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# A KINETIC DECOMPOSITION FOR SINGULAR LIMITS OF NON-LOCAL CONSERVATION LAWS

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ABSTRACT. We consider a non-local regularization of nonlinear hyperbolic conservation laws in several space variables. The regularization is motivated by the theory of phase dynamics and is based on a convolution operator. We formulate the initial value problem and begin by deriving a priori estimates which are independent of the regularization parameter. Following Hwang and Tzavaras we establish a kinetic decomposition associated with the problem under consideration, and we conclude that the sequence of solutions generated by the non-local model converges to a weak solution of the corresponding hyperbolic problem. Depending on the scaling introduced in the non-local dispersive term, this weak limit is either a classical Kruzkov solution satisfying all entropy inequalities or, more interestingly, a nonclassical entropy solution in the sense defined by LeFloch, that is, a weak solution satisfying a single entropy inequality and possibly containing undercompressive shock waves. Finally, we illustrate our analytical conclusions with numerical experiments.

## 1. Introduction

Consider the following initial value problem for the hyperbolic conservation law in $d$ spatial variables

$$u_t + \operatorname{div}(\mathbf{f}(u)) = 0, \qquad (\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}_+, \tag{1.1}$$

where the unknown function $u = u(\mathbf{x}, t)$ is a function of the spatial variables $\mathbf{x} = (x_1, \ldots, x_d)^T$ and the time variable $t > 0$, and assume the initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d \tag{1.2}$$

for some given $u_0 : \mathbb{R}^d \to \mathbb{R}$. Here, $\mathbf{f} = (f_1, \ldots, f_d)^T : \mathbb{R} \to \mathbb{R}^d$ denotes the smooth, vector-valued flux of the conservation law.

The initial value problem for equations (1.1) with "generic nonlinearities" $\mathbf{f}$ do have global smooth solutions for all initial data, and typical examples of discontinuous solutions are provided by planar shock waves. Furthermore, it is known that nonclassical undercompressive shock waves that do not satisfy Lax shock inequalities may arise by singular limits. An undercompressive shock w is, by definition, such that characteristics around it cross the shock trajectory. The theory of such shock waves was extensively developed these last ten years [11, 12, 20, 21] and various *local* perturbations of conservation laws and their singular limits were analyzed. This theory of nonclassical shocks was developed in close connection with actual models of continuum physics that do admit undercompressive shocks (or subsonic phase boundaries), especially in the context of phase transition dynamics [1, 9, 19, 32, 33] thin film flows [4], and multiphase flow in porous media [34].

The crucial modeling question that naturally arises is to select the dissipative effects that drive nonclassical waves in a given physical setting. Mathematically, we need a regularization of the conservation law (1.1) and, in the present paper, motivated by the theory of phase transition dynamics we investigate the effects of *non-local* terms.

We denote the regularized problem in the abstract form

$$u_t^\varepsilon + \operatorname{div}(\mathbf{f}(u^\varepsilon)) = R[\varepsilon; u^\varepsilon], \quad (\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}_+, \tag{1.3}$$

where $\varepsilon > 0$ is a small parameter. One of the simplest dissipation term leading to nonclassical waves in the singular limit $\varepsilon \to 0$ is defined (for arbitrary functions $w : \mathbb{R}^d \to \mathbb{R}$) by

$$R[\varepsilon; w](\mathbf{x}) := \varepsilon \Delta w(\mathbf{x}) + \lambda(\varepsilon)\, \varepsilon^2 \sum_{i=1}^{d} w_{x_i x_i x_i}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{1.4}$$

in which $\lambda$ is a fixed parameter. This scaling of the diffusion and dispersion terms is motivated by elementary arguments, for instance by formally searching for traveling wave solutions. See again [20, 21] for background material on this subject.

Now, in phase transition problems, e.g. for liquid-vapor transitions governed by the Navier-Stokes-Korteweg equations [2, 9], the physical density formally corresponds to our unknown $u^\varepsilon$. While the second-order term in (1.4) takes into account the effects of viscosity, the third-order term models the effects of capillary forces near the phase boundary. For the model problem (1.3) with $R$ defined in (1.4), undercompressive shock waves are indeed known to form in the limit $\varepsilon \to 0$; see [3, 11, 16].

A closer look at the modeling in phase transition theory reveals that, physical phenomena are *non-local* in nature (see for instance the textbook [28]) and that the widely used local versions should be regarded as approximations of non-local ones. Specifically, in the present paper we consider the following class of non-local regularizations (1.3):

$$R[\varepsilon; w](\mathbf{x}) := \varepsilon \Delta w(\mathbf{x}) + \lambda(\varepsilon) \sum_{i=1}^{d} \Big( (\phi_\varepsilon * w_{x_i})(\mathbf{x}) - w_{x_i}(\mathbf{x}) \Big), \quad \mathbf{x} \in \mathbb{R}^d. \tag{1.5}$$

Here, given a function $v \in L^1_{loc}(\mathbb{R}^d)$, its convolution $\phi_\varepsilon * v$ is defined by

$$(\phi_\varepsilon * v)(\mathbf{x}) := \int_{\mathbb{R}^d} \phi_\varepsilon(\mathbf{x} - \mathbf{y}) v(\mathbf{y})\, d\mathbf{y},$$

where the kernel function $\phi : \mathbb{R}^d \to \mathbb{R}$ is given and $\phi_\varepsilon$ is defined by

$$\phi_\varepsilon(\mathbf{x}) := \frac{1}{\varepsilon^d}\, \phi\left(\frac{\mathbf{x}}{\varepsilon}\right). \tag{1.6}$$

The kernel function models long-range interactions, as discussed in [30]. In [10, 29], it has been checked analytically and numerically that for one-dimensional equations the non-local regularization (1.5) leads to undercompressive waves if $\lambda(\varepsilon) = \gamma$ is a constant.

The analysis of the singular limit $\varepsilon \to 0$ was addressed by the compensated compactness method for *one-dimensional* conservation laws by Schonbek [31], Hayes and LeFloch [11], LeFloch and Natalini [23] (for a class of nonlinear diffusion), Kondo and LeFloch [17] (for globally Lipschitz flux and $L^2$ data), and Coclite and Karlsen [5] (for the Camassa-Holm regularization). The generalization of this technique to non-local regularizations was recently addressed by Rohde [29]. For a generalization to certain systems of two conservation laws, see [12].

It is important to observe that the scaling allowed by the argument of compensated compactness does cover the regime of nonclassical shock waves. We note also that proofs require a priori estimates in $L^p$ spaces, and that the method of compensated compactness does not in itself allow for a generalization to several space dimensions.

Next, DiPerna's measure-valued solutions [8] provided the framework to justify singular limits to *multi-dimensional* conservation laws toward Kruzkov's classical entropy solutions [18]. This was demonstrated by Correia and LeFloch [6, 7] and Kondo and LeFloch [17]. Later, Hwang and Tzavaras [15] realized that the kinetic formulation [25, 26] allows to cover the limiting case corresponding to the scaling associated with nonclassical shocks, and successfully analyzed singular limits for various (local) regularizations [13, 14].

Our purpose in the present paper is to add to this body of works and provide a proof of convergence for solutions to the *non-local* and *multi-dimensional* model (1.3), (1.5) for scalings allowing for *nonclassical* shocks. As we will see several new features and technical difficulties arise to carry out the desired program. Following Kondo and LeFloch [17] (and, later, [15]) we only rely on a natural $L^2$-type energy estimate and, consequently, assume that the flux grows at most linearly in the large. Following Hwang and Tzavaras [15], we rely on a version of the averaging lemma established by Perthame and Souganidis [27], and we determine a suitable decomposition of the dissipation terms associated with the non-local regularization under study. We also show that depending on the parameter $p$ in the scaling $\lambda(\varepsilon) = \mathcal{O}(\varepsilon^p)$ we can identify the limit $u$ as either a Kruzkov's classical entropy solution [18] of (1.1) or else as a nonclassical entropy solution in the sense introduced by LeFloch [21].

An outline of this paper is as follows. In Section 2, we specify our assumptions and state the existence of smooth solutions to (1.3) for fixed $\varepsilon$. Section 3 contains basic $\varepsilon$-independent a priori estimates and, in Section 4, we state and give the proof of our main result; see Theorem 4.1. Finally, the aforementioned discussion of the limiting solution and several numerical experiments are presented in Section 5.

## 2. Background material

We collect here assumptions and properties of the problem under study that we need for our investigation. We consider the problem (1.3), (1.2), (1.5) and let $\lambda \in C^0(\mathbb{R})$. From now on we assume that

$$\begin{aligned}&\text{the flux } \mathbf{f} \text{ is smooth and } \textit{generically nonlinear}, \text{ i.e. } \mathbf{f}' \text{ satisfies}\\ &\text{for each } \mathbf{n} \in \mathcal{S}^d \text{ and } \alpha \in \mathbb{R} \text{ and a.e. } \xi \in \mathbb{R}, \quad \alpha + \mathbf{f}'(\xi)\cdot\mathbf{n} \neq 0.\end{aligned} \tag{2.1}$$

($\mathcal{S}^d$ being the set of unit vectors in $\mathbb{R}^d$) and the initial condition satisfies

$$u_0 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{2.2}$$

Moreover, the kernel function

$$\begin{aligned}&\phi \in C^1(\mathbb{R}^d) \text{ in (1.6) is an even function satisfying } \phi \geq 0,\\ &\operatorname{supp}(\phi) \subset [-1,1], \qquad \int_{\mathbb{R}^d} \phi(\mathbf{x})\, d\mathbf{x} = 1.\end{aligned} \tag{2.3}$$

Based on the above assumptions, a global existence result of classical solutions to the regularized problem was established by Rohde [29].

**Theorem 2.1.** *Suppose that the assumptions* (2.1)–(2.3) *hold. Given a bounded interval $\Lambda$ of allowed parameter values $\lambda$, there exists a constant $K_\Lambda$ such that for all $\varepsilon$ and all $u_0 : \mathbb{R}^d \to \mathbb{R}$ with $\|u_0\|_{L^2(\mathbb{R}^d)} \leq K_\Lambda$, the initial value problem (1.3), (1.2),* *(1.5) admits a global-in-time, classical solution $u^\varepsilon : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ which is also unique in this class.*

In addition, the solution decays sufficiently fast at spatial infinity, so that the integrals arising in the forthcoming calculations will be finite. We now consider a sequence $\{u^\varepsilon\}_{\varepsilon>0}$ of classical solutions for our non-local problem. The basic tool to establish the pre-compactness of this sequence will be the velocity averaging lemma in the form established by Perthame and Souganidis in [27] which allows for a *full space-derivative* in the source term. We present it here in a version restricted to our particular situation.

**Theorem 2.2** (Averaging lemma)**.** *Let $\{f_n\}$ and $\{g_{i,n}\}$, $i = 1, \dots, d$ be sequences of solutions to the transport equation*

$$\partial_t f_n + \mathbf{a}(\xi) \cdot \nabla_x f_n = \sum_{i=1}^{d} \partial_{x_i} \partial_\xi^k g_{i,n} \tag{2.4}$$

*where $k$ is an arbitrary integer and the smooth velocity coefficient $\mathbf{a} = \mathbf{a}(\xi)$ satisfies the non-degeneracy condition* (2.1)*.*
*Then, provided $\{f_n\}$ is bounded in $L^q_{loc}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$ for some $1 < q < \infty$, and that $\{g_{i,n}\}$ is pre-compact in $L^q_{loc}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$, the average*

$$\int f_n(\boldsymbol{x}, t, \xi)\Psi(\xi) d\xi \quad \textit{is pre-compact in } L^q_{loc}(\mathbb{R}^d \times \mathbb{R}_+),$$

*for any $\Psi \in C_0^\infty(\mathbb{R})$.*

## 3. A priori estimates

To achieve the desired compactness properties we need establish first several a priori estimates. In particular, we are able to relate the $L^2$-norm of the non-local dissipative term $\phi_\varepsilon * u^\varepsilon - u^\varepsilon$ to the $L^2$-norm of the gradient of $u^\varepsilon$.

**Lemma 3.1** (Energy estimate)**.** *Suppose that the assumptions* (2.1)–(2.3) *hold. Then, for all $t > 0$ and any solution $\{u^\varepsilon\}_{\varepsilon>0}$ of (1.3), (1.5):*

$$\frac{1}{2}\|u^\varepsilon(\cdot, t)\|^2_{L^2(\mathbb{R}^d)} + \varepsilon \sum_{i=1}^{d} \|\partial_{x_i} u^\varepsilon\|^2_{L^2(\Omega_t)} = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{3.8}$$

*Proof.* To establish (3.8) we multiply (1.3), (1.5) by $u^\varepsilon$, and integrate with respect to the space and time variables:

$$\begin{aligned} &\frac{1}{2}\|u^\varepsilon(\cdot, t)\|^2_{L^2(\mathbb{R}^d)} + \varepsilon \sum_{i=1}^{d} \|\partial_{x_i} u^\varepsilon\|^2_{L^2(\Omega_t)} \\ &\qquad = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)} + \lambda \sum_{i=1}^{d} \int_0^t \int_{\mathbb{R}^d} u^\varepsilon(\mathbf{x}, s) \left((\phi_\varepsilon * u^\varepsilon(\cdot, s))(\mathbf{x}) - u^\varepsilon(\mathbf{x}, s)\right)_{x_i} d\mathbf{x} ds. \end{aligned}$$

Note that the classical solution $u^\varepsilon$ and its derivatives decay to 0 for $|\mathbf{x}| \to \infty$ such that the boundary integrals in the last relation vanish.

To conclude it remains to observe that the last term in the above identity vanishes. Actually, let $w \in C^1(\mathbb{R}^d)$ be a function tending to 0 for $|\mathbf{x}| \to \infty$ and $j \in \{1, \dots, d\}$.

Since $\phi_\varepsilon$ is even we can write

$$\begin{aligned}\int_{\mathbb{R}^d} w(\mathbf{x})\left((\phi_\varepsilon * w)(\mathbf{x})\right)_{x_i} d\mathbf{x} &= -\int_{\mathbb{R}^d} w(\mathbf{y}) \int_{\mathbb{R}^d} \phi_\varepsilon(\mathbf{x}-\mathbf{y}) w_{x_i}(\mathbf{x})\, d\mathbf{x}d\mathbf{y} \\ &= -\int_{\mathbb{R}^d} w(\mathbf{y})(\phi_\varepsilon * w_{x_i})(\mathbf{y})\, d\mathbf{y} \\ &= -\int_{\mathbb{R}^d} w(\mathbf{y})\left((\phi_\varepsilon * w)(\mathbf{y})\right)_{y_j} d\mathbf{y}.\end{aligned} \tag{3.9}$$

The last integral is simply minus the original one, and therefore

$$\int_{\mathbb{R}^d} w(\mathbf{x})\left((\phi_\varepsilon * w)(\mathbf{x})\right)_{x_i} d\mathbf{x} = 0, \quad i = 1, \dots, d. \tag{3.10}$$

□

In the next lemma we estimate the convolution term in terms of derivatives of $u^\varepsilon$. The scaling with respect to $\varepsilon$ exhibited here will be important in the subsequent analysis.

**Lemma 3.2** (Uniform estimate for the non-local operator)**.** *Suppose that the assumptions* (2.1)–(2.3) *hold. Then, there is a constant $C > 0$ such that for any s $w \in C^1(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d)$ such that*

$$\|\phi_\varepsilon * w - w\|_{L^2(\mathbb{R}^d)} \le C\,\varepsilon \sum_{i=1}^{d} \|\partial_{x_i} w\|_{L^2(\mathbb{R}^d)}.$$

*The constant $C$ depends $\phi$ but is independent of the parameter $\varepsilon$.*

*Proof.* Let $\mathbf{x} \in \mathbb{R}^d$ be arbitrary but fixed. Set $B_\varepsilon(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^d \mid |\mathbf{x}-\mathbf{y}| \le \varepsilon\}$ and consider the function $I : \mathbb{R}^d \to \mathbb{R}$ with

$$I(\mathbf{x}) := (\phi_\varepsilon * w)(\mathbf{x}) - w(\mathbf{x}) = \int_{\mathbb{R}} \phi_\varepsilon(\mathbf{x}-\mathbf{y})(w(\mathbf{x}) - w(\mathbf{y}))d\mathbf{y}.$$

We now rely on Assumption 1 (ii), the fact that $\mathrm{supp}(\phi_\varepsilon) \subset B_\varepsilon(0)$, and the following Morrey-type inequality whose proof is elementary:

$$|w(\mathbf{x}) - w(\mathbf{y})| \le C_1 \varepsilon^{1-\frac{d}{2}} \left(\int_{B_{2\varepsilon}(\mathbf{x})} |\nabla w(\mathbf{z})|^2 d\mathbf{z}\right)^{1/2},$$

for $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{y} \in B_\varepsilon(\mathbf{x})$. This leads us to the following estimate

$$\begin{aligned}|I(\mathbf{x})| &\le \int_{B_\varepsilon(\mathbf{x})} |\phi_\varepsilon(\mathbf{x}-\mathbf{y})|\,|(w(\mathbf{x}) - w(\mathbf{y}))|d\mathbf{y} \\ &\le C_1 \varepsilon^{1-\frac{d}{2}} \int_{B_\varepsilon(\mathbf{x})} |\phi_\varepsilon(\mathbf{x}-\mathbf{y})| \left(\int_{B_{2\varepsilon}(\mathbf{x})} |\nabla w(\mathbf{z})|^2 d\mathbf{z}\right)^{1/2} d\mathbf{y} \\ &= C_2 \varepsilon^{1-\frac{d}{2}} \left(\int_{B_{2\varepsilon}(\mathbf{x})} |\nabla w(\mathbf{z})|^2 d\mathbf{z}\right)^{1/2}.\end{aligned}$$

For $\mathbf{x} = (x_1, \dots, x_d)^T$ and $Q_\epsilon(\mathbf{x}) := (x_1 - 2\epsilon, x_1 + 2\epsilon) \times \ldots \times (x_d - 2\epsilon, x_d + 2\epsilon)$, we introduce the one-to-one map $\mathbf{g}(\cdot, \mathbf{x}) : \mathbb{R}^d \to Q_\epsilon(\mathbf{x})$ whose components $g_1, \dots, g_d$ are defined by

$$z_j := g_j(\tilde{\mathbf{z}}, \mathbf{x}) := x_j + \epsilon s(\tilde{z}_j)$$

and $\mathbf{z} = (z_1, \ldots, z_d)^T$. By $s \in C^1(\mathbb{R})$ we denote an arbitrary function mapping $\mathbb{R}$ to $(-1,1)$ with $s' \neq 0$ and $s' \in L^1(\mathbb{R})$.
We integrate $|I(\mathbf{x})|^2$ and obtain

$$\int_{\mathbb{R}^d} |I(\mathbf{x})|^2 d\mathbf{x} \le C_2 \varepsilon^{2-d} \int_{\mathbb{R}^d} \left( \int_{Q_\varepsilon(\mathbf{x})} |\nabla w(\mathbf{z})|^2 d\mathbf{z} \right) d\mathbf{x}$$
$$= C_2 \varepsilon^{2-d} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\nabla w(g(\widetilde{\mathbf{z}}, \mathbf{x}))|^2 \, d\mathbf{x} \right) \varepsilon^d s'(\tilde{z}_1) \cdots s'(\tilde{z}_d) \, d\widetilde{\mathbf{z}}.$$

In fact, the value of the inner integral in the last term is independent of $\widetilde{\mathbf{z}}$ which leads to

$$\int_{\mathbb{R}^d} |I(\mathbf{x})|^2 \, d\mathbf{x} = C_3 \varepsilon^2 \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\nabla w(\mathbf{x})|^2 d\mathbf{x} \right) s'(\tilde{z}_1) \cdots s'(\tilde{z}_d) \, d\widetilde{\mathbf{z}} \le C_4 \varepsilon^2 \|\nabla w\|^2_{L^2(\mathbb{R}^d)}.$$

## 4. Convergence analysis

4.1. **Main convergence theorem.** Following Hwang and Tzavaras's [15], we now prove that there exists a function $u \in L^2(\mathbb{R} \times \mathbb{R}_+)$ such that for (a subsequence of) the family $\{u^\varepsilon\}_{\varepsilon>0}$

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - u\|_{L^p_{loc}(\mathbb{R}\times\mathbb{R}_+)} = 0 \qquad (p \in [1,2)).$$

Provided the flux is globally Lipschitz, this function is a weak solution of (1.1) and, depending on the scaling parameter $\lambda$ in (1.5), can be either a classical entropy solution [18] of (1.1) or a nonclassical entropy solution [21].

Our main result is:

**Theorem 4.1** (Convergence for the non-local and multi-dimensional model)**.** *Suppose that the assumptions* (2.1)–(2.3) *hold and that* $\lambda(\varepsilon) = \mathcal{O}(1)$*. Let* $\{u^\varepsilon\}_{\varepsilon>0}$ *be a family of regular solutions to (1.3), (1.5) satisfying the initial condition (1.2). Then, there exists a limiting function* $u \in L^2(\mathbb{R}^d \times \mathbb{R}_+)$ *and a subsequence of* $\{u^\varepsilon\}_{\varepsilon>0}$ *such that*

(i) $\{u^\varepsilon\}_{\varepsilon>0}$ *converges to* $u$ *(as* $\varepsilon \to 0$*) strongly in* $L^p_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$*,* $p \in [1,2)$*.*
(ii) *Provided the* $u$*-derivative of the flux* $\mathbf{f}$ *is globally bounded, the limiting function is a weak solution of (1.1).*

The proof of this result will follow directly by the subsequent two lemmas. In the first lemma, we derive an equivalent kinetic re-formulation for the nonlinear scalar conservation law (1.3), (1.5). Following Lions, Perthame, and Tadmor [25, 26], the idea is to replace the nonlinear problem by a (formally) linear equation enlarging the space of free variables. Then we demonstrate that $\{u^\varepsilon\}_{\varepsilon>0}$ is a Cauchy sequence in $L^p_{loc}(\mathbb{R} \times \mathbb{R}_+)$, $p \in [1,2)$, and finally derive the result.

The following notation will be useful. The indicator function $\mathbb{1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathbb{1}(u,\xi) = \begin{cases} 1, & 0 < \xi \le u, \\ -1, & u \le \xi < 0, \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

**Lemma 4.2** (Kinetic decomposition for the non-local model)**.** *Suppose that the assumptions* (2.1)–(2.3) *hold and that* $\lambda(\varepsilon) = \mathcal{O}(1)$*, and let* $\{u^\varepsilon\}_{\varepsilon>0}$ *be regular solutions to (1.3), (1.5) with initial condition (1.2):*

(i) *For each* $\varepsilon > 0$ *there exist distributions*

$$\pi_i^\varepsilon \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}) \quad (i = 1, \dots, d), \qquad k^\varepsilon \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$$

*such that the function*

$$\chi^\varepsilon(\boldsymbol{x}, t, \xi) := \mathbb{1}(u^\varepsilon(\boldsymbol{x}, t), \xi), \qquad (\boldsymbol{x}, t, \xi) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}. \tag{4.2}$$

*satisfies the kinetic representation*

$$\partial_t \chi^\varepsilon + \mathbf{f}'(\xi) \cdot \nabla \chi^\varepsilon = \partial_\xi k^\varepsilon + \sum_{i=1}^d \partial_{x_i} \pi_i^\varepsilon \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}). \tag{4.3}$$

(ii) *The family of distributions* $\{\pi_i^\varepsilon\}_{\varepsilon>0}$ *satisfies for* $i = 1, \dots, d$

$$\pi_i^\varepsilon \to 0 \quad \text{in} \quad L^2(\mathbb{R}^d \times \mathbb{R}_+; H^{-1}(\mathbb{R})) \quad \text{as} \quad \varepsilon \to 0, \tag{4.4}$$

*and the family* $\{k^\varepsilon\}_{\varepsilon>0}$ *is uniformly bounded in* $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$, *with*

$$\{k^\varepsilon\}_{\varepsilon>0} \text{ is pre-compact in } W^{-1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}) \text{ for } p \in \left[1, \frac{s}{s-1}\right], \; s > d + 2. \tag{4.5}$$

The expressions of the distributions $\pi_i^\varepsilon$ and $k^\varepsilon$ will be given explicitly in our proof below. Note also that the statement (4.4) can be equivalently rewritten as

$$\pi_i^\varepsilon =: \tilde{\pi}_i^\varepsilon + \partial_\xi \hat{\pi}_i^\varepsilon, \qquad \tilde{\pi}_i^\varepsilon, \hat{\pi}_i^\varepsilon \to 0 \text{ in } L^2(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}). \tag{4.6}$$

We are now in a position to apply the averaging lemma, as follows.

**Lemma 4.3** (Strong convergence property)**.** *Suppose that the assumptions* (2.1)–(2.3) *hold and* $\lambda(\varepsilon) = \mathcal{O}(1)$ *are satisfied, and let* $\{u^\varepsilon\}_{\varepsilon>0}$ *be regular solutions to (1.3), (1.5) with initial condition (1.2). Then, a subsequence of* $\{u^\varepsilon\}_{\varepsilon>0}$ *is a Cauchy sequence in* $L^1_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$ *and, furthermore, there exists a function* $u \in L^2(\mathbb{R}^d \times \mathbb{R}_+)$ *such that* $\lim_{\varepsilon\to 0} \|u^\varepsilon - u\|_{L^p_{loc}(\mathbb{R}^d \times \mathbb{R}_+)} = 0$ *for* $p \in [1, 2)$.

4.2. **Proofs of the lemmas and of the main theorem.** *Proof of Lemma 4.2.* We begin by establishing the entropy-like indentity

$$\begin{aligned} -\int_{\mathbb{R}^d}\int_0^T\int_{\mathbb{R}} &\left(\chi^\varepsilon(\mathbf{x}, t, \xi)\varphi_t(\mathbf{x}, t) + \sum_{i=1}^d \chi^\varepsilon(\mathbf{x}, t, \xi) f_i'(\xi)\varphi_{x_i}(\mathbf{x}, t)\right)\eta'(\xi)\, d\xi dt d\mathbf{x} \\ &= -\int_{\mathbb{R}^d}\int_0^T \sum_{i=1}^d H_i^\varepsilon(\mathbf{x}, t)\eta'(u^\varepsilon)\varphi_{x_i}(\mathbf{x}, t)\, dt d\mathbf{x} \\ &\quad -\int_{\mathbb{R}^d}\int_0^T G^\varepsilon(\mathbf{x}, t)\varphi(\mathbf{x}, t)\eta''(u^\varepsilon)\, dt d\mathbf{x}, \end{aligned} \tag{4.7}$$

where we defined

$$H_i^\varepsilon(\mathbf{x}, t) := \varepsilon \partial_{x_i} u^\varepsilon(\mathbf{x}, t) + \lambda\left(\phi_\varepsilon(\mathbf{x}) * u^\varepsilon(\mathbf{x}, t) - u^\varepsilon(\mathbf{x}, t)\right), \tag{4.8}$$

$$G^\varepsilon(\mathbf{x}, t) := \sum_{i=1}^d \left(\varepsilon(\partial_{x_i} u^\varepsilon(\mathbf{x}, t))^2 + \lambda\left(\phi_\varepsilon(\mathbf{x}) * u^\varepsilon(\mathbf{x}, t) - u^\varepsilon(\mathbf{x}, t)\right)\partial_{x_i} u^\varepsilon(\mathbf{x}, t)\right). \tag{4.9}$$

Let $(\eta, \mathbf{q})$ be an entropy pair to (1.1). By an entropy pair we mean smooth functions $\eta : \mathbb{R} \to \mathbb{R}$ and $\mathbf{q} : \mathbb{R} \to \mathbb{R}^d$ that satisfy the compatibility condition $\eta' f_i' = q_i'$,

$i = 1, \dots, d$. Note that convexity of $\eta$ is not required. Inserting (1.5) in (1.3) and multiplying with $\eta'(u^\varepsilon)$, we obtain

$$\begin{aligned}\partial_t \eta(u^\varepsilon) + \operatorname{div} \mathbf{q}(u^\varepsilon) &= \varepsilon \sum_{i=1}^d \partial_{x_i} \left(\eta'(u^\varepsilon)\partial_{x_i} u^\varepsilon\right) - \varepsilon \eta''(u^\varepsilon) \sum_{i=1}^d (\partial_{x_i} u^\varepsilon)^2 \\ &+ \lambda \sum_{i=1}^d \left[\eta'(u^\varepsilon)\left(\phi_\varepsilon * u^\varepsilon - u^\varepsilon\right)\right]_{x_i} - \lambda \eta''(u^\varepsilon)\left(\phi_\varepsilon * u^\varepsilon - u^\varepsilon\right) \sum_{i=1}^d \partial_{x_i} u^\varepsilon.\end{aligned} \tag{4.10}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ be a test function and regard $\eta \in C_0^\infty(\mathbb{R})$ as a test function, too. With the indicator function $\mathbb{1}$ from (4.1) we can rewrite the entropy pair as follows

$$\begin{aligned}\eta(u) - \eta(0) &= \int_{\mathbb{R}} \mathbb{1}(u,\xi)\eta'(\xi)\, d\xi, \\ q_j(u) - q_j(0) &= \int_{\mathbb{R}} \mathbb{1}(u,\xi) f_j'(\xi)\eta'(\xi)\, d\xi.\end{aligned}$$

Now we integrate (4.10) with respect to space and time and obtain the desired equation (4.7).

Next, we show that $G^\varepsilon$, which is defined in (4.9), is uniformly bounded in $L^1(\mathbb{R}^d \times \mathbb{R}_+)$. We have

$$\begin{aligned}&\int_{\mathbb{R}^d} \int_0^T \left|G^\varepsilon(\mathbf{x},t)\right| dt d\mathbf{x} = \int_{\mathbb{R}^d} \int_0^T \Big|\sum_{i=1}^d \left(\varepsilon(\partial_{x_i} u^\varepsilon)^2 + \lambda\left(\phi_\varepsilon * u^\varepsilon - u^\varepsilon\right)\partial_{x_i} u^\varepsilon\right)\Big| \, dt d\mathbf{x} \\ &\le \int_{\mathbb{R}^d} \int_0^T \varepsilon \sum_{i=1}^d \left|\partial_{x_i} u^\varepsilon\right|^2 dt d\mathbf{x} + \lambda \int_{\mathbb{R}^d} \int_0^T \Big|\left(\phi_\varepsilon * u^\varepsilon - u^\varepsilon\right) \sum_{i=1}^d \partial_{x_i} u^\varepsilon\Big| \, dt d\mathbf{x} \\ &=: J_1^\varepsilon + J_2^\varepsilon\,.\end{aligned}$$

We can estimate $J_1^\varepsilon$ by a constant independent of $\varepsilon$, because with Lemma 3.1 we have $\varepsilon \sum_{i=1}^d \left(\partial_{x_i} u^\varepsilon(\mathbf{x},t)\right)^2 \in L^1(\mathbb{R}^d \times \mathbb{R}_+)$. Using again Lemma 3.1, Lemma 3.2, and Hlder's inequality we obtain

$$\begin{aligned}J_2^\varepsilon &= \lambda \Big\|\left(\phi_\varepsilon * u^\varepsilon - u^\varepsilon\right) \sum_{i=1}^d \partial_{x_i} u^\varepsilon\Big\|_{L^1(\mathbb{R}^d \times \mathbb{R}_+)} \\ &\le \lambda \left\|\phi_\varepsilon * u^\varepsilon - u^\varepsilon\right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} \Big\|\sum_{i=1}^d \partial_{x_i} u^\varepsilon\Big\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} \\ &\le \lambda \left(C\varepsilon \sum_{i=1}^d \left\|\partial_{x_i} u^\varepsilon\right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)}\right)\left(\sum_{i=1}^d \left\|\partial_{x_i} u^\varepsilon\right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)}\right),\end{aligned}$$

thus

$$J_2^\varepsilon \le \lambda C \varepsilon \frac{1}{\sqrt{\varepsilon}} \frac{1}{\sqrt{\varepsilon}} \le C,$$

which leads to

$$\int_{\mathbb{R}^d} \int_0^T \left|G^\varepsilon(\mathbf{x},t)\right| \, dt d\mathbf{x} < \infty. \tag{4.11}$$

We proceed with the derivation of the kinetic formulation (4.3) for the nonlinear scalar conservation law (1.3), (1.5). Define the mappings $k^\varepsilon$ and $\pi_i^\varepsilon$, $i=1,\dots,d$, by

(4.12)
$$k^\varepsilon : \begin{cases} \mathcal{D}(\mathbb{R}^d\times\mathbb{R}_+)\times\mathcal{D}(\mathbb{R}) & \to \quad \mathbb{R} \\ \varphi\otimes\eta' & \mapsto \begin{cases} \langle k^\varepsilon, \varphi(\mathbf{x},t)\otimes\eta'(\xi)\rangle \\ \quad = \int_{\mathbb{R}^d}\int_0^T G^\varepsilon(\mathbf{x},t)\varphi(\mathbf{x},t)\eta'(u^\varepsilon(\mathbf{x},t))\,dtd\mathbf{x} \end{cases} \end{cases}$$

and

(4.13)
$$\pi_i^\varepsilon : \begin{cases} \mathcal{D}(\mathbb{R}^d\times\mathbb{R}_+)\times\mathcal{D}(\mathbb{R}) & \to \quad \mathbb{R} \\ \varphi\otimes\eta' & \mapsto \begin{cases} \langle \pi_i^\varepsilon, \varphi(\mathbf{x},t)\otimes\eta'(\xi)\rangle \\ \quad = \int_{\mathbb{R}^d}\int_0^T H_i^\varepsilon(\mathbf{x},t)\varphi(\mathbf{x},t)\eta'(u^\varepsilon(\mathbf{x},t))\,dtd\mathbf{x}. \end{cases} \end{cases}$$

Since the space generated by $\varphi\otimes\eta'$ is dense in $\mathcal{D}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$, we can extend the mappings (4.12) and (4.13) to test functions $\theta\in\mathcal{D}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$. The two mappings $k^\varepsilon$ and $\pi_i^\varepsilon$ are linear and continuous. As the linearity is obvious we only have to verify the continuity.
For $\theta\in\mathcal{D}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$ we have

$$|\langle k^\varepsilon,\theta_n\rangle| = \left|\int_{\mathbb{R}^d}\int_0^T G^\varepsilon(\mathbf{x},t)\theta(\mathbf{x},t,u^\varepsilon(\mathbf{x},t))\,dtd\mathbf{x}\right| \le \sup_{\mathbf{x},t,\xi}|\theta_n(\mathbf{x},t,\xi)|\,\|G^\varepsilon\|_{L^1(\mathbb{R}^d\times\mathbb{R}_+)}\,.$$

The $L^1$-bound (4.11) implies the boundedness.

Applying the estimate $|\langle\pi_i^\varepsilon,\theta\rangle| \le C\sqrt{\varepsilon}\,\|\theta\|_{L^2(\mathbb{R}^d\times\mathbb{R}_+)}$, which we deduce in (4.16) below, the continuity of $\pi_i^\varepsilon$ follows analogously. Accordingly $k^\varepsilon$ and $\pi_i^\varepsilon$, $i=1,\dots,d$, are distributions in $\mathcal{D}'(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$.
We compute the partial derivatives of these distributions as follows:

$$\begin{aligned}\langle\partial_\xi k^\varepsilon,\varphi(\mathbf{x},t)\otimes\eta'(\xi)\rangle &= -\langle k^\varepsilon,\varphi(\mathbf{x},t)\otimes\eta''(\xi)\rangle \\ &= -\int_{\mathbb{R}^d}\int_0^T G^\varepsilon(\mathbf{x},t)\varphi(\mathbf{x},t)\eta''(u^\varepsilon(\mathbf{x},t))\,dtd\mathbf{x}\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\langle\partial_{x_i}\pi_i^\varepsilon,\varphi(\mathbf{x},t)\otimes\eta'(\xi)\rangle &= -\langle\pi_i^\varepsilon,\varphi_{x_i}(\mathbf{x},t)\otimes\eta'(\xi)\rangle \\ &= -\int_{\mathbb{R}^d}\int_0^T H_i^\varepsilon(\mathbf{x},t)\varphi_{x_i}(\mathbf{x},t)\eta'(u^\varepsilon(\mathbf{x},t))\,dtd\mathbf{x}.\end{aligned} \tag{4.15}$$

Equipped with (4.14) and (4.15) we can rewrite the entropy equality (4.7) as

$$\begin{aligned}&\langle\partial_t\chi^\varepsilon+\mathbf{f}'(\xi)\cdot\nabla\chi^\varepsilon,\varphi(\mathbf{x},t)\otimes\eta'(\xi)\rangle\\ &=\langle\partial_\xi k^\varepsilon,\varphi(\mathbf{x},t)\otimes\eta'(\xi)\rangle+\sum_{i=1}^d\langle\partial_{x_i}\pi_i^\varepsilon,\varphi(\mathbf{x},t)\otimes\eta'(\xi)\rangle\,.\end{aligned}$$

With the argument above, namely $\varphi\otimes\eta'$ is dense in $\mathcal{D}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$, we obtain for all $\theta\in\mathcal{D}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$

$$\langle\partial_t\chi^\varepsilon+\mathbf{f}'(\xi)\cdot\nabla\chi^\varepsilon,\theta(\mathbf{x},t,\xi)\rangle=\langle\partial_\xi k^\varepsilon,\theta(\mathbf{x},t,\xi)\rangle+\sum_{i=1}^d\langle\partial_{x_i}\pi_i^\varepsilon,\theta(\mathbf{x},t,\xi)\rangle\,,$$

which is the desired kinetic formulation (4.3).
We proceed to verify the regularity statements in (ii) concerning the mappings $\pi_i^\varepsilon$ and

$k^\varepsilon$. Starting with $\pi_i^\varepsilon$, $i = 1, \dots, d$, and applying Hlder's inequality, Lemma 3.1 and Lemma 3.2, we obtain

$$\begin{aligned}
&|\langle \pi_i^\varepsilon, \theta \rangle| \\
&= \left| \int_{\mathbb{R}^d} \int_0^T H_i^\varepsilon(\mathbf{x}, t) \theta(\mathbf{x}, t, u^\varepsilon(\mathbf{x}, t)) \, dt d\mathbf{x} \right| \\
&\leq \left\| \big( \varepsilon \partial_{x_i} u^\varepsilon + \lambda \left( \phi_\varepsilon * u^\varepsilon - u^\varepsilon \right) \big) \theta(\mathbf{x}, t, u^\varepsilon(\mathbf{x}, t)) \right\|_{L^1(\mathbb{R}^d \times \mathbb{R}_+)} \\
&\leq \Big( \left\| \varepsilon \partial_{x_i} u^\varepsilon \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} + \lambda \left\| \phi_\varepsilon * u^\varepsilon - u^\varepsilon \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} \Big) \left\| \theta(\mathbf{x}, t, u^\varepsilon(\mathbf{x}, t)) \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)},
\end{aligned}$$

thus

(4.16)

$$\begin{aligned}
&|\langle \pi_i^\varepsilon, \theta \rangle| \\
&\leq \left( \sqrt{\varepsilon} \sqrt{\varepsilon} \left\| \partial_{x_i} u^\varepsilon \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} + C\varepsilon \sum_{i=1}^d \left\| \partial_{x_i} u^\varepsilon \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} \right) \left\| \theta(\mathbf{x}, t, u^\varepsilon(\mathbf{x}, t)) \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} \\
&\leq \left( \sqrt{\varepsilon} C_1 + C_2 \varepsilon \frac{1}{\sqrt{\varepsilon}} \right) \left\| \theta(\mathbf{x}, t, u^\varepsilon(\mathbf{x}, t)) \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} \\
&= C\sqrt{\varepsilon} \left\| \theta(\mathbf{x}, t, u^\varepsilon(\mathbf{x}, t)) \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)}
\end{aligned}$$

for all $\theta \in \mathcal{D}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$.

Furthermore, we have

$$\begin{aligned}
\left\| \theta(\mathbf{x}, t, u^\varepsilon(x, t)) \right\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} &= \int_{\mathbb{R}^d} \int_0^T \theta^2(\mathbf{x}, t, u^\varepsilon) \, dt d\mathbf{x} \\
&= \int_{\mathbb{R}^d} \int_0^T \int_{-\infty}^{u^\varepsilon(\mathbf{x}, t)} 2\theta\theta_\xi \, d\xi dt d\mathbf{x} \\
&\leq 2 \int_{\mathbb{R}^d} \int_0^T \left( \int_{-\infty}^{u^\varepsilon} \theta^2 d\xi \right)^{1/2} \left( \int_{-\infty}^{u^\varepsilon} (\theta_\xi)^2 d\xi \right)^{1/2} dt d\mathbf{x},
\end{aligned}$$

so

$$\begin{aligned}
\left\| \theta(\mathbf{x}, t, u^\varepsilon(x, t)) \right\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} &\leq \int_{\mathbb{R}^d} \int_0^T \left( \int_{-\infty}^{u^\varepsilon} \theta^2 d\xi \right) + \left( \int_{-\infty}^{u^\varepsilon} (\theta_\xi)^2 d\xi \right) dt d\mathbf{x} \\
&\leq \int_{\mathbb{R}^d} \int_0^T \left( \int_{-\infty}^{\infty} \theta^2 d\xi \right) + \left( \int_{-\infty}^{\infty} (\theta_\xi)^2 d\xi \right) dt d\mathbf{x} \\
&= \left\| \theta \right\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}_+; H_0^1(\mathbb{R}))}.
\end{aligned}$$

This leads to

$$|\langle \pi_i^\varepsilon, \theta \rangle| \leq C\sqrt{\varepsilon} \left\| \theta \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}_+; H^1(\mathbb{R}))}, \tag{4.17}$$

which is (4.4).

Next, we consider the term $k^\varepsilon$ and verify (4.5). We recall that the $L^1$-bound (4.11) implies

$$|\langle k^\varepsilon, \theta\rangle| = \left| \int_{\mathbb{R}^d} \int_0^T G^\varepsilon(\mathbf{x},t)\theta(\mathbf{x},t,u^\varepsilon(\mathbf{x},t))dtd\mathbf{x} \right|$$
$$\leq \sup_{\mathbf{x},t,\xi} |\theta(\mathbf{x},t,\xi)| \, \|G^\varepsilon\|_{L^1(\mathbb{R}^d\times\mathbb{R}_+)}$$

for all $\theta \in \mathcal{D}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$. Thus $k^\varepsilon$ lies in particular in the space of bounded measures $\mathcal{M}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$, the dual space of $C^0(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$ (the space of continuous functions, which vanish at infinity). Now let $V \subset \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}$ be a bounded set with $\overline{V} \subset \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}$. Since $C^0_0(\overline{V}) \subset C^0(\overline{V})$ we have

$$\mathcal{M}(\overline{V}) \cong \left(C^0(\overline{V})\right)' \subset \left(C^0_0(\overline{V})\right)'.$$

The Sobolev embedding theorem implies that $W^{1,s}_0(V)$ is compactly embedded in $C^0_0(\overline{V})$ for every bounded open subset $V \subset \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}$ and $s > d+2$. We deduce for the dual space the compact embedding

$$\left(C^0_0(\overline{V})\right)' \subset W_0^{-1,\frac{s}{s-1}}(V)$$

for $s > d+2$. Let $V_k$ be an increasing sequence of sets covering the whole of $\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}$ and argue for every $V_k$, $k \in \mathbb{N}$ as above. Thus we find (4.5). □

*Proof of Lemma 4.3.* For some arbitrary but fixed $\Psi \in C^\infty_0(\mathbb{R})$ we define the sequence $\{v^\varepsilon\}_{\varepsilon>0}$ through

$$v^\varepsilon(\mathbf{x},t) := \int_{\mathbb{R}} \chi^\varepsilon(\mathbf{x},t,\xi)\Psi(\xi)\,d\xi = \int_{\mathbb{R}} \mathbb{1}(u^\varepsilon,\xi)\Psi(\xi)\,d\xi.$$

We will show at first that there is a function $v \in L^p_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$ so that $v^\varepsilon \to v$ converges in $L^p_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$, $1 \leq p < \frac{s}{s-1}$, $s > d+2$ (in the sense of a subsequence).

In summary, from Lemmas 4.2 and (4.6) we have the kinetic formulation

$$\text{(4.18)} \qquad \partial_t\chi^\varepsilon + \mathbf{f}'(\xi)\cdot\nabla\chi^\varepsilon = \partial_\xi k^\varepsilon + \sum_{i=1}^d \partial_{x_i}\left(\tilde{\pi}^\varepsilon_i + \partial_\xi\hat{\pi}^\varepsilon_i\right) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}),$$

where $\tilde{\pi}^\varepsilon_i, \hat{\pi}^\varepsilon_i \to 0$ in $L^2(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$, $k^\varepsilon$ is bounded in $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$ and pre-compact in $W^{-1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$, $1 \leq p < \frac{s}{s-1}$, $s > d+2$.

To make use of the averaging lemma 2.2 we have to show that we can understand (4.18) as being in the form (2.4) identifying $\chi^\varepsilon$ with $f_n$ and setting $\mathbf{a} = \mathbf{f}'$. Obviously, with notation (4.2), we find for the indicator function $\chi^\varepsilon = \mathbb{1}(u^\varepsilon,\xi) \in L^q_{loc}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$, $q \in [1,2]$. For the right-hand side in (4.18) we observe:

(i) Since $\hat{\pi}^\varepsilon_i \to 0$ in $L^2(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$ it follows that $\hat{\pi}^\varepsilon_i$ is pre-compact in $L^2(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$ and therefore $\hat{\pi}^\varepsilon_i$ is pre-compact in $L^q_{loc}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$, $q \in [1,2]$. Thus $\partial_{x_i}\hat{\pi}^\varepsilon_i$ is pre-compact in $W^{-1,q}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$, $q \in [1,2]$. In the same manner it follows that $\partial_\xi\hat{\pi}^\varepsilon_i$ is pre-compact in $W^{-1,q}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$, $q \in [1,2]$.

(ii) By Lemma 4.2 the distribution $k^\varepsilon$ is in fact pre-compact in $W^{-1,p}_{loc}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R})$ for $1 \leq p < \frac{s}{s-1}$, $s > d+2$. Choosing $q = p$ we can therefore see the term $k^\varepsilon$ as $\partial_{x_i}\hat{\pi}^\varepsilon_i$.

(iii) Since $\tilde{\pi}_i^\varepsilon \to 0$ in $L^2(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$ we can conclude as in (i) that $\tilde{\pi}_i^\varepsilon$ is pre-compact in $L^2_{loc}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$ and accordingly pre-compact in $W^{-1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$, $1 \le p \le 2$. Thus we can see the term $\tilde{\pi}_i^\varepsilon$ as $\partial_\xi \hat{\pi}_i^\varepsilon$ in (ii).

With the aid of (i)-(iii) we can now apply the averaging lemma and achieve

$$v^\varepsilon = \int_{\mathbb{R}} \chi^\varepsilon(\mathbf{x}, t, \xi)\Psi(\xi)\, d\xi = \int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\Psi(\xi)\, d\xi \quad \text{is pre-compact in } L^p_{loc}(\mathbb{R}^d \times \mathbb{R}_+),$$

$1 < p < \frac{s}{s-1}$. As a consequence a subsequence of $\{v^\varepsilon\}_{\varepsilon>0}$ converges to some $v$ in $L^p_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$. In addition we can deduce

$$\lim_{\varepsilon\to 0} \|v^\varepsilon - v\|_{L^q_{loc}(\mathbb{R}^d\times\mathbb{R}_+)} = 0 \quad \text{for} \quad 1 \le q \le p. \tag{4.19}$$

Next, we show that a subequence of $\{u^\varepsilon\}_{\varepsilon>0}$ is a Cauchy sequence in $L^1_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$ with limit $v \in L^2(\mathbb{R}^d \times \mathbb{R}_+)$.
We have

$$v^\varepsilon = \int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\Psi(\xi)\, d\xi \;\; \in \;\; L^2(\mathbb{R}^d \times \mathbb{R}_+)$$

since

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_0^T \left|\int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\Psi(\xi)\, d\xi\right|^2 dt d\mathbf{x} &\le \|\Psi\|^2_{C^0} \int_{\mathbb{R}^d}\int_0^T \left|\int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\, d\xi\right|^2 dt d\mathbf{x} \\
&= \|\Psi\|^2_{C^0} \int_{\mathbb{R}^d}\int_0^T |u^\varepsilon|^2\, dt d\mathbf{x} \\
&\le C\, \|u^\varepsilon\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}_+)} \le C.
\end{aligned}$$

We show that

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - v^\varepsilon\|_{L^1(\mathbb{R}^d\times\mathbb{R}_+)} = 0. \tag{4.20}$$

Let $R > 0$ and $\Psi \in C_0^\infty(\mathbb{R})$ such that $\Psi = 1$ on $(-R, R)$ and $0 \le \Psi \le 1$. Then, we obtain

$$\begin{aligned}
\left|u^\varepsilon - \int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\Psi(\xi)\, d\xi\right| &= \left|\int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\, d\xi - \int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\Psi(\xi)\, d\xi\right| \\
&= \left|\int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\, (1 - \Psi(\xi))\; d\xi\right| \\
&\le \int_{-\infty}^{-R} |\mathbb{1}(u^\varepsilon, \xi)|\; d\xi + \int_R^\infty |\mathbb{1}(u^\varepsilon, \xi)|\; d\xi \\
&= (u^\varepsilon + R)^- + (u^\varepsilon - R)^+,
\end{aligned}$$

which leads to

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_0^T \left|u^\varepsilon - \int_{\mathbb{R}} \mathbb{1}(u^\varepsilon, \xi)\Psi(\xi)\, d\xi\right| dt d\mathbf{x} &\le \int_{\mathbb{R}^d}\int_0^T (u^\varepsilon + R)^- + (u^\varepsilon - R)^+\, dt d\mathbf{x} \\
&\le \int\int_{|u^\varepsilon|>R} |u^\varepsilon|\, dt d\mathbf{x} \\
&\le \frac{1}{R}\int_{\mathbb{R}^d}\int_0^T |u^\varepsilon|^2\, dt d\mathbf{x} \le \frac{C}{R}
\end{aligned}$$

since $u^\varepsilon \in L^2(\mathbb{R}^d \times \mathbb{R}_+)$. Choosing $R = \frac{1}{\varepsilon}$ we immediatly get (4.20), and $\{u^\varepsilon\}_{\varepsilon>0}$ is a Cauchy sequence in $L^1_{loc}(\mathbb{R}^d \times \mathbb{R}_+)$.

We use (4.19), (4.20) and consider the estimate

$$\|u^\varepsilon - v\|_{L^1(K)} \le \|u^\varepsilon - v^\varepsilon\|_{L^1(K)} + \|v^\varepsilon - v\|_{L^1(K)} \to 0 \quad \text{as} \quad \varepsilon \to 0, K \subset\subset \mathbb{R}^d \times \mathbb{R}_+.$$

We conclude that $\{u^\varepsilon\}_{\varepsilon>0}$ converges to the limit $v \in L^2(\mathbb{R}^d \times \mathbb{R}_+)$ because $u$ is bounded in $L^\infty((0,T); L^2(\mathbb{R}^d)) \subset L^2(\mathbb{R}^d \times \mathbb{R}_+)$ and $L^2(\mathbb{R}^d \times \mathbb{R}_+)$ is a Banach space. Therefore the first statement of the lemma follows with $u := v$.

It remains to show that $\lim_{\varepsilon\to 0} \|u^\varepsilon - u\|_{L^p_{loc}(\mathbb{R}^d\times\mathbb{R}_+)} = 0$ holds for $p \in [1, 2)$. This is a direct consequence of interpolation theory: For $r \in (1, 2)$ there is a $\Theta$ so that

$$\frac{1}{r} = \frac{\Theta}{2} + \frac{1-\Theta}{1}$$

and with the interpolation inequality for $L^p$−norms we can estimate

$$\begin{aligned}
\|u^\varepsilon - u\|_{L^r(K)} &\le \|u^\varepsilon - u\|^{\Theta}_{L^2(K)} \|u^\varepsilon - u\|^{1-\Theta}_{L^1(K)} \\
&\le \left(\|u^\varepsilon\|_{L^2(K)} + \|u\|_{L^2(K)}\right)^{\Theta} \|u^\varepsilon - u\|^{1-\Theta}_{L^1(K)} \\
&\le C \|u^\varepsilon - u\|^{1-\Theta}_{L^1(K)}
\end{aligned}$$

for every $K \subset\subset \mathbb{R}^d \times \mathbb{R}_+$. We conclude that $\|u^\varepsilon - u\|_{L^r(K)} \le C \|u^\varepsilon - u\|^{1-\Theta}_{L^1(K)} \to 0$ if $\varepsilon \to 0$. The lemma is proven. □

*Proof of Theorem 4.1.* Lemma 4.3 gives directly the convergence of $\{u^\varepsilon\}_{\varepsilon>0}$ and the existence of a limiting function $u$. It remains to check that the limit $u$ is a weak solution to the conservation law (1.1). The approximates $u^\varepsilon$ satisfy for all $\varphi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+)$

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_0^T u^\varepsilon(\mathbf{x},t)\varphi_t + \sum_{i=1}^d f_i(u^\varepsilon(\mathbf{x},t))\varphi_{x_i}\, dtd\mathbf{x} \\
= \quad & -\int_{\mathbb{R}^d}\int_0^T \varepsilon u^\varepsilon \Delta\varphi\, dtd\mathbf{x} + \lambda \int_{\mathbb{R}^d}\int_0^T (\phi_\varepsilon * u^\varepsilon - u^\varepsilon) \sum_{i=1}^d \varphi_{x_i}\, dtd\mathbf{x} \\
=: \quad & D_1^\varepsilon[\varphi] + D_2^\varepsilon[\varphi].
\end{aligned} \tag{4.21}$$

To show that the right-hand side in (4.21) vanishes for $\varepsilon \to 0$ note that $D_1^\varepsilon[\varphi] \to 0$ due to the uniform $L^2$-estimate from Lemma 3.1, and $D_2^\varepsilon[\varphi] \to 0$ follows from

$$\begin{aligned}
|D_2^\varepsilon[\varphi]| &\le \lambda \int_{\mathbb{R}^d}\int_0^T \left|(\phi_\varepsilon * u^\varepsilon - u^\varepsilon) \sum_{i=1}^d \varphi_{x_i}\right| dtd\mathbf{x} \\
&\le \lambda \|\phi_\varepsilon * u^\varepsilon - u^\varepsilon\|_{L^2(\mathbb{R}^d\times\mathbb{R}_+)} \left\|\sum_{i=1}^d \varphi_{x_i}\right\|_{L^2(\mathbb{R}^d\times\mathbb{R}_+)} \le C\sqrt{\varepsilon}\,,
\end{aligned}$$

using Lemma 3.2. □

## 5. Characterization of the limiting solutions

5.1. **The case of classical entropy solutions.** In this last part we discuss the limit function $u$ whose existence was established in Theorem 4.1. Depending on the scaling parameter $\lambda$ the limit can be either a Kruzkov entropy solution [18] or a nonclassical entropy solution [21]. The result below improves upon [29] where more restrictive conditions on the scaling were imposed. For such results of convergence to classical entropy solutions for local regularization, we refer to [6, 7, 17], which, instead of the kinetic formulation, rely on DiPerna's measure-valued solutions.

**Theorem 5.1** (The case of classical entropy solutions)**.** *Suppose that the assumptions* (2.1)–(2.3) *hold and the u-derivative of the flux* $\mathbf{f}$ *is globally bounded. Consider the scaling* $\lambda(\varepsilon) = o(1)$. *Let* $\{u^\varepsilon\}_{\varepsilon>0}$ *be a family of regular solutions to (1.3), (1.5) satisfying the initial condition (1.2). Then, the associated limit u is a Kruzkov's classical entropy solution of (1.1), i.e.,*

$$\partial_t \eta(u) + \operatorname{div} \boldsymbol{q}(u) \le 0 \quad \textit{in} \quad \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+)$$

*holds for all entropy pairs* $(\eta, \boldsymbol{q})$ *such that* $\eta$ *is convex and grows at most linearly in the large.*

For the proof of this statement we need the following lemma of the theory of kinetic formulations of conservation laws. (See Perthame [26] for a proof.)

**Lemma 5.2.** *The following statements are equivalent.*

(i) $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ *is a Kruzkov solution of (1.1).*

(ii) *There is a nonnegative locally bounded measure* $m \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$ *with*

$$\partial_t \chi + \mathbf{f}'(\xi) \cdot \nabla \chi = \partial_\xi m \quad \textit{in} \quad \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}),$$

*in which* $\chi(\xi) = \mathbb{1}(u, \xi)$.

*Proof of Theorem 5.1.* First of all, recall the relation (4.18). We have to pass to the limit $\varepsilon \to 0$ in (4.18). We can pass to the limit in the LHS of (4.18) since

$$\chi^\varepsilon = \mathbb{1}(u^\varepsilon, \xi) \rightharpoonup \chi = \mathbb{1}(u, \xi) \quad \text{in} \quad L^p_{loc}(\mathbb{R}^d \times \mathbb{R}_+; L^1(\mathbb{R})), 1 \le p < 2, \tag{5.1}$$

which follows from

$$\begin{aligned} |\langle \chi^\varepsilon - \chi, \theta \rangle| &= \left| \int_{\mathbb{R}^d} \int_0^T \int_{\mathbb{R}} (\chi^\varepsilon(\mathbf{x}, t, \xi) - \chi(\mathbf{x}, t, \xi)) \theta(\mathbf{x}, t, \xi) \, d\xi dt d\mathbf{x} \right| \\ &\le \int_{\mathbb{R}^d} \int_0^T \|\theta(\mathbf{x}, t, \cdot)\|_{L^\infty(\mathbb{R})} |u^\varepsilon - u| \, dt d\mathbf{x} \\ &\le \|u^\varepsilon - u\|_{L^p(\mathbb{R}^d \times \mathbb{R}_+)} \|\theta\|_{L^q(\mathbb{R}^d \times \mathbb{R}_+; L^\infty(\mathbb{R}))} . \end{aligned}$$

Furthermore, since $k^\varepsilon$ in (4.18) is bounded in $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$ and pre-compact in $W^{-1,p}_{loc}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$, $1 \le p < \frac{s}{s-1}$, $s > d + 2$ (Lemma 4.3) it follows

$$k^\varepsilon \rightharpoonup k \quad \text{weak-}\star \text{ in } \mathcal{M}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}). \tag{5.2}$$

Therefore using $\tilde{\pi}_i^\varepsilon, \pi_i^\varepsilon \to 0$ in $L^2(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R})$ the function $\chi = \mathbb{1}(u, \xi)$ satisfies the transport equation

$$\partial_t \chi + \mathbf{f}'(\xi) \cdot \nabla \chi = \partial_\xi k \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}). \tag{5.3}$$

Finally, we will see that $k$ is a nonnegative measure. Then, Lemma 5.2 implies the statement.

Let $m$ denote the weak-$\star$ limit

$$\left( \varepsilon \sum_{i=1}^d \left( u^\varepsilon_{x_i} \right)^2 \right) \rightharpoonup m \quad \text{in} \quad \mathcal{M}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}).$$

Similarly to the estimates of the term $J_2^\varepsilon$ in Lemma 4.2, we have

$$\begin{aligned}\Big\| G^\varepsilon - \varepsilon \sum_{i=1}^d \left(u^\varepsilon_{x_i}\right)^2 \Big\|_{L^1(\mathbb{R}^d\times\mathbb{R}_+)} &= \Big\| \lambda \sum_{i=1}^d \left(\phi_\varepsilon * u^\varepsilon - u^\varepsilon\right) u^\varepsilon_{x_i} \Big\|_{L^1(\mathbb{R}^d\times\mathbb{R}_+)} \\ &\le \lambda \left\| \phi_\varepsilon * u^\varepsilon - u^\varepsilon \right\|_{L^2(\mathbb{R}^d\times\mathbb{R}_+)} \sum_{i=1}^d \left\| u^\varepsilon_{x_i} \right\|_{L^2(\mathbb{R}^d\times\mathbb{R}_+)} \\ &\le \lambda C \varepsilon \frac{1}{\sqrt{\varepsilon}} \frac{1}{\sqrt{\varepsilon}} = C\lambda \to 0,\end{aligned}$$

whereas we apply $\lambda(\varepsilon) = o(1)$ for the convergence. Thus we have with $k = m \geq 0$ a nonnegative limit. □

5.2. **The case of nonclassical entropy solutions.** For scalings beyond $\lambda = \lambda(\varepsilon)$ the limit $k$ in (5.3) is a bounded measure but it need not be nonnegative. In general, the measure $k$ is not nonnegative if $\lambda = \gamma$ in the regularization (1.3) with (1.5). Thus, the non-local diffusion-dispersion approximation (1.3), (1.5) can generate nonclassical shocks and does not converge to the classical entropy solution.

Namely, this conclusion is supported by numerical experiments shown in Figure 1 and 2. We plot here the solutions $u^\varepsilon$ of (1.3), (1.5) for various values of the parameter $\varepsilon$. In Figure 1, the limit solution consists of an undercompressive shock and a Lax shock, whereas the limit solution in Figure 2 consists of an undercompressive shock and a rarefaction wave. The numerical computations have been performed with a straightforward finite-difference discretization.

Specifically, we used the flux $f(u) = u^3$, the parameter $\gamma = 15$ and the kernel

$$\phi(x) = \begin{cases} \dfrac{\exp\left(\frac{1}{x^2-1}\right)}{\int_{-1}^1 \exp\left(\frac{1}{y^2-1}\right) dy}, & x \in (-1,1), \\ 0, & \text{otherwise} \end{cases}$$

whereas $\phi_\varepsilon$ is given as in (1.6). We refer to LeFloch and Mohamadian [22] for further background on numerical methods for nonclassical shocks and the current state of the art.

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## References

[1] R. Abeyaratne and J.K. Knowles, Kinetic relations and the propagation of phase boundaries in solids, Arch. Ration. Mech. Anal. 114 (1991), 119–154.

[2] D.M. Anderson, G.B. McFadden, and A.A. Wheeler, Diffuse-interface methods in fluid mechanics. Annual review of fluid mechanics, Vol. 30, 139–165, Annu. Rev. Fluid Mech., 30, Annual Reviews, Palo Alto, CA, 1998.

[3] N. Bedjaoui and P.G. LeFloch, Diffusive-dispersive traveling waves and kinetic relations. I. Nonconvex hyperbolic conservation laws, J. Differential Equations 178 (2002), 574–607.

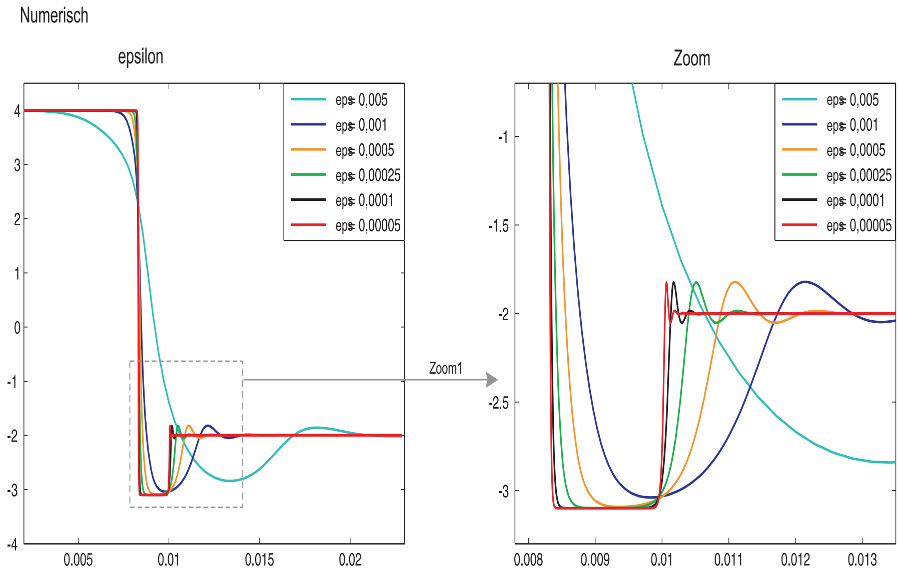


Figure 1. Numerical solution at time $t = 2,5 \cdot 10^{-4}$ for the model (1.3), (1.5) on an equidistant partition for $\varepsilon = 0.00005, \dots, 0.005$. The limiting solution contains a nonclassical undercompressive shock followed by a classical shock.

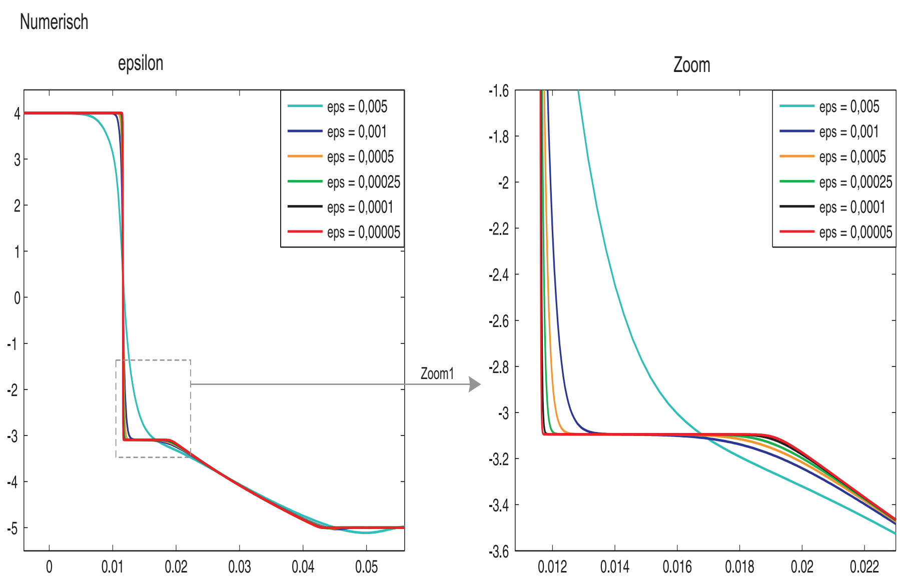


Figure 2. Numerical solution at time $t = 5,0 \cdot 10^{-4}$ for the model (1.3), (1.5) on an equidistant partition for $\varepsilon = 0.00005, \dots, 0.005$. The limiting solution contains a nonclassical undercompressive shock followed by a rarefaction wave.

[4] A.L. Bertozzi, A. Münch, and M. Shearer, Undercompressive shocks in thin film flows, Phys. D. 134 (1999), 431–464.

[5] G.M. Coclite and K.H. Karlsen, A singular limit problem for conservation laws related to the Camassa-Holm shallow water equation, Comm. Partial Differential Equations 31 (2006), 1253–1272.

[6] J.M. Correia and P.G. LeFloch, Nonlinear diffusive-dispersive limits for multidimensional conservation laws, in "Advances in Nonlinear P.D.E.'s and Related Areas", (Beijing, 1997), World Sci. Publ., River Edge, NJ, 1998, pp. 103–123. Available at: http://arxiv.org/abs/0810.1880.

[7] J.M. Correia and P.G. LeFloch, Nonlinear hyperbolic conservation laws, in "Nonlinear evolution equations and their applications" (Macau, 1998), World Sci. Publ., River Edge, NJ, 1999, pp. 21–44.

[8] R.J. DiPerna, Measure-valued solutions to conservation laws, Arch. Rational Mech. Anal. 88 (1985), 223–270.

[9] H. Fan and M. Slemrod, Dynamic flows with liquid-vapor phase transitions, in "Handbook of mathematical fluid dynamics", Vol. I, pp. 373–420, North-Holland, Amsterdam, 2002.

[10] J. Haink and C. Rohde, Local discontinuous Galerkin schemes for model problems in phase transition theory, Comm. Comput. Phys. 4 (2008), 860-893.

[11] B.T. Hayes and P.G. LeFloch, Nonclassical shocks and kinetic relations. Scalar conservation laws, Arch. Rational Mech. Anal. 139 (1997), 1–56.

[12] B.T. Hayes and P.G. LeFloch, Nonclassical shocks and kinetic relations: strictly hyperbolic systems, SIAM J. Math. Anal. 31 (2000), 941–991.

[13] S. Hwang, Nonlinear diffusive-dispersive limits for scalar multidimensional conservation laws, J. Differential Equations 225 (2006), 90–102.

[14] S. Hwang, Singular limit problem of the Camassa-Holm type equation, J. Differential Equations 235 (2007), 74–84.

[15] S. Hwang and A.E. Tzavaras. Kinetic decomposition of approximate solutions to conservation laws: Application to relaxation and diffusion-dispersion approximations, Commun. Partial Differ. Equations 27 (2002), 1229–1254.

[16] D. Jacobs, B. McKinney, and M. Shearer, Travelling wave solutions of the modified Korteweg-de Vries-Burgers equation. J. Differential Equations 116 (1995), 448–467.

[17] C.I. Kondo and P.G. LeFloch, Zero diffusion-dispersion limits for scalar conservation laws, SIAM J. Math. Anal. 33 (2002), 1320–1329.

[18] S. Kruzkov, First-order quasilinear equations with several space variables, Math. USSR Sb. 10 (1970), 217–243.

[19] P.G. LeFloch, Propagating phase boundaries: formulation of the problem and existence via the Glimm scheme, Arch. Ration. Mech. Anal. 123 (1993), 153–197.

[20] P.G. LeFloch, An introduction to nonclassical shocks of systems of conservation laws, International School on Hyperbolic Problems, Freiburg, Germany, Oct. 97, D. Kröner, M. Ohlberger and C. Rohde eds., Lect. Notes Comput. Eng., Vol. 5, Springer Verlag, 1999, pp. 28–72.

[21] P.G. LeFloch, Hyperbolic systems of conservation laws. The theory of classical and nonclassical shock waves, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2002.

[22] P.G. LeFloch and M. Mohamadian, Why many shock wave theories are necessary. Fourth-order models, kinetic functions, and equivalent equations, J. Comput. Phys. 227 (2008), 4162–4189.

[23] P.G. LeFloch and R. Natalini, Conservation laws with vanishing nonlinear diffusion and dispersion, Nonlinear Analysis 36 (1999), 213–230.

[24] P.G. LeFloch and C. Rohde, High-order schemes, entropy inequalities, and nonclassical shocks. SIAM J. Numer. Anal. 37 (2000), 2023–2060.

[25] P.-L. Lions, B. Perthame, and E.T. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations. J. Amer. Math. Soc. 7 (1994), 169–191.

[26] B. Perthame, Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure, J. Math. Pures Appl. 77 (998), 1055–1064.

[27] B. Perthame and P.E. Souganidis, A limiting case for velocity averaging, Ann. Scient. École Normale Sup. 31 (198), 591–598.

[28] E. Presutti, Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics, Theoretical and Mathematical Physics, Springer-Verlag, 2008.

[29] C. Rohde, Scalar conservation laws with mixed local and non-local diffusion-dispersion terms, SIAM J. Math. Anal. 37 (2005), 103–129.

[30] C. Rohde, On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions, ZAMM Z. Angew. Math. Mech. 85 (2005), 839–857.
[31] M.E. Schonbek, Convergence of solutions to nonlinear dispersive equations, Comm. Partial Differential Equations 7 (1982), 959–1000.
[32] M. Slemrod, Admissibility criteria for propagating phase boundaries in a van der Waals fluid, Arch. Ration. Mech. Anal. 81 (1983), 301–315.
[33] L. Truskinovsky, Kinks versus shocks, in "Shock induced transitions and phase structures in general media", R. Fosdick, E. Dunn, and M. Slemrod ed., IMA Vol. Math. Appl., Vol. 52, Springer-Verlag, New York (1993), pp. 185–229.
[34] C.J. Van Duijn, L.A. Peletier and I.S. Pop. A new class of entropy solutions of the Buckley-Leverett equation, SIAM J. Math. Anal. 39 (2007), 507–536.

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