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ISSN **1611-4176**

Existence of global weak solution for boundary value problems for compressible fluid models with nonlocal capillary tensor

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Abstract

This work is devoted to proving the global existence of a weak solution for a general isothermal model of capillary fluids, which can be used as a phase transition model. Using results of Lions in [20] on the compressible Navier-Stokes system we show the global stability of weak solutions for an initial-boundary-value problem with isentropic pressure and generaliued pressure. To illustrate the analysis a number of numerical experiments on rising bubbles in a closed container are displayed.

1 Introduction

1.1 Presentation of the model

The correct mathematical description of liquid-vapor phase interfaces and their dynamical behavior in compressible fluid flow has a long history. We are concerned with compressible fluids endowed with internal capillarity. One of the first models which takes into consideration the variation of density on the interface between two phases, originates from the XIXth century work by Van der Waals and Korteweg [18]. It was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [17, 23]. Korteweg suggests a modification of the Navier-Stokes system to additionally account for phase transition phenomena by introducing a term of capillarity. He assumed that the thickness of the interfaces was not null as in the *sharp interface approach.* This is called the *diffuse interface approach.*

Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but on the gradient of the density. In terms of the free energy, this principle takes the form of a generalized Gibbs relation, see [23]. In the present paper, we follow a new approach introduced by Coquel, Rohde *et al.* in [5]. They remark that the local diffuse

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interface approach requires more regular solutions than in the original sharp interface approach. Indeed the interfaces are assumed of nonzero thickness, so that the density varies continuously across the interface, whereas in the sharp interface models, the density may have jumps. We present here an alternative model with a capillarity term which does not involve spatial derivatives of the density. Let $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, be a bounded open set and the time $T > 0$. We denote by $n = n(s)$ the outward normal vector on $\partial\Omega$. The model reads:
(NSK)

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in } (0,T) \times \Omega, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - 2\mu\Delta u - \lambda \nabla \operatorname{div} u + \nabla(P(\rho)) = \kappa \rho \nabla D[\rho] & \text{in } (0,T) \times \Omega, \\ (\rho_{t=0}, u_{t=0}) = (\rho_0, u_0) & \text{in } \Omega, \end{cases}$$

where $\rho \geq 0$ denotes the density of the fluid and $u \in \mathbb{R}^N$ the velocity. We assume that the viscosity coefficients μ and λ satisfy $\mu > 0$ and $\lambda + \frac{2}{N}\mu > 0$ and that the capillarity coefficient κ is nonnegative. Finally, P stands for the pressure function. We supplement the system with the following condition at the boundary:

$$u(t,x) = 0 \quad \text{on} \quad (0,T) \times \partial\Omega. \tag{1.1}$$

As System (NSK) is relevant for the study of phase transitions, we expect the pressure to be of Van der Waals type, namely

$$P : (0,b) \to (0,+\infty)$$
$$P(\rho) = \frac{RT_* \rho}{b - \rho} - a\rho^2$$

where a, b, R, T_* are positive constants, R being the specific gas constant. However, we shall concentrate for the analysis on isentropic pressure laws: $P(\rho) = a\rho^\gamma$ for some $a > 0$ and $\gamma \geq 1$ and on the case of vanishing density at infinity, namely $\bar{\rho} = 0$. In the last part of the paper the system with Van-der-Waals pressure will be considered numerically.
The term $\kappa\rho\nabla D[\rho]$ accounts for the capillarity effects close to phase transitions. The classical Korteweg's capillarity term is $D[\rho] = \Delta\rho$ (see [18]). Based on Korteweg's original ideas Coquel, Rohde *et al.* in [5] and Rohde in [22] choose a nonlocal capillarity term D which penalizes rapid variations in the density field close to the interfaces. This choice of capillarity term allows to get solutions with jumps, i.e. with sharp interfaces. Here we generalize this approach, in this contribution we consider the choice:

$$D[\rho](x) = D_{global}[\rho](x) = \int_\Omega \phi(x,y)(\rho(y) - \rho(x))\, dy. \tag{1.2}$$

Here the kernel function $\phi : \Omega^2 \to [0,+\infty)$ satisfies:

- $\phi \in W^{1,1}(\Omega^2)$, $\int_\Omega \phi(x,y)\, dy \geq c > 0 \quad \forall x \in \Omega$, $\phi \geq 0$,
- $\phi(x,y) = \phi(y,x) \quad \forall x, y \in \Omega$.

1.2 Energy spaces

Before tackling the global stability theory for the system (NSK), let us formally derive the uniform bounds available on (ρ, u). Let Π (free energy) be defined by:

$$\Pi(s) = s\Big(\int_0^s \frac{P(z)}{z^2} dz\Big), \tag{1.3}$$

so that $P(s) = s\Pi^{'}(s) - \Pi(s)$, $\Pi^{'}(\bar{\rho}) = 0$ and if we renormalize the mass equation:

$$\partial_t \Pi(\rho) + \mathrm{div}(u\Pi(\rho)) + P(\rho)\mathrm{div}(u) = 0 \quad \text{in } \mathcal{D}^{'}((0,T)\times\Omega).$$

Notice that Π is convex whenever P is nondecreasing. Multiplying the equation of momentum conservation by u and integrating by parts over Ω, we obtain the following energy estimate:

$$\begin{aligned}\int_\Omega \Big(\frac{1}{2}\rho|u|^2 + \Pi(\rho) + E_{global}[\rho]\Big)\,dx + \int_0^t\int_\Omega \big(2\mu D(u):D(u) + \lambda|\mathrm{div}u|^2\big)\,dx\,ds \\ \leq \int_\Omega \Big(\frac{|m_0|^2}{2\rho_0} + \Pi(\rho_0) + E_{global}[\rho_0]\Big)\,dx,\end{aligned} \tag{1.4}$$

where we have $m_0 = u_0\rho_0$ and

$$E_{global}[\rho](x) = \frac{\kappa}{4}\int_\Omega \phi(x,y)(\rho(y)-\rho(x))^2\,dy.$$

The only non-standard term is the energy term E_{global} which comes from the product of u with the capillarity term $\kappa\rho\nabla D[\rho]$. Indeed we have:

$$\begin{aligned}&\kappa\int_\Omega u(t,x)\rho(t,x)\cdot\nabla\Big(\int_\Omega \phi(x,y)(\rho(t,y)-\rho(t,x))\,dy\Big)\,dx \\ &= -\kappa\int_\Omega \mathrm{div}(u(t,x)\rho(t,x))\Big(\int_\Omega \phi(x,y)(\rho(t,y)-\rho(t,x))\,dy\Big)\,dx \\ &\qquad - \kappa\int_{\partial\Omega}\mathrm{div}(u(t,x)\cdot n(t)\rho(t,x))\Big(\int_\Omega \phi(x,y)(\rho(t,y)-\rho(t,x))\,dy\Big)\,dx \\ &= \kappa\int_\Omega \frac{\partial}{\partial t}\rho(t,x)\Big(\int_\Omega \phi(x,y)(\rho(t,y)-\rho(t,x))\,dy\Big)\,dx \\ &= -\frac{d}{dt}\int_\Omega E_{global}[\rho(t,\cdot)](x)\,dx.\end{aligned}$$

To derive the last equality we use the relation:

$$\begin{aligned}\frac{d}{dt}\int_\Omega E_{global}[\rho(t,\cdot)](x)\,dx &= \frac{\kappa}{2}\int_\Omega\int_\Omega \phi(x,y)(\rho(t,y)-\rho(t,x))\frac{\partial}{\partial t}\rho(t,y)\,dy\,dx \\ &\quad + \frac{\kappa}{2}\int_\Omega\int_\Omega \phi(x,y)(\rho(t,x)-\rho(t,y))\frac{\partial}{\partial t}\rho(t,x)\,dy\,dx \\ &= \kappa\int_\Omega\int_\Omega \phi(x,y)(\rho(t,y)-\rho(t,x))\frac{\partial}{\partial t}\rho(t,y)\,dy\,dx.\end{aligned}$$

In the sequel we will note

$$\mathcal{E}(\rho,\rho u)(t)=\int_\Omega\left(\frac{1}{2}(\rho|u|^2)(t,x)+\Pi(\rho)(t,x)+E_{global}[\rho(t,.)](x)\right)\,dx. \tag{1.5}$$

We are interested to use the above inequality energy to determine the functional space we must work with. So if we expand $\int_\Omega E_{global}[\rho(t,.)](x)\,dx$ and use the properties of the kernel, we get

$$\int_\Omega E_{global}[\rho(t,\cdot)](x)\,dx\geq\frac{\kappa}{2}\left(c\int_\Omega\rho^2(t,x)\,dx-\int_\Omega\int_\Omega\phi(x,y)\rho(t,x)\rho(t,y)\,dx\,dy\right). \tag{1.6}$$

Remark 1 *Here to control ρ in $L^\infty(L^2)$ by the equation (1.6), we will assume in the sequel that $\phi\in L^\infty_x(L^2_y+L^\infty_y)$.*

From the mass equation we obtain that ρ is bounded in $L^\infty(0,T;L^1(\Omega))$ if we assume that $\rho_0\in L^1$. From the equations (1.4), (1.6) and as $\phi\geq 0$, to control ρ in $L^\infty(L^2)$ we only need to bound the term $\int_\Omega\int_\Omega\phi(x,y)\rho(t,x)\rho(t,y)\,dx\,dy$. In this goal we have:

$$\left|\int_\Omega\int_\Omega\phi(x,y)\rho(t,x)\rho(t,y)\,dx\,dy\right|\leq\|\phi\|_{L^\infty(L^\infty+L^2)}\|\rho\|_{L^\infty(L^1)}(\|\rho\|_{L^\infty(L^1)}+\|\rho\|_{L^\infty(L^2)}).$$

By a simple boobstrap, we get a control of ρ in $L^\infty(0,T;L^2(\Omega))$ (a property which turns out to be important to taking advantage of the theory of renormalized solutions, indeed ρ in $L^\infty(0,T;L^2(\Omega))$ implies that $\rho\in L^2_{loc}(\mathbb{R}^+\times\Omega)$ so that one may use the theorem of Di Perna-Lions on renormalized solutions, see [19]).

Remark 2 *We assume here that $P(\rho)=\rho^\gamma$ with $\gamma>1$. In the last section we will study general pressures.*

In view of (1.4), we can specify initial conditions on $\rho_{/t=0}=\rho_0$ and $\rho u_{/t=0}=m_0$ where we assume that:

- $\rho_0\geq 0$ a.e in Ω, $\rho_0\in L^1(\Omega)\cap L^s(\Omega)$ with $s=\max(2,\gamma)$,
- $m_0=0$ a.e on $\rho_0=0$,
- $\dfrac{|m_0|^2}{\rho_0}$ (defined to be 0 on $\rho_0=0$) is in $L^1(\Omega)$.

(1.7)

We deduce the following a priori bounds which give us the energy space in which we will work:

- $\rho\in L^\infty(0,T;L^1(\Omega)\cap L^s(\Omega))$,
- $\rho|u|^2\in L^\infty(0,T;L^1(\Omega))$,
- $\nabla u\in L^2((0,T)\times\Omega)^N$.

We will use this uniform bound in our result of compactness. Let us emphasize at this point that the above a priori bounds do not provide any control on $\nabla\rho$ in contrast to the case of $D[\rho]=\Delta\rho$ studied in [6] and [12].

1.3 Notion of weak solutions

We now explain what we mean by renormalized weak solutions, weak solutions, and bounded energy weak solution of problem (NSK).
Multiplying mass equation by $b^{'}(\rho)$, we obtained the so-called renormalized equation (see [19])

$$\frac{\partial}{\partial t}b(\rho) + \mathrm{div}(b(\rho)u) + (\rho b^{'}(\rho) - b(\rho))\mathrm{div}u = 0 \tag{1.8}$$

with

$$b \in C^0([0,+\infty)) \cap C^1((0,+\infty)), \;\; |b^{'}(t)| \leq ct^{-\lambda_0}, \;\; t \in (0,1], \;\; \lambda_0 < 1 \tag{1.9}$$

and growth conditions at infinity:

$$|b^{'}(t)| \leq ct^{\lambda_1}, \;\; t \geq 1, \;\; \text{where } \; c > 0, \; -1 < \lambda_1 < \frac{s}{2} - 1. \tag{1.10}$$

Definition 1.1 *A couple (ρ, u) is called a renormalized weak solution of problem (NSK) whenever*

- *the equation of momentum holds in $\mathcal{D}^{'}(\mathbb{R}^+ \times \Omega)$,*
- *the equation (1.8) holds in $\mathcal{D}^{'}(\mathbb{R}^+ \times \Omega)$ for any function b verifying (1.9) and (1.10).*

Definition 1.2 *Let the couple (ρ_0, u_0) satisfy*

- $\rho_0 \in L^1(\Omega)$, $\Pi(\rho_0) \in L^1(\Omega)$, $E_{global}[\rho_0] \in L^1(\Omega)$, $\rho_0 \geq 0$ *a.e in* Ω,
- $\rho_0 u_0 \in (L^1(\Omega))^d$ *and* $\rho_0 |u_0|^2 1_{\rho_0 > 0} \in L^1(\Omega)$,
- $\rho_0 u_0 = 0$ *whenever* $x \in \{\rho_0 = 0\}$,

where the quantity Π is defined in (1.3). We have the following definitions:

1. *A couple (ρ, u) is called a weak solution of problem (NSK) on $\mathbb{R}^+$ if:*
   (a) $\rho \in L^r(L^r(\Omega))$ *for* $s \leq r \leq \infty$,
   (b) $P(\rho) \in L^\infty(L^1(\Omega))$, $\rho \geq 0$ *a.e in* $\mathbb{R}^+ \times \Omega$,
   (c) $\nabla u \in L^2(L^2(\Omega))$, $\rho |u|^2 \in L^\infty(L^1(\Omega))$,
   (d) *Mass equation holds in* $\mathcal{D}^{'}(\mathbb{R}^+ \times \Omega)$,
   (e) *Momentum equation holds in* $\mathcal{D}^{'}(\mathbb{R}^+ \times \Omega)^N$,
   (f) $\lim_{t \to 0^+} \int_\Omega \rho(t)\varphi = \int_\Omega \rho_0 \varphi$, $\forall \varphi \in \mathcal{D}(\Omega)$,
   (g) $\lim_{t \to 0^+} \int_\Omega \rho u(t) \cdot \phi = \int_\Omega \rho_0 u_0 \cdot \phi$, $\forall \phi \in \mathcal{D}(\Omega)^N$.
2. *A couple (ρ, u) is called a bounded energy weak solution of problem (NSK) if in addition to (1d), (1e), (1f), (1g) we have:*
   - *The quantity $\mathcal{E}_0$ is finite and inequality (1.4) with $\mathcal{E}$ defined by (1.5) and with $\mathcal{E}_0$ in place of $\mathcal{E}(\rho(0), \rho u(0))$ holds a.e in $\mathbb{R}^+$.*

1.4 Mathematical results

We wish to prove global stability results for (NSK) with $D[\rho] = \phi * \rho - \rho$ in functional spaces very close to energy spaces. In the non-capillary case and $P(\rho) = a\rho^\gamma$, P.-L. Lions in [20] proved the global existence of weak solutions (ρ, u) to (NSK) with $\kappa = 0$ (that is for the compressible isothermal system of Navier-Stokes) for $\gamma > \frac{N}{2}$ if $N \geq 4$, $\gamma \geq \frac{3N}{N+2}$ if $N \in \{2, 3\}$ and initial data (ρ_0, m_0) such that:

$$\rho_0,\ \ \rho_0^\gamma,\ \ \frac{|m_0|^2}{\rho_0} \in L^1(\Omega),$$

where we agree that $m_0 = 0$ on $\{x \in \Omega/\ \rho_0(x) = 0\}$. More precisely, he obtains the existence of global weak solutions (ρ, u) to (NSK) with $\kappa = 0$ such that for all $t \in (0, +\infty)$:

- $\rho \in L^\infty(0, T; L^\gamma(\Omega))$ and $\rho \in C([0, T], L^p(\Omega))$ if $1 \leq p < \gamma$,
- $\rho \in L^q((0, T) \times \Omega)$ for $q = \gamma - 1 + \frac{2\gamma}{N} > \gamma$,
- $\rho |u|^2 \in L^\infty(0, T; L^1(\Omega))$ and $Du \in L^2((0, T) \times \Omega)$.

Notice that the main difficulty for proving Lions' theorem consists in exhibiting strong compactness properties of the density ρ in $L^p_{loc}(\mathbb{R}^+ \times \Omega)$ spaces required to pass to the limit in the pressure term $P(\rho) = a\rho^\gamma$. Let us mention that Feireisl in [9] generalized the result to any $\gamma > \frac{N}{2}$ in establishing that we can obtain renormalized solution without imposing that $\rho \in L^2_{loc}(\mathbb{R}^+ \times \Omega)$ (a property that was needed in Lions' approach in dimension $N \in \{2, 3\}$ giving the further condition $\gamma - 1 + \frac{2\gamma}{N} \geq 2$). For this he introduces the concept of oscillation defect measure evaluating the loss of compactness. We refer to the book of Novotný and Straškraba for more details (see [21]).
Let us mention here that the existence of strong solution with $D[\rho] = \Delta\rho$ is known since the work by Hattori and Li in [14], [15] in the whole space $\mathbb{R}^N$. In [6], Danchin and Desjardins study the well-posedness of the problem for the isothermal case with constant coefficients in critical Besov spaces and in [13] we study the existence of strong solutions in critical space for the scaling of the system for the non-isothermal system. We recall too the results by Rohde in [22] who obtains the existence and uniqueness in finite time for two-dimensional initial data in $H^4(\mathbb{R}^2) \times H^4(\mathbb{R}^2)$ (less regular data in any dimension $N \geq 2$ have been considered recently in [10]).

In the present paper, we aim at showing the global stability of weak solutions in the energy spaces for the system (NSK). This work is composed of three parts, the first one concerns estimates on the density. We aim at getting more integrability on the density in order to pass to the weak limit in the term of pressure and of capillarity. The second part is the passage to the weak limit in the non-linear terms of the density and the velocity according to Lions' methods. The idea is to use renormalized solution to test the weak limit on convex test functions. For the time being, we focus on the case of pressure laws of type $P(\rho) = a\rho^\gamma$. The result can be easily extended to more general pressures following [1]. The third part is devoted to numerical experiments for a Van-Der-Waals pressure.

Remark 3 *In the sequel we will assume construct by a mollifying processus a sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$ of global approximated solutions for the system (NSK). In the appendix we will briefly explain how we can build such approximated solutions by adding term of viscosity to the system (NSK). The sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$ are regular bounded energy weak solutions (it means that they check uniformly the energy bound), in addition, we assume that the sequence ρ_n is bounded in $L^r((0,T)\times\Omega))\cap L^1((0,T)\times\Omega)$ for some $r > \max(\gamma, 2)$. In the Theorem 2.2, we will prove this extra condition on the sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$. Indeed via some energy methods we can get a gain of integrability on the density.*

Theorem 1.1 *Let $N \in \{2, 3\}$. Let $\gamma > 1$. Let the couple (ρ_0^n, u_0^n) satisfy:*

- *ρ_0^n is uniformly bounded in $L^1(\Omega) \cap L^s(\Omega)$ with $s = \max(\gamma, 2)$ and $\rho_0^n \geq 0$ a.e in Ω,*
- *$\rho_0^n|u_0^n|^2$ is uniformly bounded in $L^1(\Omega)$,*
- *$\rho_0^n u_0^n = 0$ whenever $x \in \{\rho_0 = 0\}$.*

In addition we suppose that ρ_0^n converges in $L^1(\Omega)$ to ρ_0.
Then, up to a subsequence, (ρ_n, u_n) converges strongly to a weak solution (ρ, u) of the system (NSK) satisfying the initial condition (ρ_0, u_0) as in (1.7). Moreover we have the following convergence:

- *$\rho_n \to_n \rho$ in $C([0,T], L^p(\Omega)) \cap L^r((0,T) \times K)$ for all $1 \leq p < s$, $1 \leq r < q$, with $q = s + \frac{N\gamma}{2} - 1$ and where K is an arbitrary compact.*

In addition we have:

- *$\rho_n u_n \to \rho u$ in $L^p(0,T; L^r(\Omega))$ for all $1 \leq p < \infty$ and $1 \leq r < \frac{2s}{s+1}$,*
- *$\rho_n (u_i)_n (u_j)_n \to \rho_n u_i u_j$ in $L^p(0,T; L^1(\Omega))$ for all $1 \leq p < \infty$, $1 \leq i, j \leq N$;*

Remark 4 *We can check easily that with the previous theorem, we have shown that if we have built a sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$ of global approximate solutions verifying the properties of Remark 3, then there exists global weak solutions for the system (NSK).*

2 Existence of weak solution for an isentropic pressure law

2.1 A priori estimates on the density

In this part we are interested by getting a gain of integrability on the density and we consider the case where $P(\rho) = a\rho^\gamma$. This will enable us to pass to the weak limit in the pressure and the Korteweg terms. It is expressed by the following theorem:

Remark 5 *In the sequel we will speak of regular approximate solutions for our system, it means solutions built by a mollifying processus. Generally we add to the initial system a viscosity term to get global regular solutions to our system. In this paper we are interesting only in showing stability results.*

Theorem 2.2 *Let $N \in \{2,3\}$ and $\gamma \geq 1$. Let (ρ, u) be a regular approximate bounded energy weak solution of the system (NSK) with $\rho \geq 0$ and $\rho \in L^\infty(L^1 \cap L^{s+\varepsilon})$ where we define ε below. Then we have :*

$$\int_{(0,T)\times\Omega} \big(\rho^{\gamma+\varepsilon} + \rho^{2+\varepsilon}\big)\, dx\, dt \leq M \ \ \textit{for any} \ \ 0 < \varepsilon \leq \frac{4}{N}\gamma - 1.$$

with M only depends on the initial conditions and on the time T.

Proof:

We will begin with the case where $N = 3$ and we treat after the specific case $N = 2$.

Case $N = 3$:

Applying the operator $(-\Delta)^{-1}\mathrm{div}$ to the momentum equation yields

$$a\rho^\gamma = \frac{\partial}{\partial t}(-\Delta)^{-1}\mathrm{div}(\rho u) + (-\Delta)^{-1}\partial^2_{i,j}(\rho u_i u_j) + (2\mu+\lambda)\mathrm{div} u \\ - \kappa(-\Delta)^{-1}\mathrm{div}\big(\rho \nabla D[\rho]\big). \tag{2.11}$$

Next we have:

$$(-\Delta)^{-1}\mathrm{div}\big(\rho\nabla D[\rho]\big) = (-\Delta)^{-1}\mathrm{div}\Big(\rho\big(\int_\Omega \nabla_x\phi(x,y)(\rho(y)-\rho(x))dy \\ - \int_\Omega \phi(x,y)dy\, \nabla\rho\big)\Big).$$

In the sequel we will set: $c^1(x) = \int_\Omega \phi(x,y)dy$ and:

$$R(x) = (-\Delta)^{-1}\mathrm{div}\big(\rho(\int_\Omega \nabla_x\phi(x,y)(\rho(y)-\rho(x))dy\big) \\ - (-\Delta)^{-1}\big(\int_\Omega \nabla_x\phi(x,y)dy \cdot \nabla\rho\big).$$

so that multiplying by ρ^ε with $0 < \varepsilon \leq \min(\frac{1}{N}, \frac{2}{N}\gamma - 1)$, we get

$$a\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}c^1\rho^{2+\varepsilon} = -\kappa\rho^\varepsilon R + \rho^\varepsilon(-\Delta)^{-1}\partial^2_{ij}(\rho u_i u_j) + \frac{\partial}{\partial t}\big(\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)\big) \\ - [\frac{\partial}{\partial t}\rho^\varepsilon](-\Delta)^{-1}\mathrm{div}(\rho u) + (2\mu+\lambda)\mathrm{div} u. \tag{2.12}$$

We now rewrite the previous equality as follows:

$$a\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}c^1\rho^{2+\varepsilon} = -\kappa\rho^\varepsilon R + \rho^\varepsilon(-\Delta)^{-1}\partial^2_{ij}(\rho(u_i)(u_j)) + \frac{\partial}{\partial t}\big(\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)\big) \\ + \mathrm{div}[u\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)] + (2\mu+\lambda)\mathrm{div} u - (\rho)^\varepsilon u\cdot\nabla(-\Delta)^{-1}\mathrm{div}(\rho u) \\ + (1-\varepsilon)(\mathrm{div} u)\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)\ . \tag{2.13}$$

Next we integrate (2.13) in time on $[0,T]$ and in space. We get:

$$\begin{aligned}&\int_{(0,T)\times\Omega}(a\rho^{\gamma+\varepsilon}+c\frac{\kappa}{2}\rho^{2+\varepsilon})dx\,dt\leq\int_{(0,T)\times\Omega}\Big(\frac{\partial}{\partial t}[\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)]\\&+(\mu+\zeta)(\text{div}u)\rho^{\varepsilon}+(1-\varepsilon)(\text{div}u)\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)+\rho^{\varepsilon}[R_iR_j(\rho u_iu_j)\\&-u_iR_iR_j(\rho u_j)]+\text{div}[u\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)]-\kappa\rho^{\varepsilon}R\Big)dx\,dt\,,\end{aligned}\tag{2.14}$$

where R_i is the classical Riesz transform. Now we want to control the term $\int_0^T\int_\Omega\big(\rho^{\gamma+\varepsilon}+\frac{\kappa}{2}c\rho^{2+\varepsilon}\big)dx\,dt$. As ρ is positive, it will enable us to control $\|\rho\|_{L^{\gamma+\varepsilon}_{t,x}}$ and $\|\rho\|_{L^{2+\varepsilon}_{t,x}}$. This may be achieved by bounding each term on the right side of (2.14).
We start by treating the term $\int_{(0,T)\times\Omega}\frac{\partial}{\partial t}[\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)]$. So we need to control $\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)$ in $L^\infty(0,T;L^1(\Omega))$ and $\rho_0^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_0u_0)$ in $L^1(\Omega)$ because:

$$\begin{aligned}\int_{(0,T)\times\Omega}\frac{\partial}{\partial t}[\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)](t,x)dt\,dx=&\int_\Omega[\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)](t,x)dx\\&-\int_\Omega[\rho_0^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_0u_0)](x)dx.\end{aligned}$$

We recall that ρ, ρ^2, ρ^γ and $\rho|u|^2$ are bounded in $L^\infty(L^1)$ while Du is bounded in $L^2((0,T)\times\Omega)$ and u is bounded in $L^2(0,T;L^{\frac{2N}{N-2}}(\Omega))$ by Sobolev embedding. In particular by Hölder inequalities we get that ρu is bounded in $L^\infty(0,T,(L^{\frac{2\gamma}{\gamma+1}}\cap L^{\frac{4}{3}})(\Omega))$. Thus we get by using Hölder inequalities and Sobolev embedding: $\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)\in L^\infty(0,T,L^1\cap L^\alpha)$ with:

$$\frac{1}{\alpha}=\frac{\varepsilon}{s}+\min(\frac{\gamma+1}{2\gamma},\frac{3}{4})-\frac{1}{N}<1.$$

The fact that $\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)\in L^\infty(0,T,L^1)$ is obtained by interpolation because $\rho\in L^\infty(L^1)$ and by using less integrability in Sobolev embedding. Next we have the same type of estimates for $\|\rho_0^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_0u_0)\|_{L^1(\Omega^N)}$.
Finally (2.14) is rewritten on the following form in using Green formula:

$$\begin{aligned}\int_0^T\int_\Omega\big(\rho^{\gamma+\varepsilon}+\frac{\kappa}{2}\rho^{2+\varepsilon}\big)dxdt\leq C\big(1+\int_0^T\int_\Omega\big[\,|\text{div}u|\rho^{\varepsilon}(1+|(-\Delta)^{-1}\text{div}(\rho u)|\,]\\+\rho^{\varepsilon}|R_iR_j(\rho u_iu_j)-u_iR_iR_j(\rho u_j)|+\kappa\rho^{\varepsilon}R\big]dt\,dx\big).\end{aligned}$$

Now we will treat each term of the right hand side. We treat all the terms with the same type of estimates than P.-L. Lions in [20] , excepted the capillarity term.
We start with the term $|\text{div}u|\rho^{\varepsilon}|(-\Delta)^{-1}\text{div}(\rho u)|$. If $\gamma<6$ then we may write

$$|\text{div}u|\in L^2(L^2),\ \ \rho^{\varepsilon}\in L^\infty(L^{\frac{s}{\varepsilon}}),\ \ \rho\,u\in L^2(0,T,L^r(\Omega))$$

with $\frac{1}{r}=\frac{1}{s}+\frac{N-2}{2N}$ and by Sobolev embedding $|(-\Delta)^{-1}\text{div}(\rho u)|\in L^2(L^{s'})$ with $\frac{1}{s'}=\frac{1}{r}-\frac{1}{N}$ (this is possible only if $r<N$). We are in a critical case for the Sobolev embedding (i.e. $r\geq N$) only when $\gamma\geq6$. So by Hölder inequalities we get $|\text{div}u|\rho^{\varepsilon}|(-\Delta)^{-1}\text{div}(\rho u)|\in L^1(L^{s_1})$ with: $\frac{1}{s_1}=\frac{1}{s'}+\frac{\varepsilon}{s}+\frac{1}{2}=1-\frac{2}{N}+\frac{1+\varepsilon}{s}\leq1$ as we have $s>\frac{N}{2}$. Moreover by interpolation $|\text{div}u|\rho^{\varepsilon}|(-\Delta)^{-1}\text{div}(\rho u)|$ belongs to $L^1(0,T;L^1(\Omega))$.

Let us now treat the case $\gamma \geq 6$ where we choose case $\varepsilon = \frac{2}{N}\gamma - 1$. We have:

$$\begin{aligned}\||\mathrm{div}u|\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho u)|\|_{L^1} &\leq \|Du\|_{L^2(L^2)}\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}}\|\rho u\|_{L^{\frac{2(\gamma+\varepsilon)}{\gamma-\varepsilon}}(L^{\frac{6(\gamma+\varepsilon)}{5\gamma-\varepsilon}})},\\ &\leq C\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}}\|\rho u\|_{L^{\frac{10\gamma-6}{\gamma+3}}(L^{\frac{3(10\gamma-6)}{13\gamma+3}})} \leq C\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}}\|\rho u\|^{\frac{\gamma+3}{5\gamma-3}}_{L^2(L^{\frac{6\gamma}{\gamma+6}})}\|\rho u\|^{\frac{2(2\gamma-3}{5\gamma-3}}_{L^\infty(L^2)},\\ &\leq C\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}}\|\rho u\|^{\frac{5\gamma}{5\gamma-3}}_{L^2(L^{\frac{6\gamma}{\gamma+6}})}\|\rho u\|^{\frac{2(2\gamma-3)}{5\gamma-3)}}_{L^\infty(L^{\frac{2\gamma}{\gamma+1}})},\\ &\leq C\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}},\end{aligned}$$

since we have $\frac{1}{2}+\frac{\varepsilon}{\gamma+\varepsilon}+\frac{\gamma-\varepsilon}{2(\gamma+\varepsilon)} = 1$, $\frac{1}{2}+\frac{\varepsilon}{\gamma+\varepsilon}+\frac{5\gamma-\varepsilon}{6(\gamma+\varepsilon)}-\frac{1}{3} = 1$, and $\frac{6(\gamma+\varepsilon)}{5\gamma-\varepsilon} = 3\frac{10\gamma-6}{13\gamma+3} < 3$.

We now want to treat the term: $\rho^\varepsilon R$, so we have: $\rho(\int_\Omega \nabla_x\phi(x,y)(\rho(y)-\rho(x))dy \in L^\infty(L^1\cap L^{\frac{s}{2}})$ by Hölder inequalities and the fact that we have $\rho \in L^\infty(L^1)$ and $\nabla\phi \in L^\infty(L^1)$.
After we get that $\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}\big(\rho(\int_\Omega \nabla_x\phi(x,y)(\rho(y)-\rho(x))dy\big) \in L^\infty(L^{r_1})$ with: $\frac{1}{r_1} = \frac{\varepsilon}{s}+\frac{2}{s}-\frac{1}{N} = \frac{2+\varepsilon}{s}-\frac{1}{N} < 1$. We conclude that $\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}\big(\rho(\int_\Omega \nabla_x\phi(x,y)(\rho(y)-\rho(x))dy\big)$ is $L^\infty(L^1)$ by using interpolation when $N \in \{2,3\}$. We proceed similarly for the other part of R.

We have after the term $(\mathrm{div}(u))\rho^\varepsilon$. We recall that ρ^ε is in $L^\infty(L^{\frac{1}{\varepsilon}}\cap L^{\frac{s}{\varepsilon}})$. If $\varepsilon \geq \frac{1}{2}$ (i.e. $s \geq \frac{3}{4}N$), the bound is obvious because $\frac{1}{2}+\varepsilon \geq 1$ and $\frac{1}{2}+\frac{\varepsilon}{s} < 1$, we can then conclude by interpolation. On the other hand, this rather simple term presents a technical difficulty when $\varepsilon \leq \frac{1}{2}$ since we do not know in that case if $\mathrm{div}u\,\rho^\varepsilon \in L^1(times(0,T)\times\Omega)$. One way to get round the difficulty is to multiply (2.11) by $\rho^\varepsilon 1_{\{\rho\geq 1\}}$. Then we obtain an estimate on $\rho^{s+\varepsilon}1_{(\rho\geq 1)}$ in $L^1((0,T)\times\Omega)$ as $\rho^\varepsilon 1_{\{\rho\geq 1\}}|\mathrm{div}u| \leq \rho|\mathrm{div}u| \in L^1((0,T)\times\Omega)$ (where $\varepsilon \leq \frac{1}{2}$) and we can conclude since $0 \leq \rho^{s+\varepsilon}1_{\{\rho<1\}} \leq \rho$ on $(0,T)\times\Omega$ and $\rho \in L^\infty(L^1)$.
We end with the following term $\rho^\varepsilon(R_iR_j(\rho\, u_iu_j) - u_iR_iR_j(\rho\, u_j))$. In the same way than in the previous inequalities we have $\rho^\varepsilon R_iR_j(\rho\, u_iu_j)$ is bounded in $L^1(0,T;L^1(\Omega))$. Indeed we have by Hölder inequalities and the fact that R_i is continuous from L^p in L^p with $1 < p < \infty$: $\frac{1}{s}+2\frac{N-2}{2N}+\frac{\varepsilon}{s} = 1-\frac{2}{N}+\frac{1+\varepsilon}{s} \leq 1$ (because $s > \frac{N}{2}$). And we conclude by interpolation. We treat the term $\rho^\varepsilon u_iR_iR_j(\rho\, u_j)$ similarly.

We have to treat now the case $N = 2$ where we have to modify the estimates when we are in critical cases for Sobolev embedding.

Case $N = 2$:

In the case $N = 2$ most of the proof given above stay exact except for the slightly more delicate terms $\rho^e\mathrm{div}u|(-\Delta)^{-1}\mathrm{div}(\rho u)|$ and $\rho^\varepsilon(R_iR_j(\rho\, u_iu_j) - u_iR_iR_j(\rho\, u_j))$.
We start with the term $|\rho^e\mathrm{div}u(-\Delta)^{-1}\mathrm{div}(\rho u)|$. In our previous estimate it was possible to use Sobolev embedding on the term $(-\Delta)^{-1}\mathrm{div}(\rho u)|$ only if $r \geq N$ (see above the notation), so in the case where $N = 2$ we are in a critical case for the Sobolev embedding when $\gamma \geq 2$. This may be overcome by using that, by virtue of Sobolev embedding, we have:

$$\||\mathrm{div}u|\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho u)|\|_{L^1} \leq C\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}\|\rho u\|_{L^{2(\gamma+\varepsilon)}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}.$$

Indeed by Hölder inequality, we have:

$$\frac{1}{2}+\frac{\varepsilon}{\gamma+\varepsilon}+\frac{\gamma+\varepsilon+1}{2(\gamma+\varepsilon)}-\frac{1}{2}=\frac{1}{2}+\frac{2\varepsilon+1}{2\varepsilon+2\gamma}\leq 1=\frac{1}{2}+\frac{\varepsilon}{\gamma+\varepsilon}+\frac{1}{2(\gamma+\varepsilon})\leq 1,$$

thus:

$$\frac{1}{2}+\frac{\varepsilon}{\gamma+\varepsilon}+\frac{1}{2(\gamma+\varepsilon)}\leq 1.$$

Moreover we have as $\rho u=\sqrt{\rho}\sqrt{\rho}u$, hence:

$$\|\rho u\|_{L^{2(\gamma+\varepsilon)}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}\leq C\|\rho\|^{\frac{1}{2}}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}$$

and thus:

$$\||\mathrm{div}u|\rho^{\varepsilon}|(-\Delta)^{-1}\mathrm{div}(\rho u)|\|_{L^1(L^1)}\leq C\|\rho\|^{\varepsilon+\frac{1}{2}}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}.$$

Next we are interested by the term $\rho^{\varepsilon}(R_iR_j(\rho\,u_iu_j)-u_iR_iR_j(\rho\,u_j))$. We use the fact that u is bounded in $L^2(0,T;\dot{H}^1)$ and thus in $L^2(0,T;BMO)$. Then by the Coifman-Rochberg-Weiss commutator theorem in [3], we have for almost all $t\in[0,T]$:

$$\|R_iR_j(\rho\,u_iu_j)-u_iR_iR_j(\rho\,u_j)\|_{L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}\leq C\|u\|_{L^2(BMO)}\|\rho\,u\|_{L^{2(\gamma+\varepsilon)}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}.$$

So we have:

$$\|\rho^{\varepsilon}(R_iR_j(\rho\,u_iu_j)-u_iR_iR_j(\rho\,u_j))\|_{L^1}\leq C\|\rho\|^{\varepsilon+\frac{1}{2}}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}.$$

In view of the previous inequalities we get finally:

$$\|\rho\|^{\gamma+\varepsilon}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}\leq C(1+\|\rho\|^{\frac{1}{2}+\varepsilon}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})})$$

and the $L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})$ bound on ρ is proven since $\frac{1}{2}+\varepsilon<\gamma+\varepsilon$. □

2.2 Compactness results in the case of isentropic pressure

So let follow the Theorem 2.2 and assume that $\gamma\geq 1$ such that if $(\rho_n,u_n)_{n\in\mathbb{N}}$ is a sequence of approximate weak solutions which has been constructed by a mollifying process with suitable regularity to justify the formal estimates like the energy estimate (1.4) and the Theorem 2.2 then ρ_n is uniformly bounded in $L^q((0,T)\times\Omega)$ with $q=\gamma+1-\frac{2\gamma}{N}$. We can observe that in this case $q>s=\max(\gamma,2)$. We will see that it will be very useful in the sequel to justify the passage to the weak limit in some terms to get a gain of integrability on the density. Indeed the key point in proving the existence of weak solutions is the passage to the limit in the term of pressure and in the term of capillarity.
First, we assume that $(\rho_n,u_n)_{n\in\mathbb{N}}$ has the initial data of the Theorem 1.1 with uniform bounds.
In addition:

- ρ_n is bounded uniformly in $L^{\infty}(0,T;L^1\cap L^s(\Omega))\cap C([0,T];L^p(\Omega))$ for $1\leq p<\max(2,\gamma)$,

- $\rho_n \geq 0$ a.e. and ρ_n is bounded uniformly in $L^q(0,T,\Omega)$ for some $q > s$,
- ∇u_n is bounded in $L^2(0,T;L^2(\Omega))$, $\rho_n|u_n|^2$ is bounded in $L^\infty(0,T;L^1(\Omega))$,
- u_n is bounded in $L^2(0,T;L^{\frac{2N}{N-2}}(\Omega))$ for $N \geq 3$.

We will explain in the appendix how to obtain such sequence of approximate solutions. Passing to the weak limit in the previous bound in extracting subsequence if necessary, one can assume that:

- $\rho_n \rightharpoonup \rho$ weakly in $L^s((0,T)\times\Omega)$,
- $u_n \rightharpoonup u$ weakly in $L^2(0,T,\dot{H}^1(\Omega))$,
- $\rho_n^\gamma \rightharpoonup \overline{\rho^\gamma}$ weakly in $L^r((0,T)\times\Omega)$ for $r = \frac{q}{\gamma} > 1$,
- $\rho_n^2 \rightharpoonup \overline{\rho^2}$ weakly in $L^{r_1}((0,T)\times\Omega)$ for $r_1 = \frac{q}{2} > 1$.

Notation 1 *We will always write in the sequel $\overline{B(\rho)}$ to mean the weak limit of the sequence $B(\rho_n)$ bounded in appropriate space that we will precise.*

We recall that the main difficulty will be to pass to the limit in the pressure term and the capillary term. The idea of the proof will be to test the convergence of the sequence $(\rho_n)_{n\in\mathbb{N}}$ on convex functions B in order to use their properties of lower semi-continuity with respect to the weak topology in $L^1(\Omega^N)$. In this goal we will use the theory of renormalized solutions introduced by Di Perna and Lions in [7]. So we will obtain strong convergence of ρ_n in appropriate spaces.

Proof of Theorem 1.1

Before getting into the heart of the proof, we first recall that we obtain easily the convergence in distribution sense of $\rho_n u_n$ to ρu and $\rho_n(u_n)_i(u_n)_j$ to $\rho u_i u_j$. We refer to the classical result by Lions (see [20]) or the book of Novotný and Straškraba [21].
Our goal is to compare $\overline{B(\rho)}$ and $B(\rho)$ for certain concave functions B . From the mass equation we have obtained:

$$\partial_t(\overline{B(\rho)} - B(\rho)) + \mathrm{div}(u(\overline{B(\rho)} - B(\rho))) = \overline{b(\rho)\mathrm{div}(u)} - b(\rho)\mathrm{div}(u). \tag{2.15}$$

So before comparing $\overline{B(\rho)}$ and $B(\rho)$, we have to investigate the expression $\overline{b(\rho)\mathrm{div}(u)} - b(\rho)\mathrm{div}(u)$. By virtue of Theorem 2.2 which gives a gain of integrability we can take the function $B(x) = x^\varepsilon$, as we control for ε small enough $\rho^{s+\varepsilon}$. Our goal now is to exhibit the effective pressure $\widetilde{P}_{eff}$, and to multiply it by ρ^ε to extract $\overline{\mathrm{div} u\, b(\rho)}$. We will see in the sequel how to compare it with $b(\rho)\mathrm{div}(u)$.

Control of the term $\overline{\mathrm{div} u\, b(\rho)}$

Taking the $\mathrm{div}(\Delta)^{-1}$ of the momentum equation satisfied by the regular solution yields: with $\zeta = \lambda + 2\mu$, we obtain:

$$\frac{\partial}{\partial t}(-\Delta)^{-1}\mathrm{div}(\rho_n u_n) + (-\Delta)^{-1}\partial^2_{ij}(\rho_n u^i_n u^j_n) + [\zeta \mathrm{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}c^1\rho_n^2] = \kappa R_n \ . \tag{2.16}$$

After we multiply (2.16) by ρ_n^ε with ε that we choose small enough in $(0,1)$:

$$\begin{aligned}[\zeta \mathrm{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}c^1\rho_n^2]\rho_n^\varepsilon = \kappa\rho_n^\varepsilon R_n - \rho_n^\varepsilon\frac{\partial}{\partial t}(-\Delta)^{-1}\mathrm{div}(\rho_n u_n) \\ - \rho_n^\varepsilon(-\Delta)^{-1}\partial_{ij}(\rho_n u^i_n u^j_n).\end{aligned} \tag{2.17}$$

So if we rewrite (2.17), we have:

$$\begin{aligned}[\zeta \mathrm{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}c^1\rho_n^2]\rho_n^\varepsilon = \kappa\rho_n^\varepsilon R_n - \rho_n^\varepsilon(-\Delta)^{-1}\partial^2_{ij}(\rho_n u^i n u^j_n) \\ - \frac{\partial}{\partial t}\big((\rho_n)^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)\big) + [\frac{\partial}{\partial t}(\rho_n)^\varepsilon](-\Delta)^{-1}\mathrm{div}(\rho_n u_n),\end{aligned}$$

Next we have:

$$\begin{aligned}[\zeta \mathrm{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}c^1\rho_n^2]\rho_n^\varepsilon = \kappa\rho_n^\varepsilon R_n - \rho_n^\varepsilon(-\Delta)^{-1}\partial^2_{ij}(\rho_n u^i_n u^j_n) \\ - \frac{\partial}{\partial t}[(\rho_n)^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)] - \mathrm{div}[u_n(\rho_n)^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)] \\ + (\rho_n)^\varepsilon u_n\cdot\nabla(-\Delta)^{-1}\mathrm{div}(\rho_n u_n) + (1-\varepsilon)(\mathrm{div} u_n)(\rho_n)^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n),\end{aligned} \tag{2.18}$$

or finally:

$$\begin{aligned}[\zeta \mathrm{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}c^1\rho_n^2]\rho_n^\varepsilon = \kappa\rho_n^\varepsilon R_n \\ - \frac{\partial}{\partial t}[\rho_n^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)] - \mathrm{div}[u_n(\rho_n)^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)] \\ + (\rho_n)^\varepsilon[u_n.\nabla(-\Delta)^{-1}\mathrm{div}(\rho_n u_n) - (-\Delta)^{-1}\partial^2_{ij}(\rho u^i_n u^j_n)] \\ + (1-\varepsilon)(\mathrm{div} u_n)(\rho_n)^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n) \ .\end{aligned} \tag{2.19}$$

Now like in Lions [20] we want to pass to the limit in the distribution sense in (2.19) in order to estimate $\overline{\mathrm{div} u(\rho)^\varepsilon}$.

Passing to the weak limit in (2.19)

We shall use the following lemma by P.-L. Lions in [20]:

Lemma 1 *Let (g_n, h_n) converge weakly to (g, h) in $L^{p_1}(0,T,L^{p_2}(\Omega)) \times L^{q_1}(0,T,L^{q_2}(\Omega))$ where $1 \le p_1, p_2, q_1, q_2 \le +\infty$ satisfy $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}=1$. Assume in addition that:*

$$\frac{\partial g^n}{\partial t} \ \text{is bounded in} \ L^1(0,T,W^{-m,1}(\Omega)) \ \text{for some} \ m \ge 0 \ \text{independent of} \ n. \tag{2.20}$$

and that:

$$\|h^n - h^n(\cdot,\cdot+\xi)\|_{L^{q_1}(0,T,L^{q_2}(\Omega))} \to 0 \quad \textit{as } |\xi|\to 0, \textit{ uniformly in } n. \tag{2.21}$$

Then, $g^n h^n$ converges to gh (in the sense of distribution on $\Omega\times(0,T)$).

So we use the above lemma to pass to the weak limit in the following four non-linear terms of (2.19):

$$\begin{array}{ll} T_n^1 = u_n\rho_n^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n), & T_n^2 = \rho_n^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n), \\ T_n^3 = \rho_n^\varepsilon R_n, & T_n^4 = (\mathrm{div}u_n)(\rho_n)^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)\ . \end{array}$$

So we choose the different g_n^i and h_n^i as follows:

for T_n^1	$g_n^1 = u_n(\rho_n)^\varepsilon$	$g^1 = u\overline{\rho^\varepsilon}$	$h_n^1 = (-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$
for T_n^2	$g_n^2 = \rho_n^\varepsilon$	$g^2 = \overline{\rho^\varepsilon}$	$h_n^2 = (-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$
for T_n^3	$g_n^3 = \rho_n^\varepsilon$	$g^3 = \overline{\rho^\varepsilon}$	$h_n^3 = R_n$
for T_n^4	$g_n^4 = (\mathrm{div}u_n)(\rho_n)^\varepsilon$	$g^4 = \overline{\mathrm{div}u\,\rho^\varepsilon}$	$h_n^4 = (-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$.

To show that $u_n(\rho_n)^\varepsilon$ converges in distribution sense to $u\overline{\rho^\varepsilon}$ we apply lemma 1 with $h_n = u_n$ and $g_n = \rho_n^\varepsilon$. We now want to examine each term and apply the above lemma to pass to the limit in the weak sense.
We start with the first term T_n^1. We have that $\rho_n^\varepsilon u_n \in L^\infty(L^q)\cap L^2(L^r)$ with $\frac{1}{q} = \frac{\varepsilon}{2s}+\frac{1}{2}$ and $\frac{1}{r} = \frac{(N-2)}{2N}+\frac{\varepsilon}{s} = \frac{1}{2}-\frac{1}{N}+\frac{\varepsilon}{s}$. In addition the hypothesis (2.20) is immediately verified (use the momentum equation).
We now want to verify the hypothesis (2.21), so we have h_n^1 belongs to $L^\infty(W^{1,q'}_{loc}(\Omega^N))\cap L^2(W^{1,r'}_{loc}(\Omega^N))$ with $\frac{1}{q'} = \frac{1}{2}+\frac{1}{2s}$ and $\frac{1}{r'} = \frac{(N-2)}{2N}+\frac{1}{s} = \frac{1}{2}+\frac{1}{s}-\frac{1}{N}$. This result enables us to verify the hypothesis (2.21) by Sobolev embedding. So we can choose (with the notation of the above lemma) $q_1 = 2$ and $q_2 \in (r', \frac{Nr'}{N-r'})$, $p_1 = 2$, and $p_2 = 1-\frac{1}{q_2}$ which is possible by interpolation. Indeed we have: $\frac{1}{r'}+\frac{1}{r} = 1-\frac{2}{N}+\frac{1+\varepsilon}{s} \leq 1$. We proceed in the same way for T_n^2 and T_n^4.
We can similarly examine T_n^3, because $\rho_n^\varepsilon \in L^\infty(L^{\frac{1}{\varepsilon}}\cap L^{\frac{s}{\varepsilon}})$ and $\rho_n(\nabla\phi*\rho_n)\in L^\infty(L^1\cap L^{\frac{s}{2}})$, we can choose $p_2 = \frac{1}{\varepsilon}$, we have then $(\Delta)^{-1}\mathrm{div}\rho_n(\nabla\phi*\rho_n)\in L^\infty(0,T;W^{1,\frac{s}{2}})$ so that we can choose $q_1 = 2$, $q_2\in(1,\frac{N\frac{s}{2}}{N-\frac{s}{2}})$. We can conclude by interpolation.
Finally we have to study the last non-linear following term that we treat similarly as P.-L.Lions in [20]:

$$A_n = (\rho_n)^\varepsilon[u_n.\nabla(-\Delta)^{-1}\mathrm{div}(\rho_n u_n) - (-\Delta)^{-1}\partial^2_{ij}(\rho_n(u_i)_n(u_j)_n)].$$

We can express this term A_n as follows:

$$A_n = (\rho_n)^\varepsilon[u_n^j, R_{ij}](\rho_n u_n^i).$$

where $R_{ij} = (-\Delta)^{-1}\partial^2_{ij}$ with R_i the classical Riesz transform.
Next, we use a result by Coifman, Lions, Meyer, Semmes on this type of commutator (see [4]) to take advantage of the regularity of $[u_n^j, R_{ij}](\rho_n u_n^i)$.

Theorem 2.3 *The following map is continuous for any $N \geq 2$:*

$$\begin{aligned} W^{1,r_1}(\Omega)^N \times L^{r_2}(\Omega) \to W^{1,r_3}(\Omega)^N \\ (a,b) \to [a_j, R_iR_j]b_i \end{aligned} \tag{2.22}$$

with: $\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}$.

To pass to the weak limit in A_n we will use the previous lemma. We start with the case with $s > 3$. This quantity belongs to the space $L^1(W^{1,q})$ provided that $Du_n \in L^2(L^2)$ and $\rho u^j \in L^2(L^r)$ where $\frac{1}{r} = \frac{N-2}{2N} + \frac{1}{s} = \frac{1}{2} - \frac{1}{N} + \frac{1}{s}$ in which case $\frac{1}{q} = \frac{1}{r} + \frac{1}{2} = 1 - \frac{1}{N} + \frac{1}{s} \leq 1$. After we can use the above lemma applied to $h_n = [R_{ij}, u_n^j](\rho_n u_n^i)$ and $g_n = \rho_n^\varepsilon$. We can show easily in using again lemma 1 that h_n converges in distribution sense to $[R_{ij}, u_j](\rho_n u_i)$.

So we can take: $q_1 = 1$, $p_1 = +\infty$ and $q_2 \in (q, \frac{qN}{N-q})$, $p_2 = 1 - \frac{1}{q_2}$, this one because we can use interpolation and we can localize as we want limit in distribution sense.

In the case where $s \leq 3$, a simple interpolation argument can be used to accommodate the general case. It suffices to fix $L^{r_2}(\Omega^N)$ in the application (2.22) and use a result of Riesz-Thorin.

Finally passing to the limit in (2.17), we get:

$$\begin{aligned} &[\zeta\overline{\mathrm{div}u\,\rho^\varepsilon} - \overline{(a\rho^{\gamma+\varepsilon})} - \frac{\kappa}{2}c^1\overline{\rho^{2+\varepsilon}}] = \overline{\rho^\varepsilon}\overline{R} - \frac{\partial}{\partial t}[\overline{\rho^\varepsilon}(-\Delta)^{-1}\mathrm{div}(\rho u)] \\ &- \mathrm{div}[\overline{\rho^\varepsilon}u(-\Delta)^{-1}\mathrm{div}(\rho u)] + \overline{\rho^\varepsilon}[u.\nabla(-\Delta)^{-1}\mathrm{div}(\rho u) - (-\Delta)^{-1}\partial_{ij}(\rho u_i u_j)] \\ &\qquad + (1-\varepsilon)\overline{\mathrm{div}u\,\rho^\varepsilon}(-\Delta)^{-1}\mathrm{div}(\rho u). \end{aligned} \tag{2.23}$$

Inequality between the terms $\overline{\rho^\varepsilon}\,\mathrm{div}u$ and $\overline{\mathrm{div}u\,\rho^\varepsilon}$

Now we are interested in estimating the term $\overline{\rho^\varepsilon}\mathrm{div}u$ in order to describe the quantity $\overline{\rho^\varepsilon}\,\mathrm{div}u - \overline{\mathrm{div}u\,\rho^\varepsilon}$ before considering the quantity $\rho^\varepsilon\,\mathrm{div}u - \overline{\mathrm{div}u\,\rho^\varepsilon}$. We pass to the weak limit directly in (2.16) and we get in using again lemma 1:

$$\frac{\partial}{\partial t}(-\Delta)^{-1}\mathrm{div}(\rho u) + (-\Delta)^{-1}\partial^2_{ij}(\rho u_i u_j) + [(\mu+\xi)\mathrm{div}u - \overline{a\rho^\gamma}] = \; -\overline{R} + \frac{\kappa}{2}c^1\overline{\rho^2}. \tag{2.24}$$

Now we just multiply (2.24) with $\overline{\rho^\varepsilon}$ and we can see that each term has a distribution sense. So we get by proceeding in the same way as before:

$$\begin{aligned} &[\zeta\mathrm{div}u\,\overline{\rho^\varepsilon} - \overline{(a\rho^\gamma)}\;\overline{\rho^\varepsilon}) - \frac{\kappa}{2}c^1\overline{\rho^2}\,\overline{\rho^\varepsilon}] = \overline{\rho^\varepsilon}\overline{R} \\ &- \overline{\rho^\varepsilon}\frac{\partial}{\partial t}[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)] + \overline{\rho^\varepsilon}[u.\nabla(-\Delta)^{-1}\mathrm{div}(\rho u) - (-\Delta)^{-1}\partial_{ij}(\rho u_i u_j)] \\ &\qquad - \mathrm{div}[\rho^\varepsilon u(-\Delta)^{-1}\mathrm{div}(\rho u)] + (1-\varepsilon)\overline{\mathrm{div}u(\rho)^\varepsilon}\,(-\Delta)^{-1}\mathrm{div}(\rho u). \end{aligned} \tag{2.25}$$

Subtracting (2.25) from (2.23), we get:

$$\zeta\,\overline{\mathrm{div}u(\rho)^\varepsilon} - a\overline{\rho^{\gamma+\varepsilon}} - \frac{\kappa}{2}c^1\overline{\rho^{2+\varepsilon}} = \zeta\mathrm{div}u\,\overline{\rho^\varepsilon} - a\overline{\rho^\gamma}\,\overline{\rho^\varepsilon} - \frac{\kappa}{2}c^1\overline{\rho^2}\,\overline{\rho^\varepsilon} \quad \text{a.e}\,.$$

Next we observe that by convexity:

$$(\overline{\rho^{\gamma+\varepsilon}})^{\frac{\varepsilon}{\gamma+\varepsilon}} \geq (\overline{\rho^\varepsilon}), \quad (\overline{\rho^{\gamma+\varepsilon}})^{\frac{\gamma}{\gamma+\varepsilon}} \geq (\overline{\rho^\gamma}) \quad \text{a.e}\,.$$

So we get:

$$\overline{\mathrm{div}u\,(\rho)^\varepsilon} \geq \mathrm{div}u\,\overline{\rho^\varepsilon}. \tag{2.26}$$

Comparison between ρ and $\overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}}$

As (ρ_n, u_n) is a smooth approximate solution, applying equality (2.15) to $B(x) = x^{\varepsilon}$ yields:

$$\frac{\partial}{\partial t}(\rho_n)^{\varepsilon} + \mathrm{div}(u_n(\rho_n)^{\varepsilon}) = (1-\varepsilon)\mathrm{div}u_n(\rho_n)^{\varepsilon}. \tag{2.27}$$

Passing to the weak limit in (2.27), we get:

$$\frac{\partial}{\partial t}\overline{\rho^{\varepsilon}} + \mathrm{div}(u\,\overline{\rho^{\varepsilon}}) = (1-\varepsilon)\overline{\mathrm{div}u\rho^{\varepsilon}}. \tag{2.28}$$

Combining with (2.26) we thus get:

$$\frac{\partial}{\partial t}\overline{(\rho)^{\varepsilon}} + \mathrm{div}(u\overline{(\rho)^{\varepsilon}}) \geq (1-\varepsilon)\mathrm{div}u\overline{(\rho)^{\varepsilon}}. \tag{2.29}$$

Now we wish to conclude about the pointwise convergence of ρ_n in proving that $(\overline{\rho^{\varepsilon}})^{\frac{1}{\varepsilon}} = \rho$ and to finish we will use the following theorem (see [9] p. 34) applied to $B(x) = x^{\frac{1}{\varepsilon}}$ which is convex.

Theorem 2.4 *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions bounded in $L^1(\Omega^N)$ such that:*

$$v_n \rightharpoonup v \quad \textit{weakly in } L^1(\Omega).$$

Let $\varphi : \Omega \longrightarrow [-\infty, +\infty)$ be a upper semi-continuous strictly concave function such that $\varphi(v_n) \in L^1(\Omega)$ for any n, and:

$$\varphi(v_n) \rightharpoonup \overline{\varphi(v)} \quad \textit{weakly in } L^1(\Omega).$$

Then:

$$\varphi(v) \geq \overline{\varphi(v)}.$$

and if $\varphi(v) = \overline{\varphi(v)}$ then:

$$v_n(y) \to v(y) \;\; a.e.$$

extracting a subsequence as the case may be.

Now we want to use a type of Di Perna-Lions theorem on inequality (2.29). Our goal is to renormalize this inequality with the function $B(x) = x^{\frac{1}{\varepsilon}}$ so that one can compare ρ and $\overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}}$. Although (2.29) doesn't correspond exactly to the mass equation, we can use the same technics to renormalize the solution provided that $\rho \in L^{\infty}(L^2)$ which is the case. [1] We recall Di Perna-Lions theorem on renormalized solution for the mass equation.

Theorem 2.5 *Suppose that $\rho \in L^{\infty}(L^2)$, $\beta \in C[0,\infty) \cap C^1(0,\infty);\mathbb{R})$ and the function $b(z) = z\beta'(z) - \beta(z)$ is bounded on $[0,\infty)$ with moreover $\beta(0) = b(0) = 0$.*
We have then:

$$\frac{\partial \beta(\rho)}{\partial t} + \mathrm{div}(\beta(\rho)\,u) = (\beta(\rho) - \rho\beta'(\rho))\mathrm{div}u$$

in distribution sense.

[1] In our case it is very important that $\rho \in L^{\infty}(L^2)$, indeed it avoids to have supplementary conditions on the index γ like for the compressible Navier-Stokes system in [20].

We now want to adapt this theorem for our equation (2.29) with $\beta(x) = x^{\frac{1}{\varepsilon}}$, so we may regularize by ω_α (with $\omega_\alpha = \frac{1}{\alpha^N}\omega(\frac{\cdot}{\alpha})$ where $\omega \in C_0^\infty(\Omega^N)$, supp $\omega \in B_1$ and $\int \omega dx = 1$) and find for all $\beta \in C_0^\infty([0,+\infty))$:

$$\frac{\partial}{\partial t}(\overline{\rho^\varepsilon} * \omega_\alpha) + \mathrm{div}[u\,\overline{\rho^\varepsilon} * \omega_\alpha] \geq (1-\varepsilon)\mathrm{div}u\,\overline{\rho^\varepsilon} * \omega_\alpha + R_\alpha$$

where we have:

$$R_\alpha = \mathrm{div}[u\,\overline{\rho^\varepsilon} * \omega_\alpha] - \mathrm{div}(u\,\overline{\rho^\varepsilon}) * \omega_\alpha + (1-\varepsilon)[\mathrm{div}u\,\overline{\rho^\varepsilon}] * \omega_\alpha - (1-\varepsilon)\mathrm{div}u\,\overline{\rho^\varepsilon} * \omega_\alpha$$

We get:

$$\begin{aligned}\frac{\partial}{\partial t}(\beta(\overline{\rho^\varepsilon} * \omega_\alpha)) + \mathrm{div}[u\,\beta(\overline{\rho^\varepsilon} * \omega_\alpha)] &\geq (1-\varepsilon)\mathrm{div}u\,\overline{\rho^\varepsilon} * \omega_\alpha\,\beta'(\overline{\rho^\varepsilon} * \omega_\alpha)\\ &\quad + (\mathrm{div}u)[\beta(\overline{\rho^\varepsilon} * \omega_\alpha) - \overline{\rho^\varepsilon} * \omega_\alpha\beta'(\overline{\rho^\varepsilon} * \omega_\alpha)] + R_\alpha\beta'(\overline{\rho^\varepsilon} * \omega_\alpha)\\ &= -\varepsilon(\mathrm{div}u)\overline{(\rho)^\varepsilon}\beta'(\overline{\rho^\varepsilon}) + (\mathrm{div}u)\,\beta(\overline{\rho^\varepsilon}) + R_\alpha\beta'(\overline{\rho^\varepsilon} * \omega_\alpha)\,.\end{aligned}$$

After we pass to the limit when $\alpha \to 0$ and we see that R_α tends to 0 in using lemma on regularization in [19] p 43. This looks like a rather harmless manipulation but it's at this point that we require to control ρ in $L^2(0,T;\Omega)$. And in our case we don't need to impose $\gamma > \frac{N}{2}$ for $N \in \{2,3\}$. Hence:

$$\frac{\partial}{\partial t}(\beta(\overline{(\rho)^\varepsilon})) + \mathrm{div}[u\,\beta(\overline{(\rho)^\varepsilon})] \geq -\,\varepsilon(\mathrm{div}u)\overline{\rho^\varepsilon}\beta'(\overline{\rho^\varepsilon}) + (\mathrm{div}u)\beta(\overline{\rho^\varepsilon}).$$

We then choose $\beta = (\Psi_M)^{\frac{1}{\varepsilon}}$ where $\Psi_M = M\Psi(\frac{\cdot}{M})$, $M \geq 1$, $\Psi \in C_0^\infty([0,+\infty))$, $\Psi(x) = x$ on $[0,1]$, supp$\Psi \subset [0,2]$, and we obtain:

$$\begin{aligned}&\frac{\partial}{\partial t}(\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}) + \mathrm{div}[u\,\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}]\\ &\qquad\geq (\mathrm{div}u)\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}-1}\Psi_M'(\overline{\rho^\varepsilon})\overline{\rho^\varepsilon} + (\mathrm{div}u)\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}\\ &\qquad\geq \mathrm{div}u\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}-1}[\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}} - \Psi_M'(\overline{\rho^\varepsilon})\overline{\rho^\varepsilon}]1_{(\overline{\rho^\varepsilon}>M)}\\ &\qquad\geq -C_0|\mathrm{div}u|M^{\frac{1}{\varepsilon}}1_{(\overline{\rho^\varepsilon}>M)}.\end{aligned}$$

where $C_0 = \sup\{|\Psi(x)|^{\frac{1}{\varepsilon}-1}|\Psi(x) - x\Psi'(x)|,\ x \in [0,+\infty)\}$. Now we claim that:

$$\frac{\partial}{\partial t}(\overline{(\rho)^\varepsilon})^{\frac{1}{\varepsilon}}) + \mathrm{div}(u\,\overline{(\rho)^\varepsilon})^{\frac{1}{\varepsilon}}) \geq 0. \tag{2.30}$$

For proving that, we notice that by convexity $\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}} \leq \rho$, so we get :

$$\||\mathrm{div}u|M^{\frac{1}{\varepsilon}}1_{\overline{(\rho)^\varepsilon}>M}\|_{L_T^1(L^1(\Omega))} \leq \|\mathrm{div}u\|_{L_T^2(L^2(\Omega))}\|\rho\,1_{\rho>M^{\frac{1}{\varepsilon}}}\|_{L_T^2(L^2(\Omega))} \to 0 \;\text{ as }\; M \to +\infty.$$

We have concluded by dominated convergence. At this stage we subtract the mass equation to (2.30) and we get in setting $r = \rho - \overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}$:

$$\frac{\partial}{\partial t}(r) + \mathrm{div}(ur) \leq 0. \tag{2.31}$$

We now want to integrate and to use the fact that $r \geq 0$ to get that $r = 0$ a.a. To justify the integration we test our inequality against a cut-off function $\varphi_R = \varphi(\frac{\cdot}{R})$ where $\varphi \in C_0^\infty(\Omega)$, $\varphi = 1$ on $B(0,1)$, $\text{Supp}\varphi \subset B(0,2)$ and $R > 1$. We get:

$$\int_{[0,T]\times\Omega} \frac{\partial}{\partial t}[r(t,x)]\varphi_R(x) - u(t,x)r(t,x)\frac{1}{R}\nabla\varphi(\frac{x}{R})dt\,dx \leq 0. \tag{2.32}$$

Next we notice that:

$$|\int_{[0,T]\times\Omega} u(t,x)r(t,x)\frac{1}{R}\nabla\varphi(\frac{x}{R})dt\,dx| \leq \|u\|_{L^1(0,T;L^{\frac{2N}{N-2}}(\Omega))}\|r\|_{L^1(0,T;L^{\frac{2N}{N+2}}(\Omega))} \times \frac{1}{R}\|\nabla\varphi\|_{L^\infty(\Omega)}.$$

It implies that:

$$\int_{[0,T]\times\Omega} u(t,x)r(t,x)\frac{1}{R}\nabla\varphi(\frac{x}{R})dt\,dx \to 0 \quad \text{as} \quad R \to +\infty.$$

We have then:

$$\int_{[0,T]\times\Omega} \frac{\partial}{\partial t}r(t,x)\varphi_R(x)dt\,dx = \int_\Omega r(T,x)\varphi_R(T,x)dx - \int_\Omega r(0,x)\varphi_R(0,x)dx.$$

In order to conclude, it suffices to verify that $r(0,\cdot) = 0$. Indeed we will obtain that:

$$\lim_{R\to+\infty} \int_\Omega r(T,x)\varphi_R(T,x)dx \to \int_\Omega r(T,x)dx \leq 0 \quad \text{and} \quad r \geq 0.$$

then $r = 0$.
We know that ρ_n is uniformly bounded in $L^\infty(L^1 \cap L^s(\Omega))$, then ρ_n^ε is relatively compact in $C([0,T];L^p - w)$ with $1 < p < s$ (where $L^p - w$ denote the space L^p endowed with weak topology). As moreover $(\rho_0^\varepsilon)_n$ converges to ρ_0^ε, we deduce that $r(0) = 0$ a.a.
Now as $r = 0$ we conclude in using the Theorem 2.4 that ρ_n converges a.a to ρ and that ρ_n converges to ρ in $L^p([0,T] \times B_R)$ for all $p \in [1,q)$ and in $L^{p_1}(0,T,L^{p_2}(B_R))$ for all $p_1 \in [1,+\infty)$, $p_2 \in [1,s)$ and for all $R \in (0,+\infty)$.

Conclusion

We wish now conclude and get the convergence of our theorem in the total space. We aim at proving here the convergence of ρ_n in $C([0,T],L^p(\Omega)) \cap L^{q'}(\Omega \times (0,T))$ for all $1 \leq p < s$, $1 \leq q' < q$. We have just to show the convergence of ρ_n to ρ in $C([0,T],L^1(\Omega))$. To this end, we introduce $d_n = \sqrt{\rho_n}$ which clearly converges to $\sqrt{\rho}$ in $L^{2p_1}(0,T,L^{2p_2}(B_R)) \cap L^{2p}(B_R \times (0,T))$ to $d = \sqrt{\rho}$ for all $R \in (0,+\infty)$.
We next remark that $\rho \in C([0,T],L^1(\Omega))$ and thus $d \in C([0,T],L^2(\Omega)$. Indeed, using once more the regularization lemma in [19] we obtain the existence of a bounded $\rho_\alpha \in C([0,T],L^1(\Omega))$ smooth in x for all t satisfying:

$$\frac{\partial\rho_\alpha}{\partial t} + \text{div}(u\rho_\alpha) = r_\alpha \quad \text{in} \quad L^1((0,T)\times\Omega) \quad \text{as} \quad \alpha \to 0_+,$$

with $r_\alpha = \text{div}(u\rho_\alpha) - \text{div}(\rho u) * \omega_\alpha$ (where ω is defined as in the previous part).
$\rho_\alpha \to \rho$ in $L^1(\Omega \times (0,T))$, $\rho_\alpha/_{t=0} \to \rho/_{t=0}$ in $L^1(\Omega)$ as $\alpha \to 0_+$.
From these facts, it is straightforward to deduce that:

$$\frac{\partial}{\partial t}|\rho_\alpha - \rho_\eta| + \text{div}(u|\rho_\alpha - \rho_\eta|) = |r_\alpha - r_\eta|$$

and thus:

$$\sup_{[0,T]} \int_\Omega |\rho_\alpha - \rho_\eta| dx = \int_0^T \int_\Omega |r_\alpha - r_\eta| dx.$$

Since $\rho \in C([0,T], L^p(B_R) - w)$ (for all $R \in (0,+\infty)$, $1 < p < s$), we may then deduce that ρ_α converges to ρ in $C([0,T], L^1(\Omega))$.
Next, we observe that we can justify as we did above that d_n and d satisfy:

$$\frac{\partial d_n}{\partial t} + \text{div}(u_n d_n) = \frac{1}{2} d_n \text{div}(u_n),$$
$$\frac{\partial d}{\partial t} + \text{div}(ud) = \frac{1}{2} d\, \text{div}(u).$$

Therefore once more, d_n converges to d in $C([0,T], L^2(\Omega) - w)$.
Thus in order to conclude, we just have to show that whenever $t_n \in [0,T]$, $t_n \to t$, then $d_n(t_n) \to d(t)$ in $L^2(\Omega)$ or equivalently that:

$$\int_\Omega d_n(t_n)^2 dx = \int_\Omega \rho_n(t_n) dx \to_n \int_\Omega d(t)^2 dx = \int_\Omega \rho(t) dx.$$

This is the case since we deduce from the mass equation, integrating this equation over Ω and justifying the integration exactly like previously that:

$$\int_\Omega \rho_n(t_n) dx = \int_\Omega (\rho_0)_n dx \to_n \int_\Omega \rho_0 dx = \int_\Omega \rho(t) dx.$$

We then conclude by uniform continuity that $\|\rho_n(t_n) - \rho_n(t)\|_{L^1}$ tends to 0.

Proof of the convergence assertion on $\rho_n u_n$

We now want to show the convergence of $\rho_n u_n$ to have informations on strong convergence of u_n modulo the vacuum. We recall in this part some classical inequalities to get the convergence of $\rho_n u_n$, for more details see Lions in [20]. We use once more a mollifier $k_\alpha = \frac{1}{\alpha^N} k(\frac{\cdot}{\alpha})$ where $k \in C_0^\infty(\Omega)$ and we let $g_\alpha = g * k_\alpha$ for an arbitrary function g. We first observe that we have for all $\frac{N}{2} < p < s$:

$$|\big((\rho_n u_n)_\alpha - \rho_n u_n\big)(x)| = \Big| \int_\Omega [\rho_n(t,y) - \rho_n(t,x)] u_n(t,y) k_\alpha(x-y) dy$$
$$+ \rho_n(t,x)\big((u_n)_\alpha - u_n\big)(t,x)\Big|$$

We have in using Hölder inequalities with the measure $k_\alpha(x-y)dy$:

$$|\big((\rho_n u_n)_\alpha - \rho_n u_n\big)(x)| \leq \Big[\int_\Omega |\rho_n(t,y) - \rho_n(t,x)|^p k_\alpha(x-y) dy \Big]^{\frac{1}{p}} \big(|u_n|^{\frac{p}{p-1}}\big)_\alpha^{\frac{p-1}{p}}$$
$$+ \rho_n |(u_n)_\alpha - u_n|(t,x).$$

Hence for all $t \geq 0$

$$\int_\Omega |((\rho_n u_n)_\alpha - \rho_n u_n)(x)|dx \leq \Big[\int_\Omega dx \int_\Omega |\rho_n(t,y) - \rho_n(t,x)|^p k_\alpha(x-y)dy\Big]^{\frac{1}{p}}$$

$$\times \|(|u_n|^{\frac{p}{p-1}})_\alpha\|_{L^1}^{\frac{p-1}{p}} + \|\rho_n\|_{L^p}\|(u_n)_\alpha - u_n\|_{L^{\frac{p-1}{p}}},$$

$$\leq \Big[\sup_{|z|\leq\alpha} \|\rho_n(\cdot+z) - \rho_n\|_{L^p}\Big]\|u_n\|_{L^{\frac{p-1}{p}}} + \|\rho_n\|_{L^p}\|(u_n)_\alpha - u_n\|_{L^{\frac{p-1}{p}}}.$$

Next if we choose $p > \frac{2N}{N+2}$, so that $\frac{p}{p-1} < \frac{2N}{N-2}$ then $\|(u_n)_\alpha - u_n\|_{L^2(0,T;L^{\frac{p-1}{p}})}$ converges to 0 as α goes to 0_+ uniformly in n. In addition, the convergence on ρ_n assure that $\sup_{|z|\leq\alpha}\|\rho_n(\cdot+z) - \rho_n\|_{L^p}$ converge to 0 as α goes to 0_+ uniformly in n. Therefore in conclusion, $(\rho_n u_n)_\alpha - \rho_n u_n$ converge to 0 in $L^2(0,T;L^1)$ as α goes to 0_+ uniformly in n. Next $(\rho_n u_n)_\alpha$ is smooth in x, uniformly in n and in $t \in [0,T]$. Therefore, remarking that $\frac{\partial}{\partial t}(\rho_n u_n)_\alpha$ is bounded in a $L^2(0,T;H^m)$ for any $m \geq 0$, we deduce that $(\rho_n u_n)_\alpha$ converge to $(\rho u)_\alpha$ as n goes to $+\infty$ in $L^1((0,T)\times\Omega)$ for each α. Then using the bound on $\rho_n u_n$ in $L^\infty(L^{\frac{2s}{s+1}})$, we deduce that $\rho_n u_n$ converges to ρu in $L^1((0,T)\times\Omega)$ and we can conclude by interpolation.
The last convergence result is a consequence of the strong convergence of ρ_n and $\rho_n u_n$.
□

3 Numerical examples

In a last section we present numerical results for the two-dimensional (NSK) discretized by the Local Discontinuous Galerkin method in space and an implicit Runge-Kutta scheme in time. For the pressure in (NSK) we use the Van der Waals type function $P(\rho) = \frac{34\rho}{5(3-\rho)} - 3\rho^2$. As viscosity coefficients we take $\mu = 0.00256$ and $\lambda = 0.00171$, as capillarity coefficient $\kappa = 1.6$. For the kernel function we make the following compact choice (see also Figure 1):

$$\phi(x,y) := \frac{1}{\mathcal{N}(x,\Omega)}\psi(|x-y|), \quad x,y \in \Omega$$

with the even function

$$\psi(r) = \begin{cases} \Big(0.1 - \frac{r^2}{0.025}\Big)^2 & : \ r \in (-0.05, 0.05), \\ 0 & : \ \text{otherwise} \end{cases}$$

and

$$\mathcal{N}(x,\Omega) := \int_\Omega \psi(|x-y|)\,dy.$$

Thus the support of $\phi(x,.)$ is the intersection of Ω and a ball of radius 0.05 around x. The term $\mathcal{N}(x,\Omega)$ ensures the integral relation $\int_\Omega \phi(x,y)dy = 1$ for all $x \in \Omega$. The symmetry property $\phi(x,y) = \phi(y,x)$ is fulfilled as long as $\text{dist}(x,\partial\Omega) \geq 0.05$ holds true. Finally to account for gravitational effects we impose the term $g\rho$ for $g := (0,-0.05)$ on the right hand side of the momentum balance equation in (NSK).

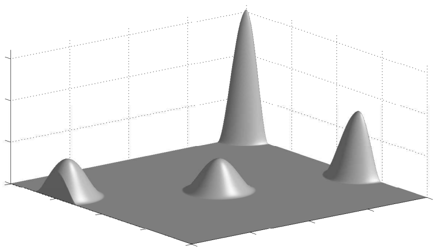


Figure 1: Kernel function at four different positions $x \in \Omega$.

We choose $\Omega := [-0.4, 0.4] \times [-2, 2]$ and divide it into 8000 triangles. The results shown below are piecewise quadratic approximations of the density component. As initial data we take zero velocity and the densities $\rho_{liquid} = 1.8071$ (Maxwell-state) resp. $\rho_{vapor} = 0.2$ (sligthly below Maxwell-state) in the phases liquid resp. vapor. The initial width of the interface between the two phases is set to 0.05. Further we make use of the tanh-function to smoothly connect the two constant states ρ_{vapor} and ρ_{liquid}.

3.1 Centered bubble

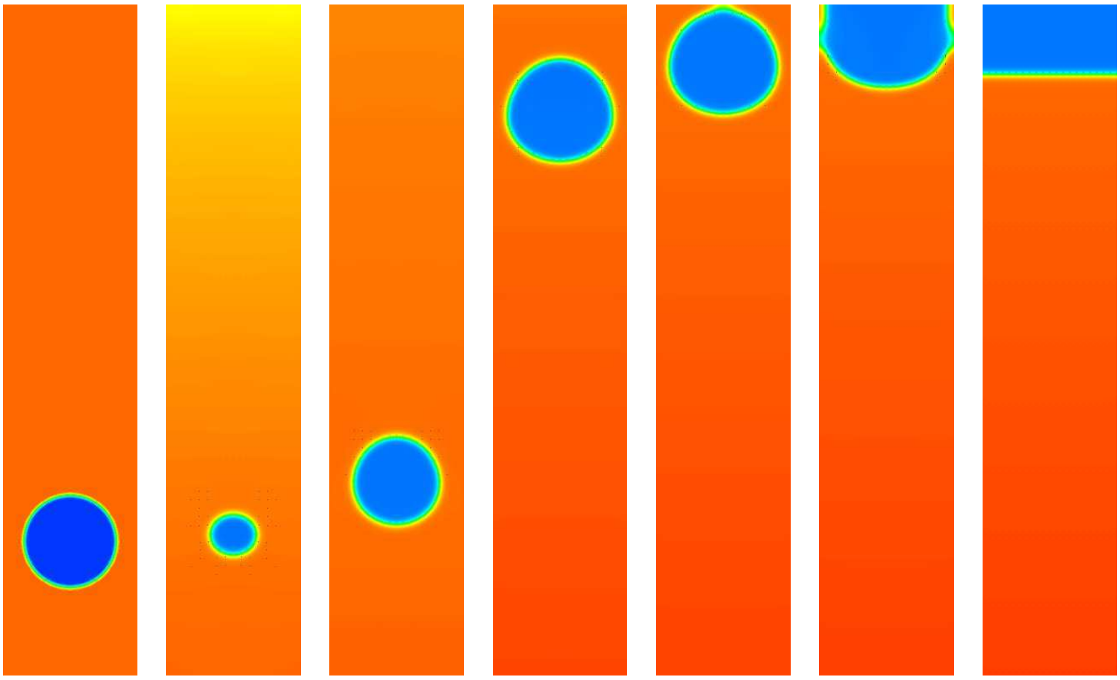
Figure 2 shows the dynamics of a single bubble of initial radius 0.3 at position $(0.0, -1.2)$. Due to the gravitation the liquid phase compresses at the bottom of the computational domain, indicated by the color field from yellow to red in Figure 2, subfigure (b). Simultaneously the vapor bubble shrinks at first. Afterwards the bubble enlarges, rises almost circular and deforms at the top of the domain Ω. The final situation at rest is a vapor layer above a liquid layer separated by a planar interface.

3.2 Bubble on the side

In a next example we shift the initial bubble to the side, here to the position $(0.4, -1.2)$ (see Figure 3). We additionally increase the radius to 0.4 such that the bubble survives the shrinking process at the beginning of the computation. One remarkable difference in contrast to the centered bubble in Figure 2 is the faster rising process, i.e., the bubble on the side reaches the top at the time $t = 47.4$ whereas the centered bubble needs almost twice the time. Furthermore note that the shape of the initially semi-circular bubble modifies skew, somehow elliptical. Moreover the bubble first touches the top in the middle region, but not at the upper right corner as one could expected.

3.3 Two rising bubbles

A last experiment shows the complex dynamics of two rising vapor bubbles in liquid (see Figure 4). Initially we choose the radius 0.35 resp. 0.25 and the coordinates $(0.0, -1.2)$ resp. $(0.08, -0.6)$ for the midpoint of the bubbles. As time goes on the two bubbles merge and thus to not rise separately but as one bubble. Before the shape of the single bubble can become circular it reaches the right boundary of Ω and climbs upwards. As in the previous example the shape of the rising bubble is skew but almost fixed (see Figure 4,



(a) $t = 0$. (b) $t = 4.2$. (c) $t = 8.4$. (d) $t = 80$. (e) $t = 92.8$. (f) $t = 97.1$. (g) $t = 200$.

Figure 2: Rising centered vapor bubble (blue) in liquid (red) at different times t.

subfigures (i) and (j)). Finally the mixture is separated into the two layers of vapor and liquid phase.

4 Appendix

This appendix is devoted proving global existence of approximate solutions for our system (NSK) checking the energy inequalities and the properties of Theorem 2.2. We use here a similar scheme as those in [9] and [21]. Here we employ a three level approximation scheme based on solving the following system of equations:

Continuity equation with vanishing viscosity

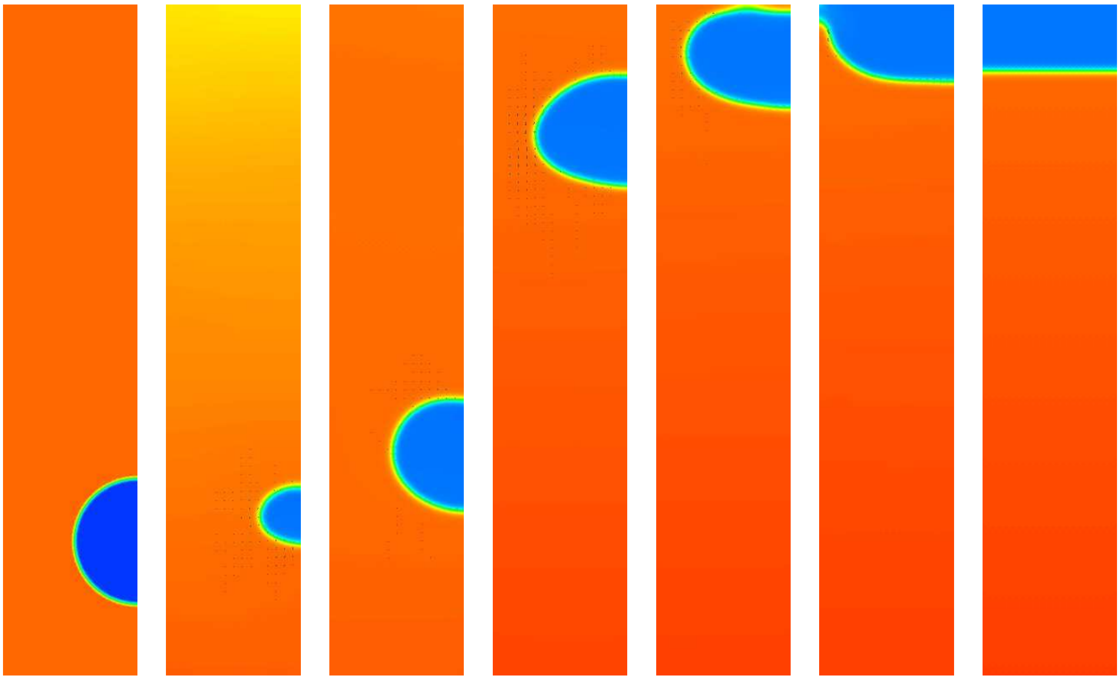
$$\partial_t \rho + \operatorname{div}(\rho u) = \varepsilon \Delta \rho \quad \text{on} \quad (0,T) \times \Omega, \quad \varepsilon > 0, \tag{4.33}$$

with the homogeneous Neumann boundary condition:

$$\nabla \rho \cdot n = 0 \quad \text{on} \quad \partial\Omega, \tag{4.34}$$

and the initial condition:

$$\rho(0) = \rho_{0,\delta} \quad \text{on} \quad \Omega. \tag{4.35}$$



(a) $t = 0$. (b) $t = 4$. (c) $t = 8$. (d) $t = 38$. (e) $t = 47.4$. (f) $t = 52$. (g) $t = 100$.

Figure 3: Rising semi-circular bubble on the side edge of Ω.

Momentum equation with artificial pressure

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div} u + \nabla(P(\rho) + \delta \rho^\beta) \\ + \varepsilon \nabla u \cdot \nabla \rho = \kappa \rho \nabla(\rho * \phi - \rho) \quad \text{on } (0,T) \times \Omega,\ \delta > 0,\ \beta > N. \tag{4.36}$$

with:

$$u = 0 \quad \text{on } \partial\Omega, \tag{4.37}$$

$$(\rho u)(0) = m_{0,\delta} \quad \text{on } \Omega. \tag{4.38}$$

We suppose that:

$$P(s) = s^\gamma,\ \gamma > \frac{N}{2}. \tag{4.39}$$

We also assume that:

$$\mu > 0,\ \lambda + \frac{2}{N}\mu \geq 0. \tag{4.40}$$

The extra term $\varepsilon \Delta \rho$ in (4.33) represents a vanishing viscosity with no specific physical meaning. From the mathematical viewpoint, however, it converts the hyperbolic equation (4.33) into a parabolic one. As a result, one can expect better regularity properties of the densities ρ constructed at this level of approximation.
The quantity $\delta \rho^\beta$ added to the momentum equation (4.36) can be considered as an artificial pressure, which was introduced to make the pressure estimates compatible with the vanishing viscosity regularization of (4.33). More precisely, the pressure estimates

based on multiplication of (4.36) by the quantity $\nabla\Delta^{-1}\rho^w$ will still remain in force for the modified system (4.33), (4.36) only if $w = 1$. Accordingly, one must take $\beta = \beta(N)$ large enough to be able to exploit the ideas of the proof of proposition 2.2. We assume then that:

$$\delta > 0 \ \text{ and } \ \beta > N. \tag{4.41}$$

In the same spirit, the new quantity $\varepsilon\nabla u \cdot \nabla\rho$ was introduced in (4.36) in order to eliminate the extra term arising in the energy inequality to save the a priori estimates. The initial data is modified as follows:

1. The density $\rho_{0,\delta} \in C^{2+\nu}(\overline{\Omega})$, $\nu > 0$, satisfies the homogeneous Neumann boundary condition:

$$\nabla\rho_{0,\delta} \cdot n_{/\partial\Omega} = 0. \tag{4.42}$$

   Furthermore, we suppose:

$$0 < \delta \leq \rho_{0,\delta}(x) \leq \delta^{-\frac{1}{2\beta}} \quad \text{for all } \ x \in \Omega, \tag{4.43}$$

2. The initial momentum $m_{0,\delta}$ are defined as

$$\begin{aligned} m_{0,\delta}(x) &= m_0 \quad \text{if } \ \rho_{0,\delta}(x) \geq 0, \\ &= 0 \quad \text{for } \ \rho_{0,\delta}(x) < \rho_0(x). \end{aligned} \tag{4.44}$$

In particular, the initial value of the modified total energy:

$$E(0) = E_\delta(0) = \int_\Omega \frac{1}{2}\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}} + P(\rho_{0,\delta}) + \frac{\delta}{\beta - 1}\rho_{0,\delta}^\beta dx, \tag{4.45}$$

is bounded by a constant independent of $\delta > 0$.
The principal strategy of the proof is to solve first the system (4.33)-(4.38) for positive values of the parameters ε and δ, then to let $\varepsilon \to 0$ to get rid of the artificial viscosity in (4.33); and finally, evoking the full strength of the pressure, we pass to the limit for $\delta \to 0$ to recover the original (NSK) system.

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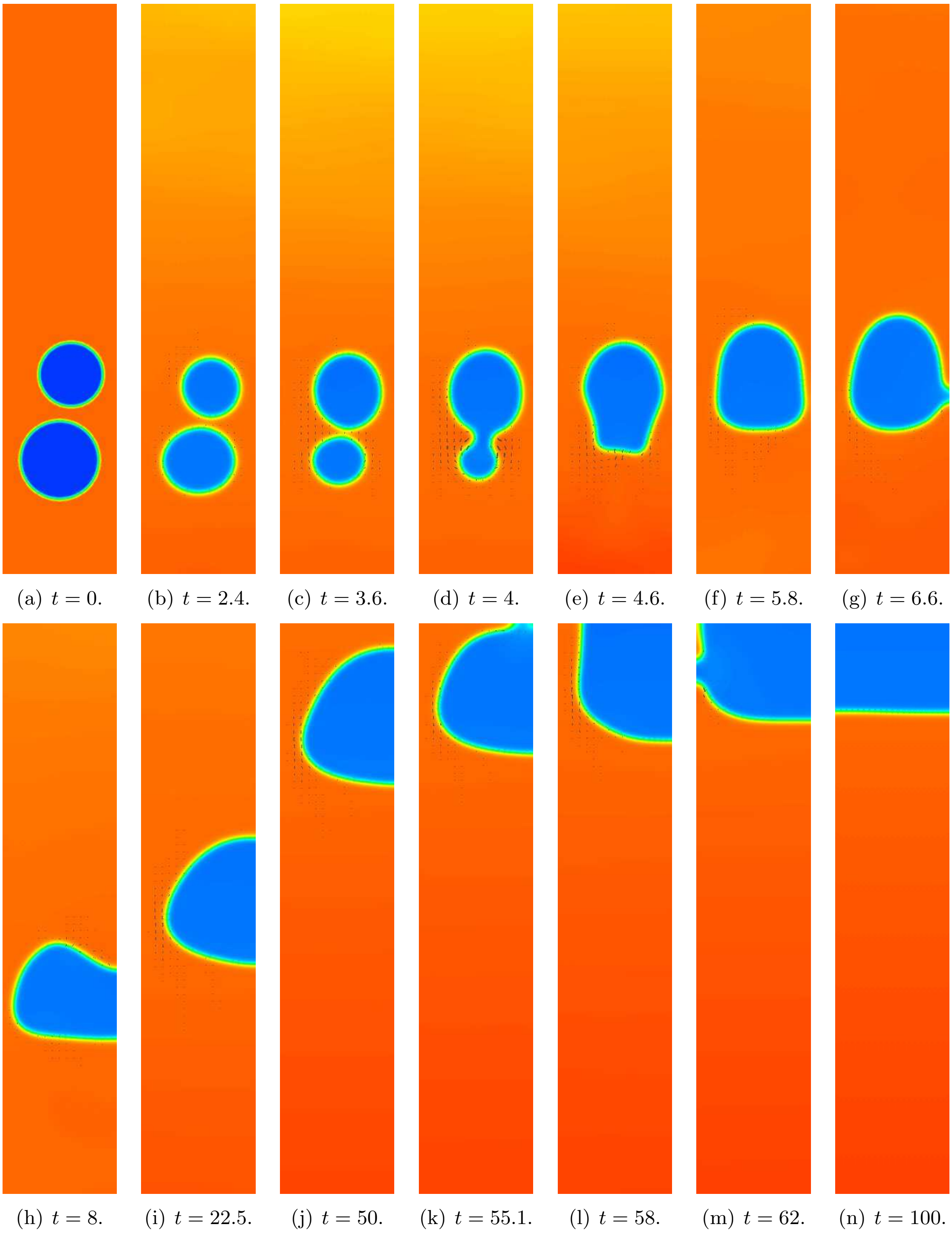

Figure 4: Complex dynamics of two rising vapor bubbles.

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