Universität Stuttgart

On a stochastic reaction-diffusion system modeling pattern formation

Jan Kelkel, Christina Surulescu

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Preprint 2008/007

Universität Stuttgart

On a stochastic reaction-diffusion system modeling pattern formation

Jan Kelkel, Christina Surulescu

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Preprint 2008/007

Institut für Angewandte Analysis und Numerische Simulation (IANS)
Fakultät Mathematik und Physik
Fachbereich Mathematik
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: `ians-preprints@mathematik.uni-stuttgart.de`
WWW: `http://preprints.ians.uni-stuttgart.de`

ISSN **1611-4176**

ON A STOCHASTIC REACTION-DIFFUSION SYSTEM MODELING PATTERN FORMATION

J. KELKEL & C. SURULESCU

ABSTRACT. Starting from the Gierer-Meinhardt setting, we propose a stochastic model to characterize pattern formation on seashells under the influence of random space-time fluctuations. We prove the existence of a positive solution for the resulting system and perform numerical simulations in order to assess the behavior of the solution in comparison with the deterministic approach.

1. INTRODUCTION

Seashells exhibit a huge variety of beautiful, highly complex patterns. This diversity, often within the same species, is one of the reasons making the study of pattern formation on molluscs such an interesting issue. One would like to know whether a single or some few mechanisms are able to generate this wide spectrum of patterns and how this can be explained in a comprehensive way.

Since the seminal paper of A. Turing [21], reaction-diffusion systems increasingly became a powerful tool for mathematically describing pattern formation. These model the dynamics of the distribution of chemicals which diffuse within a domain and interact with each other. Accordingly to the Turing approach, patterns arise as a consequence of one of the involved chemicals activating the pigment production by certain cells and its storing in the skin or shells of animals.

In his work [21] Turing showed how the interplay between diffusion and nonlinear reactions (the so-called diffusion driven instability) can lead to spatial pattern formation. Since then many examples of systems have been proposed, which are able to explain patterns based on this mechanism. For overviews and comparisons between the models we refer to the works of Murray [16], [17].

Meinhardt et al. [14], [13] have been concerned with modeling and computer simulations of patterns on seashells, as well as their interpretation in the framework of biochemical processes. Another direction is to investigate the Gierer-Meinhardt system for patterns on molluscs [7] from a qualitative viewpoint with the aid of dynamical systems methods, see e.g., [23], [24] and [8]. In [23] and [24] it is shown that the respective reaction-diffusion system is able to describe under certain conditions the generation of a special kind of patterns, the so-called spikes, while in [8] is evidentiated the possibility of time oscillations. The existence of solutions to reaction-diffusion systems has also attracted considerable interest. Thus, Rothe [18] proved the existence of a global

Date: June 10th 2008.
Key words and phrases. pattern formation, stochastic reaction-diffusion equations.

solution based on the method of semigroups of operators and later his result was further improved. A newer result on global solutions to the Gierer-Meinhard system can be found in [10].

Recently, we have shown in [11] the existence of a local solution for general initial conditions and parameters upon using an iterative approach. Furthermore, a global solution was shown to exist for suitable initial data with the aid of the method of invariant regions. The approach in [11] will also be used in the present paper to show the existence of a local solution to an adequately constructed stochastic version of the Gierer-Meinhardt system.

Reaction-diffusion systems are mostly deterministic models, meaning that at every time moment the solutions can be inferred from the data. This is actually in contradiction with phenomena happening in nature, where random influences can play an important role. Thus, the parameters of the system depend on the environment, where many different events occur continuously. Due to their complexity the latter can be conceived as random fluctuations. Moreover, the biochemical processes within the shell can also infer stochastic variations. Diffusion and chemical reactions are by their very nature random; it is only by idealising assumptions (like a very large amount of particles and/or motion in a homogeneous medium) that they receive a deterministic description, however these assumptions are rarely realistic.

Motivated by these aspects, Ishikawa and Miyajima [9] proposed a system of stochastic partial differential equations as an extension of the deterministic Gierer-Meinhardt model. They were only concerned with the numerical simulations of their system, without paying attention to essential mathematical issues regarding well posedness and positivity of solutions. On the basis of their numerical simulations they came to the conclusion that the Gierer-Meinhardt patterns are practically insensitive to random fluctuations.

Here we propose a model for pattern formation on seashells upon starting from the Gierer-Meinhardt system and allowing for random space-time fluctuations. Unlikely in [9], we also account for Brownian sheets as admissible perturbations, which is more appropriate with respect to the spatial properties of the phenomena of interest. The paper is organized as follows: we start with some modeling aspects in Section 2, followed by the mathematical setup of the problem in Section 3, then show in Section 4 the existence of a unique positive mild solution of our system and eventually perform in Section 5 numerical simulations and compare the results with those for the deterministic Gierer-Meinhardt model.

2. Some modeling aspects

The Gierer-Meinhardt model explains pattern formation on seashells through a simple mechanism based on he interplay between two chemicals, one of which is an activator stimulating the production of the melanin pigment in the patterns and -by autocatalysis- its own production. The other chemical is a fast diffusing inhibitor substance which has the role of controlling the activator production, since self-enhancement alone is not sufficient to generate stable patterns.

We extend the Gierer-Meinhardt model in order to allow for space-time random influences and consider the following system of stochastic partial differential equations (shortly SPDEs):

$$\frac{\partial u(t,\mathbf{x})}{\partial t} = d_u \triangle u(t,\mathbf{x}) + f(u,v) + \sigma_1(u,t,\mathbf{x})\frac{\partial W_1(t,\mathbf{x})}{\partial t}, \quad (t,\mathbf{x}) \in (0,T)\times\Omega \tag{1}$$

$$\frac{\partial v(t,\mathbf{x})}{\partial t} = d_v \triangle v(t,\mathbf{x}) + g(u,v) + \sigma_2(v,t,\mathbf{x})\frac{\partial W_2(t,\mathbf{x})}{\partial t}, \quad (t,\mathbf{x}) \in (0,T)\times\Omega \tag{2}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{\partial v}{\partial \boldsymbol{\nu}} = 0 \text{ on } (0,T)\times\partial\Omega \tag{3}$$

and initial conditions

$$u(0,\mathbf{x}) = u_0(\mathbf{x}), \quad v(0,\mathbf{x}) = v_0(\mathbf{x}), \quad \mathbf{x}\in\Omega. \tag{4}$$

Here $T > 0$, Ω is a bounded, regular enough subdomain of $\mathbb{R}^N$ ($N = 1, 2$) and $u(t,\mathbf{x})$ denotes the concentration of activator at time t and position $\mathbf{x}$, while the concentration of its antagonist is denoted by $v(t,\mathbf{x})$. The scalar reaction terms are the same as in the deterministic setting:

$$f(u,v) := \frac{s_1}{v}\frac{u^2}{1+S_u u^2} + s_1 b_u \tag{5}$$

$$g(u,v) := s_2\frac{u^2}{1+S_v u^2} + b_v. \tag{6}$$

The constants d_u and d_v are diffusion coefficients, s_1 and s_2 are production rates, r_u and r_v are rates of decay, and b_u and b_v are basic production rates for the activator, respectively the inhibitor. S_u is a coefficient indicating the saturation level in the system. $\sigma_1\partial_t W_1(t,\mathbf{x})$ and $\sigma_2\partial_t W_2(t,\mathbf{x})$ model the space-time random perturbations of the system. Thereby, $\partial_t W_k(t,\mathbf{x})$ is by convention the formal time derivative of the Wiener random field $W_k(t,\mathbf{x})$ ($k = 1, 2$).

In the following we refer to the equations (1)-(4) as *the stochastic Gierer-Meinhardt system*. Observe that in this setting the concentrations u, v are interpreted as stochastic processes. The corresponding mathematical framework is presented in the following section.

Most SPDE systems in literature require the coefficients of the perturbation terms not to depend on the solution. Here we allow the noise coefficients σ_1 and σ_2 to be functions of the activator and inhibitor concentrations. Their choice actually depends upon which type of stochastic influences should be accounted for. Thus, one can distinguish between *internal* and *external* noises:

If in the former case the number of molecules participating in the chemical reactions within the domain of interest is low, then the assumption of continuous concentration fields for the activator and the inhibitor (so-called *mean field approximation*) is no longer justified, which makes the validity of the deterministic reaction-diffusion system questionable. Adding some noise terms to the system allows to account for such fluctuations even in the framework of a continuum model. Thereby, the starting point is to set up a model with discrete particle numbers instead of continuous concentrations and a finite number of compartments instead of a continuous domain. Then in the limit one obtaines a system of stochastic partial differential equations.

In the case with external noises it is assumed that some control parameter of the system is affected by environmental fluctuations. For instance, if we consider a perturbation of the activator's decaying rate r_u, then we can model this by writing $r_u + \alpha_u \eta(t, \mathbf{x})$, where r_u is the constant in the classical deterministic model, whereas α_u is a constant measuring the noise intensity. This transforms the classical deterministic equation for the evolution of the activator concentration into equation (1) with $\sigma_1(u) := \alpha_u u$. The rest of the parameters can be analogously perturbed.

In this work we confine ourselves to investigate a stochastic Gierer-Meinhardt system with volatilities

$$\sigma_1(u,t) = \alpha_u(u - C_u e^{-r_u t}) \tag{7}$$

$$\sigma_2(v,t) = \alpha_v(v - C_v e^{-r_v t}), \tag{8}$$

with α_u, α_v, C_u, C_v real constants satisfying the conditions

$$u_0(\mathbf{x}) \geq C_u > 0 \quad \text{and} \quad v_0(\mathbf{x}) \geq C_v > 0 \quad \text{a.s. in } \Omega. \tag{9}$$

Thereby the positive constants C_u, C_v can be chosen arbitrarily.

Thus, the noise terms consist of a multiplicative, state-dependent part and an additive, state-independent component. The former can either be interpreted as a perturbation of a control parameter or it can be used to reflect effects caused by a small number of molecules participating in the reactions. The additive component can be seen as disturbance of the production rates b_u, b_v. This is justified by the fact that these rates describe the generation -independently of their concentration- of activator, respectively inhibitor by an unknown mechanism. One could imagine for instance that the presence of certain substances in the water plays a role in the processes of interest, which should in any case legitimate considering a stochastic process. We assume that the intensity of these perturbations decays exponentially in time.

3. Problem setting

We rewrite (1), (2) in the following form:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} + \dot{\mathbf{W}}(u,v,t,\mathbf{x}), \tag{10}$$

where $\mathbf{A}$ is the operator given by

$$\mathbf{A} = \begin{pmatrix} d_u \triangle - r_u & 0 \\ 0 & d_v \triangle - r_v \end{pmatrix} \tag{11}$$

and

$$\dot{\mathbf{W}}(u,v,t,\mathbf{x}) = \boldsymbol{\sigma}(u,t,\mathbf{x}) \partial_t \mathbf{W}(t,\mathbf{x}) := \begin{pmatrix} \sigma_1(u,t,\mathbf{x}) & 0 \\ 0 & \sigma_2(v,t,\mathbf{x}) \end{pmatrix} \begin{pmatrix} \partial_t W_1(t,\mathbf{x}) \\ \partial_t W_2(t,\mathbf{x}) \end{pmatrix}.$$

Let $(\Xi, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_t \ : \ t \in [0,T]\}$ be a family of sub-σ-algebras of $\mathcal{F}$ building a filtration. We denote by X the set of all H^1-valued $\mathcal{F}_t$-adapted processes which are continuous in $L^2(\Omega)$ and satisfy

$$\mathbb{E}\sup_{0\le t\le T}\|u(\cdot,t)\|^2_{L^2(\Omega)}+\mathbb{E}\int_0^T\|u(\cdot,t)\|^2_{H^1(\Omega)}<\infty. \tag{12}$$

Then (see e.g., [4]) X is a Banach space endowed with the norm

$$\|u\|_X=\left[\mathbb{E}\sup_{0\le t\le T}\|u(\cdot,t)\|^2_{L^2(\Omega)}+\mathbb{E}\int_0^T\|u(\cdot,t)\|^2_{H^1(\Omega)}\right]^{\frac{1}{2}}. \tag{13}$$

Let $W_i(t,\cdot)$ ($i=1,\dots,N$) be independent, $L^2(\Omega)$-valued Wiener processes satisfying

$$\begin{aligned}\mathbb{E}W_i(t,\mathbf{x}) &= 0\\ \mathbb{E}\{W_i(t,\mathbf{x})W_i(s,\mathbf{y})\} &= \min\{t,s\}r_i(\mathbf{x},\mathbf{y}) \quad \text{for } t,s\in[0,T], \quad \mathbf{x},\mathbf{y}\in\Omega.\end{aligned}$$

The r_i's denote covariance functions satisfying the following conditions:

$$\begin{aligned}r_i(\mathbf{x},\mathbf{y}) &= r_i(\mathbf{y},\mathbf{x}) \text{ for all } \mathbf{x},\mathbf{y}\in\Omega\\ r_i(\mathbf{x},\mathbf{x}) &\ge 0 \text{ for all } \mathbf{x}\in\Omega\\ |r_i(\mathbf{x},\mathbf{y})| &\le C_i \text{ for all } \mathbf{x},\mathbf{y}\in\Omega,\end{aligned} \tag{14}$$

with constants $C_i\in\mathbb{R}_+$.

We define a linear integral operator $R:\mathbf{L}^2(\Omega)\to\mathbf{L}^2(\Omega)$ by

$$\begin{aligned}(R_i\phi)(\mathbf{x}) &= \int_\Omega r_i(\mathbf{x},\mathbf{y})\phi(\mathbf{y})d\mathbf{y} \quad \text{for } \mathbf{x}\in\Omega \quad \text{and } i=1,...,N\\ R &= \text{diag }(R_1,\dots R_N).\end{aligned}$$

The R_i's are self-adjoint Hilbert-Schmidt operators on $\mathbf{L}^2(\Omega)$. They are also called *covariance operators* (see e.g., [4]). Due to (14), they also satisfy

$$\text{Trace } R_i=\int_\Omega r_i(\mathbf{x},\mathbf{x})d\mathbf{x}\le C_i|\Omega|<\infty.$$

For $i=1,\dots,N$ define

$$\|\sigma_i(u,t)\|^2_{R_i}:=\int_\Omega r_i(\mathbf{x},\mathbf{x})\sigma_i^2(u(t,\mathbf{x}))d\mathbf{x}\le C_i\|\boldsymbol{\sigma}(u,t)\|^2_{\mathbf{L}^2(\Omega)}.$$

Writing $W_t=(W_1(t,\cdot),\dots W_N(t,\cdot))^t$, we denote by $\mathcal{F}_t$ the smallest σ-algebra generated by W_s, $s\le t$. This filtration will be used in the following analyis.

4. Mathematical analysis of the stochastic Gierer-Meinhardt system

We prove the existence of a positive local solution to the system of stochastic partial differential equations in Section 3. For this purpose let us first introduce the notion of solution to be worked with.

Let us make the notations $\mathbf{w}:=\begin{pmatrix}u\\v\end{pmatrix}$ and $\mathbf{F}(\mathbf{w}):=\begin{pmatrix}f(\mathbf{w})\\g(\mathbf{w})\end{pmatrix}$.

Then using the notations in Section 3 the system (10) can be rewritten as

$$\mathbf{w}_t=\mathbf{A}\mathbf{w}+\mathbf{F}(\mathbf{w})+\dot{\mathbf{W}}(\mathbf{w}). \tag{15}$$

Let $\{\lambda_{ik}\}$ be the eigenvalues of the differential operator $\mathbf{A}$ with homogeneous Neumann boundary conditions. Let further $\{e_{ik}\}$ with $e_{ik} \in C^\infty(\Omega)$ be the corresponding orthonormal system of eigenfunctions.

Recall that the Green function for the operator $d_i\Delta - r_i$ ($i = 1, \dots N$, d_i, r_i constants) of the heat equation can be written

$$G^i(t;\mathbf{x},\mathbf{y}) = \sum_{k=1}^{\infty} e^{-\lambda_{ik}t} e_{ik}(\mathbf{x}) e_{ik}(\mathbf{y}),$$

and the corresponding Green operator G_t^i is given by

$$(G_t^i \mathbf{w}_0)(\mathbf{x}) = \int_\Omega G^i(t;\mathbf{x},\mathbf{y}) w_{0,i}(\mathbf{y}) d\mathbf{y},$$

where $w_{0,i}$ is the ith component of the given initial function.

The Green operator for the whole system can be defined analogously: $G_t = \text{diag}\,(G_t^1, \dots, G_t^N)$.

Definition 4.1. *We say that $\mathbf{w} \in \mathbf{L}^2(\Xi \times [0,T]; \mathbf{L}^2(\Omega))$ is a **mild solution** of the stochastic Gierer-Meinhardt system (15) with the boundary and initial conditions (3), (4) iff $\mathbf{w}(t,\cdot)$ is a predictable process in $\mathbf{L}^2(\Omega)$, which satisfies for almost all $(\xi, t) \in \Xi \times [0,T]$ the integral equation*

$$\mathbf{w}(t,\cdot) = G_t \mathbf{w}_0 + \int_0^t G_{t-s} \mathbf{F}(\mathbf{w}(s,\cdot)) ds + \int_0^t G_{t-s} \boldsymbol{\sigma}(\mathbf{w}(s,\cdot)) d\mathbf{W}(s,\cdot)$$

and for which holds

$$\begin{aligned} &\mathbb{E}\int_0^T \|\mathbf{F}(\mathbf{w}(t))\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^N (R_i \sigma_i(\mathbf{w}(t)), \sigma_i(\mathbf{w}(t))_{\mathbf{L}^2(\Omega)} dt \\ = \;& \mathbb{E}\int_0^T \int_\Omega |\mathbf{F}(\mathbf{w}(t,\mathbf{x}))|^2 + \sum_{i=1}^N r_i(\mathbf{x},\mathbf{x}) \sigma_i^2(\mathbf{w}(t,\mathbf{x})) dt d\mathbf{x} < \infty. \end{aligned}$$

Now we can state the main result of this paper:

Theorem 4.1. *Let the initial data u_0, v_0 satisfy the following conditions:*

(i) $u_0, v_0 : \Xi \to L^2(\Omega)$ are $\mathcal{F}_0$-measurable, square-integrable random variables;
(ii) the functions $(\xi, y) \mapsto u_0(\xi)(y)$ and $(\xi, y) \mapsto v_0(\xi)(y)$ are product-measurable;
(iii) u_0, v_0 satisfy (9);
(iv) it holds that $\mathbb{E}\|u_0\|^2 < \infty$ and $\mathbb{E}\|v_0\|^2 < \infty$.

If the saturation constants satisfy $S_u > 0$ and $S_v > 0$, then the stochastic Gierer-Meinhardt system (10) with (7), (8) and with the conditions (3), (4) has locally in time a pathwise unique mild solution $(u(t,\cdot), v(t,\cdot))$ in $H^1(\Omega) \times H^1(\Omega)$, adapted and continuous in $L^2(\Omega) \times L^2(\Omega)$.

Furthermore, the solution is for all $t \in [0,T]$ bounded from below:

(16) $$u(x,t) \geq C_u e^{-r_u t} \quad \text{and} \quad v(x,t) \geq C_v e^{-r_v t} \quad \text{a.s.}$$

For the proof of this theorem we first consider a modified version of the system, where the nonlinearity $\frac{1}{v}$ is cut off for small values of v. A lower bound (exponential decay) is deduced for the solution of the modified equations and then it is shown that it is also a solution of the original system.

Proof. (of Theorem 4.1)
Let $\psi : \mathbb{R} \to \mathbb{R}$ be given by

$$\psi(r) = \begin{cases} C_v e^{-r_v T} & \text{for } r \leq C_v e^{-r_v T} \\ r & \text{for } r > C_v e^{-r_v T} \end{cases}. \tag{17}$$

We truncate the reaction functions by

$$\hat{f}(u,v) = f(u,\psi(v)) \quad \text{and} \quad \hat{g}(u,v) = g(u,v). \tag{18}$$

The stochastic Gierer-Meinhardt system with these nonlinearities and the boundary conditions (3) as well as the initial conditions (4) will be subsequently referred to as $SGM(\hat{f},\hat{g})$.
Now the following lemma holds for the functions $\hat{f},\hat{g}$:

Lemma 4.1. *(i) There exist constants $K_{B,u}, K_{B,v}$, such that for all $u, v \in \mathbb{R}$*

$$|\hat{f}(u,v)| \leq K_{B,u} \qquad \textit{and} \qquad |\hat{g}(u,v)| \leq K_{B,v}.$$

(ii) Let $u_1, u_2, v_1, v_2 \in \mathbb{R}$. Then $\hat{f}$ and $\hat{g}$ satisfy the Lipschitz conditions

$$\begin{aligned} |\hat{f}(u_1,v_1) - \hat{f}(u_2,v_2)| &\leq K_{L,u}(|u_1-u_2| + |v_1-v_2|) \\ |\hat{g}(u_1,v_1) - \hat{g}(u_2,v_2)| &\leq K_{L,v}(|u_1-u_2| + |v_1-v_2|). \end{aligned}$$

Proof. (of Lemma 4.1)
The first claim follows immediately with $K_{B,u} = (s_1 e^{r_v T})/(C_v S_u) + s_1 b_u$ and $K_{B,v} = s_2/S_v + b_v$. For the second assertion, we first consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(x) := \frac{x^2}{1 + S_u x^2}.$$

Obviously for all $x \in \mathbb{R}$ holds

$$|\phi'(x)| \leq \frac{2\sqrt{3}}{(3+S_u)\sqrt{S_u} + 2\sqrt{3}S_u}.$$

Applying the mean value theorem yields

$$\begin{aligned} & |\hat{f}(u_1,v_1) - \hat{f}(u_2,v_2)| \\ \leq\ & |\hat{f}(u_1,v_1) - \hat{f}(u_2,v_1)| + |\hat{f}(u_2,v_1) - \hat{f}(u_2,v_2)| \\ \leq\ & \frac{s_1}{|\psi(v_1)|}\left|\frac{u_1^2}{1+S_u u_1^2} - \frac{u_2^2}{1+S_u u_2^2}\right| + s_1 \frac{u_2^2}{1+S_u u_2^2} \cdot \frac{1}{\psi(v_1)\psi(v_2)}|\psi(v_1) - \psi(v_2)| \\ \leq\ & \frac{s_1 e^{r_v T}}{C_v} \frac{2\sqrt{3}}{(3+S_u)\sqrt{S_u} + 2\sqrt{3}S_u}|u_1 - u_2| + \frac{s_1 e^{2r_v T}}{S_u C_v^2}|v_1 - v_2| \\ \leq\ & K_{L,u}(|u_1-u_2| + |v_1-v_2|). \end{aligned}$$

where in the last step we set

$$K_{L,u} := \max\left\{ \frac{2s_1 e^{r_v T}}{C_v} \frac{2\sqrt{3}}{(3+S_u)\sqrt{S_u}+2\sqrt{3}S_u}, \frac{s_1 e^{2r_v T}}{S_u C_v^2} \right\}.$$

The estimate for g follows analogously. □

Next, we consider a sequence $(u^m, v^m)_{m\in\mathbb{N}} \in X \times X$ and show that it converges to the mild solution of the modified stochastic Gierer-Meinhardt System $SGM(\hat{f}, \hat{g})$.

In [11] the deterministic Gierer-Meinhardt system has been treated by a similiar method. There it was possible to derive a bound for the nonlinearity u^2 in the reaction functions f and g by using a uniform estimate for u^m in $L^\infty(0,T;H^1(\Omega))$. However, for a stochastic system one usually expects less regularity.

Under the hypotheses of the theorem the reaction functions $\hat{f}$ and $\hat{g}$ can be estimated with the aid of Lemma 4.1.

Now let $(u^0, v^0) \in X \times X$ satisfy

$$\begin{pmatrix} u_t^0 \\ v_t^0 \end{pmatrix} = \mathbf{A} \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} + \begin{pmatrix} s_1 b_u \\ b_v \end{pmatrix} + \begin{pmatrix} \partial_t W_1(t,\mathbf{x}) \\ \partial_t W_2(t,\mathbf{x}) \end{pmatrix}, \tag{19}$$

with the boundary conditions

$$\frac{\partial u^0}{\partial \boldsymbol{\nu}} = \frac{\partial v^0}{\partial \boldsymbol{\nu}} = 0 \text{ on } (0,T)\times\partial\Omega \tag{20}$$

and the initial conditions

$$u^0(0,\mathbf{x}) = u_0(\mathbf{x}), \qquad v^0(0,\mathbf{x}) = v_0(\mathbf{x}). \tag{21}$$

Further, assume the existence of a sequence $(u^m, v^m)_{m\in\mathbb{N}} \in X \times X$ satisfying for $m \in \mathbb{N}$ the equations

$$\begin{pmatrix} u_t^{m+1} \\ v_t^{m+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} u^{m+1} \\ v^{m+1} \end{pmatrix} + \begin{pmatrix} \hat{f}(u^m, v^m) \\ \hat{g}(u^m, v^m) \end{pmatrix} + \boldsymbol{\sigma}(t, u^m, v^m) \begin{pmatrix} \partial_t W_1(t,\mathbf{x}) \\ \partial_t W_2(t,\mathbf{x}) \end{pmatrix}, \tag{22}$$

with the boundary conditions

$$\frac{\partial u^m}{\partial \boldsymbol{\nu}} = \frac{\partial v^m}{\partial \boldsymbol{\nu}} = 0 \text{ on } (0,T)\times\partial\Omega \tag{23}$$

and the initial conditions

$$u^m(0,\mathbf{x}) = u_0(\mathbf{x}), \qquad v^m(0,\mathbf{x}) = v_0(\mathbf{x}). \tag{24}$$

The reaction functions $\hat{f}$ and $\hat{g}$ are given by (18).

Clearly, from the linear theory (see e.g., [4]), the system (19)-(21) has a unique mild solution.

The following lemma states properties of the sequence $(u^m, v^m)_{m\in\mathbb{N}}$ which allow to prove the existence of a unique solution to (22), (23), (24).

Lemma 4.2. *The functions u^m and v^m are H^1-valued processes, adapted and continuous in $L^2(\Omega)$. The following estimates hold:*

$$\|u^{m+1}\|_X^2 \le K_1 \left\{ \|u_0\|_{L^2(\Omega)}^2 + K_{B,u}^2 T|\Omega| + 2T\max\{1, C_u^2|\Omega|\} C_1 \alpha_u^2 (1 + \|u^m\|_X^2) \right\} \tag{25}$$

and

$$\|v^{m+1}\|_X^2 \leq K_2 \left\{ \|v_0\|^2_{L^2(\Omega)} + K^2_{B,v} T|\Omega| + 2T \max\{1, C_v^2|\Omega|\} C_2 \alpha_v^2 (1 + \|v^m\|_X^2), \right\} \tag{26}$$

with $K_{B,u}$, $K_{B,v}$ in Lemma 4.1 and α_u, α_v as defined in (7), (8).

Proof. (of Lemma 4.2)

Induction start:

Lemma 7.1 in the Appendix immediately yields that $u^0(\cdot, t)$ and $v^0(\cdot, t)$ are H^1-valued processes, adapted and continuous in $L^2(\Omega)$, for which the estimates

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u^0\|^2_{L^2(\Omega)} + \mathbb{E} \int_0^T \|u^0\|^2_{H^1(\Omega)} \leq K_1 \left\{ \|u_0\|^2_{L^2(\Omega)} + T s_1^2 b_u^2 |\Omega| \right\} \tag{27}$$

$$\mathbb{E} \sup_{0 \leq t \leq T} \|v^0\|^2_{L^2(\Omega)} + \mathbb{E} \int_0^T \|v^0\|^2_{H^1(\Omega)} \leq K_2 \left\{ \|v_0\|^2_{L^2(\Omega)} + T b_v^2 |\Omega| \right\} \tag{28}$$

hold.

Induction hypothesis:

Assume that the lemma holds for an arbitrary $m \in \mathbb{N}$.

Inductive step:

With Lemma 4.1 we get

$$\mathbb{E} \left(\int_0^T \|\hat{f}(u^m, v^m)\|^2_{L^2(\Omega)} dt \right) \leq K^2_{B,u} T|\Omega| \tag{29}$$

$$\mathbb{E} \left(\int_0^T \|\hat{g}(u^m, v^m)\|^2_{L^2(\Omega)} dt \right) \leq K^2_{B,v} T|\Omega|. \tag{30}$$

For the noise coefficients we have

$$\|\sigma_1(u^m, t)\|^2_{R_1} = \int_\Omega r_1(x,x) \sigma_1^2(u^m(x,t), t) dx \tag{31}$$

$$\leq C_1 \|\sigma_1(u^m, t)\|^2_{L^2(\Omega)} \tag{32}$$

$$\leq 2C_1 \alpha_u^2 \max\{1, C_u^2|\Omega|\}(1 + \|u^m\|^2_{L^2(\Omega)}) \tag{33}$$

$$\|\sigma_2(v^m, t)\|^2_{R_2} = \int_\Omega r_2(x,x) \sigma_1^2(v^m(x,t), t) dx \tag{34}$$

$$\leq C_2 \|\sigma_2(v^m, t)\|^2_{L^2(\Omega)} \tag{35}$$

$$\leq 2C_2 \alpha_v^2 \max\{1, C_v^2|\Omega|\}(1 + \|v^m\|^2_{L^2(\Omega))}) \tag{36}$$

and after time integration and taking the expectation

$$\mathbb{E} \left(\int_0^T \|\sigma_1(u^m, t)\|^2_{R_1} dt \right) \leq 2T \max\{1, C_u^2|\Omega|\} C_1 \alpha_u^2 (1 + \|u^m\|_X^2) \tag{37}$$

$$\mathbb{E} \left(\int_0^T \|\sigma_2(v^m, t)\|^2_{R_2} dt \right) \leq 2T \max\{1, C_v^2|\Omega|\} C_2 \alpha_v^2 (1 + \|v^m\|_X^2). \tag{38}$$

Now we can apply Lemma 7.1 and Lemma 7.2 in the Appendix to u^{m+1} and v^{m+1}. We obtain that $u^{m+1}(\cdot,t)$ and $v^{m+1}(\cdot,t)$ are H^1-valued processes, adapted and continuous in $L^2(\Omega)$, for which the estimates

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\|u^{m+1}\|^2_{L^2(\Omega)}+\mathbb{E}\left(\int_0^T\|u^{m+1}\|^2_{H^1(\Omega)}dt\right)\\
\le\ & K_1\left\{\|u_0\|^2_{L^2(\Omega)}+\mathbb{E}[\int_0^T\|\hat{f}(u^m,v^m)\|^2_{L^2(\Omega)}dt]+\mathbb{E}[\int_0^T\|\sigma_1(u^m)\|^2_{R_1}dt]\right\}
\end{aligned}$$

resp.

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\|v^{m+1}\|^2_{L^2(\Omega)}+\mathbb{E}\left(\int_0^T\|v^{m+1}\|^2_{H^1(\Omega)}dt\right)\\
\le\ & K_2\left\{\|v_0\|^2_{L^2(\Omega)}+\mathbb{E}[\int_0^T\|\hat{g}(u^m,v^m)\|^2_{L^2(\Omega)}dt]+\mathbb{E}[\int_0^T\|\sigma_2(v^m)\|^2_{R_2}dt]\right\}
\end{aligned}$$

hold. From these the estimates (25) and (26) follow immediately.
Now that Lemma 4.2 holds for $m+1$, it holds also for all $m\in\mathbb{N}_0$. □

Lemma 4.3. *Let $0\le T\le 1$. Then there is a constant $L>0$ independent of T, such that for all $m\in\mathbb{N}$*

$$\|u^{m+1}-u^m\|^2_X+\|v^{m+1}-v^m\|^2_X\le LT(\|u^m-u^{m-1}\|^2_X+\|v^m-v^{m-1}\|^2_X) \tag{39}$$

Proof. (of Lemma 4.3)
With Lemma 4.1 it follows

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T\|\hat{f}(u^m,v^m)-\hat{f}(u^{m-1},v^{m-1})\|^2_{L^2(\Omega)}dt\right)\\
\le\ & 2K^2_{L,u}\mathbb{E}\left(\int_0^T\|u^m-u^{m-1}\|^2_{L^2(\Omega)}+\|v^m-v^{m-1}\|^2_{L^2(\Omega)}dt\right)\\
\le\ & 2K^2_{L,u}T(\|u^m-u^{m-1}\|^2_X+\|v^m-v^{m-1}\|^2_X)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T\|\hat{g}(u^m,v^m)-\hat{g}(u^{m-1},v^{m-1})\|^2_{L^2(\Omega)}dt\right)\\
\le\ & 2K^2_{L,v}\mathbb{E}\left(\int_0^T\|u^m-u^{m-1}\|^2_{L^2(\Omega)}+\|v^m-v^{m-1}\|^2_{L^2(\Omega)}dt\right)\\
\le\ & 2K^2_{L,v}T(\|u^m-u^{m-1}\|^2_X+\|v^m-v^{m-1}\|^2_X).
\end{aligned}$$

For the noise terms we further have that

$$\begin{aligned}
\mathbb{E}\left(\int_0^T\|\sigma_1(u^m,t)-\sigma_1(u^{m-1},t)\|^2_{R_1}dt\right) &\le TC_1\alpha_u^2\|u^m-u^{m-1}\|^2_X)\\
\mathbb{E}\left(\int_0^T\|\sigma_2(v^m,t)-\sigma_2(v^{m-1},t)\|^2_{R_2}dt\right) &\le TC_2\alpha_v^2\|v^m-v^{m-1}\|^2_X).
\end{aligned}$$

We set $\delta_m u := u^{m+1} - u^m$ and $\delta_m v := v^{m+1} - v^m$. Then $\delta_m u$ and $\delta_m v$ satisfy by construction

$$\delta_m u(t,\cdot) = \int_0^t G_{t-s}[\hat{f}(u^m,v^m) - \hat{f}(u^{m-1},v^{m-1})]ds + \int_0^t G_{t-s}[\sigma_1(u^m,t) - \sigma_1(u^{m-1},t)]dW(s,\cdot)$$

and

$$\delta_m v(t,\cdot) = \int_0^t G_{t-s}[\hat{g}(u^m,v^m) - \hat{g}(u^{m-1},v^{m-1})]ds + \int_0^t G_{t-s}[\sigma_2(v^m,t) - \sigma_2(v^{m-1},t)]dW(s,\cdot).$$

Applying Lemma 7.1 and Lemma 7.2 to these equations yields the estimate

$$\begin{aligned} &\mathbb{E} \sup_{0\le t\le T} \|\delta_m u\|^2 + \mathbb{E}\int_0^T \|\delta_m u\|^2_{H^1(\Omega)} \\ \le\ & K_1 \left\{ \mathbb{E}[\int_0^T \|\hat{f}(u^m,v^m) - \hat{f}(u^{m-1},v^{m-1})\|^2_{L^2(\Omega)} dt] + \mathbb{E}[\int_0^T \|\sigma_1(u^m,t) - \sigma_1(u^{m-1},t)\|^2_{R_1} dt] \right\}, \end{aligned}$$

respectively

$$\begin{aligned} &\mathbb{E} \sup_{0\le t\le T} \|\delta_m v\|^2 + \mathbb{E}\int_0^T \|\delta_m v\|^2_{H^1(\Omega)} \\ \le\ & K_2 \left\{ \mathbb{E}[\int_0^T \|\hat{g}(u^m,v^m) - \hat{g}(u^{m-1},v^{m-1})\|^2_{L^2(\Omega)} dt] + \mathbb{E}[\int_0^T \|\sigma_2(v^m,t) - \sigma_2(v^{m-1},t)\|^2_{R_2} dt] \right\} \end{aligned}$$

and thus

$$\begin{aligned} \|u^{m+1} - u^m\|^2_X &\le K_1 T[(2K^2_{L,u} + C_1\alpha_u^2)\|u^m - u^{m-1}\|^2_X + 2K^2_{L,u}\|v^m - v^{m-1}\|^2_X] \\ \|v^{m+1} - v^m\|^2_X &\le K_2 T[(2K^2_{L,v} + C_2\alpha_v^2)\|v^m - v^{m-1}\|^2_X + 2K^2_{L,v}\|u^m - u^{m-1}\|^2_X]. \end{aligned}$$

The assertion follows with $L := K_1(2K^2_{L,u} + C_1\alpha_u^2) + K_2(2K^2_{L,v} + C_2\alpha_v^2)$. □

According to lemma 4.3, (u^m, v^m) is a Cauchy sequence in $X \times X$ provided that $T < 1/L$. From this the existence of a mild solution of the system $SGM(\hat{f},\hat{g})$ follows.

For the uniqueness proof assume that (u_1, v_1) and (u_2, v_2) are mild solutions of $SGM(\hat{f},\hat{g})$. For these functions we have from Lemma 4.3 that

$$||u_1 - u_2||^2_X + ||v_1 - v_2||^2_X \le LT\left(||u_1 - u_2||^2_X + ||v_1 - v_2||^2_X\right).$$

Since $LT < 1$, it follows that $(u_1, v_1) = (u_2, v_2)$ has to hold on $X \times X$, i.e. the weak solution of $SGM(\hat{f},\hat{g})$ is (pathwise) unique.

In order to derive a lower bound for u and v, we consider the system

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathbf{A}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} s_1 b_u \\ b_v \end{pmatrix} + \begin{pmatrix} \sigma_1(t,u) & 0 \\ 0 & \sigma_2(t,v) \end{pmatrix}\begin{pmatrix} \partial_t W_1(t,\mathbf{x}) \\ \partial_t W_2(t,\mathbf{x}) \end{pmatrix}, \tag{40}$$

with σ_1, σ_2 given by (7),(8) and with the same initial- and boundary conditions as the stochastic Gierer-Meinhardt system. This system then has a pathwise unique solution, which we denote by $(\tilde{u},\tilde{v})$.

We define the functions $\hat{u}, \hat{v}$ by

$$\hat{u} := \tilde{u} - C_u e^{-r_u t}, \qquad \hat{v} := \tilde{v} - C_v e^{-r_v t}.$$

Then $(\hat{u}, \hat{v})$ is the pathwise unique solution of the system

$$\begin{pmatrix} \hat{u}_t \\ \hat{v}_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} + \begin{pmatrix} s_1 b_u \\ b_v \end{pmatrix} + \begin{pmatrix} \alpha_u \hat{u} & 0 \\ 0 & \alpha_v \hat{v} \end{pmatrix} \begin{pmatrix} \partial_t W_1(t, \mathbf{x}) \\ \partial_t W_2(t, \mathbf{x}) \end{pmatrix}. \tag{41}$$

with initial conditions

$$\hat{u}(0) = u_0 - C_u \geq 0, \qquad \hat{v}(0) = v_0 - C_v \geq 0,$$

Note that (41) does no longer contain any additive noise term.

Next we apply the comparison principle (see Theorem 7.1 in the Appendix) to (41) and the system

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \alpha_u u & 0 \\ 0 & \alpha_v v \end{pmatrix} \begin{pmatrix} \partial_t W_1(t, \mathbf{x}) \\ \partial_t W_2(t, \mathbf{x}) \end{pmatrix},$$

having the pathwise unique solution $(0, 0)$.

Indeed, it can be easily checked that the drift and noise coefficients above satisfy the hypotheses in Theorem 7.1. Moreover, by the classical maximum principle it follows that the semigroup $S(t)$ generated by the the differential operator $\mathbf{A}$ is positivity preserving, i.e.

$$\forall t \in [0, T]: \quad S(t)\phi \geq 0 \quad \text{for } \phi \in \mathbf{L}^2(\Omega) \quad \text{with } \phi \geq 0.$$

We obtain

$$\hat{u} \geq 0, \qquad \hat{v} \geq 0,$$

which by definition implies

$$\tilde{u}(t) \geq C_u e^{-r_u t} \quad \text{and} \quad \tilde{v}(t) \geq C_v e^{-r_v t}.$$

Applying again the comparison principle to (40) and $SGM(\hat{f}, \hat{g})$ finally yields the desired bound

$$u(t) \geq \tilde{u}(t) \geq C_u e^{-r_u t} \quad \text{and} \quad v(t) \geq \tilde{v}(t) \geq C_v e^{-r_v t}. \tag{42}$$

From the lower bound for v it follows that the function ψ introduced in (17) satisfies $\psi(v) \equiv v$ along the solution. Therefore, (u, v) is also a mild solution of $SGM(f, g)$. Additionally, v is the only solution of $SGM(f, g)$ which also satisfies condition (42). □

5. Numerical simulations

In this section we deal with the numerical solution of the stochastic Gierer-Meinhardt system

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix} + \begin{pmatrix} \sigma_1(t, u) & 0 \\ 0 & \sigma_2(t, v) \end{pmatrix} \begin{pmatrix} \eta_1(t, x) \\ \eta_2(t, x) \end{pmatrix} \tag{43}$$

on the domain $\Omega_T = [0, T] \times [0, 1]$ for $T > 0$ and with the previous initial and boundary conditions. The noise coefficients are given by (7), (8) and the reaction terms are those in (5), (6). Here we also treat the case of vanishing saturation parameters, i.e. $S_u = S_v = 0$. Moreover, we consider more general random fluctuations, dealing with Brownian sheets intead of the simpler space-time Wiener processes in Section 4. For convenience we recall the elementary facts about Brownian sheets in the Appendix and refer to [5], [22] for further material on this topic.

We assume $W_1(t,x)$ and $W_2(t,x)$ to be space-time Wiener processes, i.e.

$$\eta_1(t,x) = \frac{\partial^2 W_1}{\partial x \partial t}(t,x), \quad \eta_2(t,x) = \frac{\partial^2 W_2}{\partial x \partial t}(t,x),$$

where $W_1(t,x)$ and $W_2(t,x)$ denote two independent Brownian sheets.

5.1. **Discretization in space.** Our main concern is the approximation of the stochastic processes η_1 and η_2. To this aim we divide the interval $[0,1]$ with the aid of the grid points $0 = x_1 < ... < x_{N+1} = 1$ with $x_i = (i-1)\Delta x$ and $\Delta x = 1/N$. We then define

$$\hat{\eta}_1(t,x) := \frac{1}{\sqrt{\Delta x}} \sum_{i=1}^{N} \theta_{1,i}(t)\chi_i(x) \quad \hat{\eta}_2(t,x) := \frac{1}{\sqrt{\Delta x}} \sum_{i=1}^{N} \theta_{2,i}(t)\chi_i(x)$$

with

$$\theta_{1,i}(t) = \frac{1}{\sqrt{\Delta x}} \int_{x_i}^{x_{i+1}} \eta_1(t,x)dx \qquad \theta_{2,i}(t) = \frac{1}{\sqrt{\Delta x}} \int_{x_i}^{x_{i+1}} \eta_2(t,x)dx \tag{44}$$

and

$$\chi_i(x) = \begin{cases} 1 & \text{for } x_i \le x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

for $i = 1, ..., N$. The processes $\theta_{1,i}(t)$ and $\theta_{2,i}(t)$ are independent Brownian motions (scaled with the factor $1/\sqrt{\Delta x}$).

Concerning the error estimation of this approximation we can prove the following:

Lemma 5.1. *Let h be a Lipschitz-continuous function on $[0,T] \times [0,1]$. Let $L \ge 0$ be the corresponding Lipschitz constant such that*

$$|h(t,x) - h(t,y)| \le L|x-y|$$

and

$$\int_0^T \int_0^1 h^2(t,x)dxdt < \infty \tag{46}$$

hold for $t \in [0,T]$ and $x, y \in [0,1]$. Then for $\hat{\eta}_j$ $(j = 1,2)$ we have the estimate

$$E\left[\int_0^T \int_0^1 h(t,x)\eta_j(t,x)dxdt - \int_0^T \int_0^1 h(t,x)\hat{\eta}_j(t,x)dxdt\right]^2 \le TL^2(\Delta x)^2 \tag{47}$$

Proof. The proof is a straightforward adaptation of that in Lemma 2.1 in [1]. □

Next we approximate the Laplacian with the corresponding central differences

$$\triangle_i u = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} \quad \text{and} \quad \triangle_i v = \frac{v_{i+1} - 2v_i + v_{i-1}}{(\Delta x)^2} \tag{48}$$

for $i = 2, ..., N$.

This leads to a system of $2(N-1)$ ordinary stochastic differential equations

$$\begin{aligned}\frac{\partial u_i(t)}{\partial t} &= d_u \triangle_i u_i(t) + f(u_i, v_i) + \frac{\sigma_1(u_i)}{\sqrt{\Delta x}} \theta_{1,i}(t) \\ \frac{\partial v_i(t)}{\partial t} &= d_v \triangle_i v_i(t) + g(u_i, v_i) + \frac{\sigma_2(v_i)}{\sqrt{\Delta x}} \theta_{2,i}(t)\end{aligned}$$

for $i = 2, ..., N$, with initial conditions

$$u_i(0) = u_0(x_i) \quad \text{and} \quad v_i(0) = v_0(x_i)$$

and algebraic equations for the Neumann boundary conditions

$$\frac{u_2 - u_1}{\Delta x} = \frac{u_{N+1} - u_N}{\Delta x} = 0 \quad \text{and} \quad \frac{v_2 - v_1}{\Delta x} = \frac{v_{N+1} - v_N}{\Delta x} = 0. \tag{49}$$

5.2. **Discretization in time.** An important step in the analytical investigation of the stochastic Gierer-Meinhardt system was the proof that the solution of this system remains nonnegative for initial data being bounded from below.

As we now proceed with the discretization in time, we are again faced with the problem of nonnegativity, because the numerical methods usually employed do not necessarily reproduce this physically important property of the solution, as can be seen by considering e.g. the Euler-Maruyama method for a simple linear SDE, which leads with a positive probability to negative solutions.

To deal with this problem several methods have been proposed, most of them having the draw-back of not being general enough to include our equations. A notable exception is the splitting scheme proposed by E. Moro and H. Schurz in [15], which we recall below and then apply it to the stochastic Gierer-Meinhardt system.

The starting point in [15] is the stochastic differential equation for $X : [0, T] \to \mathbb{R}^d$

$$dX(t) = [\beta(X(t), t) + \gamma(X(t), t)]dt + \sigma(X(t), t)dW(t) \tag{50}$$

with functions $\gamma, \beta : \mathbb{R}^N \times [0, T] \to \mathbb{R}^N$, $\sigma : \mathbb{R}^N \times [0, T] \to \mathbb{R}^{N \times N}$ and initial conditions $X_0 \in \mathbb{R}^d$.

Let $\Phi = \Phi(t, X_0)$ be a solution of the equation

$$d\tilde{X}(t) = \beta(\tilde{X}(t), t)dt + \sigma(\tilde{X}(t), t)dW(t).$$

Applying the Euler-Maruyama method to (50) leads to

$$\begin{aligned}X(t + \Delta t) &= X(t) + \beta(X(t), t))\Delta t + \gamma(X(t), t)\Delta t + \sigma(X(t), t)\Delta W_t \\ &= \Phi(t + \Delta t, X(t)) + \gamma(X(t), t)\Delta t\end{aligned}$$

on each subinterval $[t, t + \Delta t] \subset [0, T]$. This motivates the splitting of (50) into two equations:

(1) A stochastic differential equation

$$dX^A(t) = \beta(X^A(t), t)dt + \sigma(X^A(t), t)dW(t). \tag{51}$$

This equation can be exactly solved on $[t, t+\Delta t]$ with the initial condition $X^A(t) = X(t)$. Therefore, the splitting method can only be used if the exact solution of (51) for the given functions $\beta(X^A(t), t)$ and $\sigma(X^A(t), t)$ can be obtained in a closed form.

(2) A (deterministic) ordinary differential equation

$$dX^B(t) = \gamma(X^B(t), t)dt.$$

This equation is numerically solved taking as initial conditon $X^B(t) = X^A(t + \Delta t)$. For the latter every approximate solution method for ordinary differential equations may be employed, as long as the nonnegativity of the solution is preserved. This leads to the numerical solution $X(t + \Delta t) = X^B(t + \Delta t)$ of (50) at time $t + \Delta t$.

Now we split the space discretized Gierer-Meinhardt system accordingly to the scheme proposed above. This leads to a system of stochastic differential equations on the time interval $[t, t + \Delta t]$

$$\begin{aligned}
\frac{\partial u_i^A(t)}{\partial t} &= \beta_1(u_i^A) + \frac{\sigma_1(u_i^A, t)}{\sqrt{\Delta x}}\theta_{1,i}(t) \\
\frac{\partial v_i^A(t)}{\partial t} &= \beta_2(v_i^A) + \frac{\sigma_2(v_i^A, t)}{\sqrt{\Delta x}}\theta_{2,i}(t) \text{ for } i = 2, ..., N,
\end{aligned}$$

with

$$\beta_1(u_i^A) = -r_u u_i^A \quad and \quad \beta_2(v_i^A) = -r_v v_i^A,$$

which accordingly to [12] has at $t + \Delta t$ the exact solution

$$\begin{aligned}
u_i^A(t + \Delta t) &= (u_i^A(t) - C_u e^{-r_u t}) exp\left(\left(-r_u - \frac{\alpha_u^2}{2\Delta x}\right)\Delta t + \frac{\alpha_u}{\sqrt{\Delta x}}(W_{1,i}(t + \Delta t) - W_{1,i}(t))\right) \\
&+ C_u e^{-r_u(t+\Delta t)} \\
v_i^A(t + \Delta t) &= (v_i^A(t) - C_v e^{-r_v t}) exp\left(\left(-r_v - \frac{\alpha_v^2}{2\Delta x}\right)\Delta t + \frac{\alpha_v}{\sqrt{\Delta x}}(W_{2,i}(t + \Delta t) - W_{2,i}(t))\right) \\
&+ C_v e^{-r_v(t+\Delta t)}
\end{aligned}$$

for $i = 2, ..., N$. Here $W_{1,i}$ and $W_{2,i}$ are Brownian motions with

$$(W_{1,i}(t + \Delta t) - W_{1,i}(t)), (W_{2,i}(t + \Delta t) - W_{2,i}(t)) \sim \mathcal{N}(0, \Delta t).$$

To perform this step, it is therefore sufficient to generate normal distributed random numbers. In the deterministic step, the system of ordinary differential equations

$$\begin{aligned}
\frac{\partial u_i^B(t)}{\partial t} &= \gamma_1(u_i^B, v_i^B) \\
\frac{\partial v_i^B(t)}{\partial t} &= \gamma_2(u_i^B, v_i^B) \text{ for } i = 2, ..., N
\end{aligned}$$

with

$$\gamma_1(u_i^B, v_i^B) = d_u \triangle_i u_i^B(t) + f(u_i^B, v_i^B) \quad \text{and} \quad \gamma_2(u_i^B, v_i^B) = d_v \triangle_i v_i^B(t) + g(u_i^B, v_i^B)$$

and boundary conditions

$$u_i^B(t) = u_i^A(t+\Delta t) \quad \text{and} \quad v_i^B(t) = v_i^A(t+\Delta t) \tag{52}$$

has to be solved numerically on $[t, t+\Delta t]$. Because of the boundary conditions (49), we have to set

$$\begin{aligned} u_1^B(t+\Delta t) &:= u_2^B(t+\Delta t) \quad \text{and} \quad u_{N+1}^B(t+\Delta t) := u_N^B(t+\Delta t) \\ v_1^B(t+\Delta t) &:= v_2^B(t+\Delta t) \quad \text{and} \quad v_{N+1}^B(t+\Delta t) := v_N^B(t+\Delta t). \end{aligned}$$

We approximate (52) by performing one step with the implicit Euler method, which preserves the nonnegativity (compare with [3]). For $i = 2, ..., N$ the nonlinear equations

$$\begin{aligned} \frac{u_i^B(t+\Delta t) - u_i^B(t)}{\Delta t} &= d_u \triangle_i u_i^B(t+\Delta t) + f(u_i^B(t+\Delta t), v_i^B(t+\Delta t)) \\ \frac{v_i^B(t+\Delta t) - v_i^B(t)}{\Delta t} &= d_v \triangle_i v_i^B(t+\Delta t) + g(u_i^B(t+\Delta t), v_i^B(t+\Delta t)) \end{aligned}$$

have to be solved. For the solution of this nonlinear system, we use a fixed-point iteration proposed in [6], that usually converges after few steps, provided that the size of the time step Δt is small enough. We initialize the iteration with $u_i^0 := u_i^B(t)$ and $v_i^0 := v_i^B(t)$. The following iterations are then for $l = 1, 2, ...$ given by

$$\begin{aligned} u_i^l &= u_i^{l-1} + \Delta t \left[d_u \triangle_i \frac{u_i^0 + u_i^l}{2} + f\left(\frac{u_i^0 + u_i^{l-1}}{2}, \frac{v_i^0 + v_i^{l-1}}{2} \right) \right] \\ v_i^l &= v_i^{l-1} + \Delta t \left[d_v \triangle_i \frac{v_i^0 + v_i^l}{2} + g\left(\frac{u_i^0 + u_i^{l-1}}{2}, \frac{v_i^0 + v_i^{l-1}}{2} \right) \right]. \end{aligned}$$

To define a stopping condition, we introduce the discrete residual

$$\begin{aligned} \rho_u^l &= \max_{2 \le i \le N} \left(\frac{1}{\Delta t}(u_i^l - u_i^0) - \left[d_u \triangle_i \frac{u_i^0 + u_i^l}{2} + f\left(\frac{u_i^0 + u_i^l}{2}, \frac{v_i^0 + v_i^l}{2} \right) \right] \right) \\ \rho_v^l &= \max_{2 \le i \le N} \left(\frac{1}{\Delta t}(v_i^l - v_i^0) - \left[d_v \triangle_i \frac{v_i^0 + v_i^l}{2} + g\left(\frac{u_i^0 + u_i^l}{2}, \frac{v_i^0 + v_i^l}{2} \right) \right] \right) \\ R^l &= \max\{\rho_u^l, \rho_v^l\}. \end{aligned}$$

The iteration is continued until $R^l < TOL$ holds with a given tolerance $TOL > 0$. The last approximation obtained in this way is then the numerical approximation to the solution of the stochastic Gierer-Meinhardt system at time $t + \Delta t$.

5.3. **Numerical Results.** In this paragraph we apply the previous numerical procedure to simulate the Gierer-Meinhardt system for various parameter values.

5.3.1. *Stripes parallel to the direction of growth.* We first consider the parameter set

$$\begin{pmatrix} r_u = 0.02 & s_1 = 0.02 & d_u = 0.01 & b_u = 0.03 & S_u = 0.0 & C_u = 0.0 \\ r_v = 0.06 & s_2 = 0.02 & d_v = 0.4 & b_v = 0.1072 & S_v = 0.0 & C_v = 0.0 \end{pmatrix}. \tag{53}$$

We start by examining the influence of the intensity parameters α_u and α_v and increasing them from $\alpha_u, \alpha_v = 0$ (this being the deterministic case) to $\alpha_u, \alpha_v = 0.3$. The results are shown in the figures 1 and 2. Due to the influence of noise, the stripes exhibit breaks of different lengths, after which they start anew. With increasing intensity of noise, the breaks become more frequent. Note that in the deterministic case stripes may also end, but then do not start again.

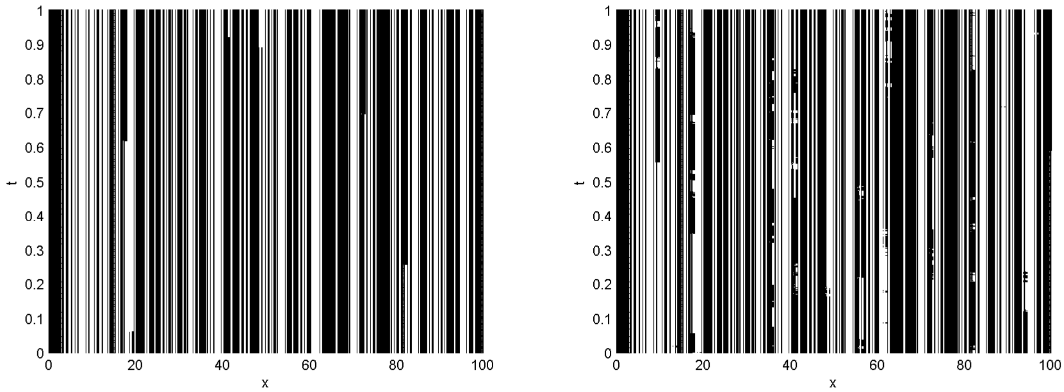

FIGURE 1. Left: pattern for the intensities $\alpha_u = \alpha_v = 0$ (deterministic system). Right: intensities $\alpha_u = \alpha_v = 0.003$.

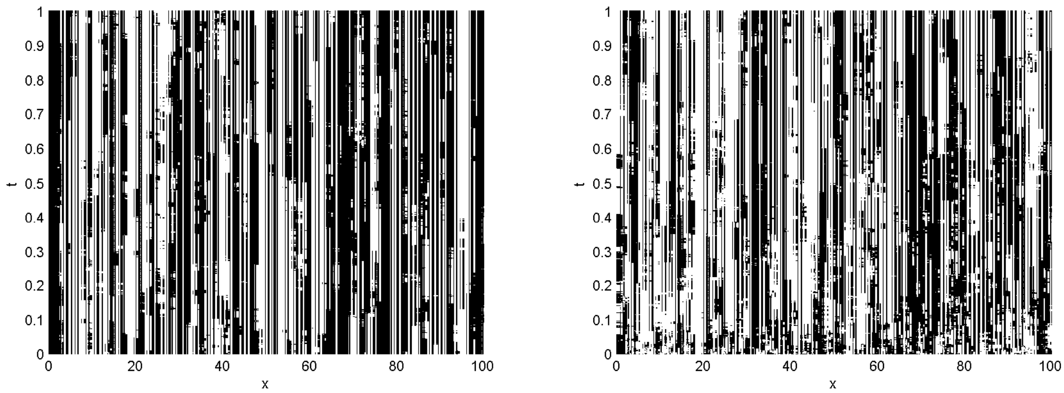

FIGURE 2. Left: pattern for the intensities $\alpha_u = \alpha_v = 0.03$. Right: intensities $\alpha_u = \alpha_v = 0.3$.

Figure 3 shows the results obtained by the variation of the saturation parameter S_u. The intensities have been set to $\alpha_u, \alpha_v = 0.03$.

The results for two further parameter sets

$$\begin{pmatrix} r_u = 0.2 & s_1 = 0.2 & d_u = 0.01 & b_u = 0.3 & S_u = 0.0 & C_u = 0.0 \\ r_v = 0.6 & s_2 = 0.2 & d_v = 0.4 & b_v = 0.82 & S_v = 0.0 & C_v = 0.0 \end{pmatrix} \tag{54}$$

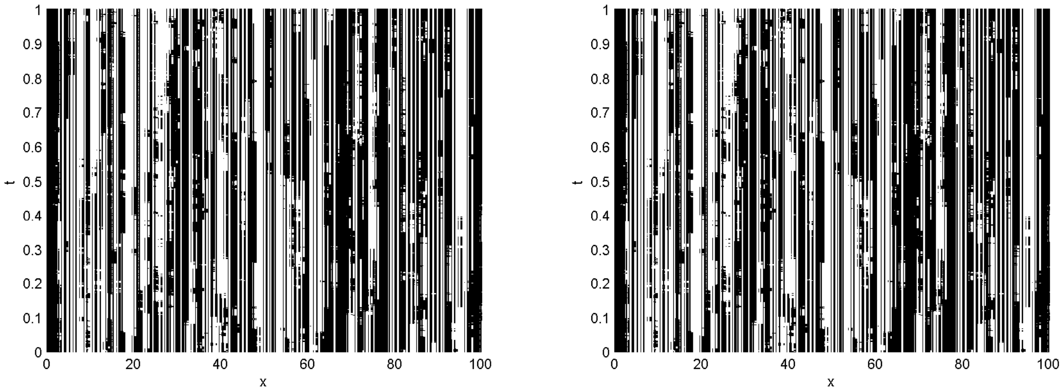

FIGURE 3. Left: pattern for the saturation parameter $S_u = 0.01$. Right: pattern for $S_u = 0.1$.

and

$$\begin{pmatrix} r_u = 0.002 & s_1 = 0.002 & d_u = 0.01 & b_u = 0.003 & S_u = 0.0 & C_u = 0.0 \\ r_v = 0.006 & s_2 = 0.002 & d_v = 0.4 & b_v = 0.1001 & S_v = 0.0 & C_v = 0.0 \end{pmatrix} \tag{55}$$

are shown in figure 4.

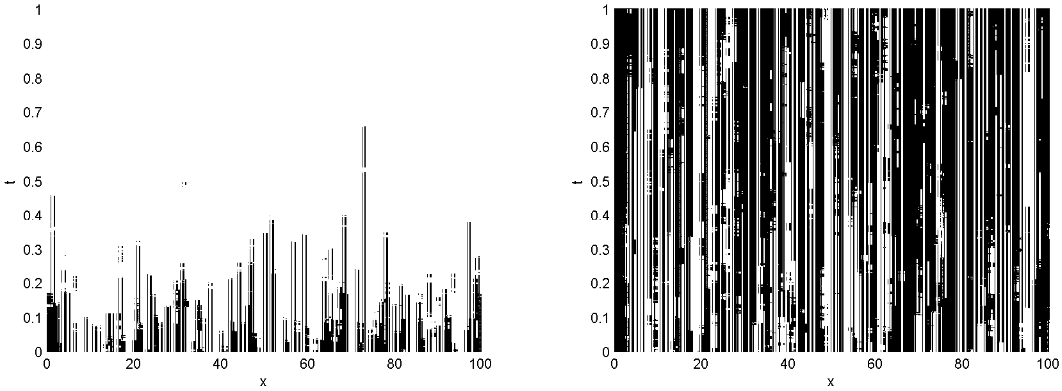

FIGURE 4. Patterns for different parameter sets. Left: set (54). Right: set (55).

Figure **??** shows a case with nonzero constants C_u and C_v in the noise coefficients. The rest of the parameters are the same as in set (53). A picture of *Lyria planicostata taiwanica* is also shown[1], in order to allow for a comparison of our simulations with real patterns.

5.3.2. *Stripes perpendicular to the direction of growth.* Here we start by choosing the parameters

$$\begin{pmatrix} r_u = 0.5 & s_1 = 1 & d_u = 0.1 & b_u = 0.3 & S_u = 0.0 & C_u = 0.0 \\ r_v = 0.3 & s_2 = 0.05 & d_v = 0.8 & b_v = 0.0 & S_v = 0.0 & C_v = 0.0 \end{pmatrix}. \tag{56}$$

[1]This picture has been taken from `http://www.specimenshells.net/1553.htm`.

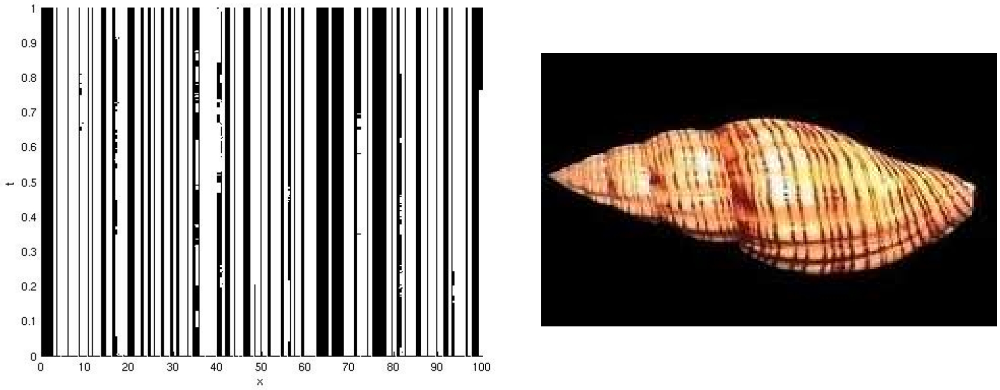

FIGURE 5. Left: pattern for set (53) with $C_u = C_v = 0.01$. Right: pattern on *Lyria planicostata taiwanica*.

Figures 6 and 7 show again the influence of the intensity parameters. In the figures 6 and 7 no qualitative difference with respect to the deterministic system is visible.The influence of noise manifests itself only through very small fluctuations. This however changes as soon as the intensities α_u, α_v are increased to 0.3. In this case a change in the frequency of the ocillation can be observed. At the same time, an additional local maximum begins to emerge.

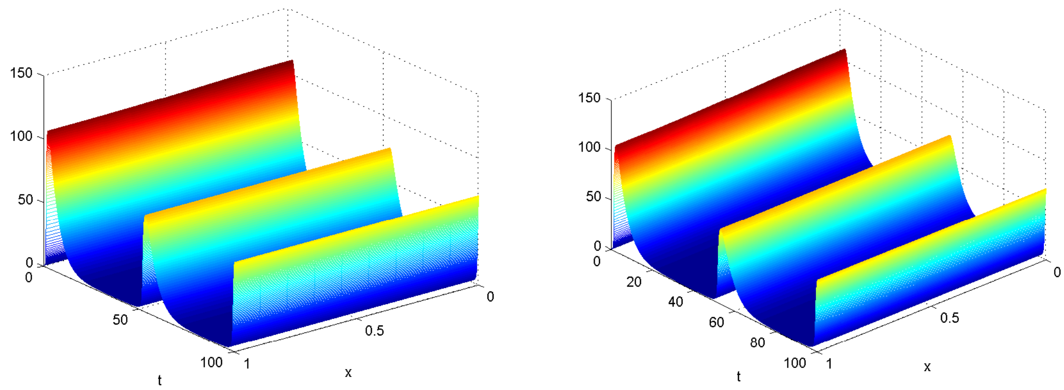

FIGURE 6. Activator field. Left: $\alpha_u = \alpha_v = 0$ (deterministic system). Right: $\alpha_u = \alpha_v = 0.003$.

Figure 8 shows a case with nonzero constants C_u and C_v in the noise coefficients. The rest of the parameters are the same as in set (56). A picture of *Amoria ellioti* is also shown[2] for the comparison with real patterns.

The results obtained by variation of the saturation parameter S_u are shown in figures 9 and 10. The intensities in each case are $\alpha_u, \alpha_v = 0.03$. The left figure shows in each case the

[2]This picture has been taken from `www.marinejewels.com.au/images/Shells`.

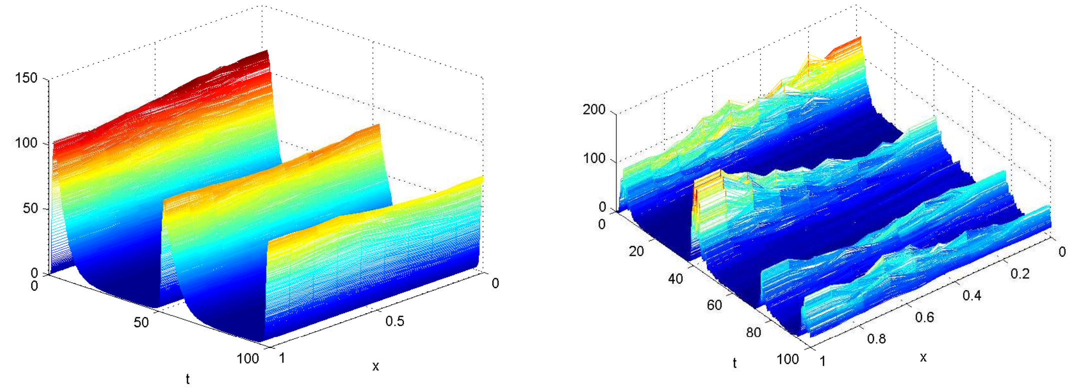

FIGURE 7. Activator field. Left: $\alpha_u = \alpha_v = 0.03$. Right: $\alpha_u = \alpha_v = 0.3$.

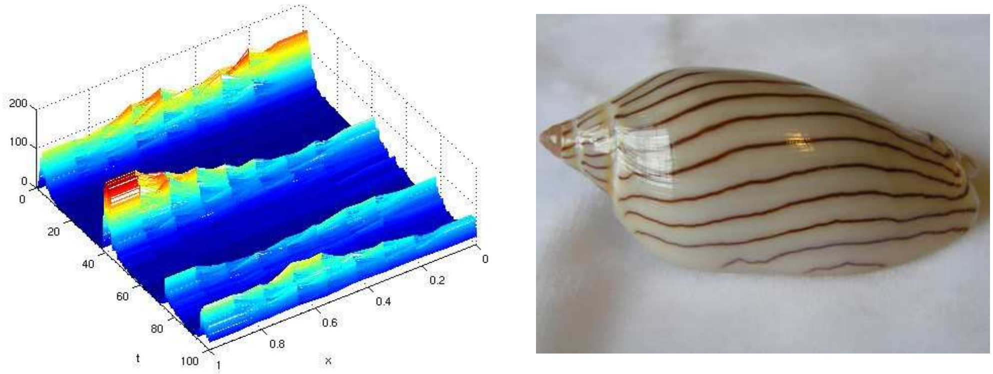

FIGURE 8. Left: activator field for the parameter set (56) and $C_u = C_v = 0.1$. Right: pattern on *Amoria elliot*i.

deterministic system for the respective choice of S_u. Unlike the deterministic system which moves into a constant state, the stochastic system exhibits visible oscillations around this state.

We conclude this section with the simulation of two additional parameter sets:

$$\begin{pmatrix} r_u = 5 & s_1 = 10 & d_u = 0.1 & b_u = 3 & S_u = 0.0 & C_u = 0.0 \\ r_v = 3 & s_2 = 0.5 & d_v = 0.8 & b_v = 0.0 & S_v = 0.0 & C_v = 0.0 \end{pmatrix} \tag{57}$$

and

$$\begin{pmatrix} r_u = 0.05 & s_1 = 0.1 & d_u = 0.1 & b_u = 0.03 & S_u = 0.0 & C_u = 0.0 \\ r_v = 0.03 & s_2 = 0.005 & d_v = 0.8 & b_v = 0.0 & S_v = 0.0 & C_v = 0.0 \end{pmatrix}. \tag{58}$$

The results are shown in the figures 11 and 12. The deterministic system for each choice of parameters is also shown for comparison.

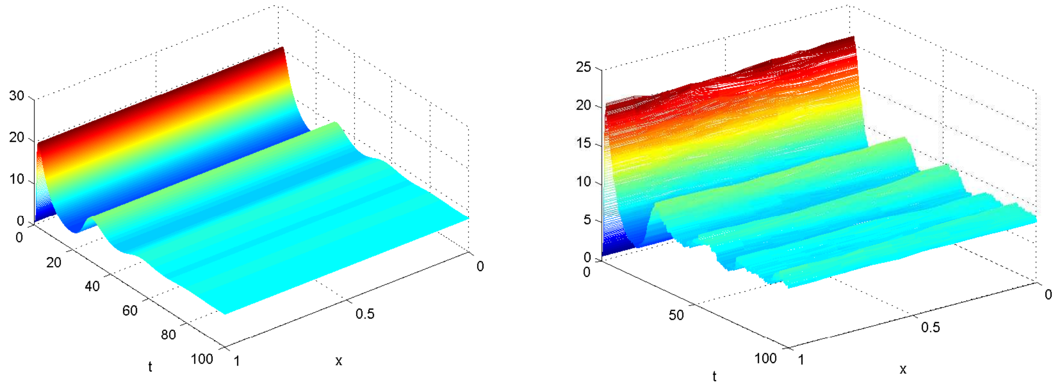

FIGURE 9. Concentration of activator after the saturation has been increased to $S_u = 0.01$. Left: deterministic. Right: stochastic.

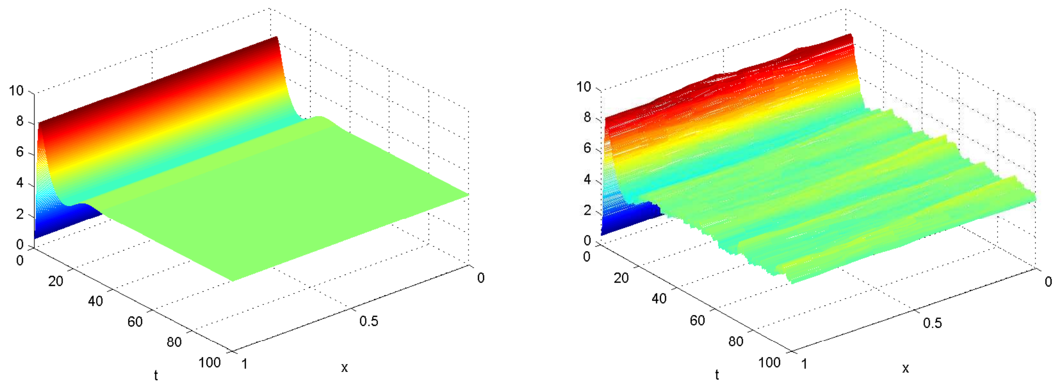

FIGURE 10. Concentration of activator after the saturation has been increased to $S_u = 0.1$. Left: deterministic. Right: stochastic.

6. CONCLUSIONS

In this paper we proposed a way to extend the Gierer-Meinhardt model for pattern formation to allow for random influences. For the resulting stochastic Gierer-Meinhard system we proved the existence of a local mild solution via an iterative method which has also been used for the existence proof in the deterministic case [11]. There it was also possible to show global existence of the solution upon using the method of invariant regions, whereas in the stochastic case the latter does not apply. The existence proof has been done for positive saturation constants (which is biologically expedient), however this seems not to be a substantial restriction, since in the deterministic case it was possible to show even global existence for nonnegative constants. For the corresponding stochastic analysis one could deduce the needed estimates with the iterative method in [19] and [20], after making some supplementary assumptions on the data.

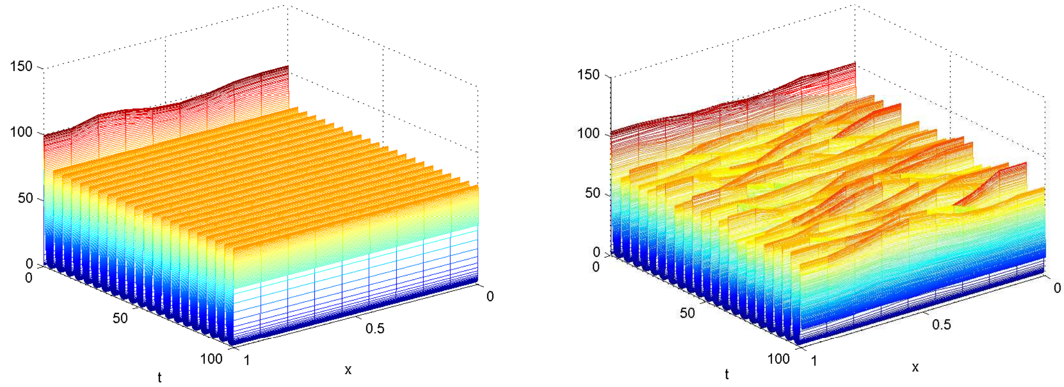

FIGURE 11. Concentration of activator for (57). Left: deterministic. Right: $\alpha_u, \alpha_v = 0.03$.

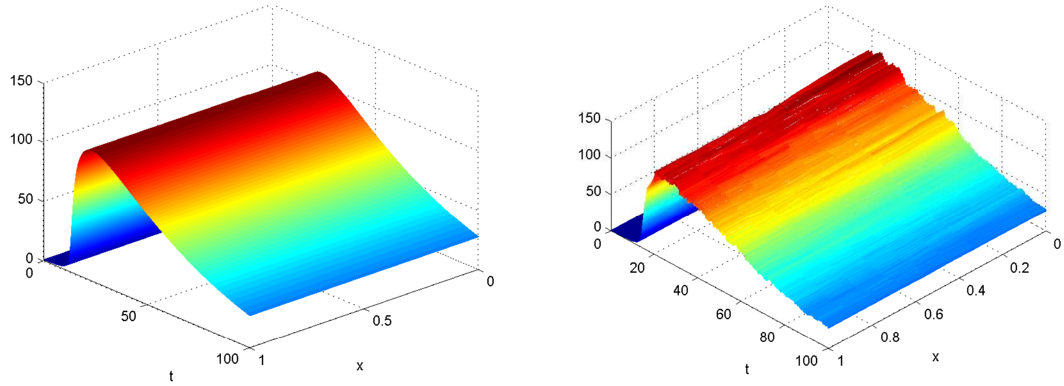

FIGURE 12. Concentration of activator for (58). Left: deterministic. Right: $\alpha_u, \alpha_v = 0.03$.

A particularly interesting issue was the positivity of the solution, since comparison principles for stochastic partial differential equations are less established than their deterministic counterparts. After having proved the positivity of the solution to the SPDE, a positivity preserving numerical procedure was needed to perform simulations. Thereby, the splitting method in [15] has proved its reliability.

The subsequent simulations have shown that space-time perturbations lead for spatial patterns to interrupted and reestablished rather than to merely 'jittered' stripes in the patterns. When the latter were generated by time oscillations there have been noticed supplementary, weaker oscillations as well as a frequency alteration in comparison with the deterministic case.

Throughout this paper we assumed that the domain of interest was not moving in time. Space-time patterns similar to those observed on real seashells have been obtained even with this simplification (see Section 5), however considering time-varying domains would allow to account for the growth of the shells. This seems to be a nontrivial issue, at least from the viewpoint of mathematical analysis. While there are some numerical simulations for growing domains (see e.g., [9]), there are by our knowledge no results concerning existence and regularity for solutions to such problems, not even in the deterministic case. Allowing the domains to move in time could also lead -together with random fluctuations- to clear qualitative differences to the deterministic setting.

Furthermore, the stochastic Gierer-Meinhard system considered in this paper is only a possibility to account for random perturbations. If internal fluctuations are to play an important role, this would lead to different stochastic equations involving noise coefficients in a nonlinear dependence of the latent variables. For these it is still open how to prove the existence of even local solutions. Nevertheless, numerical simulations could still be performed, however upon using other numerical procedures, since because of the lack of explicit solutions the splitting method employed in this paper no longer applies.

7. Appendix

7.1. **Linear SPDEs.** Consider the linear SPDE

$$\partial_t u = Au + f(t,x) + \sigma(t,x)\partial_t W(t,x), \qquad x \in \Omega, t \in (0,T), \tag{59}$$

with A a linear operator.

Its solution u can be written as $u = \hat{u} + \zeta$ with

$$\hat{u}(\cdot,t) = G_t u_0 + \int_0^t G_{t-s} f(\cdot,s)ds. \tag{60}$$

$$\zeta(\cdot,t) = \int_0^t G_{t-s}\sigma(\cdot,s)dW(\cdot,ds). \tag{61}$$

The relevant properties of $\hat{u}$ are stated in

Lemma 7.1. *(see* [4]*)*

Let $\hat{u}(x,t)$ be given by (60). Further, assume $u_0 \in L^2(\Omega)$ and let $f(t,\cdot)$ be an $\mathcal{F}_t$-adapted process in $L^2(\Omega)$, for which

$$\mathbb{E}\left\{\int_0^T \|f(\cdot,t)\|^2_{L^2(\Omega)}dt\right\} < \infty. \tag{62}$$

Then $\hat{u}(\cdot,t)$ is an adapted $H^1(\Omega)$-valued process, continuous on $L^2(\Omega)$ and satisfying the estimate

$$\mathbb{E}\sup_{0\le t\le T}\|\hat{u}(\cdot,t)\|^2_{L^2(\Omega)} + E\left(\int_0^T \|\hat{u}(\cdot,t)\|^2_{H^1}dt\right) \le C\left\{\|u_0\|^2_{L^2(\Omega)} + \mathbb{E}(\int_0^T \|f(\cdot,t)\|^2_{L^2(\Omega)}dt\right\}$$

for a constant $C > 0$.

For the noise component ζ the following result holds:

Lemma 7.2. *(see Lemma 5.2 in* [4] *for bounded covariance functions)*

Let $\zeta(t,x)$ satisfy (61) and let $\sigma(t,\cdot)$ be an $\mathcal{F}_t$-adapted process in $L^2(\Omega)$ such that

$$\mathbb{E}\left\{\int_0^T \|\sigma(\cdot,t)\|^2_{L^2(\Omega)}dt\right\} < \infty. \tag{63}$$

Then $\zeta(t,\cdot)$ is an adapted $H^1(\Omega)$-valued process, continuous in $L^2(\Omega)$, which satisfies the estimates

$$\mathbb{E}\sup_{0\le t\le T}\|v(t,\cdot)\|^2_{L^2(\Omega)} \le 16\mathbb{E}\int_0^T\int_\Omega r(x,x)\sigma^2(t,x)dxdt$$

and

$$\mathbb{E}\int_0^T \|\zeta(\cdot,t)\|^2_{H^1(\Omega)} \le \frac{1}{2}\mathbb{E}\int_0^T\int_\Omega r(x,x)\sigma^2(t,x)dxdt.$$

7.2. **A comparison principle for SPDEs.** Comparison principles are well known and useful methods for dealing with deterministic parabolic equations, since they allow a.o. to bound the solution from below. Under adequate assumptions these powerful tools carry over to the framework of stochastic partial differential equations.

In this paper we need the following particular case of Theorem 1 in [2]:

Theorem 7.1. *Let $u_0^{(l)}: \Xi \to L^2(\Omega)^m$ $(l=1,2)$ be two $\mathcal{F}_0$-measurable, square-integrale random variables such that*

$$u_0^{(1)} \le u_0^{(2)} \quad \text{P-a.s.}$$

where the relationship "$\le$" is to be understood componentwise. Further assume that $u_0^{(l)}$ is $\mathcal{F}\times\mathcal{B}(\Omega)$ product measurable as a function $(\xi,y)\mapsto u_0^{(l)}(\xi)(y)$, $l=1,2$ and let the drift and noise functions of the system satisfy the following conditions:

(a) $\|f^{(l)}(t,\xi,\mathbf{y},\mathbf{x}) - f^{(l)}(t,\xi,\mathbf{y},\mathbf{z})\|_{\mathbb{R}^N} \le C(T)\|\mathbf{x}-\mathbf{z}\|_{\mathbb{R}^N} \qquad (l=1,2)$,
for $t\in[0,T]$, $\xi\in\Xi$, $\mathbf{x},\mathbf{z}\in\mathbb{R}^N$.

(b) $\|f^{(l)}(t,\xi,\mathbf{y},\mathbf{x})\|_{\mathbb{R}^N} + \|\sigma_{ij}(t,\xi,\mathbf{y},\mathbf{x})\|_{\mathbb{R}^N} \le C(T)(1+\|\mathbf{x}\|_{\mathbb{R}^N}) \qquad (l=1,2;\ i,j=1,...,N)$
for $t\in[0,T]$, $\xi\in\Xi$, $\mathbf{x},\mathbf{z}\in\mathbb{R}^N$

(c) $|\sigma_{ij}(t,\xi,y,x)-\sigma_{ij}(t,\xi,y,z)| \le C(T)|x_i - z_i|, \quad t\in[0,T] \qquad (i,j=1,...,N)$,
for $t\in[0,T]$, $\xi\in\Xi$, $\mathbf{x},\mathbf{z}\in\mathbb{R}^N$.

Then if

$$f^{(1)}_{z^{(1)}}(t,\cdot,\mathbf{y},\mathbf{x}) \le f^{(2)}_{z^{(2)}}(t,\cdot,\mathbf{y},\mathbf{x}) \quad \text{for } \mathbf{x}\in\mathbb{R}^N \quad \text{and } z^{(1)}\le z^{(2)} \quad \textit{P-a.s.}$$

for every $t\in[0,T]$ and almost all (w.r.t. the Lebesgue measure) $\mathbf{y}\in\Omega$, we obtain for the pathwise unique mild solutions $u^{(l)}$ $(l=1,2)$ of the initial and boundary value problem with drift $f^{(l)}$ and initial function $u_0^{(l)}$ the relationship

$$u^{(1)}(t) \le u^{(2)}(t) \quad \text{P-a.s.}$$

for all $t\in[0,T]$.

Here we made the notation $\mathbf{f}^{(l)}_{\mathbf{z}}(t,\cdot,\mathbf{y},\mathbf{x}) = \mathbf{f}^{(l)}(t,\cdot,\mathbf{y},(z_1,\dots,x_i,\dots,z_N)^t)$, $i=1,\dots,N$, $l=1,2$.

7.3. **Some facts on Brownian sheets.** The Brownian sheet is an extension of the Brownian motion to one time dimension and one or more space dimensions. It is defined as follows (see Section 4.3.3 in [5]):

Definition 7.1. *Let K be the N-dimensional unit cube*

$$K = \{\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{R}^N : 0 \leq \alpha_k \leq 1, k = 1, ..., N\}.$$

Further, let $\{\phi_j\}_{j\in\mathbb{N}}$ be a complete othonormal basis of $L^2(K)$ and let $\{\beta_j\}_{j\in\mathbb{N}}$ denote a family of independent, real valued Wiener processes.

Define

$$W(t, \mathbf{x}) = \sum_{j=1}^{\infty} \beta_j(t) \int_{R(\mathbf{x})} \phi_j(\alpha) d\alpha, \tag{64}$$

with $t \in [0, T]$, $\mathbf{x} = (x_1, ..., x_N)$ and $R(\mathbf{x})$ denoting the set

$$R(\mathbf{x}) = \{\alpha \in K : 0 \leq \alpha_k \leq x_k \text{ , } k = 1, ..., N\}$$

Then the series (64) converges P-a.s. for any $t \in [0, T]$ and $\mathbf{x} \in K$. Moreover,

$$\mathbb{E}(W(t_1, \mathbf{x})W(t_2, \mathbf{y})) = \min\{t_1, t_2\} \prod_{k=1}^{N} \min\{x_k, y_k\}$$

holds for $\mathbf{x}, \mathbf{y} \in K$ and $t_1, t_2 \in [0, T]$.

The Gaussian random field $W(\cdot, \cdot)$ defined by (64) has a continuous representant, which is called a ***Brownian sheet*** *on $[0, T] \times K$.*

As it was done for the Brownian motion, one can also define stochastic integrals with respect to a Brownian sheet. Thereby, one starts with indicator functions of sets $[0, t] \times R(\mathbf{x})$ with $t \in [0, T]$ and $\mathbf{x} \in K$, for which the corresponding integral is given by

$$\int_0^T \int_K \chi_{[0,t]}(s) \chi_{R(x)}(y) dW(s, y) = W(t, x)$$

Then the stochastic integral w.r.t. a Brownian sheet can be defined for adequate real valued stochastic processes $\phi(s, \mathbf{y})$ with $s \in [0, T]$ and $\mathbf{y} \in K$, upon approximating the latter with the aid of indicator functions.

The following important property of a Brownian sheet is taken from [1]:

Lemma 7.3. *Let $W(\cdot, \cdot)$ be a Brownian sheet on $[0, T] \times [0, 1]$ and let h be a function such that*

$$\mathbb{E}\left(\int_0^T \int_0^1 h^2(t, x) dx dt\right) < \infty.$$

Then it follows that

$$\mathbb{E}\left[\int_0^T \int_0^1 h(t, x) dW(t, x)\right]^2 = \mathbb{E}\left[\int_0^T \int_0^1 h(t, x)^2 dx dt\right].$$

REFERENCES

[1] E.J. Allen, S.J. Novosel, Z. Zhang, *Finite Element and Difference Approximation of Some Linear Stochastic Partial Differential Equations*, Stochastics and Stochastic Reports 64, 117-142 (1998).
[2] S. Assing, *Comparison of Systems of Stochastic Partial Differential Equations*, Stochastic Processes and their Applications 82, 259-282 (1999).
[3] C. Bolley, M. Crouzeix, *Conservation de la positivité lors de la discrétization des problèmes d'évolution paraboliques.* R.A.I.R.O. Numer. Anal. 12(3), 237 (1978).
[4] P.L. Chow, **Stochastic Partial Differential Equations.** Chapman & Hall, 2007.
[5] G. Da Prato, J. Zabczyk, **Stochastic Equations in Infinite Dimensions**, Cambridge University Press (1992).
[6] K. Eriksson, C. Johnson, A. Logg, *Explicit time-stepping for stiff ODE's*, Chalmers Finite Element Center PREPRINT 2002-7, Göteborg (2002).
[7] A. Gierer, H. Meinhardt, *A theory of biological pattern formation*, Kybernetik 12 (1972) 30-39.
[8] M.I. Granero-Porati, A. Porati, *Temporal Organization in a morphogenetic field.*, J. Math. Biology 20, 153-157 (1984).
[9] M. Ishikawa, K. Miyajima, *Analyses of pattern formation processes in stochastic activator-inhibitor systems with saturation in growth domains*, Int. J. of Innovative Comp. Inf. and Control. 1, 3, 303-311 (2005).
[10] H. Jiang, *Global existence of solutions of an Activator-Inhibitor System*, Disc. and Cont. Dyn. Sys. 14, 737-751 (2006).
[11] J. Kelkel, C. Surulescu, *Global existence and numerical simulations for a reaction-diffusion system modeling pattern formation on seashells*, preprint, IANS, University of Stuttgart, 2008.
[12] P.E. Kloeden, E. Platen, **Numerical Solution of Stochastic Differential Equations**, Springer (1992).
[13] H. Meinhardt, **The Algorithmic Beauty of Seashells**, Springer, 2003.
[14] H. Meinhardt, M. Klingler, *A model for pattern formation on the shells of molluscs*, J. Theor. Biol 126 (1987) 63-69.
[15] E. Moro, H. Schurz, *Non-negativity preserving numerical algorithms for stochastic differential equations*, arXiv: arXiv:math/0509724v1 [math.NA] (2005).
[16] J.D. Murray, *Parameter Space for Turing instability in reaction diffusion mechanisms: a comparison of models.* J.Theor.Biol. 98, 143-163 (1982).
[17] J.D. Murray, **Mathematical Biology II.** Springer, 2003.
[18] F. Rothe, **Global Solutions of Reaction-Diffusion Systems.** Lecture Notes in Math. 1072, Springer-Verlag (1984).
[19] T. Taniguchi, *On sufficient conditions for non-explosion of solutions to stochastic differential equations.* J. Math. Analysis Appl. 153, 549-561 (1990).
[20] T. Taniguchi, *Successive approximation to solutions of stochastic differential equations.* J. Diff. Equations 96, 152-169 (1992).
[21] A.M. Turing, *The chemical basis of morphogenesis*, Phil Trans. Roy. Soc. London B 237, 37 (1952).
[22] J.B. Walsh, *An Introduction to stochastic Partial Differential Equations*, Lecture Notes in Mathematics 1180, 265-439, Springer-Verlag, Berlin (1986).
[23] J. Wei, M. Winter, *Spikes for the two-dimensional Gierer-Meinhardt system: the weak coupling case*, J. Nonlinear Sci. 11(6), 415-458 (2001).
[24] J. Wei, M. Winter, *Spikes for the Gierer-Meinhardt system in two dimensions: the strong coupling case*, J. Differential Equations, 178(2), 478-518 (2002).

Jan Kelkel
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: jan.kelkel@mathematik.uni-stuttgart.de

Christina Surulescu
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: christina.surulescu@ians.uni-stuttgart.de
WWW: http://www.ians.uni-stuttgart.de/LstAngMath/Surulescu/

Erschienene Preprints ab Nummer 2008/001

Komplette Liste: `http://preprints.ians.uni-stuttgart.de`

2008/001 *Perfahl, H., Sändig, A.-M.:* A Continuum-Mechanical Approach to Avascular Solid Tumor Growth

2008/002 *Giesselmann, J.:* A convergence result for finite volume schemes on 2-dimensional Riemannian manifolds

2008/003 *Rohde, C., Surulescu, C.:* Mathematische Modellierung und Analyse von biologischen Prozessen

2008/004 *Lalegname, A., Sändig, A.-M., Sewell, G.:* Analytical and numerical treatment of a dynamic crack model

2008/005 *Kettemann, A.:* A mathematical model for mesenchymal and chemosensitive cell dynamics in tissue networks

2008/006 *Kelkel, J., Surulescu, C.:* Global existence and numerical simulations for a reaction-diffusion system modeling pattern formation on seashells

2008/007 *Kelkel, J., Surulescu, C.:* On a stochastic reaction-diffusion system modeling pattern formation