Universität Stuttgart

Singularities of the Stokes System in Polygons

Pascal Märkl, Anna-Margarete Sändig

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Preprint 2008/009

Universität Stuttgart

Singularities of the Stokes System in Polygons

Pascal Märkl, Anna-Margarete Sändig

Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation

Preprint 2008/009

Institut für Angewandte Analysis und Numerische Simulation (IANS)
Fakultät Mathematik und Physik
Fachbereich Mathematik
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: `ians-preprints@mathematik.uni-stuttgart.de`
WWW: `http://preprints.ians.uni-stuttgart.de`

ISSN **1611-4176**

Contents

Abstract

We consider stationary Stokes flows in two dimensional polygonal domains under different conditions on parts of the boundary. It is well known that Stokes flows can develop infinite velocity gradients and infinite pressures near corners or points where the boundary conditions change. The velocity and pressure fields can be decomposed additively into a singular and a more regular part, where the singular part characterizes the optimal regularity of the solutions.

The goal of the paper is to calculate the singular terms for different conditions on parts of the boundary by a quite elementary method. Analogously to [14] where the singular terms for different boundary value problems for the Lamé equations are calculated and similar to [10], where the Dirichlet problem for the Stokes system is handled, we use special ansatzes and spherical coordinates to derive a generalized boundary eigenvalue problem. The resulting eigenvalues and eigensolutions generate the singular terms. The eigenvalues are computed with MATLAB, the eigensolutions are described by analytical formulas.

1 Introduction

It is well known [2],[12],[13] that Stokes flows in two or three dimensional domains Ω can have infinite velocity gradients and infinite pressures near corners and edges or points and curves where the boundary conditions change. The velocity and pressure fields can be decomposed additively into a singular and a more regular part.

Here we consider the two dimensional stationary case with different boundary conditions. In this case the decomposition reads in a domain with one corner point O

$$\vec{U} = \begin{pmatrix} u_1 \\ u_2 \\ p \end{pmatrix} = \sum_{\nu} c_{\alpha_\nu} \eta \begin{pmatrix} r^{\alpha_\nu} f(r,\omega) \\ r^{\alpha_\nu} g(r,\omega) \\ r^{\alpha_\nu - 1} h(r,\omega) \end{pmatrix} + \vec{U}_{\text{reg}} = \vec{U}_{\text{sing}} + \vec{U}_{\text{reg}}.$$

The first two components denote the components of the velocity field, the third the pressure field, (r, ω) are the polar coordinates, η is a cut-off function in a neighborhood of the corner point O. Starting from a weak solution $(\vec{u}, p) \in [H^1(\Omega)]^2 \times L^2(\Omega)$, the regular part should be in $[H^2(\Omega)]^2 \times H^1(\Omega)$ and for the (in general complex) exponents α_ν in the singular part should hold $\operatorname{Re} \alpha_\nu \in (0,1)$.
The goal of the paper is to calculate the singular terms for different conditions on parts of the boundary for so-called simple eigenvalues α_ν by a quite elementary method. Analogously to [14] where the singular terms for different boundary value problems for the Lamé equations are calculated and similar to [10], where the Dirichlet problem for the Stokes system is handled, we use special ansatzes and spherical coordinates to derive a generalized boundary eigenvalue problem. The resulting eigenvalues and eigensolutions generate the singular terms. The knowledge of the singular terms allows to determine the optimal regularity of weak solutions. It holds that $\vec{U} \in H^{1+\text{Re}\alpha_0 - \epsilon}(\Omega)$, where α_0 is the eigenvalue with smallest real part in the intervall $(0, 1)$.

The paper is organized as follows: We start with the formulation of the problem in section 2. Section 3 is devoted to the analytical derivation of the singular terms. In section 4 the computation of the singular exponents is explained and two examples are presented. The distribution of the eigenvalues is illustrated in Appendix A , whereas in Appendix B the MATLAB-program for the computation of complex eigenvalues is listed.

2 Formulation of the problem

Flows in incompressible, viscous fluids are described by the system of Navier-Stokes equations. It reads for a given volume force density $\vec{f}$, a constant mass density ρ and a viscosity parameter μ:

Find the velocity field $\vec{v}$ and the pressure field p such that

$$\begin{aligned}\frac{\partial}{\partial t}\vec{v} + \sum_i v_i \frac{\partial}{\partial x_i}\vec{v} = \vec{f} - \operatorname{grad}\frac{p}{\rho} + \frac{\mu}{\rho}\,\Delta\,\vec{v},\\ \operatorname{div}\vec{v} = 0\,.\end{aligned} \tag{2.1}$$

Boundary value problems for the stationary Stokes equations. If we only consider very viscous fluids, we can neglect the nonlinear term $\sum_i v_i \frac{\partial}{\partial x_i}\vec{v}$ in (2.1) and achieve the linear Stokes equations

$$\begin{aligned}\frac{\partial}{\partial t}\vec{v} = \vec{f} - \operatorname{grad}\frac{p}{\rho} + \frac{\mu}{\rho}\,\Delta\,\vec{v}\\ \operatorname{div}\vec{v} = 0.\end{aligned} \tag{2.2}$$

Furthermore, we restrict our consideration to the stationary two dimensional Stokes system in a polygonal domain Ω:

$$\begin{aligned}-\frac{\mu}{\rho}\,\Delta\,\vec{v} + \operatorname{grad}\frac{p}{\rho} = \vec{f}\\ \operatorname{div}\vec{v} = 0\,.\end{aligned} \tag{2.3}$$

To complete the mathematical description of the 2D stationary Stokes flows we define boundary conditions on $\partial\Omega$. We introduce an orthonormal basis $\mathcal{B} = \{\vec{n}, \vec{t}\}$ on every point of the smooth pieces of the boundary $\partial\Omega$, where $\vec{n} = (n_1, n_2)^{\mathrm{T}}$ is the normal vector and $\vec{t} = (-n_2, n_1)^{\mathrm{T}}$ the corresponding tangent vector. We study the following boundary conditions ([12, p.13]):

1. We prescribe a velocity field on $\Gamma_D \subset \partial\Omega$ (Dirichlet boundary condition).
2. We prescribe the surface tensions on $\Gamma_N \subset \partial\Omega$ (Neumann boundary condition).
3. We mix the conditions 1 and 2, that means we prescribe the velocity field in normal respectively tangential direction and the surface tension in tangential respectively normal direction on $\Gamma_M \subset \partial\Omega$ (mixed boundary condition).

We demand

$$\begin{aligned}\overline{\Gamma_D \cup \Gamma_N \cup \Gamma_M} = \partial\Omega\,,\\ \Gamma_D \cap \Gamma_N = \emptyset\,,\, \Gamma_D \cap \Gamma_M = \emptyset\,,\, \Gamma_N \cap \Gamma_M = \emptyset\,.\end{aligned}$$

Now, we specify the boundary conditions for the 2D stationary Stokes equations.

Dirichlet boundary conditions:

$$\vec{v} = \vec{\phi} \quad \text{on } \Gamma_D\,, \tag{2.4}$$

Neumann boundary conditions:

$$-\frac{p}{\rho}\vec{n} + \frac{\mu}{\rho}\big(\operatorname{grad}\vec{v} + (\operatorname{grad}\vec{v})^{\mathrm{T}}\big)\,\vec{n} = \vec{\psi} \quad \text{on } \Gamma_N\,, \tag{2.5}$$

mixed boundary conditions:

$$\left.\begin{aligned} \vec{v}\cdot\vec{n} &= \phi_n \\ \left(-\frac{p}{\rho}\vec{n} + \frac{\mu}{\rho}\left(\operatorname{grad}\vec{v} + \left(\operatorname{grad}\vec{v}\right)^{\mathrm{T}}\right)\vec{n}\right)\cdot\vec{t} &= \psi_t \end{aligned}\right\} \quad \text{on } \Gamma_M \tag{2.6}$$

or

$$\left.\begin{aligned} \vec{v}\cdot\vec{t} &= \phi_t \\ \left(-\frac{p}{\rho}\vec{n} + \frac{\mu}{\rho}\left(\operatorname{grad}\vec{v} + \left(\operatorname{grad}\vec{v}\right)^{\mathrm{T}}\right)\vec{n}\right)\cdot\vec{n} &= \psi_n \end{aligned}\right\} \quad \text{on } \Gamma_M\,. \tag{2.7}$$

Weak formulation. We start with the definition of some function spaces. Let $L^2(\Omega)$ be the standard Lebesgue space of quadratic integrable functions.

Definition 2.1 (Sobolev spaces)**.** *Let u be Lebesgue measurable, $k \in \mathbb{N}_0$ and α a multi-index. Then the Sobolev space $W^{k,2}(\Omega) = H^k(\Omega)$ is defined by*

$$W^{k,2}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \middle| \|u\|_{W^{k,2}(\Omega)} := \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^2 \, dx \right)^{\frac{1}{2}} < \infty \right\}$$

with

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,.$$

Definition 2.2 (Sobolev-Slobodeskij spaces)**.** *Let u be Lebesgue measurable, $\Omega \subset \mathbb{R}^2$, α a multi-index and $s \in \mathbb{R}^+$. Decompose $s = k + \lambda$ with $k \in \mathbb{N}_0$ and $0 < \lambda < 1$. Then the space*

$$H^s(\Omega) := W^{s,2}(\Omega) := \left\{ u \in W^{k,2}(\Omega) \middle| I_\alpha(u) := \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{2+2\lambda}} \, dx\, dy < \infty \quad \text{for all } \alpha \text{ with } |\alpha| = k \right\}$$

endowed with the norm

$$\|u\|_{W^{s,2}(\Omega)} := \left(\|u\|^2_{W^{k,2}(\Omega)} + \sum_{|\alpha|=k} I_\alpha(u) \right)^{\frac{1}{2}}$$

is a Sobolev-Slobodeskij space.

In order to formulate weak solutions we demand that the velocity fields belong to a space $W \subset \left[H^1(\Omega)\right]^2$, which is defined by the vanishing essential boundary conditions. The pressure field p should be contained in $L^2(\Omega)$. In general, the weak formulation reads:

For $f \in W'$ find $(\vec{v}, p) \in \left[H^1(\Omega)\right]^2 \times L^2(\Omega)$ such that for all $(\vec{w}, q) \in W \times L^2(\Omega)$ holds

$$\begin{aligned} \int_\Omega \frac{\mu}{\rho} \nabla\vec{v} : \nabla\vec{w}\, dx - \int_\Omega \frac{p}{\rho} \operatorname{div}\vec{w}\, dx + \int_{\partial\Omega} \left(-\frac{\mu}{\rho}\frac{\partial\vec{v}}{\partial\vec{n}}\cdot\vec{w} + \frac{p}{\rho}\vec{n}\cdot\vec{w} \right) d\sigma &= \int_\Omega \vec{f}\cdot\vec{w}\, dx \\ \int_\Omega (\operatorname{div}\vec{v})\, q\, dx &= 0\,. \end{aligned} \tag{2.8}$$

The scalar product of two matrices A, B is defined by $A : B = \sum_{i,j} a_{ij} b_{ij}$.

Regularity If the right hand sides are smoother, e.g. $f \in \left[L^2(\Omega)\right]^2$ and $\vec{\phi} \in \left[H^{\frac{3}{2}}(\partial\Omega)\right]^2$, $\vec{\psi} \in \left[H^{\frac{1}{2}}(\partial\Omega)\right]^2$, the boundary conditions do not change and the domain Ω is smooth enough, then a weak solution belongs to $\left[H^2(\Omega)\right]^2 \times H^1(\Omega)$, see [16, p.33, Prop. 2.2]. It is well known that this regularity result does not hold if the domain has corners or if changing boundary conditions occur. But, the regularity can be described in these cases by a decomposition of the 2D solution field

$$\vec{V}(x,y) = \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \\ p(x,y) \end{pmatrix} .$$

into a singular and a regular term:

$$\vec{V} = \vec{V}_{\text{sing}} + \vec{V}_{\text{reg}} = \sum_{i,j} r_j^{\alpha_{i,j}} S_{i,j}(\alpha_{i,j}, r_j, \omega_j) + \vec{V}_{\text{reg}} . \tag{2.9}$$

Here, the corner points are indicated by j with the corresponding polar coordinates (r_j, ω_j) the exponents $\alpha_{i,j}$ are eigenvalues of a Sturm-Liouville problem, $S_{i,j}$ are the generalized eigenvector fields and the regular part $\vec{V}_{\text{reg}}$ belongs to $\left[H^2(\Omega)\right]^2 \times H^1(\Omega)$.
The knowledge of the singular terms allows to determine the Sobolev-Slobodeskij spaces in which the weak solutions are contained. Thus we can formulate the regularity problem in the following way:

Definition 2.3 (Regularity problem for the 2D stationary Stokes equations)**.** *Determine an optimal $s \in \mathbb{R}$ with $s \geq 0$, so that the leading singularity is in the Sobolev-Slobodeskij space $\left[H^{1+s}(\Omega)\right]^2 \times H^s(\Omega)$.*

The main goal of this paper is the derivation and computation of the singular terms. There are some papers about this problem, see [2], [12], [10]. M. Dauge discussed in [2] only Dirichlet boundary conditions and M. Orlt used in his PhD thesis [12] a quite complicated method . We use ideas from Kozlov, Maz'ya and Rossmann [10] to determine the singular terms more elementary, consider additionally more boundary conditions compared with [10] and compute the singular terms.

3 Analytical derivation of the singular terms

In this section we discuss when singular terms, see (2.9), occur near corners and which structure they have. First, we localize the boundary value problems near a corner point and consider so-called model problems in an infinite cone. By using special ansatz methods and the transformation in spherical coordinates we get a generalized eigenvalue problem. We will see that the singular terms have almost throughout the structure

$$\vec{U}_{\text{sing}} = c_\alpha \begin{pmatrix} r^\alpha f(\omega) \\ r^\alpha g(\omega) \\ r^{\alpha-1} h(\omega) \end{pmatrix} \tag{3.1}$$

with $0 < \operatorname{Re}\alpha < 1$. The singular terms $\vec{U}_{\text{sing}}$ are generated by eigenpairs of a generalized eigenvalue problem. We calculate the exponents α for different boundary conditions and determine the correspondent local eigensolutions.

3.1 Localization and the model problem

In this subsection we describe how to localize the boundary value problem for the Stokes equations near corners. We need the localization to get the singular terms (3.1). After that we include the local singular terms into the global solution for the domain Ω.

3.1.1 Localization near a corner

We consider a polygonal domain with N corner points O_i, $i \in \{1, \dots, N\}$. As corner point we denote also such boundary points where the boundary conditions change. Let $\vec{V} = (v_1, v_2, p)^\top$ be a weak solution of the stationary Stokes equations in the polygonal domain. We multiply $\vec{V}$ by a cut-off function η_i, which is defined on an ε-ball $B_\varepsilon(O_i)$ around the corner O_i as follows (see Fig. 1) :

$$\begin{aligned} \eta_i &\in C_0^\infty(B_\varepsilon(O_i)) \\ 0 &\le \eta_i \le 1 \\ \eta_i &\equiv 1 \text{ on } B_{\frac{\varepsilon}{2}}(O_i)\,. \end{aligned}$$

We assume that the support of η_i contains only one corner point, namely O_i. Inserting $\vec{u_i} = \eta_i \vec{v}$

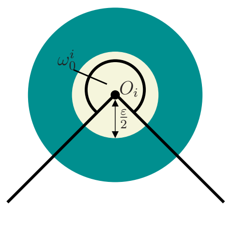

Figure 1: Illustration of the cut-off function near the corner O_i.

and $\tilde{p}_i = \eta_i p$ into (2.3) and into the Dirichlet, Neumann or mixed boundary conditions we get boundary value problems for $\vec{u_i} = \eta_i \vec{v}$ and $\tilde{p}_i = \eta_i p$, where the derivatives are to understand in the distribution sense. Thus, the partial differential equations for $\vec{u_i}, \tilde{p}_i$ read

$$\begin{aligned} -\frac{\mu}{\rho}\,\Delta\,\vec{u_i} + \operatorname{grad}\frac{\tilde{p}_i}{\rho} &= -\eta_i \frac{\mu}{\rho}\,\Delta\,\vec{v} - 2\operatorname{grad}\vec{v}\operatorname{grad}\eta_i - \vec{v}\,\Delta\,\eta_i =: \vec{F}_i \\ \operatorname{div}\vec{u_i} &= \underbrace{\eta_i \operatorname{div}\vec{v}}_{=0} + \vec{v}\cdot\operatorname{grad}\eta_i =: g_i. \end{aligned} \tag{3.2}$$

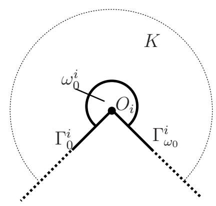

Figure 2: Infinite cone K_i with opening angle ω_0^i.

Here are $\vec{F_i} \in (L^2(\Omega))^2, g_i \in H^1(\Omega)$. Considering a zero-extension outside $B_\varepsilon(0_i)$, a new boundary value problem is defined in an infinite cone

$$K_i = \{(r, \omega), 0 < r < \infty, 0 < \omega < \omega_0^i\}$$

which coincides with the original problem near the corner O_i . If we assume that the right hand side $\vec{f}$ of (2.3) belongs to $(L^2(\Omega))^2$, then $\vec{F_i} \in (L^2(K_i))^2, g_i \in H^1(K_i)$.
The corresponding boundary conditions are defined on the edges $\Gamma_0^i\,(\omega = 0)$ and $\Gamma_{\omega_0}^i\,(\omega = \omega_0^i)$ of the cone K_i, see Fig.2. We allow only one condition per edge and distingiush between two mixed boundary conditions, Γ_{M_1} and Γ_{M_2}, whereas Γ_{M_1} means that we prescribe Dirichlet conditions in the normal direction and Neumann conditions in the tangential direction. Γ_{M_2} describes the converse case:

$$\begin{aligned}
\vec{u_i} &= \eta_i \vec{\phi} \quad \text{on } \Gamma_0, \Gamma_{\omega_0} \quad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{D}}\,, \\
-\tfrac{\tilde{p}_i}{\rho}\vec{n} + \tfrac{\mu}{\rho}\Big(\operatorname{grad}\vec{u_i} + (\operatorname{grad}\vec{u_i})^{\mathrm{T}}\Big)\vec{n} &= \eta_i\vec{\psi} + (\vec{v}\otimes\operatorname{grad}\eta_i)\vec{n} \quad \text{on } \Gamma_0, \Gamma_{\omega_0} \quad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{N}}\,, \\
\left.\begin{aligned} \vec{u_i}\cdot\vec{n} &= \eta_i\phi_n \\ \Big(-\tfrac{\tilde{p}_i}{\rho}\vec{n} + \tfrac{\mu}{\rho}\left(\operatorname{grad}\vec{u_i} + (\operatorname{grad}\vec{u_i})^{\mathrm{T}}\right)\vec{n}\Big)\cdot\vec{t} &= \eta_i\psi_t + (\vec{v}\otimes\operatorname{grad}\eta_i)\vec{n}\cdot\vec{t} \end{aligned}\right\} & \begin{aligned} &\text{on } \Gamma_0, \Gamma_{\omega_0} \\ &\quad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{M}_1}\,, \end{aligned} \\
\left.\begin{aligned} \vec{u_i}\cdot\vec{t} &= \eta_i\phi_t \\ \Big(-\tfrac{\tilde{p}_i}{\rho}\vec{n} + \tfrac{\mu}{\rho}\left(\operatorname{grad}\vec{u_i} + (\operatorname{grad}\vec{u_i})^{\mathrm{T}}\right)\vec{n}\Big)\cdot\vec{n} &= \eta_i\psi_n + (\vec{v}\otimes\operatorname{grad}\eta_i)\vec{n}\cdot\vec{n} \end{aligned}\right\} & \begin{aligned} &\text{on } \Gamma_0, \Gamma_{\omega_0} \\ &\quad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{M}_2}\,. \end{aligned}
\end{aligned} \tag{3.3}$$

Note that the right hand sides of the boundary conditions have the same smoothness as the right hand sides of the original boundary value problem in Ω. We denote

$$\vec{U_i} = \eta_i\vec{V} = \eta_i \begin{pmatrix} v_1 \\ v_2 \\ p \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \tilde{p} \end{pmatrix}\,. \tag{3.4}$$

The solution $\vec{U_i}$ of the boundary value problem (3.2),(3.3) in K_i can be decomposed in a homogeneous and a particular solution because of the linearity of the operators. So we get

$$\vec{U_i} = \vec{U}_{i\mathrm{hom}} + \vec{U}_{i\mathrm{part}}\,.$$

The solution $\vec{U}_{i\mathrm{hom}}$ can be splitted additively into a regular part $\vec{U}_{i\mathrm{hom,reg}}$ and a singular part $\vec{U}_{i\mathrm{hom,sing}}$ [10]. In the next section we shall calculate the singular part. Note that the particular solution can be chosen regular enough. Therefore, it holds

$$\vec{U_i} = \vec{U}_{i\mathrm{sing}} + \vec{U}_{i\mathrm{reg}}\,. \tag{3.5}$$

3.1.2 Decomposition of the global solution

In the following we describe how to connect the local solutions $\vec{U}_i$ in the infinite cones K_i with the global solution $\vec{V}$ in Ω.
Due to (3.4) and (3.5) it holds

$$\begin{aligned}\vec{V} &= (1-\sum_{i=1}^{N}\eta_i)\vec{V} + \sum_{i=1}^{N}\eta_i\vec{V}\\ &= (1-\sum_{i=1}^{N}\eta_i)\vec{V} + \sum_{i=1}^{N}\vec{U}_i\\ &= (1-\sum_{i=1}^{N}\eta_i)\vec{V} + \sum_{i=1}^{N}(\vec{U}_{i\mathrm{sing}} + \vec{U}_{i\mathrm{reg}})\\ &= \sum_{i=1}^{N}\vec{U}_{i\mathrm{sing}} + \vec{V}_{\mathrm{reg}}\\ &= \vec{V}_{\mathrm{sing}} + \vec{V}_{\mathrm{reg}}.\end{aligned}$$

It remains to calculate nontrivial solutions $\vec{U}_{i\mathrm{hom,sing}}$ of the following homogenous boundary value problem in the infinite cone K_i.

$$\begin{aligned}
\left.\begin{aligned} -\tfrac{\mu}{\rho}\,\Delta\,\vec{u} + \mathrm{grad}\,\tfrac{\tilde{p}}{\rho} &= 0\\ \mathrm{div}\,\vec{u} &= 0\end{aligned}\right\} &\quad \text{in } K_i\,,\\
\vec{u} = 0 &\quad \text{on } \Gamma_0, \Gamma_{\omega_0} \quad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{D}}\,,\\
-\tfrac{\tilde{p}}{\rho}\vec{n} + \tfrac{\mu}{\rho}\left(\mathrm{grad}\,\vec{u} + (\mathrm{grad}\,\vec{u})^{\mathrm{T}}\right)\vec{n} = 0 &\quad \text{on } \Gamma_0, \Gamma_{\omega_0} \quad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{N}}\,,\\
\left.\begin{aligned} \vec{u}\cdot\vec{n} &= 0\\ \left(-\tfrac{\tilde{p}}{\rho}\vec{n} + \tfrac{\mu}{\rho}\left(\mathrm{grad}\,\vec{u} + (\mathrm{grad}\,\vec{u})^{\mathrm{T}}\right)\vec{n}\right)\cdot\vec{t} &= 0\end{aligned}\right\} &\quad \text{on } \Gamma_0, \Gamma_{\omega_0} \quad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{M}_1}\,,\\
\left.\begin{aligned} \vec{u}\cdot\vec{t} &= 0\\ \left(-\tfrac{\tilde{p}}{\rho}\vec{n} + \tfrac{\mu}{\rho}\left(\mathrm{grad}\,\vec{u} + (\mathrm{grad}\,\vec{u})^{\mathrm{T}}\right)\vec{n}\right)\cdot\vec{n} &= 0\end{aligned}\right\} &\quad \text{on } \Gamma_0, \Gamma_{\omega_0} \quad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{M}_2}\,.
\end{aligned} \tag{3.6}$$

3.2 The generalized eigenvalue problem

In order to solve the homogeneous boundary value problem (3.2), (3.3) in $K_i = K$ we write the operators in polar coordinates and introduce spherical coordinates. Finally, we use a separation ansatz which leads to a generalized eigenvalue problem.

Operators in polar coordinates The operators can be written in polar coordinates as follows:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \omega^2} \tag{3.7}$$

$$\mathrm{grad}\,\tilde{p} = \begin{pmatrix} \cos\omega\frac{\partial}{\partial r} - \frac{1}{r}\sin\omega\frac{\partial}{\partial\omega}\\ \sin\omega\frac{\partial}{\partial r} + \frac{1}{r}\cos\omega\frac{\partial}{\partial\omega}\end{pmatrix} = \begin{pmatrix}\cos\omega & -\sin\omega\\ \sin\omega & \cos\omega\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial\omega}\end{pmatrix}\tilde{p} \tag{3.8}$$

$$\operatorname{grad}\vec{u} = \begin{pmatrix} \cos\omega\frac{\partial}{\partial r}u_1 - \frac{1}{r}\sin\omega\frac{\partial u_1}{\partial\omega} & \sin\omega\frac{\partial u_1}{\partial r} + \frac{1}{r}\cos\omega\frac{\partial u_1}{\partial\omega} \\ \cos\omega\frac{\partial}{\partial r}u_2 - \frac{1}{r}\sin\omega\frac{\partial u_2}{\partial\omega} & \sin\omega\frac{\partial u_2}{\partial r} + \frac{1}{r}\cos\omega\frac{\partial u_2}{\partial\omega} \end{pmatrix} \tag{3.9}$$

$$= \left[\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix} \begin{pmatrix} u_1 & u_2 \end{pmatrix}\right]^{\mathrm{T}} \tag{3.10}$$

$$\operatorname{div}\begin{pmatrix} u_1(r,\omega) \\ u_2(r,\omega) \end{pmatrix} = \cos\omega\frac{\partial u_1}{\partial r} - \frac{1}{r}\sin\omega\frac{\partial u_1}{\partial\omega} + \sin\omega\frac{\partial u_2}{\partial r} + \frac{1}{r}\cos\omega\frac{\partial u_2}{\partial\omega}$$

$$= \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} . \tag{3.11}$$

Thus, the homogeneous problem (3.2),(3.3) reads: Find $\vec{u}, \tilde{p}$ such that

$$\left.\begin{array}{r} -\mu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\omega^2}\right)\vec{u} + \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix}\tilde{p} = \vec{0} \\ \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \end{array}\right\} \quad \text{in } K\,, \tag{3.12}$$

with the boundary conditions

if $\Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{D}}$

$$\vec{u} = 0 \quad \text{on } \Gamma_0, \Gamma_{\omega_0}\,, \tag{3.13}$$

if $\Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{N}}$

$$-\frac{\tilde{p}}{\rho}\vec{n} + \frac{\mu}{\rho}\left(\left[\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix}\begin{pmatrix} u_1 & u_2 \end{pmatrix}\right]^{\mathrm{T}} + \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix}\begin{pmatrix} u_1 & u_2 \end{pmatrix}\right)\vec{n} = 0 \quad \text{on } \Gamma_0, \Gamma_{\omega_0}\,, \tag{3.14}$$

if $\Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{M}_1}$

$$\left.\begin{array}{r} \vec{u}\cdot\vec{n} = 0 \\ \left[-\frac{\tilde{p}}{\rho}\vec{n} + \frac{\mu}{\rho}\left(\left[\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix}\begin{pmatrix} u_1 & u_2 \end{pmatrix}\right]^{\mathrm{T}} + \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix}\begin{pmatrix} u_1 & u_2 \end{pmatrix}\right)\vec{n}\right]\cdot\vec{t} = 0 \end{array}\right\} \quad \text{on } \Gamma_0, \Gamma_{\omega_0}\,, \tag{3.15}$$

if $\Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{\mathrm{M}_2}$

$$\left.\begin{array}{r} \vec{u}\cdot\vec{t} = 0 \\ \left[-\frac{\tilde{p}}{\rho}\vec{n} + \frac{\mu}{\rho}\left(\left[\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix}\begin{pmatrix} u_1 & u_2 \end{pmatrix}\right]^{\mathrm{T}} + \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\omega} \end{pmatrix}\begin{pmatrix} u_1 & u_2 \end{pmatrix}\right)\vec{n}\right]\cdot\vec{n} = 0 \end{array}\right\} \quad \text{on } \Gamma_0, \Gamma_{\omega_0}\,. \tag{3.16}$$

Spherical basis We introduce a spherical basis in $\mathbb{R}^2$ since the resulting generalized eigenvalue problems can be solved then quite elementary. The spherical basis system is a comoving coordinate system, i. e. the basis vectors depend on the position. They are defined by

$$\vec{e}_r = \begin{pmatrix} \cos\omega \\ \sin\omega \end{pmatrix}, \quad \vec{e}_\omega = \begin{pmatrix} -\sin\omega \\ \cos\omega \end{pmatrix},$$

where ω describes the polar angle. A cartesian vector $\vec{u} = (u_1, u_2)^{\mathrm{T}}$, given by the standard basis

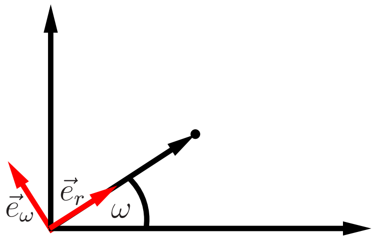

Figure 3: Illustration of the spherical basis vectors in $\mathbb{R}^2$ with polar angle ω.

$\{\vec{e}_1, \vec{e}_2\}$, can be described in the spherical basis $\{\vec{e}_r, \vec{e}_\omega\}$ with coordinates $\vec{u} = (u_r, u_\omega)^{\mathrm{T}}$ in the following way:

$$\begin{pmatrix} u_r \\ u_\omega \end{pmatrix} = H_c^s \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Vice versa we get the cartesian description of a spherical vector by the transformation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = H_s^c \begin{pmatrix} u_r \\ u_\omega \end{pmatrix} = \begin{pmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{pmatrix} \begin{pmatrix} u_r \\ u_\omega \end{pmatrix}.$$

Due to these definitions we have to read the indices of the transformation matrices from bottom to top, so H_c^s transforms a cartesian vector into spherical basis. The following relations hold:

$$H_s^c = (H_c^s)^{-1} = (H_c^s)^{\mathrm{T}} \text{ and } H_c^s = (H_s^c)^{-1} = (H_s^c)^{\mathrm{T}}.$$

Because of the term $\operatorname{grad}\vec{u}$ in the Laplacian and in the Neumann boundary condition we have to transform a tensor of rank two. By defining T_c as the tensor relative to the cartesian basis and T_s as the tensor relative to the spherical basis, the tensor transformation can be written as

$$T_s = H_c^s \, T_c \, (H_c^s)^{\mathrm{T}} = H_c^s \, T_c \, H_s^c. \tag{3.17}$$

Now, we transform the cartesian vectors of the differential equation system (3.12) and the boundary conditions (3.13)-(3.16) into the spherical ones. With the transformation rules we get the differential equation system relative to the spherical basis, i. e.

$$\begin{aligned} H_c^s \left[-\mu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \omega^2} \right) H_s^c \begin{pmatrix} u_r \\ u_\omega \end{pmatrix} + H_s^c \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \omega} \end{pmatrix} \tilde{p} \right] &= \vec{0} \\ H_s^c \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \omega} \end{pmatrix} \cdot H_s^c \begin{pmatrix} u_r \\ u_\omega \end{pmatrix} &= 0. \end{aligned}$$

The calculation of this system can be done by a computer algebra system (CAS)[1], so we achieve the Stokes equations relative to the spherical basis

$$\begin{aligned} \left(u_r + 2\frac{\partial}{\partial \omega}u_\omega - \frac{\partial^2}{\partial \omega^2}u_r \right) - r\frac{\partial}{\partial r}u_r + r^2 \left(\frac{1}{\mu}\frac{\partial}{\partial r}\tilde{p} - \frac{\partial^2}{\partial r^2}u_r \right) &= 0 \\ \left(u_\omega - 2\frac{\partial}{\partial \omega}u_r - \frac{\partial^2}{\partial \omega^2}u_\omega \right) + r \left(\frac{1}{\mu}\frac{\partial}{\partial r}\tilde{p} - \frac{\partial}{\partial r}u_\omega \right) - r^2\frac{\partial^2}{\partial r^2}u_\omega &= 0 \\ u_r + \frac{\partial}{\partial \omega}u_\omega + r\frac{\partial}{\partial r}u_r &= 0. \end{aligned} \tag{3.18}$$

[1] We used Wolfram Mathematica® Version 6.0.0.

We consider the different boundary conditions separately.

- Dirichlet boundary conditions:

$$H_s^c \begin{pmatrix} u_r \\ u_\omega \end{pmatrix} = 0 \quad \Leftrightarrow \quad \begin{pmatrix} u_r \\ u_\omega \end{pmatrix} = 0\,. \tag{3.19}$$

- Neumann boundary conditions:
  With the transformation rule (3.17) we get

$$-\tilde{p}\vec{n}_s + \mu\left(\left[H_c^s\left(H_s^c\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial\omega}\end{pmatrix}\left[\begin{pmatrix}u_r & u_\omega\end{pmatrix}H_c^s\right]\right)H_s^c\right]^{\mathrm{T}} + H_c^s\left(H_s^c\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial\omega}\end{pmatrix}\left[\begin{pmatrix}u_r & u_\omega\end{pmatrix}H_c^s\right]\right)H_s^c\right)\vec{n}_s = 0\,,$$

  where $\vec{n}_s$ is the transformed normal vector in the sperical basis. Using Mathematica for the calculations we get the Neumann boundary conditions in the spherical basis

$$-\tilde{p}\vec{n}_s + \mu\begin{pmatrix} 2\frac{\partial}{\partial r}u_r & \frac{1}{r}\left(-u_\omega + \frac{\partial}{\partial\omega}u_r\right) + \frac{\partial}{\partial r}u_w \\ \frac{1}{r}\left(-u_\omega + \frac{\partial}{\partial\omega}u_r\right) + \frac{\partial}{\partial r}u_w & \frac{2}{r}\left(u_r + \frac{\partial}{\partial\omega}u_\omega\right)\end{pmatrix}\vec{n}_s = 0\,. \tag{3.20}$$

- Mixed boundary conditions:
  We get the mixed boundary conditions in the spherical basis analogously to the Dirichlet and Neumann boundary conditions:

  - for M_1-conditions:

$$\begin{aligned} \vec{u}\cdot\vec{n}_s &= 0 \\ \left[-\tilde{p}\vec{n}_s + \mu\begin{pmatrix} 2\frac{\partial}{\partial r}u_r & \frac{1}{r}\left(-u_\omega + \frac{\partial}{\partial\omega}u_r\right) + \frac{\partial}{\partial r}u_w \\ \frac{1}{r}\left(-u_\omega + \frac{\partial}{\partial\omega}u_r\right) + \frac{\partial}{\partial r}u_w & \frac{2}{r}\left(u_r + \frac{\partial}{\partial\omega}u_\omega\right)\end{pmatrix}\vec{n}_s\right]\cdot\vec{t}_s &= 0\,. \end{aligned} \tag{3.21}$$

  - for M_2-conditions:

$$\begin{aligned} \vec{u}\cdot\vec{t}_s &= 0 \\ \left[-\tilde{p}\vec{n}_s + \mu\begin{pmatrix} 2\frac{\partial}{\partial r}u_r & \frac{1}{r}\left(-u_\omega + \frac{\partial}{\partial\omega}u_r\right) + \frac{\partial}{\partial r}u_w \\ \frac{1}{r}\left(-u_\omega + \frac{\partial}{\partial\omega}u_r\right) + \frac{\partial}{\partial r}u_w & \frac{2}{r}\left(u_r + \frac{\partial}{\partial\omega}u_\omega\right)\end{pmatrix}\vec{n}_s\right]\cdot\vec{n}_s &= 0\,. \end{aligned} \tag{3.22}$$

Generalized eigenvalue problem We consider the separation ansatz

$$\vec{U}_{\mathrm{sing}} = \begin{pmatrix} r^\alpha f(\omega) \\ r^\alpha g(\omega) \\ r^{\alpha-1}h(\omega)\end{pmatrix}$$

and transform the velocity in this ansatz to the spherical basis by

$$\begin{pmatrix}\tilde{f}(\omega)\\ \tilde{g}(\omega)\end{pmatrix} = H_c^s\begin{pmatrix} f(\omega)\\ g(\omega)\end{pmatrix}\,.$$

We obtain

$$\vec{U}_s(r,\omega) = \begin{pmatrix} r^\alpha \tilde{f}(\omega) \\ r^\alpha \tilde{g}(\omega) \\ r^{\alpha-1}h(\omega)\end{pmatrix}\,.$$

Inserting this ansatz into the equations (3.18) and the boundary conditions (3.19), (3.20), (3.21) and (3.22), it is possible to cancel the radial dependence if $r > 0$. So we get a generalized boundary eigenvalue problem for the exponent α, the velocity field $(\tilde{f}, \tilde{g})$ and the pressure field h. The singular solutions of the Stokes equations with boundary conditions in 2D are given by the solutions of the following generalized boundary eigenvalue problem:

Find parameters α (generalized eigenvalues) for which nontrivial solutions (eigensolutions) $(\tilde{f}, \tilde{g}, h)$ exist such that

in $0 < \omega < \omega_0$:

$$\begin{aligned} \tilde{f}''(\omega) &= (1-\alpha^2)\tilde{f}(\omega) + 2\tilde{g}'(\omega) + \frac{\alpha-1}{\mu}h(\omega) \\ \tilde{g}''(\omega) &= (1-\alpha^2)\tilde{g}(\omega) - 2\tilde{f}'(\omega) + \frac{1}{\mu}h'(\omega) \\ \tilde{g}'(\omega) &= -(\alpha+1)\tilde{f}(\omega)\,. \end{aligned}$$

for $\omega = 0, \omega = \omega_0$:

$$\begin{pmatrix} \tilde{f}(\omega) \\ \tilde{g}(\omega) \end{pmatrix} = 0 \qquad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_D\,, \tag{3.23}$$

$$-h(\omega)\vec{n}_s + \mu \begin{pmatrix} 2\alpha\tilde{f}(\omega) & (\alpha-1)\tilde{g}(\omega) + \tilde{f}'(\omega) \\ (\alpha-1)\tilde{g}(\omega) + \tilde{f}'(\omega) & 2(\tilde{f}(\omega) + \tilde{g}'(\omega)) \end{pmatrix} \vec{n}_s = 0 \qquad \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_N\,, \tag{3.24}$$

$$\left.\begin{aligned} \begin{pmatrix} \tilde{f}(\omega) \\ \tilde{g}(\omega) \end{pmatrix} \cdot \vec{n}_s = 0 \\ \left[-h(\omega)\vec{n}_s + \mu \begin{pmatrix} 2\alpha\tilde{f}(\omega) & (\alpha-1)\tilde{g}(\omega) + \tilde{f}'(\omega) \\ (\alpha-1)\tilde{g}(\omega) + \tilde{f}'(\omega) & 2(\tilde{f}(\omega) + \tilde{g}'(\omega)) \end{pmatrix} \vec{n}_s\right] \cdot \vec{t}_s = 0 \end{aligned}\right\} \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{M_1}\,, \tag{3.25}$$

$$\left.\begin{aligned} \begin{pmatrix} \tilde{f}(\omega) \\ \tilde{g}(\omega) \end{pmatrix} \cdot \vec{t}_s = 0 \\ \left[-h(\omega)\vec{n}_s + \mu \begin{pmatrix} 2\alpha\tilde{f}(\omega) & (\alpha-1)\tilde{g}(\omega) + \tilde{f}'(\omega) \\ (\alpha-1)\tilde{g}(\omega) + \tilde{f}'(\omega) & 2(\tilde{f}(\omega) + \tilde{g}'(\omega)) \end{pmatrix} \vec{n}_s\right] \cdot \vec{n}_s = 0 \end{aligned}\right\} \text{if } \Gamma_0, \Gamma_{\omega_0} \subset \Gamma_{M_2}\,. \tag{3.26}$$

Note, that in general there are Jordan chains of solutions belonging to the eigenvalues of different multiplicities [10].

Remark: We have started with an separation-ansatz in order to calculate the singular terms of the solutions. The question occurs, whether we have obtained all possible singular terms. The answer is positive if simple eigenvalues occur only. V.A.Kondratiev [9] has shown, that all singular terms have the following structure in a neighbourhood of corner point O_i

$$\sum_{0<\operatorname{Re}\alpha_\nu<1} c_{\alpha_\nu} r^{\alpha_\nu} S_\nu(\alpha_\nu, \ln r, \omega)\ .$$

Terms with $\ln r$ occur if the algebraic multiplicity of α_ν is greater than 1.

3.3 Analytic solutions

In this subsection we solve the generalized eigenvalue problem. First we calculate the fundamental system to the differential equations, after that we consider the boundary conditions. Then we will see the structure of the local solutions and finally we transfer the local solutions into the cartesian basis.

Fundamental system to the ordinary differential system We repeat the ordinary differential system in ω with parameter α:

$$\tilde{f}''(\omega) = (1-\alpha^2)\tilde{f}(\omega) + 2\tilde{g}'(\omega) + \frac{\alpha-1}{\mu}h(\omega) \tag{3.27}$$

$$\tilde{g}''(\omega) = (1-\alpha^2)\tilde{g}(\omega) - 2\tilde{f}'(\omega) + \frac{1}{\mu}h'(\omega) \tag{3.28}$$

$$\tilde{g}'(\omega) = -(\alpha+1)\tilde{f}(\omega)\,. \tag{3.29}$$

It is necessary to differ the four cases:

- $\alpha \neq \{0, \pm 1\}$,
- $\alpha = 0$,
- $\alpha = -1$,
- $\alpha = 1$.

The case $\alpha \neq \{0, \pm 1\}$: Because of $\alpha \neq 1$ we can express $h(\omega)$ from equation (3.27). Differentiation of the obtained equation yields to

$$h'(\omega) = \frac{\mu}{\alpha-1}\tilde{f}'''(\omega) + (1+\alpha)\mu\tilde{f}'(\omega) - 2\frac{\mu}{\alpha-1}\tilde{g}''(\omega)\,. \tag{3.30}$$

We substitute this derivative into (3.28) and get

$$\left(1 + \frac{2}{\alpha-1}\right)\tilde{g}''(\omega) = (1-\alpha^2)\tilde{g}(\omega) + (\alpha-1)\tilde{f}'(\omega) + \frac{1}{\alpha-1}\tilde{f}'''(\omega)\,. \tag{3.31}$$

Note that this equation only depends on $\tilde{f}$ and $\tilde{g}$.
With (3.29) and because of $\alpha \neq -1$ we can calculate the first and third derivative of $\tilde{f}$ depending on $\tilde{g}$ and so we get with (3.31) a linear homogeneous ordinary differential equation with constant complex coefficients:

$$\tilde{g}^{(4)}(\omega) + 2(\alpha^2+1)\tilde{g}''(\omega) + (\alpha^2-1)^2\tilde{g}(\omega) = 0\,. \tag{3.32}$$

We know from the theory of linear differential equations with constant complex coefficients (see [18, p.29ff.]) that there are four independent solutions which we can get by the ansatz

$$\tilde{g}(\omega) = \mathrm{e}^{\lambda\omega}\,. \tag{3.33}$$

The exponents λ are the roots of the characteristic polynomial

$$\lambda^4 + 2(\alpha^2+1)\lambda^2 + (\alpha^2-1)^2 = 0\,. \tag{3.34}$$

We get

$$\lambda_{1,2} = \pm i(\alpha+1), \qquad \lambda_{3,4} = \pm i(\alpha-1)\,.$$

Due to $\alpha \neq \{0, \pm 1\}$ we have four solutions of type (3.33)

$$\mathrm{e}^{\pm i(\alpha+1)}, \quad \mathrm{e}^{\pm i(\alpha-1)},$$

and with Euler's formula we achieve four linearly independent solutions for $\tilde{g}$:

$$\begin{aligned} \tilde{g}_1(\omega) &= \cos[(\alpha+1)\omega] \quad \tilde{g}_2(\omega) = \sin[(\alpha+1)\omega] \\ \tilde{g}_3(\omega) &= \cos[(\alpha-1)\omega] \quad \tilde{g}_4(\omega) = \sin[(\alpha-1)\omega]\,. \end{aligned} \tag{3.35}$$

Note that the complex sine and cosine occur. With the solutions (3.35) we get with (3.29) the correspondent solutions for $\tilde{f}$ and with (3.27) the solutions for h. So we obtain for $\alpha \neq 0, \pm 1$ the fundamental system for $\begin{pmatrix} \tilde{f} \\ \tilde{g} \\ h \end{pmatrix}$:

$$\begin{pmatrix} \sin[(\alpha+1)\omega] \\ \cos[(\alpha+1)\omega] \\ 0 \end{pmatrix}, \begin{pmatrix} -\cos[(\alpha+1)\omega] \\ \sin[(\alpha+1)\omega] \\ 0 \end{pmatrix}, \begin{pmatrix} (\alpha-1)\sin[(\alpha-1)\omega] \\ (\alpha+1)\cos[(\alpha-1)\omega] \\ 4\mu\alpha\sin[(\alpha-1)\omega] \end{pmatrix}, \begin{pmatrix} (\alpha-1)\cos[(\alpha-1)\omega] \\ -(\alpha+1)\sin[(\alpha-1)\omega] \\ 4\mu\alpha\cos[(\alpha-1)\omega] \end{pmatrix}.$$

The case $\alpha = 0$: For $\alpha = 0$ we can do the same steps as in the first case until equation (3.32). Then we get for the characteristic polynomial two roots of multiplicity two,

$$\lambda_{1,2} = i, \qquad \lambda_{3,4} = -i\,.$$

For each root λ_k of multiplicity two we know according to [18, Theorem 2.2, p.31] that $\mathrm{e}^{\lambda_k \omega}$ and $\omega \mathrm{e}^{\lambda_k \omega}$ both are solutions of the differential equation. So we achieve four independent solutions

$$\begin{aligned} \tilde{g}_1(\omega) &= \cos(\omega) \qquad \tilde{g}_2(\omega) = \sin(\omega) \\ \tilde{g}_3(\omega) &= \omega\cos(\omega) \quad \tilde{g}_4(\omega) = \omega\sin(\omega) \end{aligned} \tag{3.36}$$

and with (3.27) and (3.29) the following fundamental system

$$\begin{pmatrix} \sin(\omega) \\ \cos(\omega) \\ 0 \end{pmatrix}, \begin{pmatrix} -\cos(\omega) \\ \sin(\omega) \\ 0 \end{pmatrix}, \begin{pmatrix} \omega\sin(\omega) - \cos(\omega) \\ \omega\cos(\omega) \\ -2\mu\cos(\omega) \end{pmatrix}, \begin{pmatrix} -\omega\cos(\omega) - \sin(\omega) \\ \omega\sin(\omega) \\ -2\mu\sin(\omega) \end{pmatrix}.$$

The case $\alpha = -1$: Here, equation (3.29) reads

$$\tilde{g}'(\omega) = 0\,, \tag{3.37}$$

so we miss the relationship with $\tilde{f}$. Hence we start with equation (3.31) and set $\alpha = -1$. We get

$$\tilde{f}'(\omega) = -4\tilde{f}'''(\omega)\,. \tag{3.38}$$

Differential equation (3.38) has three linearly independent solutions

$$\begin{aligned} \tilde{f}_1(\omega) &= 1\,, \\ \tilde{f}_2(\omega) &= \cos(2\omega)\,, \\ \tilde{f}_3(\omega) &= \sin(2\omega)\,. \end{aligned}$$

Additionally from equation (3.37) follows

$$\tilde{g}_4(\omega) = 1\,.$$

From equations (3.27) and (3.28) we can calculate the correspondent solutions $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$ and h_1, h_2, h_3 as well $\tilde{f}_4$ and h_4. A fundamental system reads

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(2\omega) \\ 0 \\ 2\mu\cos(\omega) \end{pmatrix}, \begin{pmatrix} \sin(2\omega) \\ 0 \\ 2\mu\sin(\omega) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The case $\alpha = 1$: Here we consider the original differential system (3.27)-(3.29). We achieve

$$\tilde{f}''(\omega) = 2\tilde{g}'(\omega) \tag{3.39}$$

$$\tilde{g}''(\omega) = -2\tilde{f}'(\omega) + \frac{1}{\mu}h'(\omega) \tag{3.40}$$

$$\tilde{g}'(\omega) = -2\tilde{f}(\omega)\,. \tag{3.41}$$

Inserting (3.41) into (3.39) yields

$$\tilde{f}''(\omega) = -4\tilde{f}(\omega)\,. \tag{3.42}$$

This is a differential equation of order two and it has two linearly independent solutions, namely

$$\tilde{f}_1(\omega) = \sin(2\omega)\,,$$
$$\tilde{f}_2(\omega) = \cos(2\omega)\,.$$

With (3.40) and (3.41) we get for $\tilde{g}$ and h

$$\tilde{g}_1(\omega) = \cos(2\omega) + c_1$$
$$h_1(\omega) = c_2$$

and

$$\tilde{g}_2(\omega) = -\sin(2\omega) + c_3$$
$$h_2(\omega) = c_4\,.$$

So we get a fundamental system for $\alpha = 1$

$$\begin{pmatrix}\sin(2\omega)\\ \cos(2\omega)\\ 0\end{pmatrix}, \begin{pmatrix}\cos(2\omega)\\ -\sin(2\omega)\\ 0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\,.$$

Summary If we consider fundamental systems for the three cases $\alpha \neq \{0, \pm 1\}$, $\alpha = -1$ and $\alpha = 1$, we can combine them (the systems for $\alpha = \pm 1$ can be achieved by inserting α in the system for $\alpha \neq \{0, \pm 1\}$). So we get

Theorem 3.1. *The following vectors form a fundamental system of solutions to* (3.27) - (3.29)*;*
for $\alpha \neq 0$:

$$\begin{pmatrix}\sin[(\alpha+1)\omega]\\ \cos[(\alpha+1)\omega]\\ 0\end{pmatrix}, \begin{pmatrix}-\cos[(\alpha+1)\omega]\\ \sin[(\alpha+1)\omega]\\ 0\end{pmatrix}, \begin{pmatrix}(\alpha-1)\sin[(\alpha-1)\omega]\\ (\alpha+1)\cos[(\alpha-1)\omega]\\ 4\mu\alpha\sin[(\alpha-1)\omega]\end{pmatrix}, \begin{pmatrix}(\alpha-1)\cos[(\alpha-1)\omega]\\ -(\alpha+1)\sin[(\alpha-1)\omega]\\ 4\mu\alpha\cos[(\alpha-1)\omega]\end{pmatrix},$$

for $\alpha = 0$:

$$\begin{pmatrix}\sin(\omega)\\ \cos(\omega)\\ 0\end{pmatrix}, \begin{pmatrix}-\cos(\omega)\\ \sin(\omega)\\ 0\end{pmatrix}, \begin{pmatrix}\omega\sin(\omega)-\cos(\omega)\\ \omega\cos(\omega)\\ -2\mu\cos(\omega)\end{pmatrix}, \begin{pmatrix}-\omega\cos(\omega)-\sin(\omega)\\ \omega\sin(\omega)\\ -2\mu\sin(\omega)\end{pmatrix}.$$

The general solution of the differential equation system for $\alpha \neq 0$ can be written as

$$\begin{pmatrix}\tilde{f}(\omega)\\ \tilde{g}(\omega)\\ h(\omega)\end{pmatrix} = c_1\begin{pmatrix}\sin[(\alpha+1)\omega]\\ \cos[(\alpha+1)\omega]\\ 0\end{pmatrix} + c_2\begin{pmatrix}-\cos[(\alpha+1)\omega]\\ \sin[(\alpha+1)\omega]\\ 0\end{pmatrix}$$
$$+ c_3\begin{pmatrix}(\alpha-1)\cos[(\alpha-1)\omega]\\ -(\alpha+1)\sin[(\alpha-1)\omega]\\ 4\mu\alpha\cos[(\alpha-1)\omega]\end{pmatrix} + c_4\begin{pmatrix}(\alpha-1)\sin[(\alpha-1)\omega]\\ (\alpha+1)\cos[(\alpha-1)\omega]\\ 4\mu\alpha\sin[(\alpha-1)\omega]\end{pmatrix}, \tag{3.43}$$

the general solution for $\alpha = 0$ reads

$$\begin{pmatrix} \tilde{f}(\omega) \\ \tilde{g}(\omega) \\ h(\omega) \end{pmatrix} = c_1 \begin{pmatrix} \sin(\omega) \\ \cos(\omega) \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -\cos(\omega) \\ \sin(\omega) \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} \omega\sin(\omega) - \cos(\omega) \\ \omega\cos(\omega) \\ -2\mu\cos(\omega) \end{pmatrix} + c_4 \begin{pmatrix} -\omega\cos(\omega) - \sin(\omega) \\ \omega\sin(\omega) \\ -2\mu\sin(\omega) \end{pmatrix} . \quad (3.44)$$

Relevance of the solutions for $\alpha = 0$ We transform the general solution of the differential equation system back to cartesian coordinates to specify the case $\alpha = 0$. With the transformation for the velocity

$$\begin{pmatrix} f(\omega) \\ g(\omega) \end{pmatrix} = H_s^c \begin{pmatrix} \tilde{f}(\omega) \\ \tilde{g}(\omega) \end{pmatrix} ,$$

we get the cartesian solution

$$\vec{U}(r,\omega) = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -\cos^2(\omega) \\ \omega - \cos(\omega)\sin(\omega) \\ -\frac{2}{r}\mu\cos(\omega) \end{pmatrix} + c_4 \begin{pmatrix} -\omega - \cos(\omega)\sin(\omega) \\ -\sin^2(\omega) \\ -\frac{2}{r}\mu\sin(\omega) \end{pmatrix} .$$

We are interested in solutions from $[H^1(\Omega)]^2 \times L^2(\Omega)$. Hence we have to set $c_3 = c_4 = 0$ and so the solution for $\alpha = 0$ only describes translations in the two dimensional space.

3.3.1 Solving the boundary eigenvalue problem

The general solution of the differential equation system for $\alpha \neq 0$ is given by (3.43). Now, we consider the different boundary conditions on Γ_0 and Γ_{ω_0} in order to clarify for which α there exist nontrivial solutions.

Theorem 3.2 (Eigenvalue condition)**.** *We get with the specification of boundary conditions on Γ_0 and Γ_{ω_0} a homogeneous system of linear equations for the constants in the general solution* (3.43)*, more precisely we get*

$$A(\alpha) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 , \quad (3.45)$$

where the 4×4-matrix $A(\alpha)$ depends on the chosen boundary conditions. There exist nontrivial solutions, if

$$\det A(\alpha) = 0 . \quad (3.46)$$

Possible parameters α are determined by this eigenvalue condition.

Proof. The different boundary conditions are defined on the two sides Γ_0, Γ_{ω_0} of the cone K. So we have four equations for four constants c_1, c_2, c_3, c_4. The resulting matrix A depends on the parameter α. We know from the linear algebra that a system of linear equations has a nontrivial solution if the determinant vanishes. Therewith we get the eigenvalue condition for the parameter α. □

In general there is no unique solution α of (3.46). Hence we mark the several solutions with an index ν. A local solution depends on the algebraic multiplicity $m(\alpha_\nu)$ and the geometric multiplicity $d(\alpha_\nu)$ of the general eigenvalue α_ν. For the used boundary conditions in this paper a general eigenvalue α_ν has an algebraic multiplicity $m(\alpha_\nu) \leq 2$ ([12, p.58]), so we distinguish between the cases $m(\alpha_\nu) = 1$ and $m(\alpha_\nu) = 2$.

The case $\mathbf{m}(\alpha_\nu) = 1$ We know that the geometric multiplicity $d(\alpha_\nu)$ is not less than one and therefore

$$d(\alpha_\nu) = 1\,.$$

So the solution of (3.12) - (3.16) in the infinite cone relative to the spherical basis reads for a general eigenvalue α_ν

$$\vec{U}^{\alpha_\nu}(r,\omega) = r^{\alpha_\nu}\left[c_1^{\alpha_\nu}\begin{pmatrix}\sin[(\alpha_\nu+1)\omega]\\ \cos[(\alpha_\nu+1)\omega]\\ 0\end{pmatrix} + c_2^{\alpha_\nu}\begin{pmatrix}-\cos[(\alpha_\nu+1)\omega]\\ \sin[(\alpha_\nu+1)\omega]\\ 0\end{pmatrix} \right.$$
$$\left. + c_3^{\alpha_\nu}\begin{pmatrix}(\alpha_\nu-1)\cos[(\alpha_\nu-1)\omega]\\ -(\alpha_\nu+1)\sin[(\alpha_\nu-1)\omega]\\ \frac{4}{r}\mu\alpha_\nu\cos[(\alpha_\nu-1)\omega]\end{pmatrix} + c_4^{\alpha_\nu}\begin{pmatrix}(\alpha_\nu-1)\sin[(\alpha_\nu-1)\omega]\\ (\alpha_\nu+1)\cos[(\alpha_\nu-1)\omega]\\ \frac{4}{r}\mu\alpha_\nu\sin[(\alpha_\nu-1)\omega]\end{pmatrix}\right]\,. \qquad (3.47)$$

The case $\mathbf{m}(\alpha_\nu) = 2$ In this case α_ν is a general eigenvalue of algebraic multiplicity two and the local solution $\vec{U}^{\alpha_\nu}$ depends on the geometric multiplicity $d(\alpha_\nu)$. For $d(\alpha_\nu) = 2$, i. e. algebraic equals geometric multiplicity, we achieve two linearly independent eigenfunctions $\vec{U}_1^{\alpha_\nu}(r,\omega)$ and $\vec{U}_2^{\alpha_\nu}(r,\omega)$ for the eigenvalue α_ν. Here the solution is given by the superposition of both eigenfunctions.
For $d(\alpha_\nu) = 1$ associate eigenfunction occur (a detailed description can be found in [12]). Here a part of the solution can be written as

$$\vec{U}^{\alpha_\nu}(r,\omega) = c_1 r^{\alpha_\nu} E_0^{\alpha_\nu}(\omega) + c_2 r^{\alpha_\nu}\left(E_1^{\alpha_\nu}(\omega) + \ln(r) E_0^{\alpha_\nu}(\omega)\right)\,,$$

where $E_0^{\alpha_\nu}$ is the eigenfunction and $E_1^{\alpha_\nu}$ the associate eigenfunction of the eigenvalue α_ν.

Local solution Due to the results above we can describe the local solutions dependent on the properties of the general eigenvalues α_ν, it is

$$\vec{U}(r,\omega) = \sum_{\alpha_\nu} c_{\alpha_\nu}\vec{U}_{m(\alpha_\nu),d(\alpha_\nu)}(r,\omega)\,,$$

where

$$\vec{U}_{m(\alpha_\nu),d(\alpha_\nu)} = \begin{cases}\vec{U}^{\alpha_\nu}(r,\omega) & \text{if } m(\alpha_\nu)=1\\ c_1\vec{U}_1^{\alpha_\nu}(r,\omega) + c_2\vec{U}_2^{\alpha_\nu}(r,\omega) & \text{if } m(\alpha_\nu)=d(\alpha_\nu)=2\\ c_1 r^{\alpha_\nu}E_0^{\alpha_\nu}(\omega) + c_2 r^{\alpha_\nu}\left(E_1^{\alpha_\nu}(\omega)+\ln(r)E_0^{\alpha_\nu}(\omega)\right) & \text{if } m(\alpha_\nu)=2\,, d(\alpha_\nu)=1\,.\end{cases}$$

In the case of $\operatorname{Re}\alpha_\nu \geq 1$ the solution $\vec{U}_{m(\alpha_\nu),d(\alpha_\nu)}$ is regular, that means it belongs to $\left[H^2(\Omega)\right]^2 \times H^1(\Omega)$. In the case of $\operatorname{Re}\alpha_\nu \leq 0$ the solution is not in $\left[H^1(\Omega)\right]^2 \times L^2(\Omega)$. Hence we consider in the following eigenvalues which achieve in the strip

$$0 \leq \operatorname{Re}\alpha_\nu < 1\,. \qquad (3.48)$$

Note that for $\operatorname{Re}\alpha_\nu = 0$ only translations occur which are regular. The number $S(\omega_0)$ of general eigenvalues in this strip is dependent on the apex angle ω_0. $S(\omega_0)$ equals the number of singular functions near the corner. So we can also describe the local solution by

$$\vec{U}(r,\omega) = \sum_{0<\alpha_\nu<1} c_{\alpha_\nu}\vec{U}_{m(\alpha_\nu),d(\alpha_\nu)}(r,\omega) + \vec{U}_{\text{reg}}(r,\omega)\,.$$

3.3.2 Different boundary conditions

In the following we calculate the eigenvalues and eigenfunctions of the general boundary eigenvalue problem for different boundary conditions. We first specify the resulting mathematical conditions for Γ_0 and Γ_{ω_0} and then the roots of $\det A = 0$. After that we solve the resulting system of linear equations assuming the case $m(\alpha_\nu) = 1$. Finally, we transform the solution for the chosen boundary conditions into the cartesian basis.

Dirichlet-Dirichlet boundary conditions We prescribe Dirichlet boundary conditions for the velocity on Γ_0 and Γ_{ω_0}, that means

$$\begin{aligned} \vec{u}(r,0) &= 0 \\ \vec{u}(r,\omega_0) &= 0\,. \end{aligned}$$

From (3.43) it follows

$$c_1 \begin{pmatrix} 0 \\ 1 \\ \sin[(\alpha+1)\omega_0] \\ \cos[(\alpha+1)\omega_0] \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ -\cos[(\alpha+1)\omega_0] \\ \sin[(\alpha+1)\omega_0] \end{pmatrix} + c_3 \begin{pmatrix} \alpha-1 \\ 0 \\ (\alpha-1)\cos[(\alpha-1)\omega_0] \\ -(\alpha+1)\sin[(\alpha-1)\omega_0] \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ \alpha+1 \\ (\alpha-1)\sin[(\alpha-1)\omega_0] \\ (\alpha+1)\cos[(\alpha-1)\omega_0] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and so the homogeneous system of linear equations reads

$$A=A(\alpha)=\begin{pmatrix} 0 & -1 & \alpha-1 & 0 \\ 1 & 0 & 0 & \alpha+1 \\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] & (\alpha-1)\sin[(\alpha-1)\omega_0] \\ \cos[(\alpha+1)\omega_0] & \sin[(\alpha+1)\omega_0] & -(\alpha+1)\sin[(\alpha-1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.49}$$

Nontrivial solutions exist if the determinant of the system vanishes, so we get the eigenvalue condition

$$\alpha^2 \sin^2(\omega_0) - \sin^2(\omega_0\alpha) = 0\,. \tag{3.50}$$

These calculations were made with Mathematica and the eigenvalue condition was confirmed by the results of [12].
If we consider an eigenvalue α_{ω_0} which satisfies the eigenvalue condition (3.50) with algebraic multiplicity $m(\alpha_{\omega_0}) = 1$, we can solve the system of linear equations (3.49) for $c_1, \ldots, c_4$. By the inverse transformation of the velocity field into cartesian coordinates, i. e.

$$\begin{pmatrix} f(\omega) \\ g(\omega) \end{pmatrix} = H_s^c \begin{pmatrix} \tilde{f}(\omega) \\ \tilde{g}(\omega) \end{pmatrix}$$

we achieve the solution for Dirichlet-Dirichlet boundary conditions

$$\begin{aligned} \vec{U}(r,\omega) = r^{\alpha_{\omega_0}} \Bigg[& A_1 \begin{pmatrix} \alpha_{\omega_0} \sin[(\alpha_{\omega_0}-2)\omega] - (2+\alpha_{\omega_0})\sin(\alpha_{\omega_0}\omega) \\ \alpha_{\omega_0}(\cos[(\alpha_{\omega_0}-2)\omega] - \cos(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\sin[(\alpha_{\omega_0}-1)\omega] \end{pmatrix} \\ & + A_2 \begin{pmatrix} \alpha_{\omega_0}(\cos[(\alpha_{\omega_0}-2)\omega] - \cos(\alpha_{\omega_0}\omega) \\ \alpha_{\omega_0}\sin[(\alpha_{\omega_0}-2)\omega] + (\alpha_{\omega_0}-2)\sin(\alpha_{\omega_0}\omega) \\ -\frac{4}{r}\mu\alpha_{\omega_0}\cos[(\alpha_{\omega_0}-1)\omega] \end{pmatrix} \Bigg] \end{aligned} \tag{3.51}$$

with constants (dependent on apex angle ω_0)

$$\begin{aligned} A_1 &= (\alpha_{\omega_0}-1)\sin(\omega_0)\sin(\omega_0\alpha_{\omega_0})\,, \\ A_2 &= ((\cos(\omega_0)\sin(\alpha_{\omega_0}\omega_0) + \alpha_{\omega_0}\cos(\alpha_{\omega_0}\omega_0)\sin(\omega_0))\,. \end{aligned}$$

Dirichlet-Neumann boundary conditions In this case we prescribe Neumann boundary conditions on Γ_0 and Dirichlet boundary conditions on ${\Gamma_{\omega_0}}$[2] for the velocity field. Due to (3.24) we get

$$-h(0)\vec{n}_s + \mu \begin{pmatrix} 2\alpha\tilde{f}(0) & (\alpha-1)\tilde{g}(0)+\tilde{f}'(0) \\ (\alpha-1)\tilde{g}(0)+\tilde{f}'(0) & 2(\tilde{f}(0)+\tilde{g}'(0)) \end{pmatrix} \vec{n}_s = 0$$
$$\vec{u}(r,\omega_0) = 0\,.$$

The normal vector relative to the spherical basis is given by $\vec{n}_s = (0,-1)^{\mathrm{T}}$ and from (3.43) it follows

$$\tilde{f}'(\omega) = c_1(\alpha+1)\cos[(\alpha+1)\omega] + c_2(\alpha+1)\sin[(\alpha+1)\omega] - c_3(\alpha-1)^2\sin[(\alpha-1)\omega] + c_4(\alpha-1)^2\cos[(\alpha-1)\omega]$$
$$\tilde{g}'(\omega) = -c_1(\alpha+1)\sin[(\alpha+1)\omega] + c_2(\alpha+1)\cos[(\alpha+1)\omega] - c_3(\alpha^2-1)\cos[(\alpha-1)\omega] - c_4(\alpha^2-1)\sin[(\alpha-1)\omega]\,.$$

With the general solution (3.43) for $\alpha \neq 0$ we achieve the system of linear equation for Dirichlet-Neumann boundary conditions after some calculations (changing of rows, multiplication with factors)

$$A = A(\alpha) = \begin{pmatrix} 0 & -1 & \alpha+1 & 0 \\ 1 & 0 & 0 & \alpha-1 \\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] & (\alpha-1)\sin[(\alpha-1)\omega_0] \\ \cos[(\alpha+1)\omega_0] & \sin[(\alpha+1)\omega_0] & -(\alpha+1)\sin[(\alpha-1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Nontrivial solution have to satisfy the eigenvalue condition

$$\alpha^2\sin^2(\omega_0) + \sin^2(\alpha\omega_0) - 1 = 0 \tag{3.52}$$

If α_{ω_0} is an eigenvalue with $m(\alpha_{\omega_0}) = 1$, the local solution reads

$$\vec{U}(r,\omega) = r^{\alpha_{\omega_0}} \left[A_3 \begin{pmatrix} \alpha_{\omega_0}(-\sin[(\alpha_{\omega_0}-2)\omega] + \sin(\alpha_{\omega_0}\omega)) \\ -\alpha_{\omega_0}\cos[(\alpha_{\omega_0}-2)\omega] + (\alpha_{\omega_0}-2)\cos(\alpha_{\omega_0}\omega)) \\ -\frac{4}{r}\mu\alpha_{\omega_0}\sin[(\alpha_{\omega_0}-1)\omega] \end{pmatrix} + A_4 \begin{pmatrix} \alpha_{\omega_0}\cos[(\alpha_{\omega_0}-2)\omega] - (\alpha_{\omega_0}+2)\cos(\alpha_{\omega_0}\omega) \\ \alpha_{\omega_0}(-\sin[(\alpha_{\omega_0}-2)\omega] + \sin(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\cos[(\alpha_{\omega_0}-1)\omega] \end{pmatrix} \right] \tag{3.53}$$

with

$$A_3 = (\cos(\omega_0)\cos(\alpha_{\omega_0}\omega_0) - \alpha_{\omega_0}\sin(\alpha_{\omega_0}\omega_0)\sin(\omega_0))\;,$$
$$A_4 = (\alpha_{\omega_0}-1)\sin(\omega_0)\cos(\omega_0\alpha_{\omega_0})\,.$$

Neumann-Neumann boundary conditions Here we prescribe Neumann conditions on each side of the cone K. Due to (3.24) we have

$$-h(0)\vec{n}_s^1 + \mu \begin{pmatrix} 2\alpha\tilde{f}(0) & (\alpha-1)\tilde{g}(0)+\tilde{f}'(0) \\ (\alpha-1)\tilde{g}(0)+\tilde{f}'(0) & 2(\tilde{f}(0)+\tilde{g}'(0)) \end{pmatrix} \vec{n}_s^1 = 0$$
$$-h(\omega_0)\vec{n}_s^2 + \mu \begin{pmatrix} 2\alpha\tilde{f}(\omega_0) & (\alpha-1)\tilde{g}(\omega_0)+\tilde{f}'(\omega_0) \\ (\alpha-1)\tilde{g}(\omega_0)+\tilde{f}'(\omega_0) & 2(\tilde{f}(\omega_0)+\tilde{g}'(\omega_0)) \end{pmatrix} \vec{n}_s^2 = 0\,,$$

where $\vec{n}_s^1$ and $\vec{n}_s^2$ are the unit normalvectors of Γ_0 and Γ_{ω_0} in spherical coordinates

$$\vec{n}_s^1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}\,,\quad \vec{n}_s^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{3.54}$$

Hence we get for Neumann-Neumann boundary conditions the homogeneous system of linear

[2] Because of the symmetry we achieve the same results if we prescribe the versed boundary conditions, that means Dirichlet boundary conditions on Γ_0 and Neumann boundary conditions on Γ_{ω_0}. Hence we don't consider this case of boundary conditions.

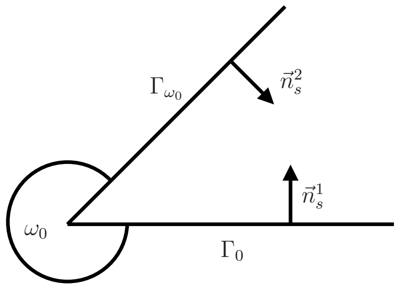

Figure 4: Normal vectors $\vec{n}_s^1, \vec{n}_s^2$ of the edges Γ_0 and Γ_{ω_0} respectively.

equations after some calculations

$$A = A(\alpha) = \begin{pmatrix} 0 & -1 & \alpha+1 & 0 \\ 1 & 0 & 0 & \alpha-1 \\ \cos[(\alpha+1)\omega_0] & \sin[(\alpha+1)\omega_0] & -(\alpha-1)\sin[(\alpha-1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] \\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] & (\alpha+1)\sin[(\alpha-1)\omega_0] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

The resulting eigenvalue condition reads

$$\alpha^2 \sin^2(\omega_0) - \sin^2(\omega_0 \alpha) = 0 . \tag{3.55}$$

Note that this is the same eigenvalue condition as in the case of Dirichlet-Dirichlet boundary conditions, but the solutions of the boundary value problem in K are different. Let α_{ω_0} be an eigenvalue with $m(\alpha_{\omega_0}) = 1$ then the solution is given by

$$\begin{aligned} \vec{U}(r,\omega) = r^{\alpha_{\omega_0}} \Bigg[& A_5 \begin{pmatrix} \alpha_{\omega_0}(\sin[(\alpha_{\omega_0}-2)\omega] - \sin(\alpha_{\omega_0}\omega)) \\ \alpha_{\omega_0}\cos[(\alpha_{\omega_0}-2)\omega] + (2-\alpha_{\omega_0})\cos(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\sin[(\alpha_{\omega_0}-1)\omega] \end{pmatrix} \\ & -A_6 \begin{pmatrix} \alpha_{\omega_0}\cos[(\alpha_{\omega_0}-2)\omega] - (\alpha_{\omega_0}+2)\cos(\alpha_{\omega_0}\omega) \\ \alpha_{\omega_0}(\sin[(\alpha_{\omega_0}-2)\omega] - \sin(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\cos[(\alpha_{\omega_0}-1)\omega] \end{pmatrix} \Bigg] \end{aligned} \tag{3.56}$$

with

$$\begin{aligned} A_5 &= (\cos(\omega_0)\sin(\alpha_{\omega_0}\omega_0) + \alpha_{\omega_0}\cos(\alpha_{\omega_0}\omega_0)\sin(\omega_0)) \ , \\ A_6 &= (\alpha_{\omega_0}-1)\sin(\omega_0)\sin(\omega_0\alpha_{\omega_0}) . \end{aligned}$$

Dirichlet-M$_1$ boundary conditions For M$_1$ mixed boundary conditions (see (3.21)) we split the condition on a side of the cone in tangential and normal direction. That means, we prescribe on Γ_0 Dirichlet boundary conditions in normal direction and Neumann boundary conditions in tangential direction. We obtain from (3.25) the following conditions

$$\begin{aligned} \vec{u}(r,0) \cdot \vec{n}_s^1 &= 0 \\ -h(0)\vec{n}_s^1 \cdot \vec{t}_s^{\,1} + \mu \left[\begin{pmatrix} 2\alpha\tilde{f}(0) & (\alpha-1)\tilde{g}(0) + \tilde{f}'(0) \\ (\alpha-1)\tilde{g}(0) + \tilde{f}'(0) & 2(\tilde{f}(0) + \tilde{g}'(0)) \end{pmatrix} \vec{n}_s^1 \right] \cdot \vec{t}_s^{\,1} &= 0 \\ \vec{u}(r,\omega_0) &= 0 . \end{aligned}$$

For the formulation of the mixed boundary conditions it is necessary to define additionally the tangent vector orthogonal to the normal vector as

$$\vec{t}_s^{\,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

So the homogeneous system of linear equation reads

$$A = A(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \alpha+1 \\ 1 & 0 & 0 & \alpha-1 \\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] & (\alpha-1)\sin[(\alpha-1)\omega_0] \\ \cos[(\alpha+1)\omega_0] & \sin[(\alpha+1)\omega_0] & -(\alpha+1)\sin[(\alpha-1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence we get the eigenvalue condition

$$\alpha \sin(2\omega_0) - \sin(2\omega_0\alpha) = 0\,. \tag{3.57}$$

The solution for an eigenvalue α_{ω_0} with $m(\alpha_{\omega_0}) = 1$ is given by

$$\begin{aligned} \vec{U}(r,\omega) = r^{\alpha_{\omega_0}} \Bigg[&(\alpha_{\omega_0} - 1)\cos[(\alpha_{\omega_0} - 1)\omega_0] \begin{pmatrix} -\cos(\alpha_{\omega_0}\omega) \\ \sin(\alpha_{\omega_0}\omega) \\ 0 \end{pmatrix} \\ &+ \cos((\alpha_{\omega_0} + 1)\omega_0) \begin{pmatrix} \alpha_{\omega_0}\cos[(\alpha_{\omega_0} - 2)\omega] - \cos(\alpha_{\omega_0}\omega) \\ -\alpha_{\omega_0}\sin[(\alpha_{\omega_0} - 2)\omega] - \sin(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\cos[(\alpha_{\omega_0} - 1)\omega] \end{pmatrix} \Bigg]\,. \end{aligned} \tag{3.58}$$

Neumann-M_1 boundary conditions Instead of the Dirichlet-M_1 boundary conditions we prescribe Neumann boundary conditions on Γ_{ω_0}, that means

$$\begin{aligned} \vec{u}(r,0)\cdot\vec{n}_s^1 &= 0 \\ -h(0)\vec{n}_s^1\cdot\vec{t}_s^{\,1} + \mu\left[\begin{pmatrix} 2\alpha\tilde{f}(0) & (\alpha-1)\tilde{g}(0) + \tilde{f}'(0) \\ (\alpha-1)\tilde{g}(0) + \tilde{f}'(0) & 2(\tilde{f}(0) + \tilde{g}'(0)) \end{pmatrix}\vec{n}_s^1\right]\cdot\vec{t}_s^{\,1} &= 0 \\ -h(\omega_0)\vec{n}_s^2 + \mu\begin{pmatrix} 2\alpha\tilde{f}(\omega_0) & (\alpha-1)\tilde{g}(\omega_0) + \tilde{f}'(\omega_0) \\ (\alpha-1)\tilde{g}(\omega_0) + \tilde{f}'(\omega_0) & 2(\tilde{f}(\omega_0) + \tilde{g}'(\omega_0)) \end{pmatrix}\vec{n}_s^2 &= 0\,. \end{aligned}$$

With the same nomenclature as above we get the following system of linear equations

$$A = A(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \alpha+1 \\ 1 & 0 & 0 & \alpha-1 \\ \cos[(\alpha+1)\omega_0] & \sin[(\alpha+1)\omega_0] & -(\alpha-1)\sin[(\alpha-1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] \\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] & (\alpha+1)\sin[(\alpha-1)\omega_0] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence the eigenvalue condition reads

$$\alpha \sin(2\omega_0) + \sin(2\omega_0\alpha) = 0\,. \tag{3.59}$$

For an eigenvalue α_{ω_0} with $m(\alpha_{\omega_0}) = 1$ we get the solution for Neumann-M_1 boundary conditions

$$\begin{aligned} \vec{U}(r,\omega) = r^{\alpha_{\omega_0}} \Bigg[&(\alpha_{\omega_0} - 1)\sin[(\alpha_{\omega_0} - 1)\omega_0] \begin{pmatrix} -\cos(\alpha_{\omega_0}\omega) \\ \sin(\alpha_{\omega_0}\omega) \\ 0 \end{pmatrix} \\ &+ \sin((\alpha_{\omega_0} + 1)\omega_0) \begin{pmatrix} \alpha_{\omega_0}\cos[(\alpha_{\omega_0} - 2)\omega] - \cos(\alpha_{\omega_0}\omega) \\ -\alpha_{\omega_0}\sin[(\alpha_{\omega_0} - 2)\omega] - \sin(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\cos[(\alpha_{\omega_0} - 1)\omega] \end{pmatrix} \Bigg]\,. \end{aligned} \tag{3.60}$$

M_1-M_1 boundary conditions In this case we consider the M_1 boundary conditions on both edges, i. e. we prescribe respectively Dirichlet boundary conditions in normal direction and Neumann boundary conditions in tangential direction:

$$\begin{aligned} \vec{u}(r,0)\cdot\vec{n}_s^1 &= 0 \\ -h(0)\vec{n}_s^1\cdot\vec{t}_s^{\,1} + \mu\left[\begin{pmatrix} 2\alpha\tilde{f}(0) & (\alpha-1)\tilde{g}(0) + \tilde{f}'(0) \\ (\alpha-1)\tilde{g}(0) + \tilde{f}'(0) & 2(\tilde{f}(0) + \tilde{g}'(0)) \end{pmatrix}\vec{n}_s^1\right]\cdot\vec{t}_s^{\,1} &= 0 \\ \vec{u}(r,0)\cdot\vec{n}_s^2 &= 0 \\ -h(\omega_0)\vec{n}_s^2\cdot\vec{t}_s^{\,2} + \mu\left[\begin{pmatrix} 2\alpha\tilde{f}(\omega_0) & (\alpha-1)\tilde{g}(\omega_0) + \tilde{f}'(\omega_0) \\ (\alpha-1)\tilde{g}(\omega_0) + \tilde{f}'(\omega_0) & 2(\tilde{f}(\omega_0) + \tilde{g}'(\omega_0)) \end{pmatrix}\vec{n}_s^2\right]\cdot\vec{t}_s^{\,2} &= 0\,. \end{aligned}$$

We get the following system of linear equations

$$A = A(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \alpha+1 \\ 1 & 0 & 0 & \alpha-1 \\ \cos[(\alpha+1)\omega_0] & \sin[(\alpha+1)\omega_0] & -(\alpha+1)\sin[(\alpha-1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] \\ \cos[(\alpha+1)\omega_0] & \sin[(\alpha+1)\omega_0] & -(\alpha-1)\sin[(\alpha-1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows the correspondent eigenvalue condition

$$\sin^2(\omega_0) - \sin^2(\omega_0\alpha) = 0\,. \tag{3.61}$$

The solution for an eigenvalue α_{ω_0} with $m(\alpha_{\omega_0}) = 1$ is given by

$$\begin{aligned} \vec{U}(r,\omega) = r^{\alpha_{\omega_0}} \Bigg[&(\alpha_{\omega_0}+1)\sin[(\alpha_{\omega_0}-1)\omega_0] \begin{pmatrix} -\cos(\alpha_{\omega_0}\omega) \\ \sin(\alpha_{\omega_0}\omega) \\ 0 \end{pmatrix} \\ &+ \sin((\alpha_{\omega_0}+1)\omega_0) \begin{pmatrix} \alpha_{\omega_0}\cos[(\alpha_{\omega_0}-2)\omega] - \cos(\alpha_{\omega_0}\omega) \\ -\alpha_{\omega_0}\sin[(\alpha_{\omega_0}-2)\omega] - \sin(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\cos[(\alpha_{\omega_0}-1)\omega] \end{pmatrix} \Bigg]. \end{aligned} \tag{3.62}$$

Dirichlet-M$_2$ boundary conditions Here we prescribe M$_2$ boundary conditions on Γ_0, i. e. Neumann boundary conditions in normal direction and Dirichlet boundary conditions in tangential direction:

$$\begin{aligned} \vec{u}(r,0)\cdot\vec{t}_s^{\,1} &= 0 \\ -h(0)\vec{n}_s^1\cdot\vec{n}_s^1 + \mu\left[\begin{pmatrix} 2\alpha\tilde{f}(0) & (\alpha-1)\tilde{g}(0)+\tilde{f}'(0) \\ (\alpha-1)\tilde{g}(0)+\tilde{f}'(0) & 2(\tilde{f}(0)+\tilde{g}'(0)) \end{pmatrix}\vec{n}_s^1\right]\cdot\vec{n}_s^1 &= 0 \\ \vec{u}(r,\omega_0) &= 0\,. \end{aligned}$$

So the system of linear equations reads

$$A = A(\alpha) = \begin{pmatrix} 0 & 1 & 1-\alpha & 0 \\ 0 & -1 & \alpha+1 & 0 \\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] & (\alpha-1)\sin[(\alpha-1)\omega_0] \\ \cos[(\alpha+1)\omega_0] & \sin[(\alpha+1)\omega_0] & -(\alpha+1)\sin[(\alpha-1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and the eigenvalue condition can be calculated to

$$\alpha\sin(2\omega_0) + \sin(2\omega_0\alpha) = 0\,. \tag{3.63}$$

Note that this is the same eigenvalue condition as in the case of Neumann-M$_1$ boundary conditions and so the possible eigenvalues are equal, however the solutions are different. In the case of Dirichlet-M$_2$ boundary condition we achieve for an eigenvalue α_{ω_0} with $m(\alpha_{\omega_0}) = 1$:

$$\begin{aligned} \vec{U}(r,\omega) = r^{\alpha_{\omega_0}} \Bigg[&(\alpha_{\omega_0}-1)\sin[(1-\alpha_{\omega_0})\omega_0] \begin{pmatrix} \sin(\alpha_{\omega_0}\omega) \\ \cos(\alpha_{\omega_0}\omega) \\ 0 \end{pmatrix} \\ &+ \sin((\alpha_{\omega_0}+1)\omega_0) \begin{pmatrix} \alpha_{\omega_0}\sin[(\alpha_{\omega_0}-2)\omega] - \sin(\alpha_{\omega_0}\omega) \\ \alpha_{\omega_0}\cos[(\alpha_{\omega_0}-2)\omega] + \cos(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\sin[(\alpha_{\omega_0}-1)\omega] \end{pmatrix} \Bigg]. \end{aligned} \tag{3.64}$$

Neumann-M$_2$ boundary conditions We proceed analogously to the case of Dirichlet-M$_2$ boundary conditions, but instead of Dirchlet boundary conditions we prescribe now Neumann boundary conditions on Γ_{ω_0}. We get

$$\begin{aligned} \vec{u}(r,0)\cdot\vec{t}_s^{\,1} &= 0 \\ -h(0)\vec{n}_s^1\cdot\vec{t}_n^1 + \mu\left[\begin{pmatrix} 2\alpha\tilde{f}(0) & (\alpha-1)\tilde{g}(0)+\tilde{f}'(0) \\ (\alpha-1)\tilde{g}(0)+\tilde{f}'(0) & 2(\tilde{f}(0)+\tilde{g}'(0)) \end{pmatrix}\vec{n}_s^1\right]\cdot\vec{n}_s^1 &= 0 \\ -h(\omega_0)\vec{n}_s^2 + \mu\begin{pmatrix} 2\alpha\tilde{f}(\omega_0) & (\alpha-1)\tilde{g}(\omega_0)+\tilde{f}'(\omega_0) \\ (\alpha-1)\tilde{g}(\omega_0)+\tilde{f}'(\omega_0) & 2(\tilde{f}(\omega_0)+\tilde{g}'(\omega_0)) \end{pmatrix}\vec{n}_s^2 &= 0\,. \end{aligned}$$

Hence we obtain

$$A = A(\alpha) = \begin{pmatrix} 0 & 1 & 1-\alpha & 0 \\ 0 & -1 & \alpha+1 & 0 \\ \cos[(\alpha+1)\omega_0] & \sin[(\alpha+1)\omega_0] & -(\alpha-1)\sin[(\alpha-1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] \\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] & (\alpha+1)\sin[(\alpha-1)\omega_0] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and the correspondent eigenvalue condition yields to

$$\alpha \sin(2\omega_0) - \sin(2\omega_0\alpha) = 0\,. \tag{3.65}$$

This is the same eigenvalue condition as in the case of Dirichlet-M_1 boundary conditions but here the solution for an eigenvalue α_{ω_0} with $m(\alpha_{\omega_0}) = 1$ is given by

$$\begin{aligned} \vec{U}(r,\omega) = r^{\alpha_{\omega_0}} \Bigg[& -(\alpha_{\omega_0} - 1)\cos[(\alpha_{\omega_0} - 1)\omega_0] \begin{pmatrix} \sin(\alpha_{\omega_0}\omega) \\ \cos(\alpha_{\omega_0}\omega) \\ 0 \end{pmatrix} \\ & + \cos((\alpha_{\omega_0} + 1)\omega_0) \begin{pmatrix} \alpha_{\omega_0}\sin[(\alpha_{\omega_0} - 2)\omega] - \sin(\alpha_{\omega_0}\omega) \\ \alpha_{\omega_0}\cos[(\alpha_{\omega_0} - 2)\omega] + \cos(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\sin[(\alpha_{\omega_0} - 1)\omega] \end{pmatrix} \Bigg]\,. \end{aligned} \tag{3.66}$$

M_2-M_2 boundary conditions We prescribe M_2 boundary conditions on both sides of the cone K, i. e.

$$\begin{aligned} \vec{u}(r,0)\cdot\vec{t}_s^{\,1} &= 0 \\ -h(0)\vec{n}_s^1\cdot\vec{n}_s^1 + \mu\left[\begin{pmatrix} 2\alpha\tilde{f}(0) & (\alpha-1)\tilde{g}(0) + \tilde{f}'(0) \\ (\alpha-1)\tilde{g}(0) + \tilde{f}'(0) & 2(\tilde{f}(0) + \tilde{g}'(0)) \end{pmatrix}\vec{n}_s^1\right]\cdot\vec{n}_s^1 &= 0 \\ \vec{u}(r,0)\cdot\vec{t}_s^{\,2} &= 0 \\ -h(\omega_0)\vec{n}_s^2\cdot\vec{n}_s^2 + \mu\left[\begin{pmatrix} 2\alpha\tilde{f}(\omega_0) & (\alpha-1)\tilde{g}(\omega_0) + \tilde{f}'(\omega_0) \\ (\alpha-1)\tilde{g}(\omega_0) + \tilde{f}'(\omega_0) & 2(\tilde{f}(\omega_0) + \tilde{g}'(\omega_0)) \end{pmatrix}\vec{n}_s^2\right]\cdot\vec{n}_s^2 &= 0\,. \end{aligned}$$

The system of linear equations is

$$A = A(\alpha) = \begin{pmatrix} 0 & 1 & 1-\alpha & 0 \\ 0 & -1 & \alpha-1 & 0 \\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] & (\alpha-1)\sin[(\alpha-1)\omega_0] \\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] & (\alpha+1)\sin[(\alpha-1)\omega_0] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and the eigenvalue conditions read

$$\sin^2(\omega_0) - \sin^2(\omega_0\alpha) = 0\,. \tag{3.67}$$

Note that this is the same eigenvalue condition as in the case of M_1-M_1 boundary conditions. For an eigenvalue α_{ω_0} with $m(\alpha_{\omega_0}) = 1$ we achieve the solution

$$\begin{aligned} \vec{U}(r,\omega) = r^{\alpha_{\omega_0}} \Bigg[& (\alpha_{\omega_0} - 1)\sin[(1 - \alpha_{\omega_0})\omega_0] \begin{pmatrix} \sin(\alpha_{\omega_0}\omega) \\ \cos(\alpha_{\omega_0}\omega) \\ 0 \end{pmatrix} \\ & + \sin((\alpha_{\omega_0} + 1)\omega_0) \begin{pmatrix} \alpha_{\omega_0}\sin[(\alpha_{\omega_0} - 2)\omega] - \sin(\alpha_{\omega_0}\omega) \\ \alpha_{\omega_0}\cos[(\alpha_{\omega_0} - 2)\omega] + \cos(\alpha_{\omega_0}\omega)) \\ \frac{4}{r}\mu\alpha_{\omega_0}\sin[(\alpha_{\omega_0} - 1)\omega] \end{pmatrix} \Bigg]\,. \end{aligned} \tag{3.68}$$

M_1-M_2 boundary conditions In the last case we combine the two different mixed boundary conditions, that means the prescription of M_1 boundary conditions on Γ_0 and M_2 boundary conditions on Γ_{ω_0}. We achieve

$$\begin{aligned} \vec{u}(r,0)\cdot\vec{n}_s^1 &= 0\\ -h(0)\vec{n}_s^1\cdot\vec{t}_s^{\,1} + \mu\left[\begin{pmatrix} 2\alpha\tilde{f}(0) & (\alpha-1)\tilde{g}(0)+\tilde{f}'(0)\\ (\alpha-1)\tilde{g}(0)+\tilde{f}'(0) & 2(\tilde{f}(0)+\tilde{g}'(0))\end{pmatrix}\vec{n}_s^1\right]\cdot\vec{t}_s^{\,1} &= 0\\ \vec{u}(r,0)\cdot\vec{t}_s^{\,2} &= 0\\ -h(\omega_0)\vec{n}_s^2\cdot\vec{n}_s^2 + \mu\left[\begin{pmatrix} 2\alpha\tilde{f}(\omega_0) & (\alpha-1)\tilde{g}(\omega_0)+\tilde{f}'(\omega_0)\\ (\alpha-1)\tilde{g}(\omega_0)+\tilde{f}'(\omega_0) & 2(\tilde{f}(\omega_0)+\tilde{g}'(\omega_0))\end{pmatrix}\vec{n}_s^2\right]\cdot\vec{n}_s^2 &= 0\,. \end{aligned}$$

So the system of linear equations reads

$$A = A(\alpha_{\omega_0}) = \begin{pmatrix} 1 & 0 & 0 & \alpha+1\\ 1 & 0 & 0 & \alpha-1\\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha-1)\cos[(\alpha-1)\omega_0] & (\alpha-1)\sin[(\alpha-1)\omega_0]\\ \sin[(\alpha+1)\omega_0] & -\cos[(\alpha+1)\omega_0] & (\alpha+1)\cos[(\alpha-1)\omega_0] & (\alpha+1)\sin[(\alpha-1)\omega_0] \end{pmatrix}\begin{pmatrix} c_1\\ c_2\\ c_3\\ c_4\end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\end{pmatrix},$$

and we get the correspondent eigenvalue condition

$$\sin^2(\omega_0) + \sin^2(\omega_0\alpha) - 1 = 0\,. \tag{3.69}$$

If α_{ω_0} is an eigenvalue with $m(\alpha_{\omega_0}) = 1$, a solution is

$$\begin{aligned} \vec{U}(r,\omega) = r^{\alpha_{\omega_0}}&\Bigg[(\alpha_{\omega_0}-1)\cos[(\alpha_{\omega_0}-1)\omega_0]\begin{pmatrix} -\cos(\alpha_{\omega_0}\omega)\\ \sin(\alpha_{\omega_0}\omega)\\ 0\end{pmatrix}\\ &+\cos((\alpha_{\omega_0}+1)\omega_0)\begin{pmatrix} \alpha_{\omega_0}\cos[(\alpha_{\omega_0}-2)\omega] - \cos(\alpha_{\omega_0}\omega)\\ -\alpha_{\omega_0}\sin[(\alpha_{\omega_0}-2)\omega] - \sin(\alpha_{\omega_0}\omega))\\ \frac{4}{r}\mu\alpha_{\omega_0}\cos[(\alpha_{\omega_0}-1)\omega]\end{pmatrix}\Bigg]\,. \end{aligned} \tag{3.70}$$

Remark: The case $m(\alpha_\nu) = 2$ is discussed in [12].

3.3.3 Regularity results

We characterize the regularity of the singular terms of the original boundary value problems in the domain Ω starting from a weak solution $(\vec{u},p)^\top \in [H^1(\Omega)]^2 \times L^2(\Omega)$, compare (2.8).

Theorem 3.3. *Let α_{ω_0} be an simple eigenvalue with $\operatorname{Re}\alpha_{\omega_0} \in (0,1)$ and assume that it realizes an eigenvalue with smallest real part. The corresponding leading singular solution in K can be written as*

$$\vec{U}_{sing}(r,\omega) = r^{\alpha_{\omega_0}}\begin{pmatrix} f(\omega,\alpha_{\omega_0})\\ g(\omega,\alpha_{\omega_0})\\ \frac{1}{r}h(\omega,\alpha_{\omega_0})\end{pmatrix} = \begin{pmatrix} u_{1,sing}\\ u_{2,sing}\\ \tilde{p}\end{pmatrix},$$

where $(f,g,h)^\top(\omega,\alpha_{\omega_0})$ desribes the angular dependent part of the solution. Consider an arbitrary small, but fixed $\varepsilon > 0$.
Then

$$\tilde{\eta}_i\vec{u}_{sing} \in \left[H^{1+\operatorname{Re}\alpha_{\omega_0}-\varepsilon}(\Omega)\right]^2 \quad \text{and} \quad \tilde{\eta}_i\tilde{p}_{sing} \in H^{\operatorname{Re}\alpha_{\omega_0}-\varepsilon}(\Omega)\,.$$

where $\tilde{\eta}_i \in C_0^\infty(B_{\tilde{\varepsilon}}(O_i))$ is a cut-off function with $\tilde{\eta}_i \equiv 1$ on $\operatorname{supp}\eta_i$ and $\tilde{\varepsilon}_i > \varepsilon_i$.

Proof. We use the theorem (1.4.5.3) of [5, p.35] which reads:
Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary Γ is a curvilinear polygon. Assume that $0 \in \Gamma$. Let V be a neighbourhood of O such that

$$V \cap \bar{\Omega} \subset \{r\cos(\omega), r\sin(\omega) | r \geq 0, a \leq \omega \leq b\}$$

with $b - a < 2\pi$. Finally let u be a function which is smooth in $\bar{\Omega}\setminus\{0\}$ and which coincides with

$$r^\alpha\varphi(\omega)$$

in $V \cap \Omega$, where $\varphi(\omega) \in C^\infty([a,b])$. Then

$$u \in W^{s,p}(\Omega) \quad \text{for } \operatorname{Re}\alpha > s - \frac{2}{p}$$

und

$$u \notin W^{s,p}(\Omega) \quad \text{for } \operatorname{Re}\alpha \leq s - \frac{2}{p}$$

when $\operatorname{Re}\alpha$ is not an integer.

Setting $O = O_i$ each component of $\tilde{\eta}\vec{U}$ satisfies the premises of the above theorem of Grisvard and so we achieve with $p = 2$

$$\tilde{\eta}_i \vec{u}_{\text{sing}} \in [H^s(\Omega)]^2 \quad \text{mit } s < \operatorname{Re}\alpha_{\omega_0} + 1$$

and

$$\tilde{\eta}_i \tilde{p}_{\text{sing}} \in H^s(\Omega) \quad \text{mit } s < \operatorname{Re}\alpha_{\omega_0} .$$

With arbitrary $\varepsilon > 0$ the assertion is shown. □

4 Computation of the singular exponents

In section 3 we have discussed the structure of the singular terms (3.1). Now, we compute the real and complex eigenvalues and point out the dependence of the exponent α_{ω_0} on the apex angle ω_0. The eigenvalues α_{ω_0} are given by the correspondent eigenvalue condition for each boundary value problem and we have to compute the roots of the different transcendental equations (3.50).

4.1 Distribution of the eigenvalues

We describe the procedure how to get the distribution of the eigenvalues for Dirichlet-Dirichlet boundary conditions in detail. The eigenvalue condition (3.50) reads

$$\alpha^2 \sin^2(\omega_0) - \sin^2(\alpha\omega_0) = 0$$

and the real eigenvalues can be easily generated with MATLAB by an implicite plot. The complex roots are more difficult to compute. In this case we use a result of [14], which states that the eigenvalue curves are continuous and that the transitions of the real eigenvalues to the complex eigenvalues are given by two conditions, the eigenvalue condition and its derivative with respect to α. The derivative of the eigenvalue condition (3.50) reads

$$2\alpha \sin^2(\omega_0) - 2\omega_0 \sin(\omega_0\alpha)\cos(\omega_0\alpha) = 0$$

and figure 5 shows the graphs. Here we can read off the intersections and start the computation of the complex solutions with the MATLAB-program `find_complex_ev` (see appendix B). Figure 6 illustrates the distribution of the eigenvalues. We are interested in singularities with an exponent $0 < \operatorname{Re}\alpha_{\omega_0} < 1$. We see that for Dirichlet-Dirichlet boundary conditions an apex angle greater than π generates such exponents.
This procedure can be done for each eigenvalue condition. Note that the eigenvalue conditions can coincide for different boundary conditions. Thus for the Neumann-Neumann boundary value problem we have the same eigenvalue condition as for the Dirichlet-Dirichlet boundary condition and so we get the same distribution as in figure 6. The figures of the distributions of other eigenvalue conditions are in the appendix A, real eigenvalues are plotted in blue and the red lines indicate the real parts of the conjugate pair of comlex eigenvalues. Figure A.1 shows the distribution of the eigenvalues in the case of Dirichlet-Neumann boundary conditions. Singularities appear for corners with an apex angle greater than $\pi/4$. The distribution of the eigenvalues for M_1-M_1 and M_2-M_2 illustrates figure A.2 and the distribution of eigenvalues for Dirichlet-M_2 and Neumann-M_1 boundary conditions is shown in figure A.3. In both cases singularities exist for an apex angle greater than $\pi/2$.

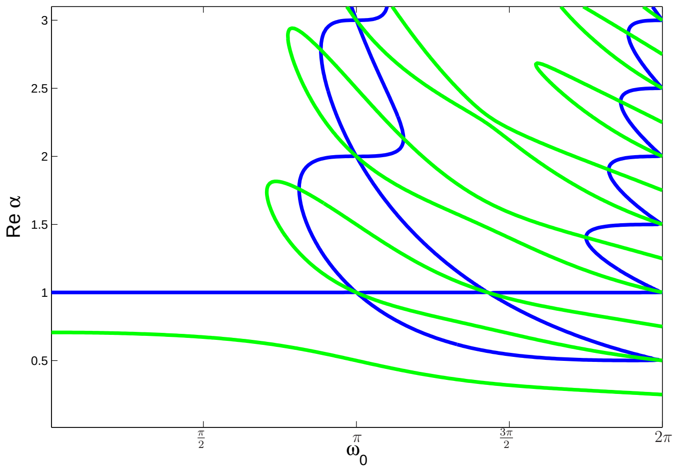

Figure 5: The roots of the eigenvalue condition and its derivative with respect to α.

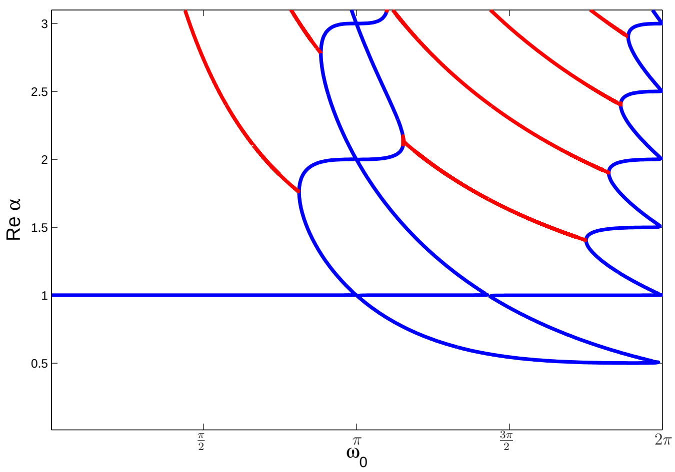

Figure 6: Eigenvalues for D-D and N-N boundary conditions

4.2 Examples

We consider two examples, first a lake and second a bay in the sea.

Lake Figure 7 shows a lake where the water flows counterclockwise, so we set Neumann boundary conditions on each side. At the bottom of the lake we see a reentrant corner like a kind of obstacle.

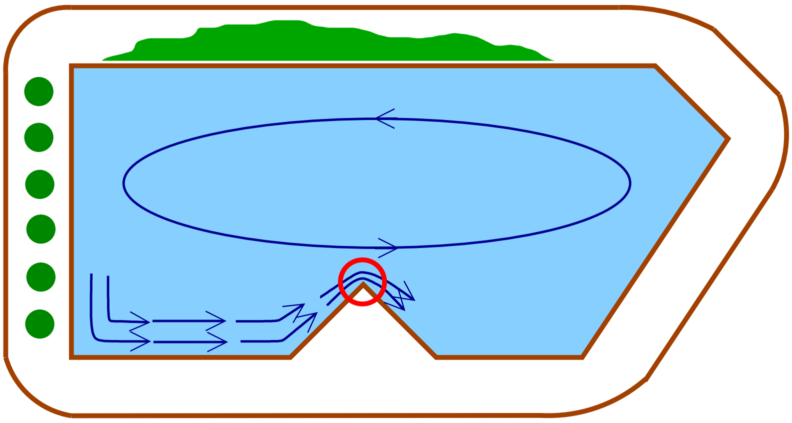

Figure 7: A lake with a reentrant corner.

Since the apex angle is greater than π we get a singularity, that means an infinite pressure and an infinite velocity gradient, near the corner marked with a red circle. So we have to be careful swimmimg near such corners.

Bay in the sea The bay is shown in figure 8 at the left side. We assume that the flow of the water is zero on its boundary, that means there are zero Dirichlet boundary conditions supposed. The water flow in the sea goes from top to down and then to the right and here we prescribe Neumann boundary conditions. The apex angles of the boundary corners of the bay are less

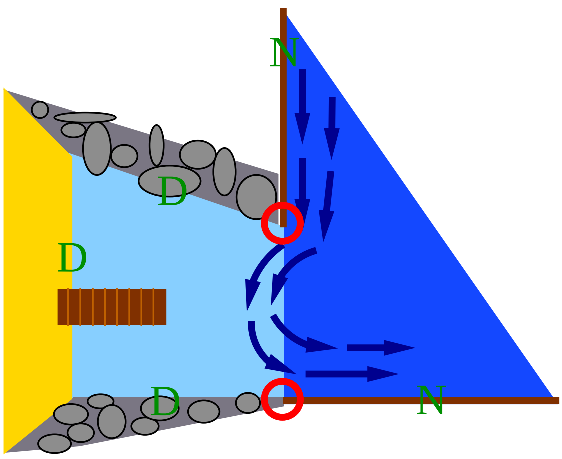

Figure 8: Bay in the sea with singular points

than π. Therefore there are no singularities, where the boundary conditions do not change. In those boundary points which are the intersections of the boundary of the sea and of the bay the boundary conditions change, that means there are corners of Dirichlet-Neumann type. Therefore, we get two singularities at the red marked singular points. Note one of the singularities occurs at the bottom „corner“ with apex angle π.

A Distributions of eigenvalues

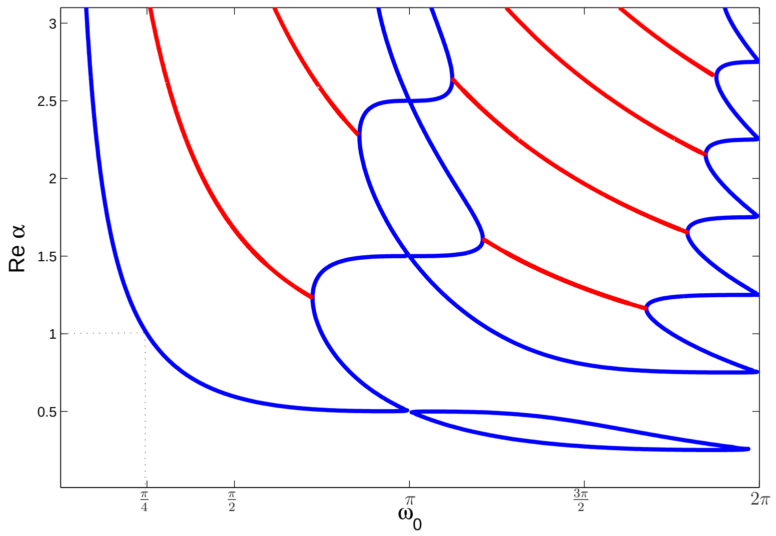

Figure A.1: Distribution of the eigenvalues for Neumann-Dirichlet boundary conditions.

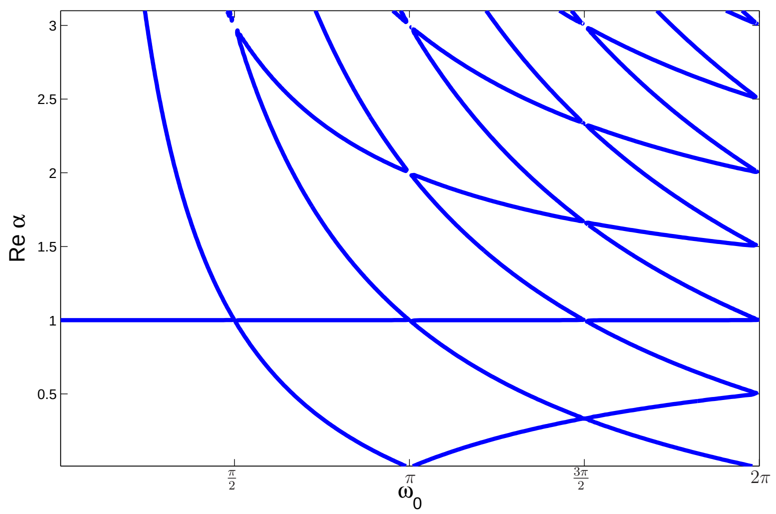

Figure A.2: Distribution of the eigenvalues for M_1-M_1 and M_2-M_2 boundary conditions. No complex eigenvalues occur.

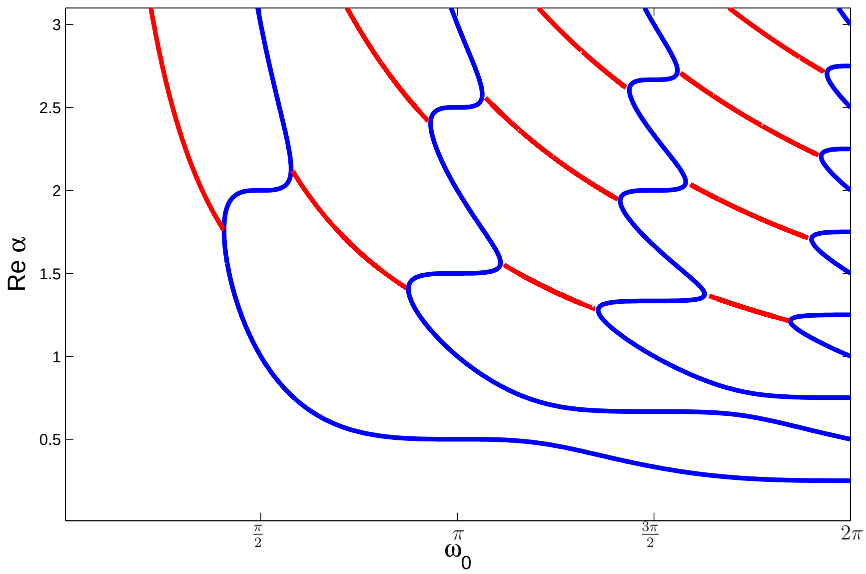

Figure A.3: Distribution of the eigenvalues for Dirichlet-M_2 and Neumann-M_1 boundary conditions.

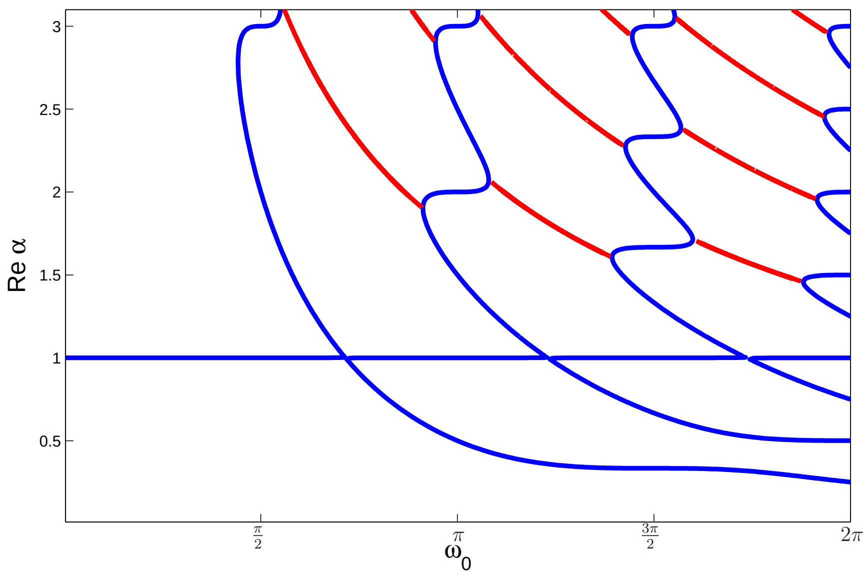

Figure A.4: Distribution of the eigenvalues for Dirichlet-M_1 and Neumann-M_2 boundary conditions.

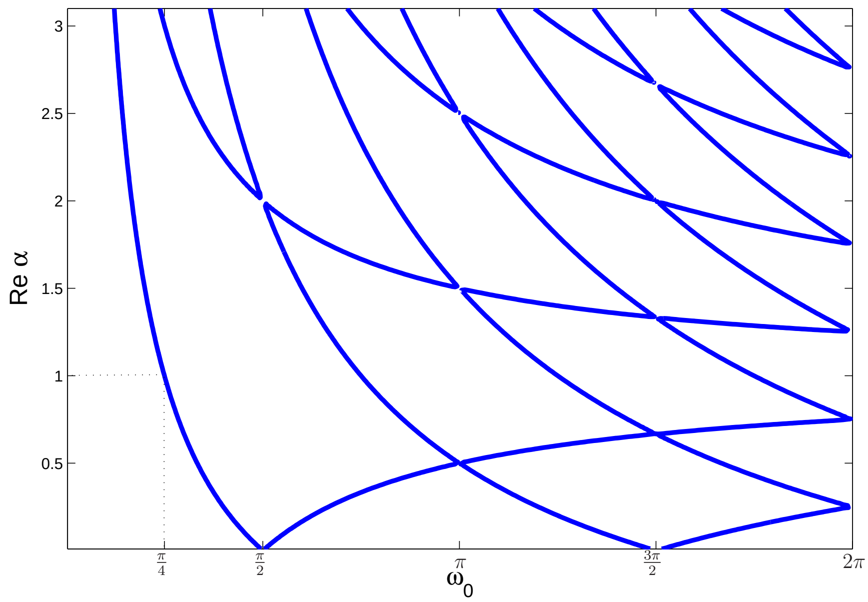

Figure A.5: Distribution of the eigenvalues for M_1-M_2 boundary conditions.

B Matlab program for the computation of complex eigenvalues of the eigenvalue condition

```
function output = find_complex_ev(areas, ew_func)
%function output = find_complex_ev(areas, ew_func)
%
%Description:
%   The function computes the complex roots of an eigenvalue condition
%   (or an arbitrary twodimensional function) in the given areas for the
%   apex angle w0.
%
%Input:
%   areas:              Matrix with areas in which the roots will be searched.
%                       The initial value of alpha must be located at w0_start.
%                       Layout of each row: w0_start, w0_end, alpha_start
%   ew_func(a,b,w0):    Eigenvalue condition as inline-function dependant on
%                       a,b, w0, whereas z = a+i*b is complex
%
%Output:
%   output:     Vector of structures, for each area one structure
%               Layout of a structure: output(i).omega0
%                                      output(i).real
%
%Copyright: Pascal Maerkl, 04.02.2008
%           University of Stuttgart, IANS



output = [];
%structure of the results
result_struct = struct('omega0',[],'real',[]);

for nr = 1:size(areas,1)
    fprintf('Area %d\n',nr);
    result = [];
    %area of the angle
        %program should be able to compute areas top down
        if areas(nr,1) <= areas(nr,2)
            w0 = areas(nr,1):0.001:areas(nr,2);
        else
            w0 = areas(nr,1):-0.001:areas(nr,2);
```

```
        end
    %search in as area +-0.1 round the initial value of the real part of alpha
    a = areas(nr,3)-0.1:0.001:areas(nr,3)+0.1;
    %initial values of the imaginary part
    b = 0:0.001:0.1;
    %create search grid
    [a,b] = meshgrid(a,b);

    %ONLY FOR READOUT
    total = size(w0,2);
    fprintf('\tProgress: %5d/%5d',0,total)
    %ONLY FOR READOUT

    %loop over the angle area
    for k=1:size(w0,2)
        %evaluate the eigenvalue condition
        y = ew_func(a,b,w0(k));

        %compare the roots with an accuracy < 10^(-2)
        l = 2;
        point = find((abs(real(y))<=10^(-l))&(abs(imag(y))<=10^(-l)));

        %if more than 10 points satisfy the above accuracy, refine with decimal power
        while size(point,1) > 10
            l = l+1;
            point = find((abs(real(y))<=10^(-l))&(abs(imag(y))<=10^(-l)));
        end

        %if maximum 10 points are found, then save them and change the
        %search areas of the complex roots
        if size(point,1) > 0
            %save
            result = [result; [w0(k)*ones(size(point,1),1), a(point), b(point)]];
            %next root will be searched in an area +-0.1 round the old root
            %because of the continuity
            a = a(point)-0.1:0.001:a(point)+0.1;
            b = b(point)-0.1:0.001:b(point)+0.1;
            %create new search grid
            [a,b] = meshgrid(a,b);
        end

        %ONLY FOR READOUT
        fprintf('\b\b\b\b\b\b\b\b\b\b\b%5d/%5d',k,total)
        %ONLY FOR READOUT

    end
    %filter real solutions
    if max(size(result)) > 0
         im = find(result(:,3)~=0);

         %ONLY FOR READOUT
         fprintf('\n\t%d complex roots are found.\n\n',size(im,1))
         %ONLY FOR READOUT

         %save the results of the area in the result structure and in the output variable
         result_struct.omega0 = result(im,1);
         result_struct.real = result(im,2);
         output = [output;result_struct];
    else
         %ONLY FOR READOUT
         fprintf('\n\tNO complex roots are found.\n\n')
         %ONLY FOR READOUT
    end
end
```

References

[1] J. Banasiak and G.F. Roach. On Corner Singularities of Solutions to Mixed Boundary-Value Problems for Second-Order Elliptic and Parabolic Equations. *Proceedings: Mathematical and Physical Sciences*, 433(1887):209–217, 1991.

[2] M. Dauge. Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. Part I: linearized equations. *SIAM J. Math. Anal.*, 20(1):74–97, January 1989.

[3] L.C. Evans. *Partial Differential Equations*. American Mathematical Society, 1998.

[4] M. Feistauer. *Mathematical methods in fluid dynamics*, volume 67 of *Pitman monographs and surveys in pure and applied mathematics*. Harlow : Longman Scientific & Technical [u.a.], 1993.

[5] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman, 1985.

[6] P. Grisvard. *Singularities in boundary value problems*. Springer-Verlag, 1992.

[7] G. Kirchhoff. Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. *Journal Reine Angewandte Mathematik*, 40:51–58, 1850.

[8] M. Kofler and H.-G. Gräbe. *Mathematica: Einführung, Anwendung, Referenz*. Addison-Wesley München, 2002.

[9] V.A. Kondratiev. *Boundary value problems for elliptic equations in domains with conical or angular points*, volume 16, pages 209–292. Trudy Moskovkogo Mat. Obschtsva, 1967.

[10] V.A. Kozlov, V.G. Maz'ya, and J. Rossmann. *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*. American Mathematical Society, 2001.

[11] L.E. Malvern. *Introduction to the mechanics of a continuous medium*. Prentice-Hall, Inc., 1969.

[12] M. Orlt. *Regularitätsuntersuchungen und FEM-Fehlerabschätzungen für allgemeine Randwertprobleme der Navier-Stokes-Gleichungen*. PhD thesis, Universität Rostock, 1998.

[13] M. Orlt and A.-M. Sändig. Regularity of viscous Navier-Stokes flows in non smooth domains. *Boundary Value Problems and Integral Equations in Non Smooth Domains, Proceedings of the conference at the CIRM, Luminy, ed. by M. Costabel, M. Dauge, S. Nicaise, Marcel Dekker Inc.*, pages 185–201, 1995.

[14] A.-M. Sändig, U. Richter, and R. Sändig. Regularity of boundary value problems for the Lamé equations in a polygonal domain. *Rostocker Math. Kolloquium*, 36:21–50, 1989.

[15] A.M. Sändig. Regularity results for linear elliptic boundary value problems in polygons. *Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation, Universität Stuttgart*, (10), 2006.

[16] R. Temam. *Navier-Stokes Equations*. North-Holland Publishing Company - Amsterdam · New York · Oxford, 1977.

[17] C. Truesdell and K.R. Rajagopal. *An Introduction to the Mechanics of Fluids*. Birkhäuser, 2000.

[18] H. Wenzel and P. Meinhold. *Gewöhnliche Differentialgleichungen*. B.G. Teubner Verlagsgesellschaft Leipzig, 1994.

[19] K. Wieghardt. *Theoretische Strömungslehre*. Teubner, 1974.

Pascal Märkl
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: Pascal.Maerkl@ians.uni-stuttgart.de

Anna-Margarete Sändig
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: Anna.Saendig@ians.uni-stuttgart.de

Erschienene Preprints ab Nummer 2008/001

Komplette Liste: `http://preprints.ians.uni-stuttgart.de`

2008/001 *Perfahl, H., Sändig, A.-M.:* A Continuum-Mechanical Approach to Avascular Solid Tumor Growth

2008/002 *Giesselmann, J.:* A convergence result for finite volume schemes on 2-dimensional Riemannian manifolds

2008/003 *Rohde, C., Surulescu, C.:* Mathematische Modellierung und Analyse von biologischen Prozessen

2008/004 *Lalegname, A., Sändig, A.-M., Sewell, G.:* Analytical and numerical treatment of a dynamic crack model

2008/005 *Kettemann, A.:* A mathematical model for mesenchymal and chemosensitive cell dynamics in tissue networks

2008/006 *Kelkel, J., Surulescu, C.:* Global existence and numerical simulations for a reaction-diffusion system modeling pattern formation on seashells

2008/007 *Kelkel, J., Surulescu, C.:* On a stochastic reaction-diffusion system modeling pattern formation

2008/008 *Béchet, É., Moës, N., Wohlmuth, B. I.:* A Stable Lagrange Multiplier Space for Stiff Interface Conditions within the Extended Finite Element Method

2008/009 *Märkl, P., Sändig, A.-M.:* Singularities of the Stokes System in Polygons