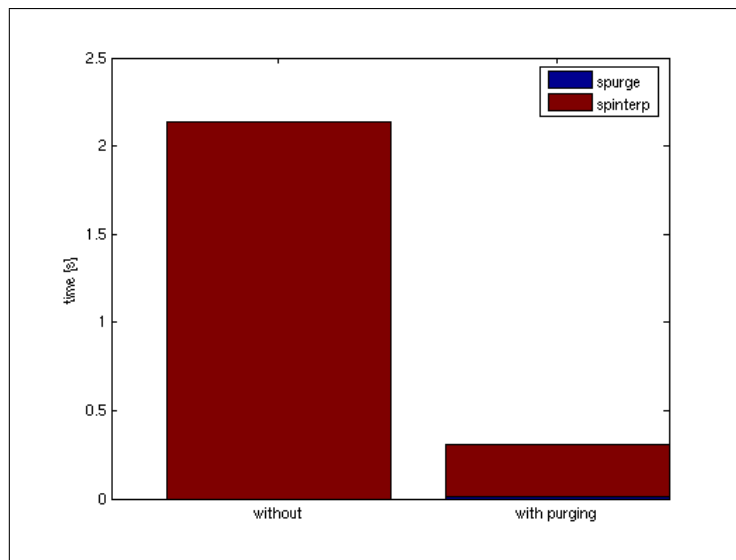
```
err_ndt =
    0.0061
```

```
err_wdt =
    0.0061
```

---

The result is quite impressive: Without losing accuracy (which is no surprise considering the very low drop tolerance of  $100 * \text{eps}$  compared to the relative error tolerance  $1e - 3$ ), for the 100 sampled points, a speedup by a factor of about 7 is achieved (including the cost of the `sppurge` function).

```
bar([NaN t1; t2 t3], 'stacked');  
legend('sppurge', 'spinterp');  
set(gca, 'XTickLabel', {'without', 'with_purging'});  
ylabel('time [s]');
```



**See Also** `spset`.

---

## spquad

Compute integral value of sparse grid interpolant.

### Syntax

```
Q = spquad(Z)
```

### Description

`Q = spquad(Z)` Computes the integral over the sparse grid domain. The sparse grid data must be given as a structure `Z` containing the hierarchical surpluses (computed with `spvals`).

The following additional option is available with `spquad` that is set by adding a field to the structure `Z`:

- `selectOutput [ integer {1} ]` Set the output variable number if an interpolant with multiple output variables was constructed with `spvals`. This determines which output variable the integral is computed for.

## Examples

Assume a sparse grid interpolant of the Matlab peaks function has been computed for the domain  $[0,2]^2$  using the following commands:

```
options = spset('DimensionAdaptive', 'on', 'RelTol', 1e-4, ...  
    'GridType', 'Chebyshev');  
z = spvals(@(x,y) peaks(x,y), 2, [0,2; 0,2], options);
```

Then, we can compute the integral for this domain simply like this:

```
spquad(z)
```

```
ans =  
    9.9553
```

For comparison, let us compute the integral value with Matlab's `dblquad`:

```
dblquad(@(x,y) peaks(x,y), 0, 2, 0, 2)
```

```
ans =  
    9.9553
```

**See Also** `spvals`.

---

## spset

Create/alter a sparse grid interpolation `OPTIONS` structure.

### Syntax

```
OPTIONS = spset('NAME1', VALUE1, 'NAME2', VALUE2, ...)  
OPTIONS = spset(OLDOPTS, 'NAME1', VALUE1, ...)  
OPTIONS = spset(OLDOPTS, NEWOPTS)
```

### Description

`spset` with no input arguments displays all property names and their possible values.

`OPTIONS = spset('NAME1', VALUE1, 'NAME2', VALUE2, ...)` creates an options structure `OPTIONS` in which the named properties have the specified values. Any unspecified properties have default values. It is sufficient to type only the leading characters that uniquely identify the

---

property. Case is ignored for property names.

`OPTIONS = spset(OLDOPTS, 'NAME1', VALUE1, ...)` alters an existing structure `OLDOPTS`.

`OPTIONS = spset(OLDOPTS, NEWOPTS)` combines an existing options structure `OLDOPTS` with a new options structure `NEWOPTS`. Any new properties overwrite corresponding old properties.

The properties configurable with `spset` are listed in Table 3.2 and Table 3.3.

## Examples

Since `spset` offers many possibilities to alter the behavior of the sparse grid interpolant construction, we provide several typical examples in the following.

### Example 1: Basic usage of `spset`

As an example for a typical task requiring the modification of the sparse grid options structure, we construct an interpolant with a specified number of function evaluations. The dimension-adaptive approach permits to do this in an elegant manner. The following code constructs an interpolant with about 100 nodes, since both `MinPoints` as well as `MaxPoints` are set to 100.

```
f = @(x,y) exp(x+y);
z = spvals(f, 2, [], spset('DimensionAdaptive', 'on', ...
                          'MinPoints', 100, 'MaxPoints', 100))
```

```
z =
    vals: {[129x1 double]}
  gridType: 'Clenshaw-Curtis'
        d: 2
    range: []
  estRelError: 8.3520e-04
  estAbsError: 0.0053
  fevalRange: [1 7.3891]
  minGridVal: [0 0]
  maxGridVal: [1 1]
    nPoints: 129
    fevalTime: 0.0546
  surplusCompTime: 0.0058
    indices: [1x1 struct]
    maxLevel: [5 5]
  activeIndices: [5x1 uint32]
  activeIndices2: [13x1 uint32]
        E: [1x19 double]
        G: [19x1 double]
        G2: [19x1 double]
  maxSetPoints: 5
    dimAdapt: 1
```

Table 3.2: Properties configurable with `spset` (part I).

| <i>Property</i>                  | <i>Value {default}</i>   | <i>Description</i>  |
|----------------------------------|--|---|
| <code>GridType</code>            | {Clenshaw-Curtis}   Maximum   NoBoundary   Chebyshev   Gauss-Patterson | Sparse grid type and basis functions to use by <code>spvals</code> . For an illustration of the grid types, run <code>cmpgrids</code> .   |
| <code>RelTol</code>              | positive scalar {1e-2}   | A relative error tolerance that applies to all hierarchical surpluses $w^n$ of the current deepest level $n$ of the sparse grid interpolation formula. The grid is further refined until all hierarchical surpluses are less than $\max(\text{RelTol} \cdot (\max(\text{fevalRange}) - \min(\text{fevalRange})), \text{AbsTol})$ , with <code>fevalRange</code> containing all results evaluating <code>FUN</code> up to that point.  |
| <code>AbsTol</code>              | positive scalar {1e-6}   | Absolute error tolerance, used by the error criterion stated under the property <code>RelTol</code> .   |
| <code>Vectorized</code>          | on   {off}   | Indicates if <code>FUN</code> is available for vectorized evaluation. Vectorized coding of <code>FUN</code> can significantly reduce the computation time used by <code>spvals</code> . For an example using a vectorized function, please see <code>spdemo</code> .  |
| <code>MinDepth</code>            | integer {2}  | Minimum interpolation depth, specifies the minimum number of hierarchical interpolation levels $N$ to compute.<br><b>Remark:</b> <code>MinDepth</code> has no effect if the dimension-adaptive grid refinement is switched on. An example is provided below.  |
| <code>MaxDepth</code>            | integer {8}  | Maximum interpolation depth, specifies the maximum number of hierarchical interpolation levels $N$ to compute.<br><b>Remark:</b> Since version 5.0, <code>MaxDepth</code> also applies to the dimension-adaptive algorithm. If <code>MaxDepth</code> is reached with respect to a coordinate direction, this direction is no longer refined further.  |
| <code>Variable Positions</code>  | 1xD vector {[]}  | Position of the ranges in the argument list when <code>FUN</code> is evaluated. By setting <code>VariablePositions</code> , <code>spvals</code> will evaluate <code>FUN</code> with respect to some of its input parameters, but not necessarily the first $D$ ones. The actual position is assigned by providing the number in the input argument list of the function <code>FUN</code> . This number must be provided for each interpolation dimension. Therefore, the value of <code>VariablePositions</code> must be a 1xD array. An example is provided below. |
| <code>NumberOf Outputs</code>    | integer {1}  | If <code>FUN</code> produces multiple outputs (where all must be scalar), indicate this here to perform the sparse grid computation for many output variables at once. Also see the example <code>spdemoarout.m</code> .  |
| <code>PrevResults</code>         | struct {[]}  | Previous sparse grid data. An existing result structure obtained from <code>spvals</code> may be provided to further refine an existing sparse grid. An example is provided below.  |
| <code>Function ArgType</code>    | {list}   vector  | Indicates whether the objective function takes the input parameters as a comma-separated list (default) or as a vector.   |
| <code>KeepFunction Values</code> | {off}   on   | If this parameter is set, a structure field <code>fvals</code> is returned, containing a cell array with the function values at the sparse grid points.   |



Table 3.3: Properties configurable with `spset` (part II).

| <i>Property</i>    | <i>Value {default}</i>       | <i>Description</i>   |
|--------------------|------------------------------|--|
| KeepGrid           | {off}   on                   | If this parameter is set, a structure field grid is returned, containing a cell array with the the sparse grid points.   |
| Dimension Adaptive | {off}   on                   | Dimension-adaptive grids try to adaptively find important dimensions and adjust the sparse grid structure accordingly. Especially in case of higher-dimensional problems, a dimension-adaptive strategy can significantly reduce the number of support nodes required to achieve a good interpolant.   |
| Dimadapt Degree    | positive scalar<br>{0.9}     | Fine-tuning parameter to alter the degree of dimensional adaptivity. A value of 1 places strong emphasis on the error estimates, and thus leads to strong dimensional adaptivity. A value of 0 disregards the error estimates, and constructs a conventional sparse grid based on the amount of work involved.   |
| MinPoints          | integer {100}                | This parameter only applies to dimension-adaptive sparse grids, and indicates the minimum number of support nodes (i.e., function evaluations to perform). An example is provided below.   |
| MaxPoints          | integer<br>{10000}           | This parameter only applies to dimension-adaptive sparse grids. The dimension-adaptive algorithm is aborted once the function evaluation count exceeds this number.  |
| Sparse Indices     | {auto}   off   on            | Manually turn the efficient sparse storage scheme (new feature since version 3.0) of the multi-index arrays on or off. The default switch <code>auto</code> uses the new scheme for the <code>ClenshawCurtis</code> , the <code>Chebyshev</code> , and the <code>Gauss-Patterson</code> grid, and the old (full) storage scheme from <code>spinterp</code> version 2.x for the <code>Maximum</code> and the <code>NoBoundary</code> grid (the sparse grid storage scheme is not supported for these two grid types).   |
| DropTol            | {auto}   off  <br>1x2 vector | During the sparse grid construction progress, the <code>spvals</code> algorithm may add sub-grids with hierarchical surpluses that are all close to zero or of negligible magnitude compared to the surpluses of other sub-grids. In particular, this occurs when additive structure is present in the objective function. To increase the performance of the <code>spinterp</code> algorithm, you may run the <code>spurge</code> algorithm that marks sub-grids to be neglected where all (absolute) hierarchical surpluses are less than $\max(\text{relDropTol} * (\max(\text{fevalRange}) - \min(\text{fevalRange})), \text{absDropTol})$ . You may specify the absolute and the relative drop tolerance as a vector <code>[absDropTol, relDropTol]</code> , or turn it off completely (= behavior of version 3.0 and earlier). The switch <code>auto</code> uses the values <code>absDropTol = 0</code> , <code>relDropTol = 100 * eps</code> , that is, by default, only a relative drop tolerance is used. |
| EnableDCT          | {on}   off                   | Enables/disables the DCT-based algorithm when constructing the Chebyshev-Gauss-Lobatto type sparse grid.   |

Note that the `MinPoints` and `MaxPoints` properties only work for dimension-adaptive grids. If we want to construct a non-adaptive grid of a certain depth, the `MinDepth` and `MaxDepth` options can be used. Recall that the number of points of a regular sparse grid can be determined a priori with the `spdim` function.

```
n = 4; d=2;
z = spvals(f, 2, [], spset('MinDepth', n, 'MaxDepth', n))
```

```
z =
      vals: {[65x1 double]}
      gridType: 'Clenshaw-Curtis'
          d: 2
      range: []
      maxLevel: 4
      estRelError: 0.0031
      estAbsError: 0.0201
      fevalRange: [1 7.3891]
      minGridVal: [0 0]
      maxGridVal: [1 1]
          nPoints: 65
          fevalTime: 0.1066
      surplusCompTime: 0.0029
          indices: [1x1 struct]
```

#### Example 2: Providing previous results

After computing an interpolant with a certain accuracy, it is often required to improve it further later on. Due to the hierarchical construction scheme, the previous results are not lost but can be passed to `spvals` for further refinement, as the following code illustrates.

```
f = @(x,y) exp(x+y);
z = [];
for n = 1:4
    z = spvals(f, 2, [], spset('MinDepth', n, 'MaxDepth', n, ...
                              'PrevResults', z));
    disp(['n=' num2str(z.maxLevel) ', estimated rel. error:', ...
          num2str(z.estRelError)]);
end
```

```
Warning: MaxDepth = 1 reached before accuracies
      RelTol = 0.01 or AbsTol = 1e-06 were achieved.
The current estimated relative accuracy is 0.62246.
n = 1, estimated rel. error: 0.62246
Warning: MaxDepth = 2 reached before accuracies
      RelTol = 0.01 or AbsTol = 1e-06 were achieved.
The current estimated relative accuracy is 0.17905.
n = 2, estimated rel. error: 0.17905
```

---

```
Warning: MaxDepth = 3 reached before accuracies
RelTol = 0.01 or AbsTol = 1e-06 were achieved.
The current estimated relative accuracy is 0.011133.
n = 3, estimated rel. error: 0.011133
n = 4, estimated rel. error: 0.0031415
```

### Example 3: Using the VariablePositions property

Consider the case of a function of four parameters, e.g.

$$f(a,b,x,y) = a(x^2 + y^2) + b\exp(x+y).$$

Suppose that the parameters  $a$  and  $b$  are fixed to  $a = 0.5$  and  $b = 0.2$ , and we wish to compute an approximation of  $f$  for  $x, y \in [0, 1]^2$ . The default syntax of `spvals` would require the interpolated parameters to appear at the start of the argument list, i.e. would require an argument list  $(x, y, a, b)$  to enable the call `spvals(f, 2, [], [], a, b)`.

By using `VariablePositions`, we can use the function as it is defined above, as the following code shows.

```
f = inline('a.*(x.^2+y.^2)+b.*exp(x+y)', 'a', 'b', 'x', 'y')
a = 0.5; b = 0.2;
options = spset('VariablePositions', [3 4]);
z = spvals(f, 2, [], options, a, b)
```

```
f =
    Inline function:
    f(a,b,x,y) = a.*(x.^2+y.^2) + b.*exp(x+y)
z =
    vals: {[29x1 double]}
    gridType: 'Clenshaw-Curtis'
    d: 2
    range: []
    maxLevel: 3
    estRelError: 0.0062
    estAbsError: 0.0142
    fevalRange: [0.2000 2.4778]
    minGridVal: [0 0]
    maxGridVal: [1 1]
    nPoints: 29
    fevalTime: 0.0508
    surplusCompTime: 0.0015
    indices: [1x1 struct]
```

Since the interpolation problem is two-dimensional, we assign a  $1 \times 2$  vector to the `VariablePositions` property, indicating that the first parameter an interpolation of which is required is located at position 3 of the argument list of  $f$ , and the second one at position 4. Note that since the function  $f$  takes four input parameters, the remaining parameters are appended to the argument list of `spvals` after the options argument.

**See Also** spget, spvals.

---

## spsurfun

Evaluate the sparse grid interpolant at a single point.

spsurfun is provided for convenience to be used as an alternative to spinterp, where the point Y to be evaluated is given as a row or column vector. This functional form is often adopted by multivariate optimization algorithms in Matlab.

Note that this form allows the evaluation of the sparse grid interpolant at a single point only. Therefore, it is recommended to use spinterp instead if multiple evaluations of the interpolant can be performed simultaneously.

### Syntax

```
IP = spsurfun(Y,Z)
[IP,IPGRAD] = spsurfun(Y,Z)
```

### Description

IP = spsurfun(Y, Z) Computes the interpolated value IP at the single point [Y1, . . . , YD] for the sparse grid interpolant Z.

[IP, IPGRAD] = spsurfun(. . .) Computes the interpolated value IP and the gradient vector IPGRAD.

Two additional options are available with spsurfun that are set by adding a field to the structure Z:

- selectOutput [ integer 1 ] Set the output variable number if an interpolant with multiple output variables was constructed with spvals.
- continuousDerivatives [ 'on' | 'off' ] Enable augmented continuous derivatives for the Clenshaw-Curtis grid.

### Examples

The following code shows how to use spsurfun. Note that as opposed to the spinterp syntax, the second argument is the sparse grid interpolant, not the first one.

```
f = inline('x.^2+y.^2-z');
z = spvals(f,3,[],spset('GridType','Chebyshev'));
[ip,ipgrad] = spinterp(z, 0.5, 0.2, 0.2)
[ip,ipgrad] = spsurfun([0.5, 0.2, 0.2], z)
```

---

```
in =
    -0.1100
ipgrad =
    [3x1 double]
ip =
    -0.1100
ipgrad =
    1.0000
    0.4000
    -2.0000
```

See the help page on sparse grid optimization for an example where `spsurfun` is used with an optimization method from Mathwork's Optimization Toolbox.

**See Also** `spinterp`.

---

## spvals

Construct a sparse grid interpolant.

### Syntax

```
Z = spvals(FUN,D)
Z = spvals(FUN,D,RANGE)
Z = spvals(FUN,D,RANGE,OPTIONS)
Z = spvals(FUN,D,RANGE,OPTIONS,P1,P2, ...)
```

### Description

`Z = spvals(FUN,D)` Compute the sparse grid representation `Z` for multi-linear sparse grid interpolation of the function `FUN`. The grid is computed over the  $d$ -dimensional unit cube  $[0, 1]^D$ .

`Z = spvals(FUN,D,RANGE)` In addition to the syntax above, the interpolation box dimensions may be specified. `RANGE` is a  $2 \times D$  array, e.g. to compute the sparse grid representation over the domain  $[0, 1] \times [2, 4] \times [1, 5]$  of `FUN`, `RANGE` must be `[0, 1; 2, 4; 1, 5]`. If `RANGE` is empty (`=[]`), it is assumed to be  $[0, 1]^D$ .

`Z = spvals(FUN,D,RANGE,OPTIONS)` Computes the sparse grid representation as above, but with default interpolation properties replaced by values in `OPTIONS`, an argument created with `spset`.

`Z = spvals(FUN,D,RANGE,OPTIONS,P1,P2, ...)` Passes the parameters `P1,P2,...` to the objective function `FUN`.

## Examples

The following examples demonstrate the generation of sparse grid interpolants under a variety of different parameters. The extensive configurability of `spvals` is achieved via the `spset` function. Additional examples of constructing interpolants of external functions, models with several output parameters, ODEs, etc. are provided in the Advanced Topics section of this document.

We first define the test function. In the examples below, we use Branin’s test function:

$$f_{\text{BR}}(x_1, x_2) = \left( \frac{5}{\pi} x_1 - \frac{5.1}{4\pi^2} x_1^2 + x_2 - 6 \right)^2 + 10 \left( 1 - \frac{1}{8\pi} \right) \cos x_1 + 10.$$

We set the dimension to  $d = 2$ , and the interpolation domain to `range = [-5, 10; 0, 15]`.

```
fun = inline(['(5/pi*x-5.1/(4*pi^2)*x.^2+y-6).^2+10*(1-1/(8*pi))*cos(x)+10']; ...
            '10*(1-1/(8*pi))*cos(x)+10'];
d = 2;
range = [-5, 10; 0, 15];
```

Now, we compute a regular (i.e. non-adaptive) sparse grid interpolant of `fun` using the default settings of the Sparse Grid Interpolation toolbox. This will compute a piecewise linear interpolant at the Clenshaw-Curtis sparse grid.

```
z1 = spvals(fun, d, range)

z1 =
      vals: {[145x1 double]}
  gridType: 'Clenshaw-Curtis'
         d: 2
   range: [2x2 double]
 maxLevel: 5
estRelError: 0.0087
estAbsError: 2.6622
 fevalRange: [1.3697 308.1291]
 minGridVal: [0.1250 0.7500]
 maxGridVal: [0 0]
   nPoints: 145
   fevalTime: 0.2437
surplusCompTime: 0.0040
   indices: [1x1 struct]
```

For comparison, we now compute two additional interpolants, one being a regular Chebyshev-Gauss-Lobatto grid, the other one being a dimension-adaptive sparse grid interpolant of the same grid type. To do this, we must pass an according options structure to the `spvals` routine. We do not have to store this options structure- it is possible to pass a structure generated on-the-fly to the function.

```
z2 = spvals(fun, d, range, spset('GridType', 'Chebyshev'))
z3 = spvals(fun, d, range, spset('GridType', 'Chebyshev', ...
                                'DimensionAdaptive', 'on', ...
```

```
'DimAdaptDegree', 1, ...  
'MinPoints', 20))
```

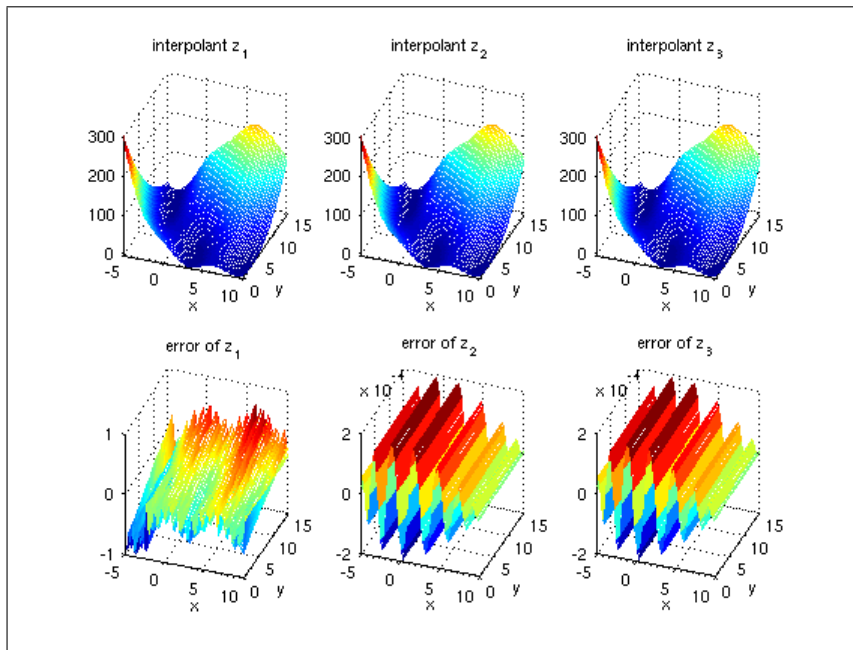
```
z2 =  
    vals: {[65x1 double]}  
    gridType: 'Chebyshev'  
    d: 2  
    range: [2x2 double]  
    maxLevel: 4  
    estRelError: 0.0095  
    estAbsError: 2.9017  
    fevalRange: [2.5620 308.1291]  
    minGridVal: [0.5000 0.2222]  
    maxGridVal: [0 0]  
    nPoints: 65  
    fevalTime: 0.1211  
    surplusCompTime: 0.0225  
    indices: [1x1 struct]
```

```
z3 =  
    vals: {[29x1 double]}  
    gridType: 'Chebyshev'  
    d: 2  
    range: [2x2 double]  
    estRelError: 0.0095  
    estAbsError: 2.9017  
    fevalRange: [2.7065 308.1291]  
    minGridVal: [0.5000 0.1464]  
    maxGridVal: [0 0]  
    nPoints: 29  
    fevalTime: 0.0468  
    surplusCompTime: 0.0094  
    indices: [1x1 struct]  
    maxLevel: [4 2]  
    activeIndices: [3x1 uint32]  
    activeIndices2: [9x1 uint32]  
    E: [1x9 double]  
    G: [9x1 double]  
    G2: [9x1 double]  
    maxSetPoints: 4  
    dimAdapt: 1
```

The following code generates a plot comparing the three interpolants. Furthermore, the error is plotted compared to the original function.

```
z = {z1, z2, z3};  
for k = 1:3  
    f_z = @(x,y) spinterp(z{k}, x, y);
```

```
error_z = @(x,y) fun(x,y) - spinterp(z{k}, x, y);  
subplot(2,3,k);  
ezmesh(f_z, [range(1,:),range(2,:)]);  
title(['interpolant z_' num2str(k)]);  
view(20,30);  
subplot(2,3,k+3);  
ezmesh(error_z, [range(1,:),range(2,:)]);  
title(['error of z_' num2str(k)]);  
view(20,30);  
end
```



**See Also** spinterp, spset.



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## SPARSE GRID INTERPOLATION TOOLBOX - LICENSE

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