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ISSN **1611-4176**

1. INTRODUCTION

Hyperbolic partial differential equations on curved manifolds occur in many applications. These include shallow water models for the atmosphere or ocean [4], [12], [17], the propagation of sound waves on curved surfaces [22] and passive tracer advection in the atmosphere. Further examples are the propagation of magneto-gravity waves in the solar tachocline [5], [9], [21] and relativistic matter flows near compact objects like black holes [8], [15].

For the numerics of these problems finite difference [8], finite volume [15], discontinuous Galerkin [11] and wave propagation methods [20] have been used. Except for the work of Amorim et. al. in [1] there is, up to the knowledge of the author, no convergence analysis in all of these cases. For convergence analysis of finite volume schemes, we will consider the following scalar model problem for non-linear hyperbolic conservation laws:

$$u_t + \nabla_g \cdot f(x,u) = 0 \text{ in } M \times \mathbb{R}_+ \tag{1}$$

$$u(x,0) = u_0(x) \text{ on } M. \tag{2}$$

Here (M, g) is a closed oriented Riemannian manifold and g is a fixed Riemannian metric on M. By $\nabla_g\cdot$ we denote the divergence operator on M induced by g. The aim of this paper is to prove a convergence rate for finite volume schemes for this model problem.

For this problem one has the notion of entropy solution, analogous to the Kruzkov definition in Euclidean space.

Definition 1. *A function $u \in L^\infty(M \times \mathbb{R}_+)$ is called an* **entropy solution** *of (1),(2) if*

$$\int_{M\times\mathbb{R}_+} \left[|u-\kappa|\varphi_t + g(x)\big(f(x,u\top\kappa) - f(x,u\bot\kappa)), \nabla_g\varphi\big)\right] dv_g\, dt \tag{3}$$

$$+ \int_M |u_0-\kappa|\varphi(\cdot,0)\, dv_g \geq 0 \quad \forall\kappa\in\mathbb{R}, \forall\varphi\in C_0^\infty(M\times\mathbb{R}_+,\mathbb{R}_+),$$

where $u\bot\kappa$ and $u\top\kappa$ denotes the minimum and maximum of u and κ respectively.

The well-posedness of this problem was investigated by Ben-Artzi and LeFloch in [2]. They show that given $u_0 \in L^\infty(M)\cap L^1(M)$ and a geometry compatible flux, i.e. $\nabla_g \cdot f(\cdot,\bar{u}) = 0$, for every $\bar{u}\in\mathbb{R}$ the problem (1),(2) has a unique entropy solution u. Furthermore for $u_0 \in L^\infty(M)\cap \mathrm{BV}(M)$ the total variation of the entropy solution is bounded for every time $t \geq 0$ in the sense that there exists $C_1 \geq 0$ depending only on $\|u_0\|_{L^\infty(M)}$ and the geometry of M such that

$$\mathrm{TV}_M(u(\cdot,t)) \leq e^{C_1 t}(1+\mathrm{TV}_M(u_0)) \text{ for all } t\geq 0.$$

In [1] it is shown that for a geometry compatible flux f and every vector-field X with $[X, f_u(\cdot,\bar{u})] = 0$ for every $\bar{u}\in\mathbb{R}$, we have

$$TV_X(u(\cdot,t)) \leq TV_X(u_0),$$

where

$$TV_X(u) := \sup_{\phi\in C^\infty(M):\|\phi\|_{L^\infty}} \int_M u\nabla_g\cdot(\phi X)\, \mathrm{d}v_g(x).$$

This implies that for $d = 1$ the entropy solution is total variation diminishing, i.e. $\mathrm{TV}_M(u(\cdot,t)) \leq TV_M(u_0)$. Furthermore they prove convergence for a class of finite volume schemes for the Cauchy-problem (1),(2). This relies on an entropy dissipation inequality for smooth entropies. We will prove a similar result for Kruzkov entropies in Lemma 13. In this paper we will prove convergence rates for these schemes. The general convergence framework by Eymard et. al. in [7] for the proof of convergence rates for finite volume schemes in Euclidean space works but requires substantial extensions to the differential geometric framework. Particularly new problems arise in the construction and properties of cut-off functions and due to the fact that on a Riemannian manifold we cannot - in general - parallel-transport one vector to the whole manifold and get a smooth vector-field (cf. the proofs of Lemmas 14 and 15). As in the Euclidean case we are able to prove convergence of order $\frac{1}{2}$ in one space dimension and convergence of order $\frac{1}{4}$ in higher space dimensions.

We refer to [19] and [20] for a treatment of the wave propagation method on curved manifolds.

We use a quite generic finite volume method. For the convergence analysis we need grids with the property that the curvature of the faces is uniformly bounded under mesh refinement and such that every point on the manifold lies only in a certain number of convex hulls of elements. In [3] different approaches to construct grids on spheres are treated and we refer to [10], [18] for geodesic grids on a sphere, which are an important class of examples where these results hold.

We make the following hypotheses on the data:

$$
(4)\qquad \begin{cases} u_0 \in L^\infty(M) \cap \mathrm{BV}(M), U_m, U_M \in \mathbb{R} : U_m \leq u_0 \leq U_M \ a.e., \\ f \in C^1(M \times \mathbb{R}, TM) \text{ such that } f(x,\bar{u}) \in T_xM \text{ for every } x \in M, \bar{u} \in \mathbb{R}, \\ \nabla_g \cdot f(\cdot,\bar{u}) = 0 \text{ for every } \bar{u} \in \mathbb{R}, \\ \text{there is a constant } C > 0 \text{ such that} \\ \left\|\nabla_g f(x,\bar{u}) - \nabla_g f(x,\bar{u}')\right\|_g \text{ for every } \bar{u}, \bar{u}' \in [U_m, U_M] \text{ and } x \in M. \end{cases}
$$

Here $\|\cdot\|_g$ denotes the operator-norm induced by g, $\nabla_g f(x,u)$ denotes the covariant derivative with respect to the first variable and $\nabla_g \cdot f(x,\bar{u})$ denotes the divergence which is the trace of the covariant derivative. The hypothesis $\mathrm{div}_g f(\cdot,\bar{u}) = 0$ is used to ensure the well-posedness of the problem and to avoid technical problems. Like in the Euclidean case it should not be necessary for the convergence rate.

The outline of this paper is as follows: In section 2 we will recall some helpful definitions and notations from differential geometry and give some results, which are necessary for the proof of the main theorem, Theorem 16. In sections 3, 4 we will present the notion of triangulation and the construction of finite volume schemes on Riemannian manifolds respectively. In section 5 we will state the main Theorem and prove it.

2. Differential geometry

2.1. **Notation and definitions.** We will consider a connected, closed, oriented d-dimensional Riemannian manifold (M,g), i.e. M is a compact, smooth, oriented manifold without boundary and g is a fixed Riemannian metric on M. This means $g(x)$ is an inner product on the tangent space T_xM of M at x. In local coordinates $(x^j)_{1\leq j\leq d}$ the partial derivatives $\partial_j = \frac{\partial}{\partial x^j}$ form a basis of the tangent space T_xM and we have the metric tensor $g_{ij}(x) := g(x)(\partial_i, \partial_j)$ with inverse g^{ij}. This enables us to define the divergence operator $\nabla_g\cdot$ by

$$
\nabla_g \cdot f(x) := \frac{1}{\sqrt{|g(x)|}}\, \partial_j \left(\sqrt{|g(x)|} f^j(x)\right)
$$

where $|g(x)| := |\det(g_{ij}(x))|$, for every smooth vector-field f on M with local representation $f = f^j \partial_j$ using the Einstein summation convention. The covariant derivative of f is given by

$$
\begin{aligned} &\nabla f : \Gamma(TM) \longrightarrow \Gamma(TM) \\ &\nabla_{Y^j\partial_j} f := f^i Y^j \Gamma^k_{ji} \partial_k + Y^j \partial_j(f^k)\partial_k, \end{aligned}
$$

where

$$
\Gamma^k_{ji} := \frac{1}{2}\left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) g^{lk}
$$

and $\Gamma(TM)$ denotes the smooth vector-fields on M, i.e. the smooth sections of the tangent bundle TM. The covariant derivative is compatible with the Riemannian metric, i.e. for vector-fields X, Y, Z we have

$$
(5)\qquad X(g(Y,Z)) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right).
$$

These definitions of divergence and covariant derivative are only well-defined in the local coordinate system, but the definitions are independent of the choice of local coordinates and so divergence and covariant derivative are well-defined all over M. Similarly for every smooth function u on M the gradient of u is defined by

$$
(\nabla_g u)^i = g^{ij} \frac{\partial u}{\partial x^j}.
$$

The Riemannian metric also defines a volume form dv_g on the manifold, a volume form dv_N on every submanifold N and a metric d_g on M. Spaces of functions of bounded variation are defined similar to the definition in Euclidean space

Definition 2.

$$\begin{aligned} TV_M(u) &:= \sup_{X \in \Gamma(TM): \|X\|_\infty \leq 1} \int_M u \nabla_g \cdot X \, dv_g, \\ BV(M) &:= \left\{ u \in L^1(M) : TV_M(u) < \infty \right\}. \end{aligned}$$

Throughout this paper we will use the following lemma, whose proof can be found in [6] for example.

Lemma 3. *Because M is compact there exists a constant $R > 0$ such that for every $x \in M$ the map $\exp_x^{-1}$ is a chart on $B_{4R}(x)$.*

Definition 4. *For a set $U \subset M$ the* **convex hull** *U^c of U is the set of all points $x \in M$ such that there are two points $y, y' \in M$ and a length minimising geodesic $\gamma_{yy'} : [0,1] \longrightarrow M$ with $\gamma_{yy'}(0) = y$, $\gamma_{yy'}(1) = y'$ and $\gamma_{yy'}(t) = x$ for some t in $[0,1]$.*

This definition is needed to define one of the properties of the grid in Definition 6.

2.2. **Parallel transport.** In the proof of Lemma 15 we will have to use parallel transport to extend vectors to local vector-fields. For $x, y \in M$ with $0 < d_g(x,y) < R$ there exists a unique minimising geodesic γ_{xy} from x to y parametrised by arc-length. So we get a well-defined mapping

$$P_{xy} : T_xM \longrightarrow T_yM$$

defined by parallel transport along this geodesic. By definition of geodesic we know that $P_{xy}(\gamma'_{xy}(0)) = \gamma'_{xy}(d_g(x,y))$. Obviously we have for $0 < d_g(x,y) < R$ the identities

$$\nabla_{g,x} d_g(x,y) = -\gamma'_{xy}(0) \quad \text{and} \quad \nabla_{g,y} d_g(x,y) = \gamma'_{xy}(d_g(x,y)). \tag{6}$$

Let v be a smooth vector-field on M then

$$\frac{d}{dt} g(\gamma_{xy}(t)) \left(P_{x\gamma_{xy}(t)}(v(x)), \gamma'_{xy}(t) \right) = 0$$

and therefore

$$g(x) \left(v(x), \nabla_{g,x} d_g(x,y) \right) = -g(y) \left(P_{xy}(v(x)), \nabla_{g,y} d_g(x,y) \right). \tag{7}$$

2.3. **Cut-off functions.** These are necessary for a doubling of variables argument in the proof of Lemma 15. Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a smooth function with supp$\psi \subset [-1, 0]$ such that

$$\int_{\mathbb{R}} \psi(x) \, dx = 1.$$

Let $\psi_\varepsilon(x) := \frac{1}{\varepsilon} \psi\left(\frac{x}{\varepsilon}\right)$. Let $\chi : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a smooth function with support in $[-1,1]$, which is even, decreasing on $[0,1]$ and fulfils

$$\int_{\mathbb{R}^d} \chi(\|x\|) = 1. \tag{8}$$

We define

$$\chi_\varepsilon : M \times M \longrightarrow \mathbb{R}, \quad (x,y) \mapsto \frac{1}{\varepsilon^d} \chi\left(\frac{d_g(x,y)}{\varepsilon} \right). \tag{9}$$

For ε sufficiently small we have for every $x \in M$ using the exponential map $\exp_x$

$$\int_M \chi_\varepsilon(x,y) \mathrm{d}v_g(x) = \int_{B_\varepsilon(0)} \frac{1}{\varepsilon^d} \chi\left(\frac{\|a\|}{\varepsilon} \right) |\det(T\exp_x^{-1})_a| \, da. \tag{10}$$

The map

$$\{(x,y) \in M^2 : d_g(x,y) < 2R\} \longrightarrow \mathbb{R} \quad (x,y) \mapsto \det(T\exp_x^{-1})_{\exp_x^{-1}(y)}$$

is smooth and equals 1 for $x = y$. Hence we have

$$\left| \det(T\exp_x^{-1})_{\exp_x^{-1}(y)} - 1 \right| \leq C d_g(x,y) \tag{11}$$

for $d_g(x,y) \le R$. Inserting this in (10) we get

$$\left| \int_M \chi_\varepsilon(x,y) \mathrm{dv}_g(y) - 1 \right| \overset{(8)}{\le} \int_{B_\varepsilon(0)} \frac{1}{\varepsilon^d} \chi\left(\frac{\|a\|}{\varepsilon}\right) |\det(T\exp_x^{-1})_a - 1|\, da \le C\varepsilon. \tag{12}$$

Furthermore when γ_{xy} denotes the length minimising geodesic from x to y parametrized by arc-length, we have using (6)

$$\nabla_{g,y}\chi_\varepsilon(x,y) = \frac{1}{\varepsilon^{d+1}}\chi'\left(\frac{d_g(x,y)}{\varepsilon}\right)\gamma'_{xy}(d_g(x,y)). \tag{13}$$

The following technical lemma will be very helpful for the proof of the theorem. Its proof will be given in the appendix.

Lemma 5. *There is a constant $C > 0$ depending only on M and g such that for every $x, y \in M$ with $d_g(x,y) < R$ and $v \in T_xM$*

$$\left| div_{g,y}(T\exp_x)_{\exp_x^{-1}(y)}(v) \right| \le C\|v\| d_g(x,y), \tag{14}$$

$$\left\| (T\exp_x)_{\exp_x^{-1}(y)}(v) \right\| \le C\|v\|. \tag{15}$$

There is another $C > 0$ such that for every $x, y \in M$, $v \in T_xM$ and $\varepsilon < R$

$$\left| \left\langle (T\exp_x)_{\exp_x^{-1}(y)}(v) - P_{xy}(v), \nabla_{g,y}\chi_\varepsilon(x,y) \right\rangle \right| \le \frac{C}{\varepsilon^{d-1}}\|v\| 1_{\{d_g(x,y)<\varepsilon\}}. \tag{16}$$

3. Triangulation

Definition 6. *A* **triangulation** *on (M,g) is a set $\mathcal{T}$ of curved polyhedra K on M such that $M = \cup_{\mathcal{T}} \bar{K}$. We impose $K_1 \cap K_2$ is a common face of K_1, K_2 or a submanifold of dimension $\le d-2$.*

The set of the faces e of a polyhedron K is denoted by ∂K and the unique polyhedron sharing the face e with K is denoted by K_e. By $n_{K,e}(x) \in T_xM$ we denote the unit outer normal to a polyhedron K in a point $x \in e$. Finally $|K|, |e|$ denote the d- and $(d-1)$-dimensional Hausdorff measures of K and e respectively.
We will need the following assumption on the triangulation: There exist $C, \bar{R}, \beta, h > 0$ and $k, N_c \in \mathbb{N}$ such that for every $K \in \mathcal{T}$ and $e \in \partial K$ the following conditions are fulfilled

$$\beta h^d \le |K|, \tag{17}$$

$$|e| \le Ch^{d-1}, \tag{18}$$

$$\#\partial K \le k, \tag{19}$$

$$\delta(K) \le h, \tag{20}$$

$$\sup_{x,i,e} |\lambda_i(e)(x)| \le \bar{R}, \tag{21}$$

$$\sup_{x\in M} \#\{i : x \in K_i^c\} \le N_c, \tag{22}$$

where $\delta(K) := \sup\{d_g(x,y) : x, y \in K\}$ and $\#$ denotes the number of elements of a certain set. Furthermore $\lambda_i(e)(x)$ denotes the $i-$th principal curvature of the face e in the point $x \in e$.

Note: The condition (22) should be fulfilled for all computational meshes and is needed for the approximation of the initial data in Lemma 7 which will be stated below and whose proof can be found in the appendix. Under mesh refinement condition (21) is satisfied for example by geodesic grids and the combined grid composed of a latitude-longitude grid away from the poles and a stereographic grid at the two polar caps which can be found in [14]. It is not satisfied by the latitude-longitude grid on the sphere which also has some other numerical drawbacks. In particular strongly differing cell sizes impose hard CFL conditions on the timestep (cf. [3]). The grids proposed in [3] also do not satisfy this condition, but satisfy it.

Lemma 7. *For h small enough and every $u \in BV(M)$ there is a constant $C > 0$ depending on M and N_c such that*

$$\|u - \bar{u}\|_{L^1(M)} \leq Ch,$$

where

$$\bar{u}(x) := \frac{1}{|K|} \int_K u(x) dv_g(x) \text{ for } x \in K,$$

so $\bar{u}$ is well-defined allmost everywhere on the manifold.

4. The scheme

For every polyhedron $K \in \mathcal{T}$ and face $e \in \partial K$ we consider a numerical flux function $f_{K,e} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that the following properties are satisfied:

$$\text{Conservation:} \quad f_{K,e}(a,b) = -f_{K_e,e}(b,a), \tag{23}$$

$$\text{Consistency:} \quad f_{K,e}(a,a) = \frac{1}{|e|} \int_e f(x,a) n_{K,e}(x) \mathrm{dv}_e(x), \tag{24}$$

$$\text{Monotonicity:} \quad f_{K,e} \quad \text{is nondecreasing in the first and nonincreasing in the second variable.} \tag{25}$$

Furthermore we impose that the $f_{K,e}$ are uniformly locally Lipschitz continuous. We will consider the following **semi-discrete scheme**:

$$(u_K^h)_t = -\frac{1}{|K|} \sum_{e \in \partial K} |e| f_{K,e}(u_K^h, u_{K_e}^h) \tag{26}$$

$$u_K^h(0) = \frac{1}{|K|} \int_K u_0(x) \mathrm{dv}_g(x) \tag{27}$$

$$u^h(x,t) = u_K^h(t) \text{ for } x \in K. \tag{28}$$

5. Proof of convergence rates

We first show that a solution of (26)-(28) exists and that it is bounded.

Lemma 8. *Assume the local existence of a solution of* (26)-(28) *and let $u_0(x) \in [U_m, U_M]$ for almost every $x \in M$, then $u_K^h(t) \in [U_m, U_M]$ for every $t \geq 0$ and $K \in \mathcal{T}$.*

Proof. It is obvious that $u_K^h(0) \in [U_m, U_M]$ for every K. First observe that for fixed K and $u_{K_e}^h \leq u_K^h$ for all $e \in \partial K$ we have

$$\begin{aligned}
(u_K^h)_t &= -\frac{1}{|K|} \sum_{e \in \partial K} |e| f_{K,e}(u_K^h, u_{K_e}^h) \\
&\overset{monotonicity}{\leq} -\frac{1}{|K|} \sum_{e \in \partial K} |e| f_{K,e}(u_K^h, u_K^h) \\
&\overset{consistency}{\leq} -\frac{1}{|K|} \sum_{e \in \partial K} \int_e f(x, u_K^h) n_{K,e} \mathrm{dv}_g(x) = 0.
\end{aligned}$$

Now we will prove that $u_K^h \leq U_M$ for all t, the proof for $u_K^h \geq U_m$ is analogous. Let

$$s := \sup\{T \geq 0 : u_K^h(t) \in [U_m, U_M] \forall t \in [0,T] \text{ and } K \in \mathcal{T}\}.$$

We have $s \geq 0$. Let $E := \max |e|$. Assume $s < \infty$. Due to continuity we have $u_K^h(s) \in [U_m, U_M] \, \forall K$. Because the $f_{K,e}$ are locally Lipschitz continuous, it exists $\delta > 0$ such that a solution $\{u_K^h\}_{K \in \mathcal{T}}$ of (1),(2) exists in $[0, s+\delta)$. Let $A := \sup\{u_K^h(t) : t \leq s + \frac{\delta}{2}\}$ and L the uniform Lipschitz constant of the $f_{K,e}$ on $[-A, A]$. Because $s < \infty$ there are $a_1, \varepsilon > 0$ and $K_1 \in \mathcal{T}$ such that $a_1 < \min\left(\frac{\delta}{2}, \frac{1}{kLE}\right)$ and

$$u_{K_1}^h(s + a_1) = U_M + \varepsilon. \tag{29}$$

Now we will prove by induction that there exist $0 < a_n \leq a_1$ and $K_n \in \mathcal{T}$ such that

$$u^h_{K_n}(s+a_n) \geq U_M + \frac{\varepsilon}{(a_1 kLE)^{n-1}}. \tag{30}$$

The induction starts with (29). If (30) is fulfilled there has to be an $a_{n+1} < a_n$ such that

$$u^h_{K_n}(s+a_{n+1}) \geq U_M \text{ and}$$
$$(u^h_{K_n})_t(s+a_{n+1}) \geq \frac{\varepsilon}{a_n(a_1 kLE)^{n-1}} \geq \frac{\varepsilon}{a_1(a_1 kLE)^{n-1}}.$$

Thus due to the monotonicity and Lipschitz property of the $f_{K,e}$ there must be a $K_{n+1} \in \mathcal{T}$ such that

$$u^h_{K_{n+1}}(s+a_{n+1}) \geq U_M + \frac{\varepsilon}{(a_1 kLE)^n}.$$

There are only finitely many $K \in \mathcal{T}$ so there is a subsequence a_{k_l} and some $K \in \mathcal{T}$ such that

$$u^h_K(s+a_{k_l}) \overset{l\longrightarrow\infty}{\longrightarrow} \infty,$$

because all a_{k_l} are smaller than a_1 this is a contradiction to the continuity of u^h_K on $[0, s+\delta)$. So $s = \infty$. □

As an immediate consequence of Lemma 8 and the local Lipschitz continuity of the numerical fluxes we have:

Corollary 9. *There exists a global solution of the system* (26)-(28).

The next step is to prove a TVD estimate in the $d = 1$ and a weak BV-estimate in the $d \geq 2$ case. For brevity we introduce the following notation: for real numbers a, b we define

$$C(a,b) := \{(c,d) \in [a\bot b, a\top b]^2 : (b-a)(d-c) \geq 0,\}$$

where $a\top b$ and $a\bot b$ denote the maximum and minimum of a and b respectively. For every $t \geq 0$ we define

$$E(t) := \{(K,e) : K \in \mathcal{T}, e \in \partial K, u^h_K(t) > u^h_{K_e}(t)\}.$$

Lemma 10 (TVD property)**.** *Let M be 1-dimensional then the scheme (26)-*(28) *is TVD, i.e.*

$$TV_M(u^h(\cdot,t)) \leq TV_M(u_0) \text{ for all } t > 0.$$

This implies that for every $T > 0$ there exists a $C > 0$ depending only on f,u_0,M,$f_{K,e}$,T such that

$$\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \max_{(c,d)\in C(u^h_K, u^h_{K_e})} |f_{K,e}(c,d) - f_{K,e}(c,c)|\, dt \leq C.$$

Proof. We will consider times t where $\frac{d}{dt}|u^h_K - u^h_{K_e}|$ exists for all $K \in \mathcal{T}$ and $e \in \partial K$. These derivatives exist for almost every $t \geq 0$ and we have

$$\frac{d}{dt}\mathrm{TV}_M(u^h(\cdot,t)) = \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{e\in\partial K}\frac{d}{dt}|u^h_K(t) - u^h_{K_e}(t)|.$$

Now we fix one $K \in \mathcal{T}$ and observe that K has exactly two neighbours K_1,K_2.

- If $u^h_{K_1}(t) \leq u^h_K(t) \leq u^h_{K_2}(t)$ or $u^h_{K_2}(t) \leq u^h_K(t) \leq u^h_{K_1}(t)$ then $(u^h_K)_t$ occurs exactly twice with different signs in the sum and therefore vanishes.
- If $u^h_K(t) > u^h_{K_1}(t), u^h_{K_2}(t)$ the term

$$(u^h_K)_t(t) = -\sum_{e\in\partial K}\frac{|e|}{|K|}f_{K,e}(u^h_K(t), u^h_{K_e}(t)) \leq -\sum_{e\in\partial K}\frac{|e|}{|K|}f_{K,e}(u^h_K(t), u^h_K(t)) = 0$$

  occurs twice in the sum.
- If $u^h_K(t) < u^h_{K_1}(t), u^h_{K_2}(t)$ the term

$$-(u^h_K)_t(t) = \sum_{e\in\partial K}\frac{|e|}{|K|}f_{K,e}(u^h_K(t), u^h_{K_e}(t)) \leq \sum_{e\in\partial K}\frac{|e|}{|K|}f_{K,e}(u^h_K(t), u^h_K(t)) = 0$$

  occurs twice in the sum.

So we know $\mathrm{TV}_M(u^h(\cdot,t))$ is nonincreasing in time. For every $K \in \mathcal{T}$ there exist $x_K, y_K \in K$ such that

$$u_0(x_K) \geq u_K^h(0) \geq u_0(y_K).$$

Let K_1, K_2 be the neighboring elements for some $K \in \mathcal{T}$, then we define

$$\zeta_K = \begin{cases} x_K & : \quad u_K^h > u_{K_1}^h, u_{K_2}^h \\ y_K & : \quad \text{else.} \end{cases}$$

We have

$$\begin{aligned} 2\mathrm{TV}_M(u^h(\cdot,0)) &= \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |u_K^h(0) - u_{K_e}^h(0)| \\ &\leq \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |u_0(\zeta_K) - u_0(\zeta_{K_e})| \leq 2\mathrm{TV}_M(u_0). \end{aligned}$$

This proves the TVD property. For $(c,d) \in C(u_K^h, u_{K_e}^h)$ we have

$$|f_{K,e}(c,d) - f_{K,e}(c,c)| \leq L|c-d| \leq L|u_K^h - u_{K_e}^h|,$$

where L is the uniform Lipschitz constant for all $f_{K,e}$ on $[U_m, U_M]$. Using $|e| = 2$ we get

$$\begin{aligned} &\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \max_{(c,d)\in C(u_K^h, u_{K_e}^h)} |f_{K,e}(c,d) - f_{K,e}(c,c)|\, dt \\ \leq\; &\int_0^T 2L \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |u_K^h - u_{K_e}^h|\, dt \\ \leq\; &4L \int_0^T \mathrm{TV}_M(u^h(\cdot,t))\, dt \leq 4LT\mathrm{TV}_M(u_0). \end{aligned}$$

□

In the higher-dimensional case there is no TVD estimate, but we can prove a weak BV estimate which will play a similar role in the convergence proof.

Lemma 11 (weak BV-estimate)**.** *Let $d \geq 2$ be the dimension of M. For every $T > 0$ there exists $C > 0$ depending only on f,u_0,M,β,$\{f_{K,e}\}$, T, k such that*

$$\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \max_{(c,d)\in C(u_K^h, u_{K_e}^h)} |f_{K,e}(c,d) - f_{K,e}(c,c)|\, dt \leq \frac{C}{\sqrt{h}}.$$

Proof. We have

$$\begin{aligned} \int_0^T \sum_{K\in\mathcal{T}} |K| u_K^h (u_K^h)_t\, dt &= \frac{1}{2}\int_0^T \sum_{K\in\mathcal{T}} |K| \left((u_K^h)^2\right)_t\, dt \\ &= \frac{1}{2}\sum_{K\in\mathcal{T}} |K| \left((u_K^h)^2(T) - (u_K^h)^2(0)\right) \\ &\geq -\frac{1}{2}\sum_{K\in\mathcal{T}} |K| (u_K^h)^2(0) \\ &= -\frac{1}{2}\|u^h(0)\|_{L^2(M)}^2 \geq -\frac{1}{2}\|u_0\|_{L^2(M)}^2. \end{aligned} \tag{31}$$

Now we multiply (26) by $|K|u_K^h(t)$ and sum over all $K \in \mathcal{T}$

$$
\begin{aligned}
\int_0^T \sum_{K\in\mathcal{T}} |K| u_K^h (u_K^h)_t \, dt &= -\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| f_{K,e}(u_K^h, u_{K_e}^h) u_K^h \, dt \\
&\overset{(24)(4)}{=} \int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \left(f_{K,e}(u_K^h, u_K^h) - f_{K,e}(u_K^h, u_{K_e}^h) \right) u_K^h \, dt \\
&= \int_0^T \sum_{(K,e)\in E(t)} |e| \left[\left(f_{K,e}(u_K^h, u_K^h) - f_{K,e}(u_K^h, u_{K_e}^h) \right) u_K^h \right. \\
&\quad \left. + \left(f_{K_e,e}(u_{K_e}^h, u_{K_e}^h) - f_{K_e,e}(u_{K_e}^h, u_K^h) \right) u_{K_e}^h \right] dt \\
&\overset{(23)}{=} \int_0^T \sum_{(K,e)\in E(t)} |e| \left[\left(f_{K,e}(u_K^h, u_K^h) - f_{K,e}(u_K^h, u_{K_e}^h) \right) u_K^h \right. \\
&\quad \left. - \left(f_{K,e}(u_{K_e}^h, u_{K_e}^h) - f_{K,e}(u_K^h, u_{K_e}^h) \right) u_{K_e}^h \right] dt.
\end{aligned} \tag{32}
$$

Now we define $F_{K,e}(a) := f_{K,e}(a,a)$ and let $\Phi_{K,e}$ be a primitive of $a \mapsto aF'_{K,e}(a)$ satisfying $\Phi_{K,e}(0) = 0$. Let $a = u_K^h, b = u_{K_e}^h$ then every single summand has the form

$$|e| \left[a(F_{K,e}(a) - f_{K,e}(a,b)) - b(F_{K,e}(b) - f_{K,e}(a,b)) \right].$$

Integration by parts yields

$$
\begin{aligned}
\Phi_{K,e}(b) - \Phi_{K,e}(a) &= \int_a^b u F'_{K,e}(u) \, du \\
&= b(F_{K,e}(b) - f_{K,e}(a,b)) - a(F_{K,e}(a) - f_{K,e}(a,b)) \\
&\quad - \int_a^b (F_{K,e}(u) - f_{K,e}(a,b)) \, du.
\end{aligned}
$$

Due to the conservation property (23) of the numerical fluxes we have $F_{K,e} = -F_{K_e,e}$ and therefore $\Phi_{K,e} = -\Phi_{K_e,e}$. Because the flux is geometry compatible (4) we have

$$\sum_{e\in\partial K} |e| f_{K,e}(a,a) = 0 \Longrightarrow \sum_{e\in\partial K} |e| F'_{K,e}(a) = 0 \Longrightarrow \sum_{e\in\partial K} |e| \Phi_{K,e}(a) = 0$$

for every $K \in \mathcal{T}$ and $a \in \mathbb{R}$. Thus we have

$$
\begin{aligned}
\sum_{(K,e)\in E(t)} |e| \left(\Phi_{K,e}(u_K^h) - \Phi_{K,e}(u_{K_e}^h) \right) &= \sum_{(K,e)\in E(t)} |e| \left(\Phi_{K,e}(u_K^h) + \Phi_{K_e,e}(u_{K_e}^h) \right) \\
&= \sum_{(K,e)\in E(t)} |e| \left(\Phi_{K,e}(u_K^h) + \Phi_{K_e,e}(u_{K_e}^h) \right) \\
&\quad + \underbrace{\sum_{\{(K,e):u_K^h = u_{K_e}^h\}} |e| \left(\Phi_{K,e}(u_K^h) \right)}_{0} \\
&= \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \Phi_{K,e}(u_K^h) = 0.
\end{aligned}
$$

Using this in (32) implies

$$
\begin{aligned}
\int_0^T \sum_{K\in\mathcal{T}} |K| u_K^h (u_K^h)_t \, dt &= -\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| f_{K,e}(u_K^h, u_{K_e}^h) u_K^h \, dt \\
&= -\int_0^T \sum_{(K,e)\in E(t)} |e| \int_{u_K^h}^{u_{K_e}^h} \left(f_{K,e}(u,u) - f_{K,e}(u_K^h, u_{K_e}^h) \right) du \, dt \\
&= \int_0^T \sum_{(K,e)\in E(t)} |e| \int_{u_{K_e}^h}^{u_K^h} \left(f_{K,e}(u,u) - f_{K,e}(u_K^h, u_{K_e}^h) \right) du \, dt.
\end{aligned} \tag{33}
$$

For $u^h_{K_e} \le c \le d \le u^h_K$ we have due to (25)

$$\begin{aligned} \int_{u^h_{K_e}}^{u^h_K} \underbrace{\left(f_{K,e}(u^h_K, u^h_{K_e}) - f_{K,e}(u,u)\right)}_{\ge 0} du &\ge \int_c^d \left(f_{K,e}(u^h_K, u^h_{K_e}) - f_{K,e}(u,u)\right) du \\ &\ge \int_c^d \left(f_{K,e}(d,c) - f_{K,e}(u,u)\right) du. \end{aligned}$$

We will now use the following fact which can be found in [7]:

Lemma 12. *Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a monotonic Lipschitz continuous function, with Lipschitz constant $G > 0$. Then*

$$\left| \int_c^d (g(u) - g(c))\, du \right| \ge \frac{1}{2G}(g(d) - g(c))^2, \;\; \forall c, d \in \mathbb{R}.$$

Thus (25) and the Lipschitz continuity of the $f_{K,e}$ imply

$$\begin{aligned} \int_c^d \left(f_{K,e}(d,c) - f_{K,e}(u,u)\right) du &\ge \int_c^d \left(f_{K,e}(d,c) - f_{K,e}(d,u)\right) du \\ &\ge \frac{1}{2L}\left(f_{K,e}(d,c) - f_{K,e}(d,d)\right)^2 \end{aligned}$$

and

$$\begin{aligned} \int_c^d \left(f_{K,e}(d,c) - f_{K,e}(u,u)\right) du &\ge \int_c^d \left(f_{K,e}(d,c) - f_{K,e}(u,c)\right) du \\ &\ge \frac{1}{2L}\left(f_{K,e}(d,c) - f_{K,e}(c,c)\right)^2, \end{aligned}$$

where L is the uniform Lipschitz constant of the $f_{K,e}$ on $[U_m, U_M]$. Multiplying both inequalities with $\frac{1}{2}$ and adding them yields with (31) and(33)

$$\begin{aligned} \frac{1}{2}\|u_0\|^2_{L^2(M)} &\ge \int_0^T \sum_{(K,e)\in E(t)} \int_{u^h_{K_e}}^{u^h_K} |e| \left(f_{K,e}(u^h_K, u^h_{K_e}) - f_{K,e}(u,u)\right) du\, dt \\ &\ge \int_0^T \sum_{(K,e)\in E(t)} \frac{|e|}{2L} \Bigg(\max_{u^h_{K_e} \le c \le d \le u^h_K} \left(f_{K,e}(d,c) - f_{K,e}(d,d)\right)^2 \\ &\qquad + \max_{u^h_{K_e} \le c \le d \le u^h_K} \left(f_{K,e}(d,c) - f_{K,e}(c,c)\right)^2 \Bigg) dt \qquad (34) \\ &\ge \int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} \frac{|e|}{2L} \max_{(c,d)\in C(u^h_K, u^h_{K_e})} |f_{K,e}(c,d) - f_{K,e}(c,c)|^2\, dt. \end{aligned}$$

Now by Cauchy Schwartz inequality we get

$$\begin{aligned} &\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e| \max_{(c,d)\in C(u^h_K, u^h_{K_e})} |f_{K,e}(c,d) - f_{K,e}(c,c)|\, dt \\ \le\;& \left(\int_0^T \sum_{K\in\mathcal{T}} \sum_{e\in\partial K} |e|^2 \max_{(c,d)\in C(u^h_K, u^h_{K_e})} |f_{K,e}(c,d) - f_{K,e}(c,c)|^2\, dt \right)^{\frac{1}{2}} \\ &\cdot \left(\sum_{K\in\mathcal{T}} \sum_{e\in\partial K} 1 \right)^{\frac{1}{2}} \\ \le\;& C L^{\frac{1}{2}} \|u_0\|_{L^2(M)} h^{\frac{d-1}{2}} h^{-\frac{d}{2}} \frac{1}{\beta^{\frac{1}{2}}} k^{\frac{1}{2}}. \end{aligned}$$

the last line follows from (34) and the assumptions on the grid (17)-(19).

□

Next we prove a weak discrete entropy inequality for the approximate solution, which is an auxiliary result to prove a continuous entropy inequality for the approximate solution. This continuous entropy inequality is important for the main convergence proof and has a similar importance for the proof like the entropy inequality for the exact solution.

Lemma 13 (Weak discrete entropy inequality)**.** *For every $\kappa \in [U_m, U_M]$, every polyhedron $K \in \mathcal{T}$ and every test function $\varphi \in C_0^\infty(\mathbb{R}_+, \mathbb{R}_+)$ the following inequality holds*

$$\int_{\mathbb{R}_+} |K||u_K^h(t) - \kappa|\varphi_t \, dt + |K||u_K^h(0) - \kappa|\varphi(0)$$
$$- \int_{\mathbb{R}_+} \sum_{e \in \partial K} |e| \left(f_{K,e}(u_K^h \top \kappa, u_{K_e}^h \top \kappa) - f_{K,e}(u_K^h \bot \kappa, u_{K_e}^h \bot \kappa) \right) \varphi \, dt \geq 0.$$

Proof. Let $B = \sup\{t > 0 : \varphi(t) \neq 0\}$. Consider disjoint intervals $\{I_j = (a_j, b_j) : j \in \mathcal{J}\}$, where $\mathcal{J}$ is some countable index set, such that

$$A := \bigcup_{j \in \mathcal{J}} I_j = \{t \in (0, B) : u_K^h(t) > \kappa\}.$$

For all b_j we have $u_K^h(b_j) = \kappa$ or $\varphi(b_j) = 0$. For all but at most one a_j we have $u_K^h(a_j) = \kappa$. If there is an $a_* \in \{a_j : j \in \mathcal{J}\}$ with $u_K^h(a_*) \neq \kappa$ we have $a_* = 0$. To make the proof shorter we nevertheless denote one a_j by a_* satisfying $a_* = 0$ or $u_K^h(a_*) = \kappa$. Using this notation we have

$$\begin{aligned}
|K| \int_{\mathbb{R}_+} (u_K^h(t) \top \kappa) \varphi_t \, dt &= |K| \sum_j \int_{I_j} u_K^h \varphi_t \, dt + |K| \int_{\mathbb{R}_+ \setminus A} \kappa \varphi_t \, dt \\
&= |K| \sum_j \int_{I_j} (u_K^h - \kappa) \varphi_t \, dt + |K| \int_{\mathbb{R}_+} \kappa \varphi_t \, dt \\
&= |K| \left(\sum_j \left[(u_K^h - \kappa)(b_j)\varphi(b_j) - (u_K^h - \kappa)(a_j)\varphi(a_j) - \int_{I_j} (u_K^h)_t \varphi \, dt \right] - \kappa \varphi(0) \right) \\
&= -|K| u_K^h(a_*)\varphi(a_*) + \kappa |K| (\varphi(a_*) - \varphi(0)) - \int_A |K| (u_K^h)_t \varphi \, dt \\
&\geq -(u_K^h(0) \top \kappa)\varphi(0)|K| - \int_A |K| (u_K^h)_t \varphi \, dt.
\end{aligned}$$

For $t \in A$ we have by (25) and (26)

$$\begin{aligned}
|K|(u_K^h)_t \varphi &= -\sum_{e \in \partial K} |e| f_{K,e}(u_K^h \top \kappa, u_{K_e}^h) \varphi \\
&\leq -\sum_{e \in \partial K} |e| f_{K,e}(u_K^h \top \kappa, u_{K_e}^h \top \kappa) \varphi
\end{aligned}$$

while for $t \in \mathbb{R} \setminus A$ we have by (4),(24) and (25)

$$\begin{aligned}
0 &= \varphi \sum_{e \in \partial K} |e| f_{K,e}(u_K^h \top \kappa, \kappa) \\
&\leq -\sum_{e \in \partial K} |e| f_{K,e}(u_K^h \top \kappa, u_{K_e}^h \top \kappa) \varphi.
\end{aligned}$$

Thus we get

$$|K| \int_{\mathbb{R}_+} (u_K^h \top \kappa) \varphi_t \, dt + (u_K^h(0) \top \kappa) \varphi(0) |K|$$
$$- \int_{\mathbb{R}_+} \sum_{e \in \partial K} |e| f_{K,e}(u_K^h \top \kappa, u_{K_e}^h \top \kappa) \varphi \, dt \geq 0.$$

In a similar way we can prove

$$|K| \int_{\mathbb{R}_+} (u_K^h \perp \kappa)\varphi_t \, dt + (u_K^h(0) \perp \kappa)\varphi(0)|K|$$
$$- \int_{\mathbb{R}_+} \sum_{e \in \partial K} |e| f_{K,e}(u_K^h \perp \kappa, u_{K_e}^h \perp \kappa)\varphi \, dt \leq 0.$$

The Lemma follows from $|u_K^h(t) - \kappa| = (u_K^h(t) \top \kappa) - (u_K^h(t) \perp \kappa)$. □

We observe that because M is compact (4) implies that the norms $\|f\|_{L^\infty(M)}$ and $\|\nabla f\|_g$ are bounded by a constant C_2. This means particularly for every unit vector t tangent to M the following estimate for the covariant derivative in direction t holds: $\|\nabla_t f\|_g \leq C_2$ on $M \times [U_m, U_M]$.

Lemma 14 (Continuous entropy inequality)**.** *Provided the assumptions (17)-(20) on the grid with h small enough and (23)-(25) on the numerical fluxes, there is a constant $C > 0$ such that for every $\varphi \in C_0^\infty(M \times \mathbb{R}_+, \mathbb{R}_+)$ and $\kappa \in [U_m, U_M]$ we have*

$$\begin{aligned}
&\int_0^T \int_M |u^h(x,t) - \kappa|\varphi_t(x,t) \, dv_g(x)dt + \int_M |u_0(x) - \kappa|\varphi(x,0) \, dv_g(x) \\
+ &\int_0^T \int_M \big(f(x, u^h(x,t) \top \kappa) - f(x, u^h(x,t) \perp \kappa)\big) \cdot \nabla_g \varphi(x,t) \, dv_g(x) dt \\
\geq &- \int_M |u^h(x,0) - u_0(x)|\varphi(x,0) dv_g(x) \\
- &\; 2\int_0^T \sum_{K \in \mathcal{T}} \sum_{e \in \partial K} |e| \left[\max_{(c,d) \in C(u_K^h, u_{K_e}^h)} |f_{K,e}(c,d) - f_{K,e}(c,c)| + C\delta(K) \right] r_{K,e}(t) \, dt
\end{aligned}$$

with

$$r_{K,e}(t) := \frac{1}{|K||e|} \int_e \int_K \int_0^{d_g(x,y)} \|\nabla_g \varphi(\gamma_{xy}(\theta), t)\|_g \, d\theta dv_e(y) dv_g(x). \tag{35}$$

Proof. We start by using $\psi(t) := \frac{1}{|K|}\int_K \varphi(x,t) \, \mathrm{dv}_g(x)$ as test function in the weak discrete entropy inequality (Lemma 13) and summing over all $K \in \mathcal{T}$. Using that f is geometry compatible (4) and the consistency property of the numerical fluxes (24) we get $T_1 + T_2 \leq 0$ with

$$\begin{aligned}
T_1 \;\; &:= \;\; -\int_0^T \sum_{K \in \mathcal{T}} |u_K^h(t) - \kappa| \left(\int_K \varphi(x,t) \, \mathrm{dv}_g(x) \right)_t dt \\
&\quad\; - \sum_{K \in \mathcal{T}} |u_K^h(0) - \kappa| \int_K \varphi(x,0) \, \mathrm{dv}_g(x) \\
&= \;\; -\int_0^T \int_M |u^h(x,t) - \kappa|\varphi_t(x,t) \, \mathrm{dv}_g(x) dt \\
&\quad\; - \int_M |u^h(x,0) - \kappa|\varphi(x,0) \mathrm{dv}_g(x) \\
T_2 \;\; &:= \;\; \int_0^T \sum_{K \in \mathcal{T}} \sum_{e \in \partial K} \frac{|e|}{|K|} \Big(f_{K,e}(u_K^h(t) \top \kappa, u_{K_e}^h(t) \top \kappa) \\
&\quad\; - f_{K,e}(u_K^h(t) \top \kappa, u_K^h(t) \top \kappa) - f_{K,e}(u_K^h(t) \perp \kappa, u_{K_e}^h(t) \perp \kappa) \\
&\quad\; + f_{K,e}(u_K^h(t) \perp \kappa, u_K^h(t) \perp \kappa) \Big) \int_K \varphi(x,t) \, \mathrm{dv}_g(x) dt.
\end{aligned}$$

Now let

$$\begin{aligned} T_{10} &:= -\int_0^T \int_M |u^h(x,t) - \kappa| \varphi_t(x,t) \, \mathrm{dv}_g(x) dt \\ &\quad - \int_M |u_0(x) - \kappa| \varphi(x,0) \, \mathrm{dv}_g(x) \\ T_{20} &:= -\int_0^T \int_M \big(f(x, u^h(x,t) \top \kappa) \\ &\quad - f(x, u^h(x,t) \bot \kappa) \big) \nabla_g \varphi(x,t) \, \mathrm{dv}_g(x) dt. \end{aligned}$$

We are going to estimate $|T_1 - T_{10}|$ and $|T_2 - T_{20}|$. Obviously we have

$$\begin{aligned} |T_1 - T_{10}| &\leq \int_M \big| |u^h(x,0) - \kappa| - |u_0(x) - \kappa| \big| \, \varphi(x,0) \, \mathrm{dv}_g(x) \\ &\leq \int_M |u^h(x,0) - u_0(x)| \varphi(x,0) \, \mathrm{dv}_g(x). \end{aligned}$$

Due to the geometry compatibility of the numerical fluxes (4) we have

$$\begin{aligned} T_{20} &= -\int_0^T \sum_{K \in \mathcal{T}} \int_K \nabla_g \cdot \Big[\big(f(x, u_K^h(t) \top \kappa) - f(x, u_K^h(t) \bot \kappa) \big) \, \varphi(x,t) \Big] \, \mathrm{dv}_g(x) \\ &= -\int_0^T \sum_{K \in \mathcal{T}} \sum_{e \in \partial K} \int_e \big(f(x, u_K^h(t) \top \kappa) - f(x, u_K^h(t) \bot \kappa) \big) \, n_{K,e}(x) \varphi(x,t) \, \mathrm{dv}_e(x) \\ &\quad + \int_0^T \sum_{K \in \mathcal{T}} \sum_{e \in \partial K} |e| \big(f_{K,e}(u_K^h(t) \top \kappa, u_{K_e}^h(t) \top \kappa) \\ &\quad - f_{K,e}(u_K^h(t) \bot \kappa, u_{K_e}^h(t) \bot \kappa) \big) \, \frac{1}{|e|} \int_e \varphi(x,t) \, \mathrm{dv}_e(x) dt \end{aligned}$$

because the last summand is zero due to the fact that each face e is a face of exactly two polyhedra and the conservation property (23) of the numerical fluxes. Therefore we get

$$\begin{aligned} |T_2 - T_{20}| \leq \int_0^T \Big| &\sum_{K \in \mathcal{T}} \sum_{e \in \partial K} \int_e \big(f(y, u_K^h(t) \top \kappa) - f(y, u_K^h(t) \bot \kappa) \big) \\ &\cdot n_{K,e}(y) \left(\varphi(y,t) - \frac{1}{|K|} \int_K \varphi(x,t) \, \mathrm{dv}_g(x) \right) \, \mathrm{dv}_e(y) \\ + \;\; &|e| \big(f_{K,e}(u_K^h(t) \top \kappa, u_{K_e}^h(t) \top \kappa) - f_{K,e}(u_K^h(t) \bot \kappa, u_{K_e}^h(t) \bot \kappa) \big) \\ &\left(\frac{1}{|K|} \int_K \varphi(x,t) \, \mathrm{dv}_g(x) - \frac{1}{|e|} \int_e \varphi(y,t) \, \mathrm{dv}_e(y) \right) \Big| \, dt. \end{aligned} \tag{36}$$

To estimate this further we need an estimate for

$$\big| f(x, u_K^h(t) \top \kappa) \cdot n_{K,e}(x) - f_{K,e}(u_K^h \top \kappa, u_K^h \top \kappa) \big|$$

for every $x \in e$. The fact that $f \cdot n_{K,e}$ is continuous with respect to the space variable implies due to (24)

$$\begin{aligned} f_{K,e}(u_K^h \top \kappa, u_K^h \top \kappa) &= \frac{1}{|e|} \int_e f(x, u_K^h(t) \top \kappa) n_{K,e}(x) \, \mathrm{dv}_e(x) \\ &= f(\xi, u_K^h(t) \top \kappa) n_{K,e}(\xi) \end{aligned}$$

for some $\xi \in e$. Due to the bound $\bar{R}$ for the principal curvatures of the faces we have

$$\begin{aligned} &t \langle f(x, (u_K^h(t) \top \kappa)), n_{K,e}(x) \rangle_g \\ &= \langle \nabla_t f(x, (u_K^h(t) \top \kappa)), n_{K,e}(x) \rangle_g + \langle f(x, (u_K^h(t) \top \kappa)), \nabla_t n_{K,e}(x) \rangle_g \\ &\leq C_2 + d \bar{R} \|f\|_\infty. \end{aligned}$$

Thus we have

$$\begin{aligned}
&\left|f(x,(u_K^h(t)\top\kappa))n_{K,e}(x)-f_{K,e}(u_K^h\top\kappa,u_K^h\top\kappa)\right|\\
=\;&\left|f(x,(u_K^h(t)\top\kappa))n_{K,e}(x)-f(\xi,(u_K^h(t)\top\kappa))n_{K,e}(\xi)\right|\\
\le\;&\delta(e)\left(C_2+d\bar{R}\|f\|_\infty\right)\le\delta(K)\left(C_2+d\bar{R}\|f\|_\infty\right).
\end{aligned}$$

Using a similar estimate for the $\perp$ case we get from (36)

$$\begin{aligned}
|T_2-T_{20}| \;\le\; & \int_0^T\sum_{K\in\mathcal{T}}\sum_{e\in\partial K}\Big[|e|\,\big|-f_{K,e}(u_K^h(t)\top\kappa,u_K^h(t)\top\kappa)\\
& +f_{K,e}(u_K^h(t)\perp\kappa,u_K^h(t)\perp\kappa)\\
& +f_{K,e}(u_K^h(t)\top\kappa,u_{K_e}^h(t)\top\kappa)-f_{K,e}(u_K^h(t)\perp\kappa,u_{K_e}^h(t)\perp\kappa)\big|\\
& \frac{1}{|e||K|}\int_e\int_K|\varphi(x,t)-\varphi(y,t)|\,\mathrm{dv}_e(y)\mathrm{dv}_g(x)\\
+\; & \delta(K)C\int_e\left|\varphi(y,t)-\frac{1}{|K|}\int_K\varphi(x,t)\mathrm{dv}_g(x)\right|\mathrm{dv}_e(y)\Big].
\end{aligned}$$

For h small enough, $x\in K$ and $y\in e\in\partial K$ let γ_{xy} denote the unique minimising geodesic from x to y parametrised by arc length. Then we have

$$\begin{aligned}
&\frac{1}{|K||e|}\int_e\int_K|\varphi(x,t)-\varphi(y,t)|\,\mathrm{dv}_e(y)\mathrm{dv}_g(x)\\
=\;&\frac{1}{|K||e|}\int_e\int_K\left|\int_0^{d_g(x,y)}\langle\nabla_g\varphi(\gamma_{xy}(s),t),\gamma'_{xy}(s)\rangle_g\,ds\right|\mathrm{dv}_g(x)\mathrm{dv}_e(y)\\
=\;&\frac{1}{|K||e|}\int_e\int_K\int_0^{d_g(x,y)}\|\nabla_g\varphi(\gamma_{xy}(\theta),t)\|_g\;d\theta\,\mathrm{dv}_e(y)\mathrm{dv}_g(x).
\end{aligned}$$

This finally yields

$$\begin{aligned}
&|T_2-T_{20}|\le\int_0^T\sum_{K\in\mathcal{T}}\sum_{e\in\partial K}|e|\,\Big[\big|-f_{K,e}(u_K^h(t)\top\kappa,u_K^h(t)\top\kappa)\\
&+f_{K,e}(u_K^h(t)\perp\kappa,u_K^h(t)\perp\kappa)+f_{K,e}(u_K^h(t)\top\kappa,u_{K_e}^h(t)\top\kappa)\\
&-f_{K,e}(u_K^h(t)\perp\kappa,u_{K_e}^h(t)\perp\kappa)\big|+C\delta(K)\Big]r_{K,e}
\end{aligned}\tag{37}$$

with $r_{K,e}$ given in (35).

Now we want to estimate the right hand side of the above inequality (37). Due to the monotonicity (25) of the numerical fluxes we observe for $u_K^h\ge u_{K_e}^h$

$$\begin{aligned}
0\;&\le -f_{K,e}(u_K^h\top\kappa,u_K^h\top\kappa)+f_{K,e}(u_K^h\top\kappa,u_{K_e}^h\top\kappa)\\
&\le\max_{u_{K_e}^h\le c\le d\le u_K^h}\left(-f_{K,e}(d,d)+f_{K,e}(d,c)\right)\\
&=\max_{(c,d)\in C(u_K^h,u_{K_e}^h)}|f_{K,e}(c,d)-f_{K,e}(c,c)|
\end{aligned}$$

and for $u_K^h\le u_{K_e}^h$

$$\begin{aligned}
0\;&\le f_{K,e}(u_K^h\top\kappa,u_K^h\top\kappa)-f_{K,e}(u_K^h\top\kappa,u_{K_e}^h\top\kappa)\\
&\le\max_{u_K^h\le c\le d\le u_{K_e}^h}\left(+f_{K,e}(c,c)-f_{K,e}(c,d)\right)\\
&=\max_{(c,d)\in C(u_K^h,u_{K_e}^h)}|f_{K,e}(c,d)-f_{K,e}(c,c)|.
\end{aligned}$$

There are similar estimates for $\perp$ instead of $\top$ which show that

$$\begin{aligned} & \sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \, \big| f_{K,e}(u_K^h(t)\top\kappa, u_K^h(t)\top\kappa) - f_{K,e}(u_K^h(t)\top\kappa, u_{K_e}^h(t)\top\kappa) \\ - \quad & f_{K,e}(u_K^h(t)\perp\kappa, u_K^h(t)\perp\kappa) + f_{K,e}(u_K^h(t)\perp\kappa, u_{K_e}^h(t)\perp\kappa)\big| + C\delta(K)\big] \\ \leq \quad & 2\sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \left[\max_{(c,d)\in C(u_K^h, u_{K_e}^h)} |f_{K,e}(c,d) - f_{K,e}(c,c)| + C\delta(K) \right]. \end{aligned}$$

This implies together with (37)

$$\begin{aligned} |T_2 - T_{20}| \quad \leq \quad & \int_0^T 2\sum_{K\in\mathcal{T}}\sum_{e\in\partial K} |e| \Big[\max_{(c,d)\in C(u_K^h, u_{K_e}^h)} |f_{K,e}(c,d) - f_{K,e}(c,c)| \\ + \quad & C\delta(K)\Big] r_{K,e}(t) \end{aligned}$$

which implies the Lemma. □

The next Lemma is a very important step in the convergence proof. There will be different estimates for $d = 1$ and $d \geq 2$. This is due to the fact that while we have the TVD property (Lemma 10) in the $d = 1$ case, we only have the weak BV estimate (Lemma 11) in the $d \geq 2$ case. The proof will be done for $d \geq 2$ only. The proof for $d = 1$ follows from the same arguments using Lemma 10 instead of Lemma 11.

Lemma 15. *Provided the assumptions from Lemma 14 there exists a constant $C > 0$ depending only on $M, g, u_0, \{f_{K,e}\}, f, \beta, k, \bar{R}, N_c$ such that for small enough h and every test function $\alpha \in C_0^\infty(M \times \mathbb{R}_+, \mathbb{R}_+)$ the following inequality holds*

$$\begin{aligned} & \int_{M\times\mathbb{R}_+} |u^h(x,t) - u(x,t)|\alpha_t(x,t)\, dv_g(x)\, dt \\ + \quad & \int_{M\times\mathbb{R}_+} \big[f(x, u(x,t)\top u^h(x,t)) - f(x, u(x,t)\perp u^h(x,t))\big] \cdot \nabla_g \alpha(x,t)\, dv_g(x)\, dt \\ \geq \quad & \begin{cases} -Ch^{\frac{1}{2}} & : \quad d = 1 \\ -Ch^{\frac{1}{4}} & : \quad d \geq 2 \end{cases}. \end{aligned}$$

Proof. The proof is based on a doubling of variables argument. We recall the entropy inequality (3) fulfilled by the entropy solution u of (1),(2)

$$\begin{aligned} & \int_{M\times\mathbb{R}_+} |u(y,s) - \kappa|\varphi_s(y,s) \\ + \quad & [f(y, u(y,s)\top\kappa) - f(y, u(y,s)\perp\kappa)] \cdot \nabla_g \varphi(y,s)\, \mathrm{dv}_g(y) ds \\ + \quad & \int_M |u_0(y) - \kappa|\varphi(y,0)\, \mathrm{dv}_g(y) \geq 0 \end{aligned}$$

for all $\kappa \in \mathbb{R}$ and $\varphi \in C_0^\infty(M \times \mathbb{R}_+, \mathbb{R}_+)$. In (3) we set $\kappa = u^h(x,t)$ and $\varphi(y,s) = \alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)$, where χ_ε and ψ_ε are cut-off functions as defined in subsection 2.3. Now we integrate this equation with respect to x and t. In the continuous entropy inequality from Lemma 14 we set $\kappa = u(y,s)$ and $\varphi(x,t) = \alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)$ and integrate with respect to y and s. Adding both equations yields

$$
\begin{aligned}
& \int_{M^2\times\mathbb{R}_+^2} |u^h(x,t)-u(y,s)| \\
& \quad \alpha_t(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds \\
+ & \int_{M^2\times\mathbb{R}_+^2} \left[f(y,u(y,s)\top u^h(x,t)) - f(y,u(y,s)\bot u^h(x,t))\right] \\
& \quad \cdot\alpha(x,t)\nabla_{g,y}\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds \\
+ & \int_{M^2\times\mathbb{R}_+^2} \left[f(x,u(y,s)\top u^h(x,t)) - f(x,u(y,s)\bot u^h(x,t))\right] \\
& \quad \cdot\nabla_g\alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds \\
+ & \int_{M^2\times\mathbb{R}_+^2} \left[f(x,u(y,s)\top u^h(x,t)) - f(x,u(y,s)\bot u^h(x,t))\right] \\
& \quad \cdot\alpha(x,t)\nabla_{g,x}\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds \qquad (38)\\
+ & \int_{M^2\times\mathbb{R}_+} |u_0(x)-u(y,s)|\alpha(x,0)\chi_\varepsilon(x,y)\psi_\varepsilon(-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)ds \\
\geq & -\int_{M^2\times\mathbb{R}_+} |u^h(x,0)-u_0(x)|\alpha(x,0)\chi_\varepsilon(x,y)\psi_\varepsilon(-s)\, ds\mathrm{dv}_g(y)\mathrm{dv}_g(x) \\
- & \ 2\int_{M\times\mathbb{R}_+}\int_0^T \sum_{K\in\mathcal{T}}\sum_{e\in\partial K}|e|\Bigg[\max_{(c,d)\in C(u_K^h,u_{K_e}^h)}|f_{K,e}(c,d)-f_{K,e}(c,c)| \\
& \quad +C\delta(K)\Bigg]r_{K,e}(t)\,dt\,ds\mathrm{dv}_g(y).
\end{aligned}
$$

We will start with the most difficult summand. Let E_2 be the sum of the second and fourth summand, i.e.

$$
\begin{aligned}
E_2 \ := \ & \int_{M^2\times\mathbb{R}_+^2}\Big\{\left[f(y,u(y,s)\top u^h(x,t)) - f(y,u(y,s)\bot u^h(x,t))\right]\cdot\nabla_{g,y}\chi_\varepsilon(x,y) \\
& + \left[f(x,u(y,s)\top u^h(x,t)) - f(x,u(y,s)\bot u^h(x,t))\right]\cdot\nabla_{g,x}\chi_\varepsilon(x,y)\Big\} \\
& \alpha(x,t)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds.
\end{aligned}
$$

We also define

$$
\begin{aligned}
E_{2b} \ := \ & \int_{M^2\times\mathbb{R}_+^2}\Big\{\left[f(y,u(x,t)\top u^h(x,t)) - f(y,u(x,t)\bot u^h(x,t))\right]\cdot\nabla_{g,y}\chi_\varepsilon(x,y) \\
& + \left[f(x,u(x,t)\top u^h(x,t)) - f(x,u(x,t)\bot u^h(x,t))\right]\cdot\nabla_{g,x}\chi_\varepsilon(x,y)\Big\} \\
& \alpha(x,t)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds.
\end{aligned}
$$

For ease of notation we will from now on omit the $\bot$-terms for the estimate for E_2. Adding zero we get

$$
\begin{aligned}
E_{2b} \ \overset{(7),(13)}{=} \ & \int_{M^2\times\mathbb{R}_+^2} f(y,u(x,t)\top u^h(x,t))\alpha(x,t)\nabla_{g,y}\chi_\varepsilon(x,y)\psi_\varepsilon(t-s) \\
& -(T\exp_x)_{\exp_x^{-1}(y)}(f(x,u(x,t)\top u^h(x,t)))\alpha(x,t)\nabla_{g,y}\chi_\varepsilon(x,y)\psi_\varepsilon(t-s) \\
& + \Big((T\exp_x)_{\exp_x^{-1}(y)}(f(x,u(x,t)\top u^h(x,t))) - P_{xy}(f(x,u(x,t)\top u^h(x,t)))\Big)\cdot \\
& \quad \nabla_{g,y}\chi_\varepsilon(x,y)\alpha(x,t)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds.
\end{aligned}
$$

Now using integration by parts w.r.t. y the first summand vanishes because $f(\cdot,s)$ is divergence free for fixed $s\in\mathbb{R}$ (4), the absolute value of the second summand is smaller or equal $C\varepsilon$ which can be seen after integration by parts w.r.t. y by (14). To get an estimate for the third summand we observe that for every $x \in M$ and $\varepsilon < 2R$

$$\int_{B_\varepsilon(x)} \mathrm{dv}_g(y) = \int_{B_\varepsilon(0)} |\det(T\exp_x)_v| \; dv < C \sup_{x\in M, \|v\|<\varepsilon} |\det(T\exp_x)_v| \, \varepsilon^d. \tag{39}$$

This yields due to (11) and (16)

$$\begin{aligned} |\cdot| &\leq \int_{M^2\times\mathbb{R}_+^2} C\mathbb{I}_{\{d_g(x,y)<\varepsilon\}} \frac{\|f\|_\infty}{\varepsilon^{d-1}} \alpha(x,t)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds \\ &\leq \int_{M\times\mathbb{R}_+} C \sup_{x\in M,\|v\|<\varepsilon} |\det(T\exp_x)_v| \, \varepsilon\alpha(x,t)\mathrm{dv}_g(x)dt \\ &\leq C\varepsilon. \end{aligned}$$

So finally $|E_{2b}| < C\varepsilon$, and it remains to show $|E_2 - E_{2b}| < C\varepsilon$. Introducing the following notation omitting the dependence on t and s :

$$\begin{aligned} h(x,y,\tau) &:= f(\gamma_{xy}(\tau), u(y,s)\top u^h(x,t)) \cdot \gamma'_{xy}(\tau), \\ \tilde{h}(x,y,\tau) &:= f(\gamma_{xy}(\tau), u(x,t)\top u^h(x,t)) \cdot \gamma'_{xy}(\tau), \\ D(x,y) &:= h(x,y,d_g(x,y)) - h(x,y,0) - \tilde{h}(x,y,d_g(x,y)) + \tilde{h}(x,y,0), \end{aligned}$$

we have due to (6) and the definition of χ_ε in (9)

$$|E_2 - E_{2b}| \leq \Big| \int_{M^2\times\mathbb{R}_+^2} D(x,y)\alpha(x,t)\frac{1}{\varepsilon^{d+1}}\chi'\left(\frac{d_g(x,y)}{\varepsilon}\right)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds\Big| \tag{40}$$

$$\begin{aligned} |D| &\leq \left| \int_0^{d_g(x,y)} \frac{d}{d\tau}\left(h(x,y,\tau) - \tilde{h}(x,y,\tau)\right) d\tau \right| \\ &= \left| \int_0^{d_g(x,y)} \left\langle \nabla_{\gamma'_{xy}} \left(f(\cdot, u(y,s)\top u^h(x,t)) - f(\cdot, u(x,t)\top u^h(x,t))\right), \gamma'_{xy} \right\rangle d\tau \right| \\ &\overset{(4)}{=} \leq C\,|u(y,s) - u(x,t)|\, d_g(x,y). \end{aligned}$$

Inserting this in (40) and using the boundedness of χ' we have

$$\begin{aligned} |E_2 - E_{2b}| \leq\; & C \int_{M^2\times\mathbb{R}_+^2} |u(y,s) - u(x,t)| \\ & |\alpha(x,t)| \frac{1}{\varepsilon^d} \mathbb{I}_{\{d_g(x,y)<\varepsilon\}} \psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds. \end{aligned}$$

Now we cover M with finitely many geodesic balls $B_r(x_1), \ldots, B_r(x_N)$, where $r < R$ and R is the constant from Lemma 3. We furthermore restrict to the $\varepsilon < r$ case. Because the derivative of $\exp_{x_i}^{-1}$ is bounded there exists a constant $C_i > 0$ such that

$$d_g(\exp_{x_i}(a), \exp_{x_i}(b)) > C_i\|a-b\|, \;\forall a \in B_r(0), b \in B_{2r}(0) \tag{41}$$

which implies

$$\mathbb{I}_{\{d_g(\exp_{x_i}(a),\exp_{x_i}(b))<\varepsilon\}} \leq \mathbb{I}_{\{\|b-a\|<\frac{\varepsilon}{C_i}\}}.$$

Hence we have by

$$\begin{aligned} |E_2 - E_{2b}| &\leq C\sum_i \int_{\mathbb{R}_+^2}\int_{B_r(0)}\int_{B_{2r}(0)} |u(\exp_{x_i}(a),t) - u(\exp_{x_i}(b),s)| \\ &\quad\; \varepsilon^{-d}\mathbb{I}_{\{\|b-a\|<\frac{\varepsilon}{C_i}\}}\mathbb{I}_{\mathrm{supp}\alpha}\psi_\varepsilon(t-s)\,da\,db\,dt\,ds \\ &\leq C\varepsilon, \end{aligned}$$

because each $u\circ\exp_{x_i}$ has bounded variation. Finally we have

$$|E_2| \leq C\varepsilon.$$

Let

$$\begin{aligned} E_1 &:= \int_{M^2\times\mathbb{R}_+^2} |u^h(x,t)-u(y,s)|\alpha_t(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)dtds, \\ E_{1b} &:= \int_{M\times\mathbb{R}_+} |u^h(x,t)-u(x,t)|\alpha_t(x,t)\mathrm{dv}_g(x)\,dt. \end{aligned}$$

Due to (12) we have

$$\begin{aligned} |E_1-E_{1b}| \quad\leq\quad & \int_{M^2\times\mathbb{R}_+^2} |u(x,t)-u(y,s)||\alpha_t(x,t)|\chi_\varepsilon(x,y) \\ & \psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\mathrm{dv}_g(y)\,dt\,ds+C\varepsilon. \end{aligned}$$

To estimate the first part of the right hand side we again cover M with balls $B_r(x_i)$ like in the estimate for E_2. From the definition of χ_ε in (9) we know that χ is nonincreasing for positive x, this yields the following inequality

$$\begin{aligned} |E_1-E_{1b}| \quad\leq\quad & \sum_i\int_{\mathbb{R}_+^2}\int_{B_{r_i}(0)}\int_{B_{r_i}(0)} |u(\exp_{x_i}(a),t)-u(\exp_{x_i}(b),s)| \\ & |\alpha_t(\exp_{x_i}(a),t)|\frac{1}{\varepsilon^d}\chi\left(\frac{C_i\|b-a\|}{\varepsilon}\right)\psi_\varepsilon(t-s)\,da\,db\,dt\,ds+C\varepsilon \\ \leq\quad & C\varepsilon, \end{aligned}$$

because the L^1-norms of the $\frac{1}{\varepsilon^d}\chi\left(\frac{C_i\|b-a\|}{\varepsilon}\right)$ are uniformly bounded with respect to ε. The constants C_i were chosen like in (41).

Let

$$\begin{aligned} E_3 \quad:=\quad & \int_{M^2\times\mathbb{R}_+^2} \left[f(x,u(y,s)\top u^h(x,t))-f(x,u(y,s)\bot u^h(x,t))\right] \\ & \cdot\nabla_g\alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\,\mathrm{dv}_g(x)\mathrm{dv}_g(y)\,dt\,ds \\ E_{3b} \quad:=\quad & \int_{M\times\mathbb{R}_+} \left[f(x,u(x,t)\top u^h(x,t))-f(x,u(x,t)\bot u^h(x,t))\right]\cdot\nabla_g\alpha(x,t)\,\mathrm{dv}_g(x)\,dt. \end{aligned}$$

Then we have

$$\begin{aligned} |E_3-E_{3b}| \quad\leq\quad & C\int_{M^2\times\mathbb{R}_+^2} |u(y,s)-u(x,t)|\,\|\nabla_g\alpha(x,t)\| \\ & \chi_\varepsilon(x,y)\psi_\varepsilon(t-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)\,dt\,ds+C\varepsilon \\ \leq\quad & C\varepsilon, \end{aligned}$$

like in the estimate for E_1. To estimate the fifth summand on the left hand side of (38), denotedE_4, we consider the entropy inequality (3) fulfilled by u. For fixed $x\in M$ we define

$$\varphi(x,y,s):=\alpha(x,0)\chi_\varepsilon(x,y)\int_s^\infty\psi_\varepsilon(-\tau)\,d\tau.$$

and $\kappa=u_0(x)$. Then integration with respect to x yields

$$\begin{aligned} - \quad & \int_{M^2\times\mathbb{R}_+} |u(y,s)-u_0(x)|\alpha(x,0)\chi_\varepsilon(x,y)\psi_\varepsilon(-s)\mathrm{dv}_g(x)\mathrm{dv}_g(y)\,ds \\ + \quad & \int_{M^2\times\mathbb{R}_+} (f(y,u(y,s)\top u_0(x))-f(y,u(y,s)\bot u_0(x)))\cdot\nabla_{g,y}\chi_\varepsilon(x,y) \\ & \alpha(x,0)\left(\int_s^\infty\psi_\varepsilon(-\tau)d\tau\right)\mathrm{dv}_g(x)\mathrm{dv}_g(y)\,ds \\ + \quad & \int_{M^2} |u_0(y)-u_0(x)|\alpha(x,0)\chi_\varepsilon(x,y)\left(\int_0^\infty\psi_\varepsilon(-\tau)d\tau\right)\mathrm{dv}_g(x)\mathrm{dv}_g(y)\geq 0. \end{aligned}$$

We note that the first summand here is exactly $-E_4$, thus we denote the summands by $-E_4, E_5, E_6$ respectively. To estimate E_5 we define E_{5b} by

$$\begin{aligned} E_{5b} \quad := \quad & \int_{M^2\times\mathbb{R}_+}\int_s^\infty \left(f(y,u(y,s)\top u_0(y)) - f(y,u(y,s)\bot u_0(y))\right) \\ & \cdot\nabla_{g,y}\chi_\varepsilon(x,y)\alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\,\mathrm{dv}_g(x)\mathrm{dv}_g(y) \end{aligned}$$

From now on we will omit the $\bot$-terms for convenience again, they are estimated in exactly the same way as the $\top$-terms. Using integration by parts we have by (7)

$$\begin{aligned} E_{5b} \quad = \quad & -\int_{M^2\times\mathbb{R}_+}\int_s^\infty P_{yx}\left(f(y,u(y,s)\top u_0(y))\right) \\ & \cdot\nabla_{g,x}\chi_\varepsilon(x,y)\alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\,\mathrm{dv}_g(x)\mathrm{dv}_g(y) \\ = \quad & \int_{M^2\times\mathbb{R}_+}\int_s^\infty \Big((T\exp_y)_{\exp_y^{-1}(x)}(f(y,u(y,s)\top u_0(y))) - P_{yx}\left(f(y,u(y,s)\top u_0(y))\right)\Big) \\ & \cdot\nabla_{g,x}\chi_\varepsilon(x,y)\alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\,\mathrm{dv}_g(x)\mathrm{dv}_g(y) \\ & -\int_{M^2\times\mathbb{R}_+}\int_s^\infty (T\exp_y)_{\exp_y^{-1}(x)}(f(y,u(y,s)\top u_0(y)))\cdot\nabla_{g,x}\chi_\varepsilon(x,y) \\ & \alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\,\mathrm{dv}_g(x)\mathrm{dv}_g(y) \end{aligned}$$

Now we get using (16) for the first and integration by parts for the second summand

$$\begin{aligned} |E_{5b}| \quad \leq \quad & \Big|\int_{M^2\times\mathbb{R}_+}\int_s^\infty C\mathbb{I}_{\{d_g(x,y)<\varepsilon\}}\frac{1}{\varepsilon^{d-1}}\|f\|_\infty\alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\,\mathrm{dv}_g(x)\mathrm{dv}_g(y)\Big| \\ & +\Big|\int_{M^2\times\mathbb{R}_+}\int_s^\infty \mathrm{div}_{g,x}(T\exp_y)_{\exp_y^{-1}(x)}(f(y,u(y,s)\top u_0(y))) \\ & \chi_\varepsilon(x,y)\alpha(x,0)\psi_\varepsilon(-\tau)\,d\tau\,ds\,\mathrm{dv}_g(x)\mathrm{dv}_g(y)\Big| \\ & +\Big|\int_{M^2\times\mathbb{R}_+}\int_s^\infty (T\exp_y)_{\exp_y^{-1}(x)}(f(y,u(y,s)\top u_0(y))) \\ & \cdot\nabla_g\alpha(x,0)\chi_\varepsilon(x,y)\psi_\varepsilon(-\tau)\,d\tau\,ds\,\mathrm{dv}_g(x)\mathrm{dv}_g(y)\Big|. \end{aligned}$$

The first summand is smaller than $C\varepsilon$ because of (39). The second summand is smaller than $C\varepsilon$ due to (14) and the third summand has this property because, due to (15), the integrand is bounded and the support with respect to s lies in $[0,\varepsilon]$. Finally we have $|E_{5b}| \leq C\varepsilon$. Furthermore

$$|E_5 - E_{5b}| \quad \leq \quad C\int_{M^2\times\mathbb{R}_+}\int_s^\infty |u_0(x)-u_0(y)|\,\|\nabla_{g,y}\chi_\varepsilon(x,y)\|\,\psi_\varepsilon(-\tau)d\tau\,ds\mathrm{dv}_g(y)\mathrm{dv}_g(x).$$

Integrating with respect to τ and s yields

$$|E_5 - E_{5b}| \leq C\int_{M^2} |u_0(x)-u_0(y)|\,\|\nabla_{g,y}\chi_\varepsilon(x,y)\|\,\varepsilon\mathrm{dv}_g(y)\mathrm{dv}_g(x)$$

because the integral over τ is bounded by 1 and the support with respect to s lies in $[0,\varepsilon]$. Then we use the fact

$$\varepsilon\,\|\nabla_{g,y}\chi_\varepsilon(x,y)\| \leq C\varepsilon^{-d}\mathbb{I}_{\{d_g(x,y)<\varepsilon\}}.$$

We cover M with balls like in the estimate for E_2 again and a similar argument yields

$$|E_5 - E_{5b}| \leq C\varepsilon.$$

Another version of this argument implies

$$|E_6| \leq C\varepsilon.$$

So we finally have

$$|E_4| \leq C\varepsilon.$$

Now we have to find an estimate for the right hand side of (38): Keeping in mind the weak BV-estimate (11), the essential part of this estimate is an estimate for

$$\left|\int_{M\times\mathbb{R}_+} r_{K,e}(t)\, ds \mathrm{dv}_g(y)\right|,$$

where $r_{K,e}$ was defined in (35). Because the test function φ in the definition of $r_{K,e}$ now has the form

$$\varphi(x,t) = \alpha(x,t)\chi_\varepsilon(x,y)\psi_\varepsilon(t-s)$$

we have

$$\begin{aligned}
&\left|\int_{M\times\mathbb{R}_+} r_{K,e}(t)\, \mathrm{dv}_g(x) ds\right| \\
&= \frac{C}{|K||e|}\int_M\int_{\mathbb{R}_+}\int_K\int_e\int_0^{d_g(x,z)} \|\nabla_g\alpha(\gamma_{xz}(\theta),t)\|_g \\
&\quad \chi_\varepsilon(\gamma_{xz}(\theta),y)\psi_\varepsilon(t-s) d\theta \mathrm{dv}_e(z)\mathrm{dv}_g(x)\, ds\, \mathrm{dv}_g(y) \\
&+ \frac{C}{|K||e|}\int_M\int_{\mathbb{R}_+}\int_K\int_e\int_0^{d_g(x,z)} \|\nabla_1\chi_\varepsilon(\gamma_{xz}(\theta),y)\|_g \\
&\quad \alpha(\gamma_{xz}(\theta),t)\psi_\varepsilon(t-s) d\theta \mathrm{dv}_e(z)\mathrm{dv}_g(x)\, ds\, \mathrm{dv}_g(y).
\end{aligned} \tag{42}$$

Now integration over s, y yields that the first summand in (42) can be estimated by

$$\begin{aligned}
&\frac{C}{|K||e|}\int_K\int_e\int_0^{d_g(x,z)} \underbrace{\|\nabla_g\alpha(\gamma_{xz}(\theta),t)\|_g}_{\leq\|\nabla_g\alpha\|_{L^\infty(M)}}(1+C\varepsilon) d\theta \mathrm{dv}_e(z)\mathrm{dv}_g(x) \\
&\leq C(1+\varepsilon)\delta(K).
\end{aligned}$$

To estimate the second summand in (42) we observe

$$\|\nabla_1\chi_\varepsilon(\gamma_{xz}(\theta),y)\|_g \quad\leq\quad C\varepsilon^{-d-1}\mathbb{I}_{\{d_g(\gamma_{xz}(\theta),y)\leq\varepsilon\}}.$$

Due to (39) integration over s and y yields that the second summand in (42) is smaller than

$$\frac{C}{|K||e|}\int_K\int_e\int_0^{d_g(x,y)} C\varepsilon^{-1}\, d\theta \mathrm{dv}_e(z)\mathrm{dv}_g(x) \leq C\frac{\delta(K)}{\varepsilon}.$$

So we have due to the weak BV estimate Lemma 11

$$\begin{aligned}
&\int_{M\times\mathbb{R}_+}\int_0^T\sum_{K\in\mathcal{T}}\sum_{e\in\partial K}\left[\max_{(c,d)\in C(u_K^h,u_{K_e}^h)}|e||f_{K,e}(c,d)-f_{K,e}(c,c)|\right] r_{K,e}\mathrm{dv}_g(x)dtds \\
&\leq \frac{C}{\sqrt{h}}\left(\frac{h}{\varepsilon}+h+h\varepsilon\right) = C\sqrt{h}\left(\frac{1}{\varepsilon}+1+\varepsilon\right).
\end{aligned} \tag{43}$$

We observe that due to (17)-(20) we have

$$\int_0^T\sum_{K\in\mathcal{T}}\sum_{e\in\partial K} C\delta(K)|e| \leq 3C\beta^{-1}k.$$

Now it remains to estimate

$$\int_{M^2\times\mathbb{R}_+} |u^h(x,0)-u_0(x)|\alpha(x,0)\chi_\varepsilon(x,y)\psi_\varepsilon(-s)\, ds \mathrm{dv}_g(y)\mathrm{dv}_g(x).$$

Integrating with respect to y, s yields that this term is smaller than

$$C(1+C\varepsilon)\int_M |u^h(x,0)-u_0(x)|\mathrm{dv}_g(x) \leq C(1+\varepsilon)h$$

for ε, h small enough by Lemma 7. This finally implies

$$\begin{aligned} & \int_{M\times\mathbb{R}_+} |u^h(x,t)-u(x,t)|\alpha_t(x,t)\mathrm{dv}_g(x)\,dt \\ + & \int_{M\times\mathbb{R}_+} [f(x,u(x,t)\top u^h(x,t))-f(x,u(x,t)\bot u^h(x,t))]\cdot\nabla_g\alpha(x,t)\mathrm{dv}_g(x)\,dt \\ \geq & -C(\varepsilon+Ch+h\varepsilon+\frac{\sqrt{h}}{\varepsilon}+\sqrt{h}+2\sqrt{h}\varepsilon+h\varepsilon) \\ = & -C(h^{\frac{1}{4}}+h+h^{\frac{3}{2}}+h^{\frac{1}{4}}+h^{\frac{1}{2}}+h+h^{\frac{3}{2}}) \end{aligned}$$

where we set $\varepsilon = h^{\frac{1}{4}}$ for the last equality. □

Now the convergence proof is quite easy and only consists of choosing a sensible test function α in Lemma 15.

Theorem 16. *Provided the assumptions of Lemma 15 hold, then we have for every time $T>0$ a constant $C>0$ depending only on $f, u_0, M, g, T, \{f_{K,e}\}, \beta, k, N_c, \bar{R}$ such that*

$$\int_0^T\int_M |u^h(x,t)-u(x,t)|dv_g(x)\,dt \leq \begin{cases} Ch^{\frac{1}{2}} & : \quad d=1 \\ Ch^{\frac{1}{4}} & : \quad d\geq 2. \end{cases}$$

Proof. For $t\geq 0$ we define

$$\rho(t) := \begin{cases} (1-t)\exp\left(\frac{1}{t^2-1}\right) & : \quad t\leq 1 \\ 0 & : \quad t\geq 1 \end{cases}$$

and $\rho_T(t) := \rho\left(\frac{t}{2T}\right)$. We have $\rho_T(t)\in[0,e]$ and there exists $\varepsilon>0$ such that $\rho'(t) < -\varepsilon\ \forall t\in[0,\frac{1}{2}]$ and therefore $\rho_T'(t) < \frac{-\varepsilon}{2T}\ \forall t\in[0,T]$. Now we define $\alpha(x,t)=\rho_T(t)$, then we have $\alpha\in C_0^\infty(M\times\mathbb{R}_+,\mathbb{R}_+)$ and $\nabla_g\alpha = 0$ which implies

$$\begin{aligned} & \frac{-\varepsilon}{2T}\int_M\int_0^T |u^h(x,t)-u(x,t)|\mathrm{dv}_g(x)\,dt \\ \geq & \int_M\int_0^T \alpha_t(x,t)|u^h(x,t)-u(x,t)|\mathrm{dv}_g(x)\,dt \\ \geq & \begin{cases} -Ch^{\frac{1}{2}} & : \quad d=1 \\ -Ch^{\frac{1}{4}} & : \quad d\geq 2, \end{cases} \end{aligned}$$

which proves the Theorem. □

6. Appendix

6.1. **Proof of Lemma 5.** Let $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\alpha\}_{\alpha\in A}$ be finite coverings of M with $\bar{U}_\alpha\subset V_\alpha$, such that there is an orthonormal basis $\{X_i^\alpha\}_{i=1,\ldots,n}$ of vector-fields over V_α. Let R be the constant from Lemma 3, then we define for every α

$$K_\alpha := \{(x,y)\in M^2 : x\in\bar{U}_\alpha, d_g(x,y)\leq R\}, \tag{44}$$

$$\tilde{K}_\alpha := \{(x,y)\in M^2 : x\in V_\alpha, d_g(x,y)< 2R\} \tag{45}$$

and observe that each K_α is compact. Furthermore we define for $\alpha\in A, i=1,\ldots,n$

$$h_i^\alpha : \tilde{K}_\alpha \longrightarrow TM \quad (x,y)\mapsto \left(y, (T\exp_x)_{\exp_x^{-1}(y)}(X_i^\alpha(x))\right),$$

which is smooth as a function of (x,y). Hence its norm is bounded on K_α, which proves (15) because there are only finitely many $\alpha\in A$. For fixed $x\in V_\alpha$ the map $h_i^\alpha(x,\cdot)$ is a vector-field on $B_{2R}(x)$ and hence $\nabla_y h_i^\alpha(x,y)$, the covariant derivative with respect to y, is well-defined and smooth on $\tilde{K}_\alpha$. Furthermore

$$\nabla_y h_i^\alpha(x,x) = 0 \quad \forall x\in V_\alpha \tag{46}$$

(cf. [13]), which implies

$$\mathrm{div}_{g,y} h_i^\alpha(x,x) = 0,$$

because the divergence is the trace of the covariant derivative. For every α, i, j, k

$$(x,y) \mapsto \langle \nabla_{X_j(y)} h_i^\alpha(x,y), X_k(y)\rangle$$

is smooth on $\tilde{K}_\alpha$ and vanishes for $x = y$ due to (46). Its gradient is bounded on K_α and we have

$$\|\nabla_y h_i^\alpha(x,y)\| \le d^2 \sup_{j,k} |\langle \nabla_{X_j} h_i^\alpha, X_k\rangle| \le d^2 C_{i,\alpha} d_g(x,y) \tag{47}$$

for $(x,y) \in K_\alpha$, where $\|\cdot\|$ denotes the operatornorm with respect to g. This obviously implies

$$|\mathrm{div}_{g,y} h_i^\alpha(x,y)| \le d^3 C_{i,\alpha} d_g(x,y).$$

Now let $x \in M$, $y \in B_R(x)$ and $v \in T_xM$ then there is some $\alpha \in A$ such that$x \in U_\alpha$ and we have

$$v = \sum_i \langle X_i^\alpha(x), v\rangle X_i^\alpha(x).$$

Hence

$$\begin{aligned}
(T\exp_x)_{\exp_x^{-1}(y)}(v) \quad &= (T\exp_x)_{\exp_x^{-1}(y)}\left(\sum_i \langle X_i^\alpha, v\rangle X_i^\alpha\right)\\
&= \sum_i \langle X_i^\alpha(x), v\rangle (T\exp_x)_{\exp_x^{-1}(y)}(X_i^\alpha(x))\\
&= \sum_i \langle X_i^\alpha(x), v\rangle h_i^\alpha(x,y)\\
\implies \quad \left|\mathrm{div}_{g,y}(T\exp_x)_{\exp_x^{-1}(y)}(v)\right| \quad &= \left|\sum_i \langle X_i^\alpha(x), v\rangle \mathrm{div}_{g,y} h_i^\alpha(x,y)\right|\\
&\le \sum_i \langle X_i^\alpha(x), v\rangle d^3 C_{i,\alpha} d_g(x,y)\\
&\le \|v\| C d_g(x,y),
\end{aligned}$$

where $C = d^3 \sum_{i,\alpha} C_{i,\alpha}$. So we proved (14) and next we will show (16). Due to the continuity of χ' we have

$$\left|\chi'\left(\frac{d_g(x,y)}{\varepsilon}\right)\right| \le C 1_{\{d_g(x,y)<\varepsilon\}} \tag{48}$$

and define

$$\beta(t) \quad := \quad \left\langle (T\exp_x)_{\exp_x^{-1}(\gamma_{xy}(t))}(v) - P_{x\gamma_{xy}(t)}(v), \gamma_{xy}'(t)\right\rangle.$$

We recall (13), hence to prove (16) we need an estimate for $\beta(d_g(x,y))$. We have $\beta(0) = 0$ and

$$\begin{aligned}
\tfrac{d}{dt}\beta(t) \quad &= \gamma_{xy}'(t)\left\langle (T\exp_x)_{\exp_x^{-1}(\gamma_{xy}(t))}(v) - P_{x\gamma_{xy}(t)}(v), \gamma'(t)\right\rangle\\
&= \langle \nabla_{\gamma_{xy}'(t)} (T\exp_x)_{\exp_x^{-1}(\gamma_{xy}(t))}(v), \gamma'(t)\rangle
\end{aligned}$$

because $\nabla_{\gamma_{xy}'}\gamma_{xy}' = \nabla_{\gamma_{xy}'} P_{xy}(v) = 0$. Hence we have for some $\alpha \in A$

$$\left|\frac{d}{dt}\beta(t)\right| \le \sum_i \|\gamma_{xy}'\|^2 \|\nabla_{g,y} h_i^\alpha(x,y)\| \|v\| \overset{(47)}{\le} C d_g(x,y)\|v\|.$$

Hence we have $|\beta(d_g(x,y))| \le C(d_g(x,y))^2\|v\|$ and using (13) and (48)

$$\begin{aligned}
\left|\left\langle (T\exp_x)_{\exp_x^{-1}(y)}(v) - P_{xy}(v), \nabla_{g,y}\chi_\varepsilon(x,y)\right\rangle\right| \quad &\le \quad \frac{1}{\varepsilon^{d+1}} C 1_{\{d_g(x,y)<\varepsilon\}} |\beta(d_g(x,y))|\\
&\le \quad \frac{1}{\varepsilon^{d-1}} C 1_{\{d_g(x,y)<\varepsilon\}} \|v\|.
\end{aligned}$$

6.2. **Proof of Lemma 7.**

Proof. Miranda et. al. showed in [16] that there exists a sequence $(f_j)_j \in C^\infty(M)$ such that

$$\|f_j - u\|_{L^1(M)} \leq \frac{1}{j} \quad \text{and} \quad \lim_{j\to\infty} \int_M \|\nabla_g f_j\|_g \mathrm{dv}_g(x) = \mathrm{TV}_M(u) < \infty. \tag{49}$$

For every j we have

$$\begin{aligned}
\|u - \bar{u}\|_{L^1(M)} &\leq \|u - f_j\|_{L^1(M)} + \|f_j - \bar{f}_j\|_{L^1(M)} + \|\bar{f}_j - \bar{u}\|_{L^1(M)} \\
\|\bar{f}_j - \bar{u}\|_{L^1(M)} &= \sum_K \|\bar{f}_j - \bar{u}\|_{L^1(K)} \leq \sum_K \|f_j - u\|_{L^1(K)} = \|f_j - u\|_{L^1(M)} \\
\Rightarrow \|u - \bar{u}\|_{L^1(M)} &\leq 2\|u - f_j\|_{L^1(M)} + \|f_j - \bar{f}_j\|_{L^1(M)}.
\end{aligned}$$

Furthermore we have for every $K \in \mathcal{T}$

$$\begin{aligned}
\|f_j - \bar{f}_j\|_{L^1(K)} &\leq \int_K \left| f_j(x) - \frac{1}{|K|} \int_K f_j(y) \mathrm{dv}_g(y) \right| \mathrm{dv}_g(x) \\
&\leq \frac{1}{|K|} \int_{K^2} |f_j(x) - f_j(y)| \mathrm{dv}_g(y) \mathrm{dv}_g(x).
\end{aligned}$$

When the diameter of all elements is smaller than R, then for every pair of points $x, y \in K$ there is a unique minimising geodesic from x to y. It can be written as

$$\gamma : [0,1] \longrightarrow M \quad \theta \mapsto \begin{cases} \exp_y((1-\theta)\exp_y^{-1}(x)) & \text{for} \quad 0 \leq \theta \leq \frac{1}{2} \\ \exp_x(\theta \exp_x^{-1}(y)) & \text{for} \quad \frac{1}{2} \leq \theta \leq 1. \end{cases}$$

This implies

$$\begin{aligned}
\|f_j - \bar{f}_j\|_{L^1(K)} \leq\ & \frac{1}{|K|} \int_{K^2} \int_0^{\frac{1}{2}} \Big| \nabla_g f_j(\exp_y((1-\theta)\exp_y^{-1}(x))) \\
& \cdot (T\exp_y)_{(1-\theta)\exp_y^{-1}(x)} (\exp_y^{-1}(x)) \Big| \, d\theta \mathrm{dv}_g(x) \mathrm{dv}_g(y) \\
+\ & \frac{1}{|K|} \int_{K^2} \int_{\frac{1}{2}}^1 \Big| \nabla_g f_j(\exp_x(\theta \exp_x^{-1}(y))) \\
& \cdot (T\exp_x)_{\theta \exp_x^{-1}(y)} (\exp_x^{-1}(y)) \Big| \, d\theta \mathrm{dv}_g(x) \mathrm{dv}_g(y).
\end{aligned} \tag{50}$$

Due to (15) we have

$$\|(T\exp_y)_{(1-\theta)\exp_y^{-1}(x)} (\exp_y^{-1}(x))\|_g \leq C \|\exp_y^{-1}(x)\| \leq C\delta(K). \tag{51}$$

Inserting (51) in (50) implies

$$\begin{aligned}
\|f_j - \bar{f}_j\|_{L^1(K)} \leq\ & \frac{C\delta(K)}{|K|} \int_{K^2} \int_0^{\frac{1}{2}} \|\nabla_g f_j(\exp_y((1-\theta)\exp_y^{-1}(x)))\|_g \\
& d\theta \mathrm{dv}_g(x) \mathrm{dv}_g(y) \\
+\ & \frac{C\delta(K)}{|K|} \int_{K^2} \int_{\frac{1}{2}}^1 \|\nabla_g f_j(\exp_x(\theta \exp_x^{-1}(y)))\|_g \\
& d\theta \mathrm{dv}_g(x) \mathrm{dv}_g(y) \\
=\ & \frac{C\delta(K)}{|K|} \int_K \int_0^{\frac{1}{2}} \int_{(1-\theta)\exp_y^{-1}(K)} \|\nabla_g f_j(\exp_y(w))\|_g \\
& \frac{1}{(1-\theta)^d} |\det(T\exp_y)_w| \, d\theta \, dw \mathrm{dv}_g(y) \\
+\ & \frac{C\delta(K)}{|K|} \int_K \int_{\frac{1}{2}}^1 \int_{\theta \exp_x^{-1}(K)} \|\nabla_g f_j(\exp_x(v))\|_g \\
& \frac{1}{\theta^d} |\det(T\exp_x)_v| \, d\theta \, dv \mathrm{dv}_g(x)
\end{aligned}$$

where the determinants are computed with respect to orthonormal bases of the respective tangent spaces. The determinant of $(T\exp_x)_v$ is continuous and positive on the compact set $\mathcal{K} := \{(x,v) \in TM : \|v\| \leq R\}$ so there exists $C > 0$ such that

$$\frac{1}{C} < |\det(T\exp_x)_v| < C \quad \forall (x,v) \in \mathcal{K}. \tag{52}$$

We have

$$\begin{aligned}
\|f_j - \bar{f}_j\|_{L^1(K)} \leq\ & \frac{C\delta(K)}{|K|} \int_K \int_0^{\frac{1}{2}} \int_{\exp_y((1-\theta)\exp_y^{-1}(K))} \|\nabla_g f_j(z)\|_g \\
& |\det((T\exp_y^{-1})_z)|\,\mathrm{dv}_g(z) d\theta \mathrm{dv}_g(y) \\
+\ & \frac{C\delta(K)}{|K|} \int_K \int_{\frac{1}{2}}^1 \int_{\exp_x(\theta\exp_x^{-1}(K))} \|\nabla_g f_j(z)\|_g \\
& |\det((T\exp_x^{-1})_z)|\,\mathrm{dv}_g(z) d\theta \mathrm{dv}_g(x).
\end{aligned}$$

By definition of convex hull (definition 4) we get

$$\begin{aligned}
\exp_y((1-\theta)\exp_y^{-1}(K)) &\subset K^c \text{ for } 0 \leq \theta \leq 1 \text{ and } x,y \in K \\
\exp_x(\theta\exp_x^{-1}(K)) &\subset K^c \text{ for } 0 \leq \theta \leq 1 \text{ and } x,y \in K.
\end{aligned}$$

This implies by (52) due to

$$\begin{aligned}
\|f_j - \bar{f}_j\|_{L^1(K)} &\leq \frac{C\delta(K)}{|K|} \int_K \int_{K^c} \|\nabla_g f_j(z)\|_g \mathrm{dv}_g(z) \mathrm{dv}_g(y) \\
&\quad + \frac{C\delta(K)}{|K|} \int_K \int_{K^c} \|\nabla_g f_j(z)\|_g \mathrm{dv}_g(z) \mathrm{dv}_g(x) \\
&\leq C\delta(K) \|\nabla_g f_j\|_{L^1(K^c)}.
\end{aligned}$$

Finally we have due to (49) and our assumption on the triangulation

$$\begin{aligned}
\|u - \bar{u}\|_{L^1(M)} &\leq \lim_{j\to\infty} 2\|u - f_j\|_{L^1(M)} + \lim_{j\to\infty} \|f_j - \bar{f}_j\|_{L^1(M)} \\
&= \lim_{j\to\infty} \sum_K \|f_j - \bar{f}_j\|_{L^1(K)} \\
&\leq Ch \lim_{j\to\infty} \sum_K \|\nabla_g f_j\|_{L^1(K^c)} \\
&\leq ChN_c TV_M(u).
\end{aligned}$$

□

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