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Preprint 2008/011

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ISSN **1611-4176**

FINITE-VOLUME SCHEMES FOR FRIEDRICHS SYSTEMS WITH INVOLUTIONS

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ABSTRACT. In applications solutions of systems of hyperbolic balance laws often have to satisfy additional side conditions. We consider initial value problems for the general class of Friedrichs systems where the solutions are constrained by differential conditions given in the form of involutions [6]. These occur in particular in electrodynamics, electro- and magnetohydrodynamics as well as in elastodynamics. Neglecting the involution on the discrete level typically leads to instabilities.

To overcome this problem in electrodynamical applications it has been suggested in [21] to solve an extended system. Here we suggest an extended formulation to the general class of constrained Friedrichs systems. It is proven for explicit Finite-Volume schemes that the discrete solution of the extended system converges to the weak solution of the original system for vanishing discretization and extension parameter under an appropriate scaling. Moreover we show that the involution is weakly satisfied in the limit. The proofs rely on a reformulation of the extension as a relaxation-type approximation and careful use of the convergence theory for Finite-Volume methods for systems of Friedrichs type. Numerical experiments illustrate our analytical results.

1. INTRODUCTION

In this paper, we study linear systems of balance laws, namely $(m \times m)$-systems of Friedrichs type [12] with $m \in \mathbb{N}$. We consider the spatially d-dimensional case with $d = 2, 3$, space coordinates $x = (x_1, \dots, x_d)^T$, and time $t \geq 0$. For $T > 0$, let $G^1, \dots, G^d, D : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{m \times m}$ and $f : \mathbb{R}^d \times [0,T] \to \mathbb{R}^m$ be given (matrix-valued) functions. We suppose that the matrices $G^1(x,t), \dots, G^d(x,t)$ are symmetric for all $(x,t) \in \mathbb{R}^d$. Then the initial value problem for the unknown vector-valued function $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}^m$ takes the form

$$\frac{\partial}{\partial t} u(x,t) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(G^i(x,t)u(x,t)\big) + D(x,t)u(x,t) = f(x,t), \tag{1}$$

$$u(x,0) = u_0(x). \tag{2}$$

Date: 20th November 2008.

Key words and phrases. Involutionary systems, Friedrichs system, Finite-Volume scheme and Relaxation-type formulation.

Here $u_0 : \mathbb{R}^d \to \mathbb{R}^m$ denotes the initial function. Moreover we require the solution u to satisfy a linear differential side condition of the form

$$\sum_{i=1}^{d} M_i \frac{\partial}{\partial x_i} (u(x,t)) = 0, \qquad \big((x,t) \in \mathbb{R}^d \times [0,T)\big). \tag{3}$$

Here M_i, $i = 1,\ldots,d$, are constant $(m \times m)$-matrices.

Following the notion of Dafermos [6] the side condition (3) is called an **involution** for the system (1) if and only if any (weak) solution of (1), (2) satisfies (3) in the weak sense, whenever the initial data do so (see Definition 2.1). For the rest of the paper we assume that (3) is an involution. Involutions appear frequently in applications. We mention the classical Maxwell system to describe electrodynamical processes [17]. The divergence of the electrical and magnetical field is constrained in this case. The induction equations in the (in)compressible electro- and magnetohydrodynamical equations provide similar examples but with (x,t)-dependence in the flux ([5], Sect. 5 below). Solutions of the equations of linear elasticity have to satisfy compatibility conditions on the deformation gradient, which result in an involutionary condition [6, Chapter 5]. Yet another example is the linear piezoelectrical system [19]. In Sect. 5 we present some of these examples in more detail. Let us mention that involutions of course appear also in the more challenging case of nonlinear conservation laws. Again magnetohydrodynamics [5], electrohydrodynamics, nonlinear elasticity systems, but also Einstein's equations of general relativity are prominent examples.

On the analytical level an involutionary side condition is not problematic. The well-posedness for (1)-(3) is well known from [6]. By definition the involution (3) is satisfied. Also standard numerical schemes are known to converge. However, without consideration of (3) in the numerical scheme the residuum in the side condition usually grows with increasing time. In coupled processes this is a typical source of instabilities (cf. [20] and cites therein). Therefore a wide range of stabilization methods has been suggested (e.g. [1, 2, 4, 14, 21]).

The motivation for this contribution is the work of Munz et al. [21]. They introduced in particular the so-called hyperbolic Generalized Lagrangian Multiplier Finite Volume method (GLM-FV) to compute approximate solutions for Maxwell's system of linear electrodynamics. We formulate this approach for the general problem (1)-(2) with involution (3). While the original approach is motivated by a generalization of a Finite-Element type method [1] for a constrained wave equation we consider the approach as the approximation of (1)-(3) by an extended *relaxation-type system*. Relaxation approximations of systems of conservation laws have been intensively studied in the last decade (see [27] for an overview).

To be precise let $\varepsilon > 0$ and $u_0, \psi_0 : \mathbb{R}^d \to \mathbb{R}^m$ be given. Consider the following initial value problem for the unknown function $w^\varepsilon : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{2m}$ with $w^\varepsilon :=$

$(u_1^\varepsilon, \dots, u_m^\varepsilon, \psi_1^\varepsilon, \dots, \psi_m^\varepsilon)^T$ satisfying

$$\frac{\partial}{\partial t}u^\varepsilon + \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(G^i(x,t)u^\varepsilon\right) + M_i^T \frac{\partial}{\partial x_i}\psi^\varepsilon + D(x,t)u^\varepsilon = f(x,t), \tag{4}$$

$$\frac{\partial}{\partial t}\psi^\varepsilon + \sum_{i=1}^d \frac{M_i}{\varepsilon}\frac{\partial}{\partial x_i}u^\varepsilon + a\psi^\varepsilon = 0, \tag{5}$$

and

$$u^\varepsilon(x,0) = u_0^\varepsilon(x), \qquad \psi^\varepsilon(x,0) = \psi_0^\varepsilon(x). \tag{6}$$

Here $a \geq 0$ is a fixed parameter, chosen arbitrarily, and $\psi_0^\varepsilon : \mathbb{R}^d \to \mathbb{R}^m$ a given function. We will show in Section 2 that the initial value problem for the extended system (4)-(6) is well-posed. For vanishing relaxation parameter ε we prove under mild assumptions on the coefficients (see Proposition 2.7) that

$$\|u^\varepsilon - u\|_{L^2(\mathbb{R}^d\times[0,T];\mathbb{R}^m)} = \mathcal{O}(\varepsilon^{1/2}). \tag{7}$$

In Section 3 we present the Extended Generalized Lagrangian Multiplier Finite Volume Method (**EX-GLM-FV**) for the general system (1)-(3). For mesh parameter $h > 0$ this gives us the mesh function $u_h^\varepsilon : \mathbb{R}^d \times [0,T] \to \mathbb{R}^m$. The method will be analyzed in Sect. 4. By careful investigation of the convergence theory developed in [15, 25] we obtain (see Theorem 4.5)

$$\|u_h^\varepsilon - u^\varepsilon\|_{L^2(\mathbb{R}^d\times[0,T];\mathbb{R}^m)} = \mathcal{O}\left(h^{1/2}\varepsilon^{-1/4}\right). \tag{8}$$

The crucial fact is that the estimate does not depend critically on the relaxation parameter ε. This expresses the dissipative character of the approximation (4)-(6).
If we combine the estimates (7) and (8) and if we choose $\varepsilon = h^\alpha$ with α the solution of an optimization problem we get as our main result for the error of the **EX-GLM-FV** method

$$\|u_h^\varepsilon - u\|_{L^2(\mathbb{R}^d\times[0,T];\mathbb{R}^m)} = \mathcal{O}(h^{1/3}). \tag{9}$$

Moreover we establish that the involution condition is satisfied in the weak sense by our discretization, i.e.,

$$\lim_{h\to 0, \varepsilon = h^\alpha} \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^d M_i u_h^\varepsilon \cdot \frac{\partial \omega}{\partial x_i}\, dxdt = 0 \text{ for all } \omega \in C_0^\infty([0,T)\times\mathbb{R}^d;\mathbb{R}^m). \tag{10}$$

The scaling number α was chosen up to now by heuristic arguments in practical computations. The mentioned minimality problems shows that $\alpha = 2/3$ is optimal. Up to our knowledge convergence statements as (9) and (10) have not been derived for any of the existing methods to handle involutionary systems ([1, 2, 4, 14, 21]).
The assumptions, definitions, general results on Friedrichs systems and some notation are summarized in Section 2, while Section 3 is devoted to the numerical scheme. Section 4 contains the analysis of the scheme and in particular the proofs of the main convergence

theorems (Theorem 4.5, Corollaries 4.8 and 4.9). In the last section we present applications and numerical examples.
Finally we comment on related work for nonlinear systems of hyperbolic balance laws. The approach of Munz et al. has been transferred to the system of compressible magnetohydrodynamics in [7]. Discontinuous-Galerkin methods with locally divergent-free ansatz functions have been introduced in [18] and studied for the MHD equations in [2]. Even much earlier Powell suggested an (non-relaxation) extension of the magnetohydrodynamical system [22], see also [8]. A general approach can be found in [24], which has been applied to nonlinear systems in [10, 26].

2. Preliminaries

For $d, m \in \mathbb{N}$ we denote by $L^2(\mathbb{R}^d;\mathbb{R}^m)$, $H^1(\mathbb{R}^d;\mathbb{R}^m)$ the usual Lebesgue and Sobolev spaces equipped with the norms $\|\cdot\|_{L^2(\mathbb{R}^d;\mathbb{R}^m)}$, $\|\cdot\|_{H^1(\mathbb{R}^d;\mathbb{R}^m)}$, respectively [9]. $C_b^{0,1}(\mathbb{R}^d \times [0,T])$ is the set of bounded, Lipschitz continuous functions on $\mathbb{R}^d \times [0,T]$. Furthermore, for $l \in \mathbb{N}$ we need the Bochner spaces $C^l([0,T];X)$ and $L^2(0,T;X)$ where X is an arbitrary function space. The corresponding norms are denoted by $\|\cdot\|_{C^l([0,T];X)}$ and $\|\cdot\|_{L^2(0,T;X)}$. For $M : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{m\times m}$ we define

$$\|M\| = \sup_{(x,t)\in\mathbb{R}^d\times[0,T]} \|M(x,t)\|_2,$$

where $\|\cdot\|_2$ denotes the spectral norm. By $\mathcal{C} > 0$ we denote a generic constant (that can change from a line to the next!) independent on h and ε.

Definition 2.1. *We say $u \in L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^m))$ is called a **weak solution** of* (1)-(3) *if*

$$\int_{\mathbb{R}^d}\int_0^T \left(u\cdot\frac{\partial\phi}{\partial t} + \sum_{i=1}^d G^i(x,t)u\cdot\ \frac{\partial\phi}{\partial x_i} - D(x,t)u\cdot\phi + f\cdot\phi \right) dxdt$$
$$= \int_{\mathbb{R}^d} u_0(x)\cdot\phi(x,0)\,dx,$$
$$\int_{\mathbb{R}^d}\int_0^T \sum_{i=1}^d M_i u(x,t)\cdot\frac{\partial\omega(x,t)}{\partial x_i}\,dxdt = 0,$$

holds for all ϕ, $\omega \in C_0^\infty(\mathbb{R}^d\times[0,T);\mathbb{R}^m)$.

We specify all assumptions on the coefficients in Assumption 1 below. We note that in particular the regularity statement (i) can be relaxed, however, it does not lead to a better result in terms of the order of convergence.

Assumption 1. *Consider the initial value problem* (1)-(3).

(i) *The mappings* $D, G^1, \ldots, G^d : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{m\times m}$ *satisfy*

$$D, G^1, \ldots, G^d \in C^\infty(\mathbb{R}^d \times [0,T], \mathbb{R}^{m\times m}),$$
$$G^i(x,t)^T = G^i(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times [0,T]\,(i=1,\ldots,d),$$
$$\sum_{i=1}^d \left(\|\partial_t^j \partial_x^\alpha G^i\| + \|\partial_t^j \partial_x^\alpha D\|\right) < +\infty \qquad \forall \alpha \in \mathbb{N}_0^d, j \in \mathbb{N}_0.$$

(ii) *The functions* u_0 *and* f *satisfy* $u_0 \in H^1(\mathbb{R}^d;\mathbb{R}^m)$ *and* $f \in L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^m))$.

(iii) *The* $(m\times m)$*-matrices* M_i *are constant for* $i = 1,\ldots,d$.

We proceed with the presentation of the extended GLM formulation (4)-(6).

Definition 2.2. *For* $\varepsilon > 0$ *the function* $(u^\varepsilon, \psi^\varepsilon)^T \in L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))$ *is called a* ***weak solution*** *of the extended problem* (4)-(6) *if*

$$\int_{\mathbb{R}^d}\int_0^T \left(u^\varepsilon \cdot \frac{\partial \phi}{\partial t} + \sum_{i=1}^d \left(G^i(x,t)u^\varepsilon + M_i^T \psi^\varepsilon\right) \cdot \frac{\partial \phi}{\partial x_i} - D(x,t)u^\varepsilon \cdot \phi + f \cdot \phi\right) dxdt$$
$$= \int_{\mathbb{R}^d} u_0(x) \cdot \phi(x,0)\, dx,$$
$$\int_{\mathbb{R}^d}\int_0^T \left(\psi^\varepsilon \cdot \frac{\partial \omega}{\partial t} + \sum_{i=1}^d \frac{M_i}{\varepsilon} u^\varepsilon \cdot \frac{\partial \omega}{\partial x_i} - a\psi^\varepsilon \cdot \omega\right) dxdt = \int_{\mathbb{R}^d} \psi_0^\varepsilon(x) \cdot \omega(x,0)\, dx,$$

holds for all $\phi, \omega \in C_0^\infty(\mathbb{R}^d \times [0,T);\mathbb{R}^m)$.

This approach generalies the idea of Munz et al.[21] to arbitrary Friedrichs systems. The small parameter ε has to be identified with the ratio $\mathcal{C}_h/\mathcal{C}_p$ in [7]. Note that (at least formally) we recover the original formulation (1)-(3) by letting $\varepsilon \to 0$ in (4)-(6). Regarding to the system (4)-(6) we need an additional assumption for the variable ψ^ε.

Assumption 2. *For* $\varepsilon > 0$ *consider the initial value problem* (4)-(6).

(i) *Assumption 1 holds.*

(ii) *The function* ψ_0^ε *belongs to* $H^1(\mathbb{R}^d;\mathbb{R}^m)$.

To analyze (1)-(3) one can use pseudo-differential calculus. We have the following well-posedness result [[23], Chapter 2].

Theorem 2.3. *Suppose that Assumption 1 holds. Then there exists a unique weak solution* u *of* (1)-(3) *and we have* $u \in C([0,T];H^1(\mathbb{R}^d;\mathbb{R}^m))$. *In addition there exists a constant* $\mathcal{C} > 0$ *such that we have for* $t \in [0,T]$ *the estimate*

$$\|u(\cdot,t)\|_{H^1(\mathbb{R}^d;\mathbb{R}^m)} + \|u_t(\cdot,t)\|_{L^2(\mathbb{R}^d;\mathbb{R}^m)} \le \mathcal{C}\left(\|u_0\|_{H^1(\mathbb{R}^d;\mathbb{R}^m)} + \int_0^t \|f(\cdot,s)\|_{H^1(\mathbb{R}^d;\mathbb{R}^m)} ds\right).$$

Moreover, if $u_0 \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^m)$ *and* $f \in C_0^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^m)$ *then* u *is a classical solution and belongs to the space* $C_0^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^m)$*.*

The extended GLM formulation (4)-(6) is not symmetric. Therefore, we consider the change of variables $\varphi^\varepsilon := \psi^\varepsilon\sqrt{\varepsilon}$. In a blockmatrix structure (4)-(6) read

$$\frac{\partial}{\partial t}U^\varepsilon + \sum_{i=1}^d \frac{\partial}{\partial x_i}(A^{\varepsilon,i}U^\varepsilon) + BU^\varepsilon = F, \tag{11}$$

$$U^\varepsilon(x,0) = \begin{pmatrix} u_0^\varepsilon(x) \\ \psi_0^\varepsilon(x)\sqrt{\varepsilon} \end{pmatrix}, \tag{12}$$

with

$$U^\varepsilon := \begin{pmatrix} u^\varepsilon \\ \varphi^\varepsilon \end{pmatrix}; \quad A^{\varepsilon,i} := \begin{pmatrix} G^i & \frac{M_i^T}{\sqrt{\varepsilon}} \\ \frac{M_i}{\sqrt{\varepsilon}} & 0 \end{pmatrix}; \quad B := \begin{pmatrix} D & 0 \\ 0 & aI \end{pmatrix}; \quad F := \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

In particular it is clear that we are again in the framework of symmetric systems and the extended formulation leads to a hyperbolic system.

Remark 2.4. *We note that from Assumption 2 we have* $\|A^{\varepsilon,i}\| = \mathcal{O}(1/\sqrt{\varepsilon})$, $\|B\| = \mathcal{O}(1)$ *and* $\|\mathrm{div}A\| = \mathcal{O}(1)$ *with* $\mathrm{div}A := \sum_{i=1}^d (A^{\varepsilon,i})_{x_i}$*.*

We want a generic estimate (like in Theorem 2.3) for the system (11)-(12), but it is not clear how the constant will depend on ε. However, an analogous estimate holds, i.e, we have the following result.

Lemma 2.5. *Let Assumption 2 be satisfied. There exists a constant* $\mathcal{C} > 0$*, independent of* ε*, such that for* $t \in [0,T]$ *we have*

$$\begin{aligned} &\|u^\varepsilon(\cdot,t)\|^2_{H^1(\mathbb{R}^d;\mathbb{R}^m)} + \|\varphi^\varepsilon(\cdot,t)\|^2_{H^1(\mathbb{R}^d;\mathbb{R}^m)} + \varepsilon\|u_t^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^m)} + \varepsilon\|\varphi_t^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^m)} \\ &\le \mathcal{C}\left(\|u_0\|^2_{H^1(\mathbb{R}^d;\mathbb{R}^m)} + \|\varphi_0^\varepsilon\|^2_{H^1(\mathbb{R}^d;\mathbb{R}^m)} + \int_0^t \|f(\cdot,s)\|^2_{H^1(\mathbb{R}^d;\mathbb{R}^m)}ds\right). \end{aligned}$$

Proof. We compute the following energy estimates

$$\begin{aligned} \frac{d}{dt}\left(\int_{\mathbb{R}^d}\frac{|u^\varepsilon|^2}{2}dx\right) &+ \int_{\mathbb{R}^d}\left(\sum_{i=1}^d u^\varepsilon\cdot G^i\frac{\partial u^\varepsilon}{\partial x_i} + u^\varepsilon\cdot\frac{M_i^T}{\sqrt{\varepsilon}}\frac{\partial\varphi^\varepsilon}{\partial x_i}\right)dx \\ &+ \int_{\mathbb{R}^d}\left(u^\varepsilon\cdot(D+\mathrm{div}G)u^\varepsilon - u^\varepsilon\cdot f\right)dx = 0, \\ \frac{d}{dt}\left(\int_{\mathbb{R}^d}\frac{|\varphi^\varepsilon|^2}{2}\right) &+ \int_{\mathbb{R}^d}\left(\sum_{i=1}^d \varphi^\varepsilon\cdot\frac{M_i}{\sqrt{\varepsilon}}\frac{\partial u^\varepsilon}{\partial x_i} + a|\varphi^\varepsilon|^2\right)dx = 0. \end{aligned}$$

We see that thanks to Assumption 2 (symmetry of G^i) we have

$$\frac{\partial}{\partial x_i}\left((G^iu^\varepsilon)^Tu^\varepsilon\right) = 2(u^\varepsilon)^TG^i\frac{\partial u^\varepsilon}{\partial x_i} + (u^\varepsilon)^T\frac{\partial G^i}{\partial x_i}u^\varepsilon.$$

Adding the above energy estimates and using the last expression we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{|u^\varepsilon|^2}{2}+\frac{|\varphi^\varepsilon|^2}{2}\right)dx+\int_{\mathbb{R}^d}\left(u^\varepsilon\cdot(D+\frac{1}{2}\mathrm{div}G)u^\varepsilon+a|\varphi^\varepsilon|^2\right)dx=\int_{\mathbb{R}^d}f\cdot u^\epsilon\,dx.$$

Using Assumption 2 (properties of D, G) and the Cauchy-Schwarz inequality we find

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{|u^\varepsilon|^2}{2}+\frac{|\varphi^\varepsilon|^2}{2}\right)dx\le\mathcal{C}(\|D\|,\|\mathrm{div}G\|)\int_{\mathbb{R}^d}\left(|u^\varepsilon|^2+|\varphi^\varepsilon|^2\right)dx+\int_{\mathbb{R}^d}|f|^2dx$$

Applying Gronwall inequality we get finally

$$\int_{\mathbb{R}^d}\left(\frac{|u^\varepsilon|^2}{2}+\frac{|\varphi^\varepsilon|^2}{2}\right)dx\le e^{\mathcal{C}t}\left(\int_{\mathbb{R}^d}\left(|u_0|^2+|\varphi_0^\varepsilon|^2\right)dx+\int_0^t\int_{\mathbb{R}^d}|f|^2dxdt\right)$$

Reasoning as above we find analogous estimates for $\|U^\varepsilon_{x_i}\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}$ with $i=1,\dots,d$. In order to get estimates for $\|u^\varepsilon_t\|$ and $\|\varphi^\varepsilon_t\|$ we use (11), the bound for $\|U^\varepsilon\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}$ and $\|U^\varepsilon_{x_i}\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}$ and Remark 2.4. □

With Lemma 2.5 we can conclude as in Theorem 2.3 that the following theorem holds.

Theorem 2.6. *Suppose that Assumption 2 holds. Then there exists a unique weak solution U^ε of* (11),(12) *with $U^\varepsilon\in C([0,T];H^1(\mathbb{R}^d;\mathbb{R}^{2m}))$. In addition there exists a constant $\mathcal{C}>0$ independent of ε such that*

$$\begin{aligned}&\|u^\varepsilon(\cdot,t)\|_{H^1(\mathbb{R}^d;\mathbb{R}^m)}+\|\varphi^\varepsilon(\cdot,t)\|_{H^1(\mathbb{R}^d;\mathbb{R}^m)}+\sqrt{\varepsilon}\|u^\varepsilon_t(\cdot,t)\|_{L^2(\mathbb{R}^d;\mathbb{R}^m)}+\sqrt{\varepsilon}\|\varphi^\varepsilon_t(\cdot,t)\|_{L^2(\mathbb{R}^d;\mathbb{R}^m)}\\&\quad\le\mathcal{C}\left(\|u_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^m)}+\|\varphi_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^m)}+\int_0^t\|f(\cdot,s)\|_{H^1(\mathbb{R}^d;\mathbb{R}^m)}\,ds\right).\end{aligned}$$

Moreover, if $U_0^\varepsilon\in C_0^\infty(\mathbb{R}^d;\mathbb{R}^{2m})$ and $F\in C_0^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})$ then U^ε is a classical solution and lies in the space $C_0^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})$.

Now we are in a position to estimate the error $\|u^\varepsilon-u\|$. The corresponding result in the special case of electrodynamics can be found in [20].

Proposition 2.7. *Let u be the weak solution of* (1)-(3) *and $U^\varepsilon=(u^\varepsilon,\varphi^\varepsilon)^T$ the weak solution of* (11)-(12)*. Under Assumption 2, we have for all $t\in[0,T]$*

$$\begin{aligned}\|u^\varepsilon(\cdot,t)-u(\cdot,t)\|_{H^1(\mathbb{R}^n;\mathbb{R}^m)}&\le\mathcal{C}\sqrt{\varepsilon},\\ \|\psi^\varepsilon(\cdot,t)\|_{H^1(\mathbb{R}^n;\mathbb{R}^m)}&\le\mathcal{C}\|\psi_0^\varepsilon\|_{H^1(\mathbb{R}^n;\mathbb{R}^m)}.\end{aligned}$$

The constant $\mathcal{C}$ depends on data but not on ε.

Proof. Defining $\overline{u} := u^\varepsilon - u$, a direct computation yields

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d}\frac{|\overline{u}|^2}{2}dx\right) + \int_{\mathbb{R}^d}\left(\sum_{i=1}^d \overline{u}\cdot G^i\frac{\partial\overline{u}}{\partial x_i} + \overline{u}\cdot\frac{M_i^T}{\sqrt{\varepsilon}}\frac{\partial\varphi^\varepsilon}{\partial x_i}\right)dx$$
$$+\int_{\mathbb{R}^d}(\overline{u}\cdot(D+\text{div}G)\overline{u})\,dx = 0,$$
$$\frac{d}{dt}\left(\int_{\mathbb{R}^d}\frac{|\varphi^\varepsilon|^2}{2}dx\right) + \int_{\mathbb{R}^d}\left(\sum_{i=1}^d \varphi^\varepsilon\frac{M_i}{\sqrt{\varepsilon}}\frac{\partial\overline{u}}{\partial x_i} + a|\varphi^\varepsilon|^2\right)dx = 0$$

Adding the last two equations and using Assumption 2 (symmetry of G^i) we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{|\overline{u}|^2}{2}+\frac{|\varphi^\varepsilon|^2}{2}\right)dx + \int_{\mathbb{R}^d}\left(\overline{u}^T(D+\frac{1}{2}\text{div}G)\overline{u} + a|\varphi^\varepsilon|^2\right)dx = 0,$$

which implies

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{|\overline{u}|^2}{2}+\frac{|\varphi^\varepsilon|^2}{2}\right)dx \leq \mathcal{C}(\|D\|,\|\text{div}G\|)\int_{\mathbb{R}^d}|\overline{u}|^2dx.$$

Applying Gronwall inequality and noting that $\|\overline{u}_0\|_{L^2(\mathbb{R}^d;\mathbb{R}^m)} = 0$ we finally obtain

$$\int_{\mathbb{R}^d}\left(|\overline{u}(\cdot,t)|^2 + |\varphi^\varepsilon(\cdot,t)|^2\right)dx \leq \mathcal{C}\int_{\mathbb{R}^d}|\varphi_0^\varepsilon|^2dx.$$

Here we recall from (11) that $\varphi^\varepsilon = \psi^\varepsilon\sqrt{\varepsilon}$ and from Assumption 2 that $\psi_0^\varepsilon \in H^1(\mathbb{R}^d;\mathbb{R}^m)$ so we get

$$\begin{aligned}\|\overline{u}(\cdot,t)\|_{L^2(\mathbb{R}^n;\mathbb{R}^m)} &\leq \mathcal{C}\sqrt{\varepsilon},\\ \|\varphi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^n;\mathbb{R}^m)} &\leq \mathcal{C}\|\varphi_0^\varepsilon\|_{L^2(\mathbb{R}^n;\mathbb{R}^m)}.\end{aligned}$$

In a similar way we find estimates for $\|\overline{u}_{x_i}(t)\|_{L^2(\mathbb{R}^d;\mathbb{R}^m)}$, $i = 1,\ldots,d$. □

Proposition 2.7 shows that solutions of the extended formulation (11)-(12) approximate the solution of the original problem (1)-(3) as $\varepsilon \to 0$. However it is not clear how ε must be chosen (asymptotically) in computations and whether the constraint (3) is satisfied in the limit.

3. Finite-Volume Discretization

We approximate the solution of (11)-(12) by a Finite-Volume scheme on unstructured meshes. This construction follows [15, 25]. We begin with some standard generalities on Finite-Volume schemes.

Definition 3.1. *For some index set $I \subset \mathbb{N}$ let a family $\{K^i\}_{i\in I}$ of open non-empty sets be given. This family is called a triangulation if each element is a convex polyhedron, $\cup_{i\in I}K^i = \mathbb{R}^d$, and*

$$K^i \cap K^j = \qquad \forall i,j \in I \;\; i \neq j, \qquad h := \sup_{i\in I}\left\{\text{diam}(\text{K}^\text{i})\right\} < \infty.$$

We denote the family $\{K^i\}_{i\in I}$ by $\mathcal{T}_h$ and introduce the following notations for $K \in \mathcal{T}_h$

$$\begin{aligned} |K| &: \text{ area of } K,\\ e \in \partial K &: \text{ an edge of } K \text{ with length } |e|,\\ n_{e,K} = (n_{e,K}^1, \dots, n_{e,K}^d)^T &: \text{ unit outward normal to the edge } e \text{ of } K,\\ K_e &: \text{ neighboring cell of } K \text{ with } \overline{K} \cap \overline{K}_e = e. \end{aligned}$$

For $N \in \mathbb{N}$, let $0 = t^1 < t^2 \dots < t^N = T$ be a partition of the interval $[0, T]$. We denote $\Delta t^n = t^{n+1} - t^n$ for $n \in \mathcal{N} \cup \{N\}$, $\mathcal{N} = \{0, \dots, N-1\}$.
For each $n \in \{0, \dots, N\}$, $K \in \mathcal{T}_h$, and $e \in \partial K$ we define for $S : \mathbb{R}^d \to \mathbb{R}^{2m}$

$$S_K^n := \frac{1}{\Delta t^n |K|} \int_{t^n}^{t^{n+1}} \int_K S(x,t)dxdt, \qquad S_e^n := \frac{1}{\Delta t^n |e|} \int_{t^n}^{t^{n+1}} \int_e S(\zeta,t)d\zeta dt,$$
$$S_K(t) := \frac{1}{|K|} \int_K S(x,t)dx, \qquad S_e(t) := \frac{1}{|e|} \int_e S(\zeta,t)\, d\zeta.$$

For the sake of clarity we summarize all the assumptions on the mesh.

Assumption 3. *Let $\mathcal{T}_h$ be a triangulation (Def. 3.1) of $\mathbb{R}^d$. There exist constants $\eta > 0$ and $\nu > 0$ such that*

$$|K| \geq \eta h^d, \quad |\partial K| \leq \nu h^{d-1} \qquad (\forall K \in \mathcal{T}_h, \forall e \in \mathcal{E}(K)).$$

Moreover we assume that the time step Δt is constant, i.e., $\Delta t^n = \Delta t$.

Definition 3.2 (EX-GLM-FVM). *The* ***EX-GLM-FV approximation*** $U_h^\varepsilon := (u_h^\varepsilon, \varphi_h^\varepsilon)^T : \mathbb{R}^d \times [0,T) \to \mathbb{R}^{2m}$ *of* (1)-(3) *with initial data $U_0^\varepsilon = (u_0^\varepsilon, \varphi_0^\varepsilon)^T$ is given by*

$$U_h^\varepsilon(x,t) = V_k^{\varepsilon,n} \qquad \textit{for } (x,t) \in K \times [t^n, t^{n+1}).$$

The vectors $V_K^{\varepsilon,n} \in \mathbb{R}^{2m}$ are given for $n = 0$ and $K \in \mathcal{T}_h$ by

$$V_K^{\varepsilon,0} = \frac{1}{|K|} \int_K U_0^\varepsilon(x)dx,$$

and iteratively for $n \in \mathcal{N}$ by

$$V_K^{\varepsilon,n+1} = V_K^{\varepsilon,n} - \frac{\Delta t}{|K|} \sum_{e \in \partial K} |e| g_{e,K}^n(V_K^{\varepsilon,n}, V_{K_e}^{\varepsilon,n}) - \Delta t B_K^n V_K^{\varepsilon,n} + \Delta t F_K^n. \tag{13}$$

The numerical flux $g_{e,K}^n : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is defined for $K \in \mathcal{T}_h$ and $e \in \partial K$ by

$$g_{e,K}^n(U,V) = -C_{e,K}^{\varepsilon,n} V + D_{e,K}^{\varepsilon,n} U \qquad (U, V \in \mathbb{R}^m) \tag{14}$$

with

$$A_{e,K}^{\varepsilon,n} := \sum_{i=1}^d n_{e,K}^i (A^{\varepsilon,i})_e^n, \;\; C_{e,K}^{\varepsilon,n} := -O^T \Lambda^- O, \;\; D_{e,K}^{\varepsilon,n} := O^T \Lambda^+ O, \tag{15}$$

and

$$A^{\varepsilon,n}_{e,K} = O^T\Lambda^+O + O^T\Lambda^-O, \tag{16}$$

where $\Lambda^+(\Lambda^-)$ *is a diagonal matrix which entries are the positive (negative) eigenvalues of* $A^{\varepsilon,n}_{e,K}$.

As long as ε is fixed in Definition 3.2 the **EX-GLM-FV** method gives an approximate for the weak solution of (11)-(12). However, we will choose $\varepsilon = \varepsilon(h)$ with $\varepsilon(h) \to 0$ as $h \to 0$, so that the **EX-GLM-FV** method is supposed to approximate the weak solution of (1)-(3). This is the problem to solve. The crucial question here is how to determine $\varepsilon(h)$ to get (an optimal order of) convergence.

Remark 3.3. (i) *Note that thanks to the hyperbolicity of the formulation the decomposition* (16) *makes sense.*

(ii) *The Definition 3.2 leads to a consistent upwind numerical scheme. Note that we have the symmetric relation* $C^{\varepsilon,n}_{e,K} = D^{\varepsilon,n}_{e,K_e}$. *This leads to* $g^n_{e,K}(U,V) = -g^n_{e,K_e}(V,U)$ *for* $U,V \in \mathbb{R}^{2m}$ *and ensures that the scheme is conservative.*

(iii) *Using (ii) the iteration* (13) *can be written as*

$$V^{\varepsilon,n+1}_K = V^{\varepsilon,n}_K - \frac{\Delta t}{|K|}\sum_{e\in\partial K}|e|C^{\varepsilon,n}_{e,K}(V^n_K - V^{\varepsilon,n}_{K_e}) - \Delta t\big((B^n_K + \operatorname{div}A^n_K)V^{\varepsilon,n}_K - F^n_K\big). \tag{17}$$

(iv) *We note that since* $\|C^{\varepsilon,n}_{e,K}\|$ *is a function of* $A^{\varepsilon,i}$ *we have that* $\mathcal{O}(\|C^{\varepsilon,n}_{e,K}\|) = 1/\sqrt{\varepsilon}$.

4. Convergence of the GLM-FV scheme

Our main goal in this section is to show that the component u^ε_h of U^ε_h given by Definition 3.2 converges to the solution u of (1)-(3) as $h, \varepsilon \to 0$. To do this we carefully track the parameter ε in the constants that appear in the finite volume error analysis in [25]. We assume throughout this section that Assumptions 1-3 hold.

4.1. **Stability results.** We start with a result that can be seen as a local stability lemma.

Lemma 4.1. *Under the CFL condition*

$$\sup_{K\in\mathcal{T}_h, e\in\partial K}\frac{\Delta t|\partial K|\|C^{\varepsilon,n}_{e,K}\|}{|K|} < 1-\delta, \qquad \delta\in(0,1) \tag{18}$$

the solution U^ε_h *generated by the* ***EX-GLM-FVM*** *satisfies*

$$\begin{aligned}
&|V^{\varepsilon,n+1}_K|^2 - |V^{\varepsilon,n}_K|^2 + 2\Delta t(V^{\varepsilon,n+1}_K)^T[(B^n_K + \operatorname{div}A^n_K)V^{\varepsilon,n}_K - F^n_K]\\
&+\frac{\Delta t}{|K|}\sum_{e\in\partial K}|e|\big((V^{\varepsilon,n}_K)^T C^{\varepsilon,n}_{e,K}V^{\varepsilon,n}_K - (V^{\varepsilon,n}_{K_e})^T C^{\varepsilon,n}_{e,K}V^{\varepsilon,n}_{K_e}\big)\\
&\le -\delta\frac{\Delta t}{|K|}\sum_{e\in\partial K}(V^{\varepsilon,n}_{K_e} - V^{\varepsilon,n}_K)^T C^{\varepsilon,n}_{e,K}(V^{\varepsilon,n}_{K_e} - V^{\varepsilon,n}_K)|e|.
\end{aligned} \tag{19}$$

Proof. We represent $V_K^{\varepsilon,n+1}$ as the convex decomposition

$$V_K^{\varepsilon,n+1} = \sum_{e\in\partial K} \frac{|e|}{|\partial K|} V_{e,K}^{\varepsilon,n+1}. \tag{20}$$

Thereby we used (17) and

$$V_{e,K}^{\varepsilon,n+1} := V_K^{\varepsilon,n} - \frac{\Delta t|\partial K|}{|K|} C_{e,K}^{\varepsilon,n}(V_K^{\varepsilon,n} - V_{K_e}^{\varepsilon,n}) - \Delta t((B_K^n + \mathrm{div}A_K^n)V_K^{\varepsilon,n} - F_K^n).$$

We define also

$$W_{e,K}^{\varepsilon,n+1} := V_K^{\varepsilon,n} - \frac{\Delta t|\partial K|}{|K|} C_{e,K}^{\varepsilon,n}(V_K^{\varepsilon,n} - V_{K_e}^{\varepsilon,n}). \tag{21}$$

Scalar multiplication of $W_{e,K}^{\varepsilon,n+1}$ with $V_K^{\varepsilon,n}$ and the symmetry of $C_{e,K}^{\varepsilon,n}$ gives

$$\frac{1}{2}|W_{e,K}^{\varepsilon,n+1}|^2 - \frac{1}{2}|V_K^{\varepsilon,n}|^2 = -\frac{\Delta t|\partial K|}{2|K|}\big((V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n} V_K^{\varepsilon,n} - (V_{K_e}^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n} V_{K_e}^{\varepsilon,n}\big) + Q, \tag{22}$$

with

$$Q = \frac{1}{2}|W_{e,K}^{\varepsilon,n+1} - V_K^{\varepsilon,n}|^2 - \frac{\Delta t|\partial K|}{2|K|}\big((V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\big).$$

A straightforward calculation shows that

$$Q = -\frac{\Delta t|\partial K|}{2|K|}\left((V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}\left[I - \frac{\Delta t|\partial K|}{|K|} C_{e,K}^{\varepsilon,n}\right](V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\right).$$

Here I is the unitary matrix in $\mathbb{R}^{2m\times 2m}$. Using the CFL condition (18) we get

$$\begin{aligned}
Q &= -\frac{\Delta t|\partial K|}{2|K|}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n}) \\
&\quad + \frac{(\Delta t|\partial K|)^2}{2|K|^2}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})^T (C_{e,K}^{\varepsilon,n})^2(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n}) \\
&\le \frac{\Delta t|\partial K|}{2|K|}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n}) \\
&\quad + (1-\delta)\frac{\Delta t|\partial K|}{2|K|}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n}) \\
&= -\delta\frac{\Delta t|\partial K|}{2|K|}\big((V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\big).
\end{aligned} \tag{23}$$

From (20) and (21) we also see

$$V_K^{\varepsilon,n+1} = \sum_{e\in\partial K} \frac{|e|}{|\partial K|}\left(W_{e,K}^{\varepsilon,n+1} - \Delta t(B_K^n + \mathrm{div}A_K^n)V_K^{\varepsilon,n} + \Delta t F_K^n\right).$$

Scalar multiplication of the last expression with $V_K^{\varepsilon,n+1}$ yields

$$\begin{aligned}\frac{1}{2}|V_K^{\varepsilon,n+1}|^2 &= -\Delta t(V_K^{\varepsilon,n+1})^T(B_K^n+\mathrm{div}A_K^n)\cdot V_K^{\varepsilon,n}+\Delta t(V_K^{\varepsilon,n+1})^T F_K^n\\ &\quad+\frac{1}{2}\sum_{e\in\partial K}\frac{|e|}{|\partial K|}\left(|W_{e,K}^{\varepsilon,n+1}|^2-|W_{e,K}^{\varepsilon,n+1}-V_K^{\varepsilon,n+1}|^2\right)\\ &\le -\Delta t(V_K^{\varepsilon,n+1})^T(B_K^n+\mathrm{div}A_K^n)V_K^{\varepsilon,n}+\Delta t(V_K^{\varepsilon,n+1})^T F_K^n\\ &\quad+\frac{1}{2}\sum_{e\in\partial K}\frac{|e|}{|\partial K|}|W_{e,K}^{\varepsilon,n+1}|^2.\end{aligned}$$

Replacing the expression for $|W_{e,K}^{\varepsilon,n+1}|^2$ in (22) we finally obtain

$$\begin{aligned}\frac{1}{2}|V_K^{\varepsilon,n+1}|^2 &\le -\Delta t(V_K^{\varepsilon,n+1})^T(B_K^n+\mathrm{div}A_K^n)V_K^{\varepsilon,n}+\Delta t(V_K^{\varepsilon,n+1})^T F_K^n\\ &\quad-\frac{\Delta t}{2|K|}\sum_{e\in\partial K}|e|\left((V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}V_K^{\varepsilon,n}-(V_{K_e}^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}V_{K_e}^{\varepsilon,n}\right)\\ &\quad+\sum_{e\in\partial K}\frac{|e|}{|\partial K|}Q+\frac{1}{2}\sum_{e\in\partial K}\frac{|e|}{|\partial K|}|V_K^{\varepsilon,n}|^2\\ &\le \frac{1}{2}|V_K^{\varepsilon,n}|^2-\Delta t(V_K^{\varepsilon,n+1})^T(B_K^n+\mathrm{div}A_K^n)V_K^{\varepsilon,n}+\Delta t(V_K^{\varepsilon,n+1})^T F_K^n\\ &\quad-\frac{\Delta t}{2|K|}\sum_{e\in\partial K}|e|\left((V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}V_K^{\varepsilon,n}-(V_{K_e}^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}V_{K_e}^{\varepsilon,n}\right)\\ &\quad-\delta\frac{\Delta t}{2|K|}\sum_{e\in\partial K}|e|(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n}).\end{aligned}$$

For the last line we used (23). □

With Lemma 4.1 we can now prove the following global L^2-stability result as a discrete counterpart to Lemma 2.5.

Proposition 4.2. *Assume that the CFL condition* (18) *is satisfied for a given* $\delta\in(0,1)$. *Then, for* h *small enough, the* ***EX-GLM-FV*** *approximation* U_h^ε *satisfies for all* $0\le t\le T$

$$\|U_h^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}\le\mathcal{C}\left(\|U_0^\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;L^2(\mathbb{R}^d;\mathbb{R}^{2m}))}\right). \tag{24}$$

The constant $\mathcal{C}$ *depend on the data but not on* ε. *Moreover, the discrete space derivatives of* U_h^ε *satisfy the following weak estimate:*

$$\begin{aligned}&\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n})|e|\Delta t\\ &\le\frac{\mathcal{C}}{\delta}\left(\|U_0^\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;L^2(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2.\end{aligned} \tag{25}$$

The constant $\mathcal{C}>0$ *is independent of* ε, *depends only on the data of the problem.*

Proof. First, by adding the nil quantity

$$-\Delta t \sum_{e\in\partial K} V_K^{\varepsilon,n}\cdot A_{e,K}^{\varepsilon,n} V_K^{\varepsilon,n}|e| + |K|\Delta t V_K^{\varepsilon,n}\cdot \mathrm{div} A_K^n V_K^{\varepsilon,n}$$

to the R.H.S of (19), we obtain the following form of the local energy estimate:

$$\begin{aligned}\frac{|K||V_K^{\varepsilon,n+1}|^2}{2} \leq\; & \frac{|K||V_K^{\varepsilon,n}|^2}{2} - \Delta t|K|V_K^{\varepsilon,n+1}\cdot\left(\left(B_K^n+\mathrm{div}A_K^n\right)V_K^{\varepsilon,n}-F_K^n\right)\\ & -\frac{\Delta t}{2}\sum_{e\in\partial K}|e|\left(V_K^{\varepsilon,n}\cdot D_{e,K}^n V_K^{\varepsilon,n} - V_{K_e}^{\varepsilon,n}\cdot C_{e,K}^{\varepsilon,n}V_{K_e}^{\varepsilon,n}\right)\\ & +\frac{\Delta t|K|}{2}V_K^{\varepsilon,n}\cdot\mathrm{div}A_K^n V_K^{\varepsilon,n}\\ & -\frac{\delta}{2}\Delta t\sum_{e\in\partial K}|e|(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n})\cdot C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n}).\end{aligned}$$

Let $\beta := \|B\|$, $\kappa := \|\mathrm{div}A\|$. Summing the last inequality over all the elements of the triangulation, we obtain

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}|K||V_K^{\varepsilon,n+1}|^2 + \delta\Delta t\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}|e|(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n})\cdot C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n})\\ &\quad\leq \sum_{K\in\mathcal{T}_h}|K|\left(|V_K^{\varepsilon,n}|^2+\Delta t V_K^{\varepsilon,n}\cdot\mathrm{div}A_K^n V_K^{\varepsilon,n}\right)\\ &\qquad -2\Delta t\sum_{K\in\mathcal{T}_h}|K|V_K^{\varepsilon,n+1}\cdot\left(\left(B_K^n+\mathrm{div}A_K^n\right)V_K^{\varepsilon,n}-F_K^n\right)\\ &\quad\leq (1+\kappa\Delta t)\|U_h^\varepsilon(t^n,\cdot)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}\\ &\qquad +2\Delta t(\beta+\kappa)\|U_h^\varepsilon(t^{n+1},\cdot)\|_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}\|U_h^\varepsilon(t^n,\cdot)\|_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}\\ &\qquad +2\Delta t\|U_h^\varepsilon(t^{n+1},\cdot)\|_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}\left(\sum_{K\in\mathcal{T}_h}|K||F_K^n|^2\right)^{1/2}.\end{aligned}$$

Thus setting $\gamma := 1+\kappa+\beta$ and using the inequality $2ab \leq a^2+b^2$, we get:

$$\begin{aligned}&\|U_h^\varepsilon(t^{n+1},\cdot)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}(1-\Delta t\gamma) + \delta\Delta t\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}|e|(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n})\cdot C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n}-V_K^{\varepsilon,n})\\ &\quad\leq \left(1+\Delta t(2\kappa+\beta)\right)\|U_h^\varepsilon(t^n,\cdot)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})} + \Delta t\sum_{K\in\mathcal{T}_h}|K||F_K^n|^2.\end{aligned} \tag{26}$$

Thanks to the CFL-condition (18), we have, for h small enough, $\gamma\Delta t \leq 1/2$ (we remark that γ is independent of ε). Hence:

$$1 \leq (1-\Delta t\gamma)^{-1} \leq 1 + 2\gamma\Delta t.$$

We deduce from (26) that:

$$
\begin{aligned}
&\|U_h^\varepsilon(t^{n+1},\cdot)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})} + \delta\Delta t \sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K} |e|(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\cdot C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\\
&\quad\le (1+2\gamma\Delta t)^2\|U_h^\varepsilon(t^n,\cdot)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})} + \Delta t(1+2\gamma\Delta t)\sum_{K\in\mathcal{T}_h}|K||F_K^n|^2\\
&\quad\le (1+6\gamma\Delta t)\|U_h^\varepsilon(t^n,\cdot)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})} + 2\Delta t\sum_{K\in\mathcal{T}_h}|K||F_K^n|^2
\end{aligned}
$$

Iterating the last inequality, we get for any $t\in[0,T)$:

$$
\begin{aligned}
&\|U_h^\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})} + \gamma\sum_{n=0}^{N}\sum_{K\in\mathcal{T}_h}\Delta t(1+\mathcal{C}\Delta t)^{N-n}\sum_{e\in\partial K}|e|(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\cdot C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\\
&\quad\le\left(1+\mathcal{C}\frac{T}{N}\right)^N\|U_0^\varepsilon\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})} + 2\sum_{n=0}^{N}\Delta t(1+\mathcal{C}\Delta t)^{N-n}\sum_{K\in\mathcal{T}_h}|K||F_K^n|^2,
\end{aligned}
$$

with N being the integer part of $T/\Delta t$ and $\mathcal{C} := 6\gamma$. We finally obtain that $\forall t\in[0,T)$:

$$
\begin{aligned}
&\|U_h^\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})} + \gamma\sum_{n=0}^{N}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}\Delta t|e|(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\cdot C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\\
&\quad\le\exp(\mathcal{C}T)\left(\|U_0^\varepsilon\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})} + 2\|F\|^2_{L^2(0,T;L^2(\mathbb{R}^d;\mathbb{R}^{2m}))}\right).
\end{aligned}
$$

□

4.2. **A Comparison Result.** In the next step we consider the difference between the exact solution U^ε of (11)-(12) and some function $V\in L^2(\mathbb{R}^d\times(0,T);\mathbb{R}^{2m})$ in terms of residual errors. To achieve the main result we will put $V = U_h^\varepsilon$. Following [16] we introduce the two useful measures.

Definition 4.3. *Let $V\in L^2(\mathbb{R}^d\times(0,T);\mathbb{R}^{2m})$ be given. The* ***weak consistency measure***

$$\mu_V : C^1([0,T];L^2(\mathbb{R}^d;\mathbb{R}^{2m}))\cap C([0,T];H^1(\mathbb{R}^d;\mathbb{R}^{2m}))\to\mathbb{R}$$

and the ***dissipation measure***

$$\nu_V : C_b^{0,1}(\mathbb{R}^d\times[0,T])\to\mathbb{R}$$

are defined for $\pi \in C^1([0,T];L^2(\mathbb{R}^d;\mathbb{R}^{2m})) \cap C([0,T];H^1(\mathbb{R}^d;\mathbb{R}^{2m}))$ *and* $\omega \in C_b^{0,1}(\mathbb{R}^d \times [0,T])$ *by*

$$\begin{aligned} < \mu_V, \pi > \;&:=\; -\int_0^T \int_{\mathbb{R}^d} \left(V^T \partial_t \pi + \sum_{i=1}^d V^T A^{\varepsilon,i} \partial_i \pi \right) dxdt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} (V^T B^T - F^T)\pi \, dxdt - \int_{\mathbb{R}^d} (U_0^\varepsilon)^T \pi(x,0)\, dx, \\ < \nu_V, \omega > \;&:=\; -\int_0^T \int_{\mathbb{R}^d} \left(|V|^2 \partial_t \omega + \sum_{i=1}^d V^T A^{\varepsilon,i} V \partial_i \omega \right) dxdt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \left(V^T(\mathrm{div} A + B + B^T)V - 2F^T V \right) \omega \, dxdt \\ &\quad - \int_{\mathbb{R}^d} |U_0^\varepsilon|^2 \omega(x,0)\, dx. \end{aligned}$$

Proposition 4.4. *Let* $U_0^\varepsilon \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^{2m})$, $F \in C_0^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})$, $V \in L^2((0,T)\times \mathbb{R}^d;\mathbb{R}^{2m})$ *and define* $\alpha := \|B + B^T + \mathrm{div} A\|$. *Then we have*

$$\int_0^T \int_{\mathbb{R}^d} \exp(-\alpha t)|U^\varepsilon - V|^2 dxdt \;\le < \nu_V, \theta > -2 < \mu_V, \theta U^\varepsilon > \tag{27}$$

where U^ε *is the exact solution of* (11)-(12) *and* $\theta : [0,T] \to \mathbb{R}$ *is defined by* $\theta(t) = \exp(-\alpha t)(T-t)$ *with* $t \in [0,T]$.

Proof. : See Proposition 2.4 in [15]. □

4.3. **The Error Estimate.** In order to prove Theorem 4.5 we will apply the estimate of Proposition 4.4 to the approximate solution U_h^ε generated by the **EX-GLM-FV** method.

Theorem 4.5. *Under the* CFL*-condition* (18)*, the* ***EX-GLM-FV*** *approximation* U_h^ε *converges towards the solution* U^ε *of* (11)-(12) *in* $L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})$. *Moreover* U_h^ε *satisfies the following error estimate*

$$\|U^\varepsilon - U_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})} \le \mathcal{C} \left(\frac{h}{\sqrt{\varepsilon}} \right)^{1/2}. \tag{28}$$

In (28) $\mathcal{C}$ *is a positive constant that depends only on* δ, U_0, F *and* T *but not on* ε.

In order to prove Theorem 4.5 we cite a Proposition of [25] and a Lemma of [15] which are given below. We note that $U_h^\varepsilon \in L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})$ thanks to Proposition 4.2.

Lemma 4.6 (Proposition 5.1 in [25])**.** *If we choose* $V = U_h^\varepsilon$ *in Definition 4.3 we get*

$$< \mu_{U_h^\varepsilon}, \pi > = \sum_{l=1}^7 \mathcal{R}_h^l(\pi), \qquad < \nu_{U_h^\varepsilon}, \omega > \le \sum_{l=1}^7 \mathcal{E}_h^l(\omega) - \delta Q_h^\varepsilon(\omega)$$

where $\omega : \mathbb{R}^d \times [0,T] \to [0,\infty)$ *and* $\pi : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{2m}$ *are smooth functions with compact support in* x*. Here we used*

$$
\begin{aligned}
\mathcal{R}_h^1(\pi) &= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} |K|(V_K^{\varepsilon,n+1} - V_K^{\varepsilon,n})^T(\pi_K(t^{n+1}) - \pi_K^n),\\
\mathcal{R}_h^2(\pi) &= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K} \Delta t|e|(V_K^{\varepsilon,n} - V_{K_e}^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(\pi_e^n - \pi_K^n),\\
\mathcal{R}_h^3(\pi) &= \int_{\mathbb{R}^d} (U_h^\varepsilon(x,0) - U_0^\varepsilon(x))^T\pi(x,0)dx,\\
\mathcal{R}_h^4(\pi) &= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} (V_K^{\varepsilon,n})^T\left[\sum_{e\in\partial K}\Delta t|e|A_{e,K}^n\pi_e^n - \int_{t^n}^{t^{n+1}}\int_K\sum_{i=1}^d \frac{\partial}{\partial x_i}(A^{\varepsilon,i}\pi)dxdt\right],\\
\mathcal{R}_h^5(\pi) &= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} \Delta t|K|(V_K^{\varepsilon,n})^T\left[\frac{1}{\Delta t|K|}\int_{t^n}^{t^{n+1}}\int_K (\mathrm{div}A)\pi dtdx - (\mathrm{div}A)_K^n\pi_K^n\right],\\
\mathcal{R}_h^6(\pi) &= -\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} \Delta t|K|(V_K^{\varepsilon,n})^T\left[\frac{1}{\Delta t|K|}\int_{t^n}^{t^{n+1}}\int_K B^T\pi dtdx - (B_K^n)^T\pi_K^n\right],\\
\mathcal{R}_h^7(\pi) &= -\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} \Delta t|K|\left[\frac{1}{\Delta t|K|}\int_{t^n}^{t^{n+1}}\int_K F^T\pi dtdx - (F_K^n)^T\pi_K^n\right],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_h^1(\omega) &= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} |K|(|V_K^{\varepsilon,n+1}|^2 - |V_K^{\varepsilon,n}|^2)(\omega_K(t^{n+1}) - \omega_K^n),\\
\mathcal{E}_h^2(\omega) &= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K} \Delta t|e|(V_K^{\varepsilon,n} - V_{K_e}^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_K^{\varepsilon,n} + V_{K_e}^{\varepsilon,n})(\omega_e^n - \omega_K^n),\\
\mathcal{E}_h^3(\omega) &= \int_{\mathbb{R}^d} (|U_h^\varepsilon(x,0)|^2 - |U_0^\varepsilon(x)|^2)\omega(x,0)dx,\\
\mathcal{E}_h^4(\omega) &= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} (V_K^{\varepsilon,n})^T\left[\sum_{e\in\partial K}\Delta t|e|A_{e,K}^n\omega_e^n - \int_{t^n}^{t^{n+1}}\int_K\sum_{i=1}^d \frac{\partial}{\partial x_i}(A^{\varepsilon,i}\omega)dxdt\right]V_K^{\varepsilon,n},\\
\mathcal{E}_h^5(\omega) &= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} (V_K^{\varepsilon,n})^T\Big[\int_{t^n}^{t^n}\int_K \big(B + B^T + 2(\mathrm{div}A)\big)\,\omega\,dxdt\\
&\quad -|K|\Delta t(B_K^n + (B_K^n)^T + 2(\mathrm{div}A)_K^n)\omega_K^n\Big]\,V_K^{\varepsilon,n},\\
\mathcal{E}_h^6(\omega) &= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} 2\Delta t|K|(V_K^{\varepsilon,n})^T\left[\frac{1}{\Delta t|K|}\int_{t^n}^{t^{n+1}}\int_K F\omega\,dxdt - (F_K^n)\omega_K^n\right],\\
\mathcal{E}_h^7(\omega) &= -\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} 2\Delta t|K|(V_K^{\varepsilon,n+1} - V_K^{\varepsilon,n})^T\,(B_K^n + (\mathrm{divA})_{\mathrm{K}}^{\mathrm{n}})\,V_K^{\varepsilon,n} - F_K^n)\omega_K^n,
\end{aligned}
$$

and

$$Q_h^\varepsilon(\omega) = \sum_{n\in\mathcal{N}} \sum_{K\in\mathcal{T}_h} \sum_{e\in\partial K} \Delta t|e|(V_{K_e}^{\varepsilon,n} - V_K^n)^T C_{e,K}^{\varepsilon,n}(V_{K_e}^{\varepsilon,n} - V_K^{\varepsilon,n})\omega_K^n.$$

Proof. See [25]. □

Lemma 4.7 (Lemma 4.3 in [15]). *Let $z \in H^1(K;\mathbb{R}^{2m})$, then there exists a constant $\mathcal{C} > 0$ such that for $K \in \mathcal{T}_h$*

$$\int_K |z - z_k|^2 dx \leq \mathcal{C}h^2 \int_K |Dz|^2 dx, \tag{29}$$

$$\int_e |z - z_k|^2 d\zeta \leq \mathcal{C}h \int_K |Dz|^2 dx \qquad (e \in \partial K), \tag{30}$$

where $\mathcal{C}$ depends only on d and m.

Proof. See [15]. □

We conclude with the proof of Theorem 4.5.

Proof. We suppose first that $U_0^\varepsilon \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^{2m})$, $F \in C_0^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})$. Accordingly to Theorem 2.6 we have $U^\varepsilon \in C_0^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})$. Applying Proposition 4.4 and Lemma 4.6 with $\omega = \theta$ and $\pi = \theta U^\varepsilon$ we just have to estimate

$$\sum_{l=1}^{7} \left[\mathcal{E}_h^l(\theta) - 2\mathcal{R}_h^l(\theta U^\varepsilon)\right] - \delta Q_h^\varepsilon(\theta).$$

We first consider two terms that will appear many times in our calculation, namely

$$\sum_{n\in\mathcal{N}} \sum_{K\in\mathcal{T}_h} \theta^n |K||V_K^{\varepsilon,n+1} - V_K^{\varepsilon,n}|^2, \qquad \sum_{n\in\mathcal{N}} \sum_{K\in\mathcal{T}_h} |K||V_K^{\varepsilon,n+1} - V_K^{\varepsilon,n}|^2.$$

From (17) we obtain

$$\begin{aligned} |V_K^{\varepsilon,n+1} - V_K^{\varepsilon,n}|^2 \leq\; & \mathcal{C}(\Delta t)^2 |(B_K^n + \mathrm{div}A_K^n)V_K^{\varepsilon,n} - F_K^n|^2 \\ & + \mathcal{C}\frac{(\Delta t)^2}{|K|^2} \sum_{e\in\partial K} |e|^2 |C_{e,K}^{\varepsilon,n}(V_K^{\varepsilon,n} - V_{K_e}^{\varepsilon,n})|^2. \end{aligned} \tag{31}$$

Multiplying (31) by $|K|\theta^n$, summing over all the elements, using the CFL condition (18) and the stability result (Proposition 4.2) we get that

$$\begin{aligned} &\sum_{n\in\mathcal{N}} \sum_{K\in\mathcal{T}_h} \theta^n |K||V_K^{\varepsilon,n+1} - V_K^{\varepsilon,n}|^2 \\ &\leq \mathcal{C} \sum_{n\in\mathcal{N}} \sum_{K\in\mathcal{T}_h} \theta^n (\Delta t)^2 |K| \left(\|B_K^n + \mathrm{div}A_K^n\|^2 |V_K^{\varepsilon,n}|^2 + |F_K^n|^2\right) \\ &\quad + \mathcal{C} \sum_{n\in\mathcal{N}} \sum_{K\in\mathcal{T}_h} \theta^n \frac{(\Delta t)^2}{|K|} \sum_{e\in\partial K} |e|^2 \|C_{e,K}^{\varepsilon,n}\| (V_K^{\varepsilon,n} - V_{K_e}^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_K^{\varepsilon,n} - V_{K_e}^{\varepsilon,n}) \\ &\leq \mathcal{C}Q_h^\varepsilon + \mathcal{C}\Delta t \left(\|U_0^\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})} + \|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2. \end{aligned} \tag{32}$$

In an analogous way we find that

$$\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}|K||V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n}|^2\le\mathcal{C}\left(\|U_0^\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2. \tag{33}$$

In both cases $\mathcal{C}$ is independent of ε.
Term $\mathcal{E}_h^1-2\mathcal{R}_h^1$:
We define

$$\begin{aligned}\mathcal{R}_h^{1,a}(\theta U^\varepsilon) &:= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}|K|(V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n})^TU_K^\varepsilon(t^{n+1})(\theta(t^{n+1})-\theta^n),\\ \mathcal{R}_h^{1,b}(\theta U^\varepsilon) &:= \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}|K|(V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n})^T(\theta^nU_K^\varepsilon(t^{n+1})-(\theta U^\varepsilon)_K^n).\end{aligned}$$

Since U^ε is a C^1-function in time (Theorem 2.6) and applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}|\theta^nU_K^\varepsilon(t^{n+1})-(\theta U^\varepsilon)_K^n| &\le \frac{1}{|K|\Delta t}\int_{t^n}^{t^{n+1}}|\int_K(U^\varepsilon(x,t^{n+1})-U^\varepsilon(x,t))dx|\theta(t)\,dt\\ &\le \mathcal{C}\theta^n\frac{1}{|K|}\int_{t^n}^{t^{n+1}}\int_K\left|\frac{\partial U^\varepsilon}{\partial t}\right|dxdt\\ &\le \mathcal{C}\theta^n\left(\frac{\Delta t}{|K|}\right)^{1/2}\left(\int_{t^n}^{t^{n+1}}\int_K\left|\frac{\partial U^\varepsilon}{\partial t}\right|^2dxdt\right)^{1/2},\end{aligned}$$

where $\mathcal{C}$ is independent of ε.
Using the Cauchy-Schwarz inequality we get

$$\begin{aligned}|\mathcal{R}_h^{1,b}(\theta U^\varepsilon)| &\le \mathcal{C}(\Delta t)^{1/2}\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\int_{t^n}^{t^{n+1}}\int_K\left|\frac{\partial U^\varepsilon}{\partial t}\right|^2dxdt\right)^{1/2}\\ &\quad\times\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\theta^n|K||V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n}|^2\right)^{1/2},\end{aligned}$$

which gives using Theorem 2.6, (32) and the CFL condition (18)

$$\begin{aligned}|\mathcal{R}_h^{1,b}(\theta U^\varepsilon)| &\le \mathcal{C}\left(\frac{h}{\sqrt{\varepsilon}}Q_h^\varepsilon\right)^{1/2}\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)\\ &\quad+\mathcal{C}h\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2. \end{aligned}\tag{34}$$

On the other hand

$$\begin{aligned}&|\mathcal{E}_h^1-2\mathcal{R}_h^{1,a}|\\ &\le \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}|K||\theta(t^{n+1})-\theta^n||V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n}||V_K^{\varepsilon,n+1}+V_K^{\varepsilon,n}-2U_K^\varepsilon(t^{n+1})|.\end{aligned}$$

We have

$$\begin{aligned}|\theta(t^{n+1})-\theta^n| &\leq \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} |\theta(t^{n+1})-\theta(t)|dt\\ &\leq \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \|\theta'\|_\infty |t^{n+1}-t|dt\\ &\leq \mathcal{C}\Delta t.\end{aligned}$$

We get by the Cauchy-Schwarz inequality

$$\begin{aligned}&|\mathcal{E}_h^1 - 2\mathcal{R}_h^{1,a}|\\ &\leq \mathcal{C}\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\Delta t|K||V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n}|^2\right)^{1/2}\\ &\quad\times\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\Delta t|K||V_K^{\varepsilon,n+1}+V_K^{\varepsilon,n}-2U_K^\varepsilon(t^{n+1})|^2\right)^{1/2}.\end{aligned}$$

We consider now the following splitting

$$V_K^{\varepsilon,n+1}+V_K^{\varepsilon,n}-2U_K^\varepsilon(t^{n+1}) = V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n}+2(V_K^{\varepsilon,n}-(U^\varepsilon)_K^n)+2((U^\varepsilon)_K^n-U_K^\varepsilon(t^{n+1})),$$

It is easy to check that

$$|V_K^{\varepsilon,n}-(U^\varepsilon)_K^n| \leq \frac{1}{(\Delta t|K|)^{1/2}}\left(\int_{t^n}^{t^{n+1}}\int_K |U^\varepsilon - U_h^\varepsilon|^2 dxdt\right)^{1/2},$$

$$|U_K^\varepsilon(t^{n+1})-(U^\varepsilon)_K^n| \leq \left(\frac{\Delta t}{|K|}\right)^{1/2}\left(\int_{t^n}^{t^{n+1}}\int_K \left|\frac{\partial U^\varepsilon}{\partial t}\right|^2 dxdt\right)^{1/2}.$$

With these inequalities, using (33), Theorem 2.6 and the CFL condition (18) we obtain

$$\begin{aligned}&\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\Delta t|K||V_K^{\varepsilon,n+1}+V_K^{\varepsilon,n}-2U_K^\varepsilon(t^{n+1})|^2\\ &\leq \mathcal{C}\|U^\varepsilon-U_h^\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})}+\mathcal{C}\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\Delta t|K||V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n}|^2\\ &\quad+\mathcal{C}(\Delta t)^2\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\int_{t^n}^{t^{n+1}}\int_K\left|\frac{\partial U^\varepsilon}{\partial t}\right|^2 dxdt\\ &\leq \mathcal{C}\|U^\varepsilon-U_h^\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})}+\mathcal{C}h\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2,\end{aligned}$$

again using (33) we get

$$\begin{aligned}|\mathcal{E}_h^1-2\mathcal{R}_h^{1,a}| &\leq \mathcal{C}h^{1/2}\|U^\varepsilon-U_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})}\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)\\ &\quad+\mathcal{C}h\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2. \qquad (35)\end{aligned}$$

Term $\mathcal{E}_h^2 - 2\mathcal{R}_h^2$:
First we note that since θ depends only on t we have

$$\mathcal{E}_h^2 = 0. \tag{36}$$

Using the Cauchy-Schwarz inequality and the bound of $\|C_{e,K}^{\varepsilon,n}\|$ (Remark 3.3) we obtain

$$\begin{aligned}
|\mathcal{R}_h^2| &\leq \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}(\theta^n\Delta t|e|)\,\left|\left(C_{e,K}^{\varepsilon,n}(V_K^{\varepsilon,n}-V_{K_e}^{\varepsilon,n})\right)^T((U^\varepsilon)_e^n-(U^\varepsilon)_K^n)\right| \\
&\leq \mathcal{C}\left(\frac{1}{\sqrt{\varepsilon}}\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}\theta^n\Delta t|e|\,(V_K^{\varepsilon,n}-V_{K_e}^{\varepsilon,n})^T C_{e,K}^{\varepsilon,n}(V_K^{\varepsilon,n}-V_{K_e}^{\varepsilon,n})\right)^{1/2} \\
&\quad\times\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}\theta^n\Delta t|e||(U^\varepsilon)_e^n-(U^\varepsilon)_K^n|^2\right)^{1/2} \\
&\leq \mathcal{C}\left(\frac{Q_h^\varepsilon}{\sqrt{\varepsilon}}\right)^{1/2}\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}\theta^n\Delta t|e||(U^\varepsilon)_e^n-(U^\varepsilon)_K^n|^2\right)^{1/2}.
\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality and Lemma 4.7 we find

$$\begin{aligned}
|(U^\varepsilon)_e^n-(U^\varepsilon)_K^n| &= \frac{1}{|e|}\left|\int_e((U^\varepsilon)^n(x)-(U^\varepsilon)_K^n)\,dx\right| \\
&\leq \frac{1}{|e|^{1/2}}\left(\int_e|(U^\varepsilon)^n(x)-(U^\varepsilon)_K^n|^2dx\right)^{1/2} \\
&\leq \mathcal{C}\frac{h^{1/2}}{|e|^{1/2}}\left(\int_K|DU^\varepsilon|^2\,dx\right)^{1/2}.
\end{aligned}$$

Moreover, from Lemma 4.7 and since $U^\varepsilon(\cdot,t)\in H^1(\mathbb{R}^d;\mathbb{R}^{2m})$ (Theorem 2.6) we get

$$\begin{aligned}
&\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}\theta^n\Delta t|e||(U^\varepsilon)_e^n-(U^\varepsilon)_K^n|^2 \\
&\quad\leq \mathcal{C}h\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\theta^n\Delta t\int_K|D(U^\varepsilon)^n(x)|^2dx \\
&\quad\leq \mathcal{C}h\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\theta^n\int_{t^n}^{t^{n+1}}\int_K|DU^\varepsilon(x,t)|^2\,dxdt \\
&\quad\leq \mathcal{C}h\left(\|U_0^\varepsilon\|^2_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|^2_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2,
\end{aligned}$$

and finally

$$|\mathcal{R}_h^2|\leq\mathcal{C}(Q_h^\varepsilon)^{1/2}\left(\frac{h}{\sqrt{\varepsilon}}\right)^{1/2}\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right). \tag{37}$$

Term $\mathcal{E}_h^3 - 2\mathcal{R}_h^3$**:**
From a direct calculation and Lemma 4.7, we obtain

$$\begin{aligned} &|\mathcal{E}_h^3 - 2\mathcal{R}_h^3| \\ &\leq \int_{\mathbb{R}^d} |(U_h^\varepsilon(0,x) - U_0^\varepsilon(x))^T \big((U_h^\varepsilon(0,x) + U_0^\varepsilon(x))\theta(0) - 2U_0^\varepsilon(x)\theta(0)\big)|\, dx \\ &\leq T \int_{\mathbb{R}^d} |U_h^\varepsilon(0,x) - U_0^\varepsilon(x)|^2\, dx \\ &\leq \mathcal{C}h^2. \end{aligned} \tag{38}$$

Term $\mathcal{E}_h^4 - 2\mathcal{R}_h^4$**:**
We first note that
(39)

$$\int_{t^n}^{t^{n+1}} \int_K \sum_{i=1}^d \frac{\partial}{\partial x_i}\big(A^{\varepsilon,i}U^\varepsilon\theta\big) dxdt = (T_1)_K^n + (T_2)_K^n + (T_3)_K^n + (T_4)_K^n + (T_5)_K^n + (T_6)_K^n$$

where

$$\begin{aligned} (T_1)_K^n &= \int_{t^n}^{t^{n+1}} \int_K \theta(t)\mathrm{div}A(x,t)(U^\varepsilon(x,t) - U_K^\varepsilon(t))\, dxdt, \\ (T_2)_K^n &= \int_{t^n}^{t^{n+1}} \int_K \theta(t)(\mathrm{div}A(x,t) - \mathrm{div}A^n(x))U_K^\varepsilon(t)\, dxdt, \\ (T_3)_K^n &= \int_{t^n}^{t^{n+1}} \int_K \theta(t)\mathrm{div}A^n(x)U_K^\varepsilon(t)\, dxdt, \\ (T_4)_K^n &= \int_{t^n}^{t^{n+1}} \int_K \theta(t) \sum_{i=1}^d (A^{\varepsilon,i}(x,t) - (A^{\varepsilon,i})_K(t))\frac{\partial}{\partial x_i}U^\varepsilon(x,t)\, dxdt, \\ (T_5)_K^n &= \int_{t^n}^{t^{n+1}} \int_K \theta(t) \sum_{i=1}^d ((A^{\varepsilon,i})_K(t) - (A^{\varepsilon,i})_K^n)\frac{\partial}{\partial x_i}U^\varepsilon(x,t)\, dxdt, \\ (T_6)_K^n &= \int_{t^n}^{t^{n+1}} \int_K \theta(t) \sum_{i=1}^d (A^{\varepsilon,i})_K^n\frac{\partial}{\partial x_i}U^\varepsilon(x,t)\, dxdt. \end{aligned}$$

We get

$$\begin{aligned} |\mathcal{R}_h^4| \leq \sum_{n\in\mathcal{N}} \sum_{K\in\mathcal{T}_h} &\left| (V_K^{\varepsilon,n})^T \int_{t^n}^{t^{n+1}} \left(\sum_{e\in\partial K} A_{e,K}^{\varepsilon,n} U_e^\varepsilon(t)\theta(t)|e| \right) dt - (T_3)_K^n - (T_6)_K^n \right| \\ + \sum_{n\in\mathcal{N}} \sum_{K\in\mathcal{T}_h} &\big(|(V_K^{\varepsilon,n})^T(T_1)_K^n| + |(V_K^{\varepsilon,n})^T(T_2)_K^n| + |(V_K^{\varepsilon,n})^T(T_4)_K^n| + |(V_K^{\varepsilon,n})^T(T_5)_K^n|\big) \end{aligned} \tag{40}$$

It is easy to check that

$$\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\big(|(V_K^{\varepsilon,n})^T(T_1)_K^n| + |(V_K^{\varepsilon,n})^T(T_2)_K^n| + |(V_K^{\varepsilon,n})^T(T_4)_K^n| + |(V_K^{\varepsilon,n})^T(T_5)_K^n|\big)$$
$$\leq \mathcal{C}h\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})} + \|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2$$

holds where $\mathcal{C}$ does not depend on ε.
For example using the Cauchy-Schwarz inequality, Lemma 4.7, the regularity of U^ε ($U^\varepsilon(\cdot,t)\in H^1(\mathbb{R}^d;\mathbb{R}^{2m})$ by Theorem 2.6) and Proposition 4.2 we obtain

$$\begin{aligned}
&\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}|(V_K^{\varepsilon,n})^T(T_1)_K^n| \\
\leq\;& \mathcal{C}\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}|V_K^{\varepsilon,n}|\left(\int_{t^n}^{t^{n+1}}\int_K 1\cdot|U^\varepsilon(x,t)-U_K^\varepsilon(t)|dxdt\right)\\
\leq\;& \mathcal{C}\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}(\Delta t)^{1/2}|K|^{1/2}|V_K^{\varepsilon,n}|\left(\int_{t^n}^{t^{n+1}}\int_K |U^\varepsilon(x,t)-U_K^\varepsilon(t)|^2dxdt\right)^{1/2}\\
\leq\;& \mathcal{C}h\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}(\Delta t)^{1/2}|K|^{1/2}|V_K^{\varepsilon,n}|\left(\int_{t^n}^{t^{n+1}}\int_K |DU^\varepsilon|^2dxdt\right)^{1/2}\\
\leq\;& \mathcal{C}h\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\Delta t|K||V_K^{\varepsilon,n}|^2 + \int_{t^n}^{t^{n+1}}\int_K |DU^\varepsilon|^2dxdt\right)\\
\leq\;& \mathcal{C}h\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})} + \|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2.
\end{aligned}$$

Returning to the first term on the R.H.S of (40) we note that thanks to Green's formula we find

$$\sum_{e\in\partial K}\sum_{i=1}^d n_{e,K}^i(A^{\varepsilon,i})_K^n U_K^\varepsilon(t)\theta(t)|e| = 0.$$

Therefore using the last expression and the Green formula we have that

$$(T_6)_K^n = \int_{t^n}^{t^{n+1}}\big(\sum_{e\in\partial K}\sum_{i=1}^d n_{e,K}^i(A^{\varepsilon,i})_K^n U_e^\varepsilon(t)\theta(t)|e|\big)dt$$

This leads to

$$\begin{aligned}
&\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\left|(V_K^{\varepsilon,n})^T\int_{t^n}^{t^{n+1}}\left(\sum_{e\in\partial K}A_{e,K}^{\varepsilon,n}U_e^\varepsilon(t)\theta(t)|e|\right)dt-(T_3)_K^n-(T_6)_K^n\right|\\
&\quad=\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\left|(V_K^{\varepsilon,n})^T\int_{t^n}^{t^{n+1}}\left[\sum_{e\in\partial K}\left(A_{e,K}^{\varepsilon,n}-\sum_{i=1}^d n_{e,K}^i(A^{\varepsilon,i})_K^n\right)(U_e^\varepsilon(t)-U_K^\varepsilon(t))\,\theta(t)|e|\right]dt\right|\\
&\quad\leq\sum_{n\in\mathcal{N}}\int_{t^n}^{t^{n+1}}\left(\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}|V_K^{\varepsilon,n}|^2\left|A_{e,K}^{\varepsilon,n}-\sum_{i=1}^d n_{e,K}^i(A^{\varepsilon,i})_K^n\right|^2|e|\right)^{1/2}\\
&\qquad\times\left(\sum_{K\in\mathcal{T}_h}\sum_{e\in\partial K}|e||U_e^\varepsilon(t)-U_K^\varepsilon(t)|^2\right)^{1/2}\theta(t)dt.
\end{aligned}$$

We note that thanks to Lemma 4.7 and the regularity of A (Assumption 2)

$$\begin{aligned}
|U_e^\varepsilon(t)-U_K^\varepsilon(t)|^2 &\leq \mathcal{C}\int_{K\in\mathcal{T}_h}|DU^\varepsilon|^2dx,\\
|A_{e,K}^{\varepsilon,n}-\sum_{i=1}^d n_{e,K}^i(A^{\varepsilon,i})_K^n|^2 &\leq \mathcal{C}h.
\end{aligned}$$

Indeed,

$$\begin{aligned}
|(A^{\varepsilon,i})_e^n-(A^{\varepsilon,i})_K^n| &= \left|\frac{1}{\Delta t|e|}\int_{t^n}^{t^{n+1}}\int_e A^{\varepsilon,i}(\zeta,t)-A_K^{\varepsilon,i}(t)\,d\zeta dt\right|\\
&\leq \left|\frac{1}{\Delta t|e|^{1/2}}\int_{t^n}^{t^{n+1}}\left(\int_e(A^{\varepsilon,i}(\zeta,t)-A_K^{\varepsilon,i}(t))^2\,d\zeta\right)^{1/2}dt\right|\\
&\leq \left|\frac{\mathcal{C}h^{1/2}}{\Delta t|e|^{1/2}}\int_{t^n}^{t^{n+1}}\left(\int_K|DA^{\varepsilon,i}|^2\,dx\right)^{1/2}dt\right|\\
&\leq \mathcal{C}h.
\end{aligned}$$

Finally with the help of the stability result (Proposition 4.2) we obtain

$$|\mathcal{R}_h^4|\leq\mathcal{C}h\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2. \tag{41}$$

$\mathcal{E}_h^4$ can be treated in the same way as $\mathcal{R}_h^4$.
Terms $\mathcal{R}_h^5, \mathcal{R}_h^6, \mathcal{R}_h^7, \mathcal{E}_h^5, \mathcal{E}_h^6$**:**
It is easy to check that

$$\begin{aligned}
|\mathcal{R}_h^l| &\leq \mathcal{C}\frac{h}{\sqrt{\varepsilon}}\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2 \quad l=5,6,7.\\
|\mathcal{E}_h^l| &\leq \mathcal{C}h\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2 \quad l=5,6.
\end{aligned} \tag{42}$$

For example, using the Cauchy-Schwarz inequality, Lemma 4.7, the regularity of F and θ, Theorem 2.6 and the CFL condition we obtain

$$\begin{aligned}
|\mathcal{R}_h^7| &\leq \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\int_{t^n}^{t^{n+1}}\int_K |F(x,t)\cdot(\pi(x,t)-\pi_K(t)+\pi_K(t)-\pi_K^n)|\,dxdt \\
&\leq \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\left(\int_{t^n}^{t^{n+1}}\int_K F^2\,dxdt\right)^{1/2}\left(\int_{t^n}^{t^{n+1}}\int_K |\pi(x,t)-\pi_K(t)|^2\,dxdt\right)^{1/2} \\
&\quad +\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\left(\int_{t^n}^{t^{n+1}}\int_K F^2\,dxdt\right)^{1/2}\left(\int_{t^n}^{t^{n+1}}\int_K |\pi_K(t)-\pi_K^n|^2\,dxdt\right)^{1/2} \\
&\leq \mathcal{C}h\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\left(\int_{t^n}^{t^{n+1}}\int_K F^2\,dxdt\right)^{1/2}\left(\int_{t^n}^{t^{n+1}}\int_K |D\pi|^2\,dxdt\right)^{1/2} \\
&\quad +\mathcal{C}\Delta t\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\left(\int_{t^n}^{t^{n+1}}\int_K F^2\,dxdt\right)^{1/2}\left(\int_{t^n}^{t^{n+1}}\int_K \left|\frac{\partial\pi}{\partial t}\right|^2\,dxdt\right)^{1/2} \\
&\leq \mathcal{C}\left(h+\Delta t+\frac{\Delta t}{\varepsilon}\right)\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2 \\
&\leq \mathcal{C}\frac{h}{\sqrt{\varepsilon}}\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2.
\end{aligned}$$

Term $\mathcal{E}_h^7$:
Using Cauchy-Schwarz inequality and (32) we obtain

$$\begin{aligned}
|\mathcal{E}_h^7| &\leq \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h} 2\Delta t|K||(V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n})^T((B_K^n+\mathrm{div}A_K^n)V_K^{\varepsilon,n}-F_K^n)\theta^n| \\
&\leq \mathcal{C}\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\Delta t|K||V_K^{\varepsilon,n+1}-V_K^{\varepsilon,n}|^2\theta^n\right)^{1/2} \\
&\quad\times\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\Delta t|K||(B_K^n+\mathrm{div}A_K^n)V_K^{\varepsilon,n}-F_K^n|^2\theta^n\right)^{1/2} \\
&\leq \mathcal{C}\left(\Delta tQ_h^\varepsilon+(\Delta t)^2\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2\right)^{1/2} \\
&\quad\times\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\Delta t|K||(B_K^n+\mathrm{div}A_K^n)V_K^{\varepsilon,n}-F_K^n|^2\theta^n\right)^{1/2}.
\end{aligned}$$

Using the stability result (Proposition 4.2), the bound on B (Assumption 2) and θ, and the regularity of F (Assumption 2) we find

$$\begin{aligned}\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}&\Delta t|K|(|(B_K^n+\mathrm{div}A_K^n)V_K^{\varepsilon,n}|^2+|F_K^n|^2\theta^n\\&\leq\mathcal{C}\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2,\end{aligned}$$

and finally we get

$$\begin{aligned}|\mathcal{E}_h^7|\ &\leq\ \mathcal{C}(\Delta t)^{1/2}(Q_h^\varepsilon)^{1/2}\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)\\&\quad+\mathcal{C}\Delta t\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2.\end{aligned}\tag{43}$$

Now, applying Proposition 4.4 and combining the estimates (34), (35), (36), (37), (38), (41), (42) and (43) we get

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}\exp(-\alpha t)|U^\varepsilon-U_h^\varepsilon|^2dxdt+\delta Q_h^\varepsilon\\&\leq\ \mathcal{C}\frac{h}{\sqrt{\varepsilon}}\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2\\&\quad+\mathcal{C}\left(h^{1/2}\|U^\varepsilon-U_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})}+\left(\frac{h}{\sqrt{\varepsilon}}Q_h^\varepsilon\right)^{1/2}\right)\\&\quad\times\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)\end{aligned}$$

Appropriate application of Young's inequality yields to

$$\begin{aligned}&\frac{\exp(-\alpha T)}{2}\|U^\varepsilon-U_h^\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})}+\frac{\delta}{2}Q_h^\varepsilon\\&\leq\ \mathcal{C}\frac{h}{\sqrt{\varepsilon}}\left(\|U_0^\varepsilon\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2.\end{aligned}\tag{44}$$

Using the fact that $Q_h^\varepsilon\geq 0$ we conclude the theorem for smooth data.
Now we consider the sequences of smooth functions $\{F^j\}_{j\in\mathbb{N}}$, $\left\{U_0^{j,\varepsilon}\right\}_{j\in\mathbb{N}}$ with $F^j\in C_0^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})$ and $U_0^{j,\varepsilon}\in C_0^\infty(\mathbb{R}^d;\mathbb{R}^{2m})$. Assume that

$$\lim_{j\to\infty}\|F-F^j\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}=\lim_{j\to\infty}\|U_0^\varepsilon-U_0^{j,\varepsilon}\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}=0.$$

Such sequence can be constructed using mollifiers. Using the linearity of the system and the *a priori* estimate of Theorem 2.6 we conclude that $U^{j,\varepsilon}\to U^\varepsilon$ in $C([0,T];H^1(\mathbb{R}^d;\mathbb{R}^{2m}))$. Here $U^{j,\varepsilon}$ denote the weak solution of (11)-(12) with data F^j and $U_0^{j,\varepsilon}$. Let $U_h^{j,\varepsilon}$ denote the corresponding approximation from Definition 3.2. Using the linearity of the scheme and Proposition 4.2 we get $U_h^{j,\varepsilon}\to U_h^\varepsilon$ in $L^\infty(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))$. According to (44) one obtains

$$\int_0^T\int_{\mathbb{R}^d}|U_h^{j,\varepsilon}-U^\varepsilon|^2\leq\mathcal{C}\frac{h}{\sqrt{\varepsilon}}\left(\|U_0^{j,\varepsilon}\|_{H^1(\mathbb{R}^d;\mathbb{R}^{2m})}+\|F^j\|_{L^2(0,T;H^1(\mathbb{R}^d;\mathbb{R}^{2m}))}\right)^2.$$

Passing to the limit in the last inequality, one concludes the proof. □

Now we can extract from Theorem 4.5 the results announced in Section 1.

Corollary 4.8. *Suppose that Assumptions 1-3 and CFL-condition* (18) *hold. Let* u *be the weak solution of* (1)-(3) *and* $U_h^\varepsilon = (u_h^\varepsilon, \varphi_h^\varepsilon)^T$ *be the approximated solution generated by the* ***EX-GLM-FV*** *scheme. Then we have*

$$\|u - u_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})} \leq \mathcal{C}h^{1/3}. \tag{45}$$

Proof. : The triangular inequality and Theorem 4.5 give

$$\begin{aligned} \|u - u_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})} &\leq \|u - u^\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})} + \|u^\varepsilon - u_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^{2m})} \\ &\leq \mathcal{C}\left(\sqrt{\varepsilon} + \left(\frac{h}{\sqrt{\varepsilon}}\right)^{1/2}\right). \end{aligned} \tag{46}$$

We consider $\varepsilon(h) = h^\alpha$ with α determined by the solution of the problem

$$\min_{0<\alpha<1} \left\{h^{\alpha/2} + h^{1/2-\alpha/4}\right\}, \text{ with } 0 < h < 1.$$

Differentiating and equaling to 0 we get

$$\alpha^* = \frac{2}{3}.$$

Replacing in (46) we find the desired result. □

For a first order method we can not expect to get a convergence rate directly for the expression $\sum_{i=1}^d M^i(u_h^\varepsilon)_{x_i}$. However it is possible to obtain a weak convergence estimate.

Corollary 4.9. *Suppose that Assumptions 1-3 and CFL-condition* (18) *hold. Let* $U_h^\varepsilon = (u_h^\varepsilon, \varphi_h^\varepsilon)$ *be the* ***EX-GLM-FV*** *approximation from Definition 3.2. Then*

$$\left|\int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^d M^i u_h^\varepsilon \cdot \frac{\partial \phi}{\partial x_i} dxdt\right| \to 0 \quad \textit{as} \quad h \to 0\,, \forall \phi \in C_0^\infty([0,T)\times\mathbb{R}^d, \mathbb{R}^m). \tag{47}$$

Proof. : We consider $\pi = (0,\phi)^T$ in Definition 4.3 with $\phi \in C_0^\infty([0,T)\times\mathbb{R}^d,\mathbb{R}^m)$. Then

$$\begin{aligned} \left|\int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^d (\frac{M^i}{\sqrt{\varepsilon}} u_h^\varepsilon)^T \frac{\partial \phi}{\partial x_i} dxdt\right| &= \Big| - < \mu_{V_h}, \pi > - \int_0^T \int_{\mathbb{R}^d} (\varphi_h^\varepsilon)^T \partial_t \phi \, dxdt \\ &\quad + \int_0^T \int_{\mathbb{R}^d} (\varphi_h^\varepsilon)^T \phi \, dxdt - \int_{\mathbb{R}^d} (\varphi_h^\varepsilon)^T \phi(x,0)\, dx \Big|. \end{aligned}$$

From Lemma 4.6 we know

$$< \mu_{V_h}, \pi >= \sum_{l=1}^7 \mathcal{R}_h^l(\pi).$$

Moreover from Theorem 4.5, we have

$$\left|\mathcal{R}_h^l(\pi)\right| \leq \mathcal{C}\frac{h}{\sqrt{\varepsilon}} \qquad \text{for} \quad l = 2,4,5,6,7.$$

It remains to focus just on $\mathcal{R}_h^1$ and $\mathcal{R}_h^3$. We have for $\mathcal{R}_h^1$
For $\mathcal{R}_h^1$

$$\mathcal{R}_h^1(\pi) \;=\; \sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}|K|\big((\varphi_h^\varepsilon)_K^{n+1}-(\varphi_h^\varepsilon)_K^n\big)^T\big(\phi_K(t^{n+1})-\phi_K^n\big).$$

Since ϕ is a smooth function we obtain

$$|\phi_K(t^{n+1})-\phi_K^n| \leq \left(\frac{\Delta t}{|K|}\right)^{1/2}\left(\int_{t^n}^{t^{n+1}}\int_K\left|\frac{\partial\phi}{\partial t}\right|^2 dxdt\right)^{1/2}.$$

Applying now the Cauchy-Schwarz inequality we get

$$\left|\mathcal{R}_h^1(\pi)\right| \leq \mathcal{C}(\Delta t)^{1/2}\left(\sum_{n\in\mathcal{N}}\sum_{K\in\mathcal{T}_h}\Delta t|K||(\varphi_h^\varepsilon)_K^{n+1}-(\varphi_h^\varepsilon)_K^n|^2\right)^{1/2}.$$

Using (33) we find

$$\left|\mathcal{R}_h^1(\pi)\right| \leq \mathcal{C}\Delta t \leq \mathcal{C}\frac{h}{\sqrt{\varepsilon}}.$$

We consider now $\mathcal{R}_h^3(\pi)$

$$\begin{aligned}\left|\mathcal{R}_h^3(\pi)\right| \;&=\; \left|\int_{\mathbb{R}^d}\big(\varphi_h^\varepsilon(x,0)-\varphi_0^\varepsilon(x)\big)^T\phi(x,0)dx\right| \\ &\leq \left(\int_{\mathbb{R}^d}|\varphi_h^\varepsilon(x,0)-\varphi_0^\varepsilon(x)|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^d}|\phi(x,0)|^2dx\right)^{1/2} \\ &\leq \mathcal{C}\Big(\sum_{K\in\mathcal{T}_h}\int_K|(\varphi_h^\varepsilon)_K^0-\varphi_0^\varepsilon(x)|^2dx\Big)^{1/2}.\end{aligned}$$

Since $\varphi_0^\varepsilon \in H^1(\mathbb{R}^d;\mathbb{R}^m)$(Assumption 2) and using Lemma 4.7 we obtain

$$|\mathcal{R}_h^3(\pi)| \leq \mathcal{C}h.$$

Now we have altogether

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^d}\sum_{i=1}^d\left(\frac{M^i}{\sqrt{\varepsilon}}u_h^\varepsilon\right)^T\frac{\partial\phi}{\partial x_i}dxdt\right| \\ &\quad\leq \left|\int_0^T\int_{\mathbb{R}^d}\big((\varphi_h^\varepsilon)^T\partial_t\phi-(\varphi_h^\varepsilon)^T\phi\big)\,dxdt\right| + \left|\int_{\mathbb{R}^d}\phi^T(x,0)\varphi_h^\varepsilon(x,0)dx\right| + \mathcal{C}\frac{h}{\sqrt{\varepsilon}}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, the stability result (Proposition 4.2) and multiplying by $\sqrt{\varepsilon}$ we obtain

$$\left| \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^d (M^i u_h^\varepsilon)^T \frac{\partial \phi}{\partial x_i} dx dt \right| \to 0,$$

as $h, \varepsilon \to 0$. □

5. Numerical Examples

5.1. **Example 1.** [Homogeneous Maxwell equations]
One example of Friedrichs systems with involutions are provided by the Maxwell equations given in $\mathbb{R}^3 \times (0, \infty)$ by the system

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mathbf{j}, \tag{48}$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \tag{49}$$

$$\nabla \cdot \mathbf{E} = \rho, \tag{50}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{51}$$

Here $\mathbf{E} = \mathbf{E}(x,t) \in \mathbb{R}^3$, $\mathbf{B} = \mathbf{B}(x,t) \in \mathbb{R}^3$, $\mathbf{j} = \mathbf{j}(x,t) \in \mathbb{R}^3$ and $\rho = \rho(x,t) \in \mathbb{R}$ denote the electric field, the magnetic induction, the current density and the charge density respectively.
For the computations we have considered the homogeneous Maxwell equations in two space dimensions (i.e. $\mathbf{j} = 0$, $\rho = 0$, B_1, B_2 and E_3 are constant), periodic boundary conditions on the computational domain $(0,1) \times (0,1)$, $t \in [0,1]$, and used a Cartesian mesh with mesh parameter $h > 0$. The time step is calculated according to the CFL condition (18) with $\varepsilon = h^{2/3}$. We have set $a = 0$ (see eq. (4)) in order to get a completely conservative system. It is easy to check that an exact solution of (48)-(51) (which is periodic) is given by

$$\begin{aligned} E_1(x_1,x_2,t) &= -\tfrac{k_\perp}{k_\parallel} \sin(k_\perp x_2) \cos(k_\parallel x_1 - \omega t), \\ E_2(x_1,x_2,t) &= \cos(k_\perp x_2) \sin(k_\parallel x_1 - \omega t), \\ B_3(x_1,x_2,t) &= \tfrac{\omega}{k_\parallel c^2} \cos(k_\perp x_2) \sin(k_\parallel x_1 - \omega t). \end{aligned} \tag{52}$$

Here $c > 0$ is the light speed, and the longitudinal and transverse wave numbers $k_\parallel$ and $k_\perp$, respectively, are related to the frequency ω according to

$$k_\parallel^2 + k_\perp^2 = \frac{\omega^2}{c^2}.$$

For the experiments we use $c = 1$, $k_\parallel = k_\perp = 2\pi$ and $\psi_0^\varepsilon = 0$. Initial condition for E_1, E_2 and B_3 are chosen according to (52) together with the results for the application of the **EX-GLM-FV** method. The idea in this numerical example is to illustrate the rate of convergence predicted by Corollary 4.8. The results of the **EX-GLM-FVM** are presented in Table 1. The rate of convergence takes a bigger value than we predicted. This is not

		EX-GLM-FV		FV	
h	Time steps	L^2[error]	E.O.C.	L^2[error]	E.O.C.
0.1	172	8.30E-1	0.24	7.14E-1	0.49
0.05	434	7.04E-1	0.38	5.08E-1	0.68
0.025	1094	5.39E-1	0.52	3.18E-1	0.81
0.0125	2758	3.77E-1	0.61	1.81E-1	0.90
0.00625	6948	2.47E-1	0.66	9.71E-2	0.94
0.003125	17510	1.56E-1	0.64	5.05E-2	0.92

Table 1: Numerical results for the **EX-GLM-FV** method and Finite Volume Method applied to Maxwell equations for defined mesh parameters.

a surprise because the rate predicted in the case without including the involution is $1/2$ ([25]) but the numerical simulations show an experimental order of 1. Our case follows the same rule, the rate observed (2/3) is two times the rate predicted (1/3). We have not shown the error in the discrete computation of $\nabla \cdot \mathbf{E}$ because $\nabla \cdot \mathbf{E} = 0$ is preserved to machine precision.

Example 5.1 suggests that the use of the **EX-GLM-FV** method does not pay off since the higher computational cost comes with an even worse convergence rate (compared to the original FV method). The next two examples show the benefits of this approach.

5.2. **Example 2.** [Induction equation]
Another physical example of, in fact non-linear, conservation laws with involutions are the MHD equations in $\mathbb{R}^3 \times (0, \infty)$ given by

$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \nabla \cdot \big(\rho \mathbf{u} \otimes \mathbf{u} + (p + \frac{1}{2}|\mathbf{B}|^2) I - \mathbf{B} \otimes \mathbf{B}\big) &= 0, \\
\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) &= 0, \\
\nabla \cdot \mathbf{B} &= 0.
\end{aligned}$$

Here $\rho = \rho(x,t) \in \mathbb{R}$ is the density, $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^3$ is the velocity field, $\mathbf{B} = \mathbf{B}(x,t) \in \mathbb{R}^3$ the magnetic field, $p = p(\rho) \in \mathbb{R}$ the pressure and I the identity matrix. For simplicity we have written down the isentropic version. We consider the system of the MHD equations in two space dimensions and we suppose that the velocity field $\mathbf{u} = (u^1, u^2)^T$ is given. If we add the "source" term (which is zero due to $\nabla \cdot \mathbf{B} = 0$) $-\mathbf{u} \nabla \cdot \mathbf{B}$ to the induction equation we get (after some manipulations) the induction system

$$\partial_t \mathbf{B} + \partial_{x_1}(A^1 \mathbf{B}) + \partial_{x_2}(A^2 \mathbf{B}) + C\mathbf{B} = 0,$$

where

$$A^i = \begin{pmatrix} u^i & 0 \\ 0 & u^i \end{pmatrix}, \quad C = -\begin{pmatrix} \partial_{x_1} u^1 & \partial_{x_2} u^1 \\ \partial_{x_1} u^2 & \partial_{x_2} u^2 \end{pmatrix} \quad (i = 1, 2).$$

This linear system fits exactly to our setting. We use again periodic boundary conditions in the domain $(0,1)^2$, $t \in [0,1)$, a Cartesian mesh, and $\psi_0^\varepsilon = 0$. We take $a = 1$ in order to see the dissapative effect. The idea of this numerical example is to study the behavior of the discrete version of $\nabla \cdot \mathbf{B}$ generated by the **EX-GLM-FV** approximation. The velocity and initial condition are taken as in [13]. These values are

$$B_0^1(x_1, x_2) = \partial_{x_2} A(x_1, x_2), B_0^2 = -\partial_{x_1} A(x_1, x_2),$$

where

$$A(x_1, x_2) = \frac{1}{2\pi} \sin(2\pi x_1) \cos(2\pi x_2) + x_2 - x_1,$$

and

$$\mathbf{u}(x_1, x_2) = (1,1) + 0.25(\cos(2\pi x_1) + 2\sin(2\pi x_2), \sin(2\pi x_1) + 2\cos(2\pi x_2)).$$

We present the results of the error $\nabla \cdot \mathbf{B}$ in the L^2-norm at $t = 0.5$. The L^2-norm of the discrete $\nabla \cdot \mathbf{B}$, denoted by $\mathrm{div}_h \mathbf{B}_h$, for $t \in [0,1)$ is calculated as follows:

$$\mathrm{div}_h \mathbf{B}_h = \sqrt{\sum_{K \in \mathcal{T}_h} \left(\sum_{e \in \partial K} \mathbf{B}_{K_e} \cdot n_{e,K} \right)^2}.$$

From Figure 1 we see that for both methods (FV and **EX-GLM-FV**), the error in $\nabla \cdot \mathbf{B}$ converges to zero. However we observe also that the error in the **EX-GLM-FV** method remains considerably lower than in the FV method.

5.3. **Example 3.** [Inhomogeneous Maxwell equations]
Even though the **EX-GLM-FV** method damps the divergence error in Example 5.2 much better than the FV method, one might conclude that also the FV method leads to stable computations in the homogeneous case. This is not true for the inhomogeneous case with $\mathbf{j} \neq 0$ in the Maxwell equations as we will demonstrate below. In fact in almost all practical computations, the Maxwell equations are coupled to other equations via source terms. An uncontrollable increase in the divergence error can stop the computation. This problem was the motivation of Munz et al. [20] to develop the GLM. We use the same two-dimensional system as in Example 5.1 but with $\mathbf{j} \neq 0$.
Taking the divergence of equation (48) we get

$$\partial_t(\nabla \cdot \mathbf{E}) = \partial_t \rho = -\nabla \cdot \mathbf{j}. \tag{53}$$

If we consider $\rho = 0$ we find the compatibility condition $\nabla \cdot \mathbf{j} = 0$. We want to study now the behavior of the **EX-GLM-FV** method for the Maxwell equations under a small

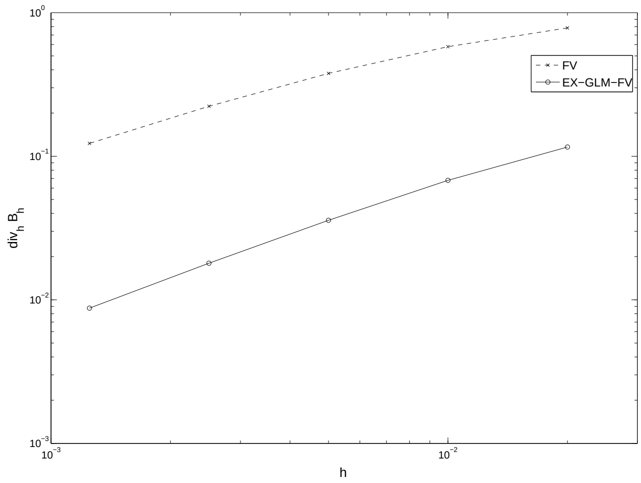

Figure 1: Comparison of the divergence error between FV and **EX-GLM-FV** methods in the induction equation at t=0.5.

perturbation on the condition $\nabla\cdot\mathbf{j}=0$. To do so we consider the following current density (for which the divergence is not zero)

$$\begin{aligned} j_1(x,y) &= -1.001\frac{k_\perp}{k_\|}\sin(k_\perp y)\cos(k_\| x), \\ j_2(x,y) &= \cos(k_\perp y)\sin(k_\| x). \end{aligned}$$

with $k_\|$ and $k_\perp$ as in Example 1. Moreover we consider an initial electrical field $\mathbf{E}_0$ such that $\nabla\cdot\mathbf{E}_0=0$. We again set $a=0$.
Accordingly to (53) we can expect a linear growth for $\mathrm{div}_h\mathbf{E}_h$ with respect to time in the case without correction. In Figure 2 we show the error in $\mathrm{div}_h\mathbf{E}_h$ for the FV and the **EX-GLM-FV** methods. It was calculated accordingly to

$$\mathrm{div}_h\mathbf{E}_h=\sqrt{\sum_{t=0}^{T}\sum_{K\in\mathcal{T}_h}\left(\sum_{e\in\partial K}\mathbf{E}^n_{K_e}\cdot n_{e,K}\right)^2\Delta t}\,.$$

We observe that the **EX-GLM-FV** method is stable under small perturbations on the side condition. On the other side we see that the FV method follows the behavior predicted by equation (53) and can lead to instabilities in the coupled process.

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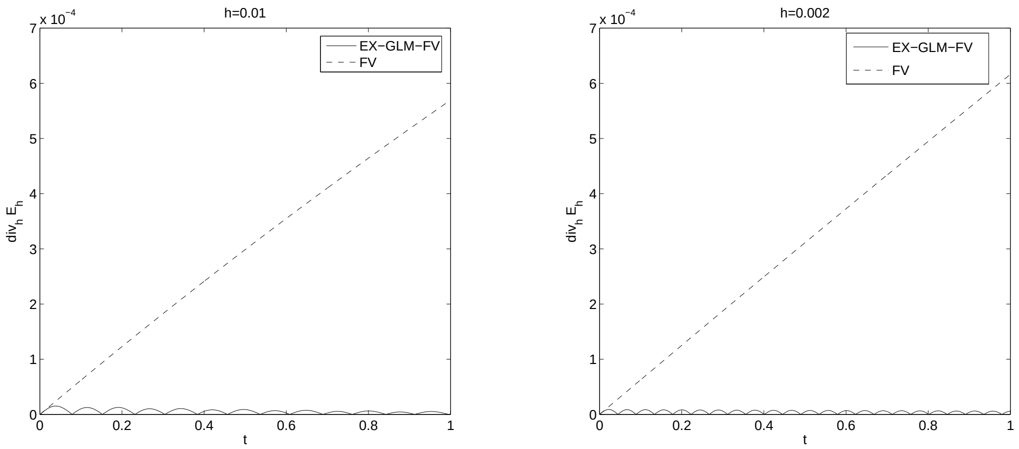

Figure 2: Comparison of the divergence error between FV and **EX-GLM-FV** method for the Maxwell equations (left figure: h=0.01; right figure: h=0.002)

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