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Homogenization of the Signorini boundary-value problem in a thick junction and boundary integral equations for the homogenized problem

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Dedicated to Professor Dr. Anna-Margarete Sändig
on the occasion of her 65th birthday

Abstract

We consider a mixed boundary-value problem for the Poisson equation in a thick junction Ω_ε which is the union of a domain Ω_0 and a large number of $\varepsilon-$periodically situated thin cylinders. The nonuniform Signorini conditions are given on the lateral surfaces of the cylinders. The asymptotic analysis of this problem is done as $\varepsilon \to 0$, i.e., when the number of the thin cylinders infinitely increases and their thickness tends to zero. We prove a convergence theorem and show that the nonuniform Signorini boundary conditions are transformed in the limiting variational inequalities in the region that is filled up by the thin cylinders as $\varepsilon \to 0$. The convergence of the energy integrals is proved as well. The existence and uniqueness of the solution to this non-standard limit problem is established. This solution can be constructed by using a penalty formulation and successive iteration. For some subclass, these problems can be reduced to an obstacle problem in Ω_0 and an appropriate postprocessing. The equations in Ω_0 finally are also treated with boundary integral equations.

Key words: homogenization; thick junction; Signorini type conditions; variational inequalities; boundary integral equations

MOS subject classification: 35B27, 35R45, 35J20, 74K30, 60N38, 65R20

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1 Introduction and statement of the problem

Boundary-value problems in thick junctions are mathematical models of widely used engineering and industrial constructions as well as many other physical and biological systems with very distinct characteristic scales. In recent years, many new results on asymptotic analysis of boundary-value problems in thick junctions have appeared (see [2]–[8]).

A thick junction (or thick multi-structure) of type $k : p : d$ is the union of some domain in $\mathbb{R}^n$, which is called the junction's body, and a large number of ε-periodically situated thin domains along some manifold on the boundary of the junction's body (see Fig. 1.1)). This manifold is called the joint zone. Here ε is a small parameter, which characterizes the distance between neighboring thin domains and their thickness. The type $k : p : d$ of a thick junction refers to the limiting dimensions of the body, the joint zone, and each of the attached thin domains, respectively.

This classification of thick junctions was given in [9]–[15], where rigorous mathematical methods were developed (homogenization, approximation, asymptotic expansions) for analyzing the main boundary-value problems in thick junctions of different types. It was pointed out that qualitative properties of solutions essentially depend on the junction type and on the conditions given on the boundaries of the attached thin domains. In addition, as it was shown in [16] that such problems loose the coercitivity as $\varepsilon \to 0$ which creates special difficulties in the investigation of the asymptotics. It should be noted that the papers [17, 18] were the first papers in this direction.

Signorini presented in [19] the first rigorous mathematical analysis of contact problems. In the Signorini boundary value problem the location of the boundary regions, where the boundary conditions (Dirichlet or Neumann) are satisfied, is part of the solution. Many problems in applied mathematics involve the Signorini boundary conditions. Applications arise in groundwater hydrology, in plasticity, in crack theory, in optimal control problems, etc. (see [20]). Many of these problems can be recast as variational inequalities (see [20]–[23]).

In this paper we homogenize the Signorini problem in a thick junction of type $3 : 2 : 1$ by using the integral identity method developed in [7, 9] and then apply boundary element analysis to the corresponding homogenized problem using the approach proposed in [31]. The homogenization of the Signorini problem in a plane thick junction of type $2 : 1 : 1$ was performed in [24]. and for different variational problems in perforated domains in [25]-[29].

The homogenized problem is formulated in terms of a variational inequality. By using a penalization, it can be approximated by a family of nonlinear problems which can be solved with successive approximation (see [35, 36]). For a subclass, our problems can be reduced to an obstacle problem in Ω_0, and the behavior of the solution in the junction's body can be found a posteriori. Since in the domain Ω_0 the linear Poisson equation is required, the formulation can also be reduced to a coupled system of boundary integral equations including one given Newton potential. The successive iteration of the penalized equations then leads to a corresponding successive solution of boundary integral equations whose numerical realization could be executed by using modern multipole or ACA boundary element methods.

1.1 Statement of the problem

Let a and h be positive numbers and N be a large positive integer. Define a small parameter $\varepsilon = \frac{a}{N}$.

A model thick junction Ω_ε of type $3:2:1$ (see Fig. 1.1) consists of the "body"

$$\Omega_0 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x' = (x_1, x_2) \in \Xi_0 = (0,a) \times (0,a), \quad -\gamma(x') < x_3 < 0\}$$

and a large number of thin curvilinear cylinders $G_\varepsilon = \bigcup_{i,j=0}^{N-1} G_\varepsilon(i,j)$,

$$G_\varepsilon(i,j) = \left\{ x \in \mathbb{R}^3 : \ 0 < x_3 < h, \quad \left(\frac{x_1}{\varepsilon} - \frac{1}{2} - i\right)^2 + \left(\frac{x_2}{\varepsilon} - \frac{1}{2} - j\right)^2 < \varrho^2(x_3) \right\}, \tag{1}$$

where the given functions γ and ϱ are smooth and positive on $[0,a] \times [0,a]$ and $[0,h]$, respectively; in addition $0 < \varrho < \frac{1}{2}$. Obviously, the thin curvilinear cylinders fill out the parallelepiped $\Omega^+ = \Xi_0 \times (0,h)$ in the limit passage as $N \to +\infty$ $(\varepsilon \to 0)$.

Remark 1. *The more general case of general curvilinear cylinders*

$$G_\varepsilon(i,j) = \{x \in \mathbb{R}^3 : \quad 0 < x_3 < h, \quad (\varepsilon^{-1}x_1 - i, \, \varepsilon^{-1}x_2 - j) \in \omega(x_3)\}, \tag{2}$$

where $\omega(x_3)$ is a plane domain, which lies in the interior of the square $\{\xi' = (\xi_1, \xi_2) : 0 < \xi_1 < 1, \, 0 < \xi_2 < 1\}$ for every $x_3 \in [0,h]$, and the surface $\{(\xi', x_3) : \xi' \in \partial\omega, \, x_3 \in [0,h]\}$ is smooth, allows the same type of analysis, however, sometimes requires additional cumbersome calculations. Hence, for simplicity, we consider here only the simpler case (1).

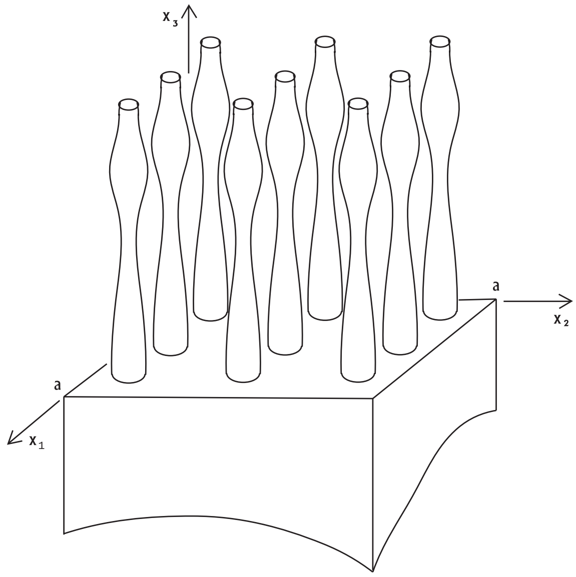

Figure 1: A model thick junction Ω_ε of type $3:2:1$.

In Ω_ε we consider the following boundary value problem:

$$\begin{aligned}
-\Delta_x u_\varepsilon(x) = f(x), \quad x \in \Omega_\varepsilon, \\
u_\varepsilon(x) = 0, \quad x \in \Gamma_\varepsilon, \\
u_\varepsilon(x) \le g(x), \quad \partial_\nu u_\varepsilon(x) \le \varepsilon d(x), \quad x \in S_\varepsilon, \\
(u_\varepsilon(x) - g(x))\,(\partial_\nu u_\varepsilon(x) - \varepsilon d(x)) = 0, \quad x \in S_\varepsilon, \\
\partial_\nu u_\varepsilon(x) = 0, \quad x \in \partial\Omega_\varepsilon \setminus (S_\varepsilon \cup \Gamma_\varepsilon),
\end{aligned} \tag{3}$$

where $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative, S_ε is the union of the lateral surfaces of the thin cylinders. The union of the upper bases of G_ε at $x_3 = h$ is denoted by Γ_ε; and f, g, d are given functions.

We assume that $f \in L^2(\Omega_1)$, where $\overline{\Omega_1} = \overline{\Omega_0} \cup \overline{\Omega^+}$, $\Omega^+ = \Xi_0 \times (0,h)$,

$$\Xi_0 = \{x : \ x' = (x_1, x_2) \in (0,a) \times (0,a), \quad x_3 = 0\}.$$

The function g belongs to the Sobolev space $H^1(\Omega^+; \Xi_h \cup \Xi_0) = \{v \in H^1(\Omega^+) : \ v|_{\Xi_h \cup \Xi_0} = 0\}$, where $\Xi_h = \{x : \ x' \in (0,a) \times (0,a), \quad x_3 = h\}$, and $d \in H^1(\Omega^+)$.

In biological application, the exterior around the cylinders is filled with biological material so that on S_ε boundary conditions of Signorini type are to be satisfied (see [21]).

Our goal is to study the asymptotic behavior of the solution u_ε to problem (3) as $\varepsilon \to 0$, i.e., when the number of the thin cylinders becomes infinite whereas their thickness tends to zero.

2 Definitions of the weak solution and its existence

In the Sobolev space $H^1(\Omega_\varepsilon; \Gamma_\varepsilon) = \left\{u \in H^1(\Omega_\varepsilon) : \ u|_{\Gamma_\varepsilon} = 0\right\}$, we define the subset

$$K_\varepsilon = \{\varphi \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon) : \quad \varphi|_{S_\varepsilon} \le g|_{S_\varepsilon} \quad \text{a.e. on} \quad S_\varepsilon\},$$

where $\psi|_S$ denotes the trace of ψ on the surface S. The set K_ε is closed and convex for every fixed value of $\varepsilon > 0$.

Let us suppose that a classical solution to problem (3) exists. We can regard that $g = 0$ in Ω_0. Multiplying the equation of problem (3) by the function $(u_\varepsilon - g)$, integrating by parts in Ω_ε and taking into account the boundary conditions for u_ε, we obtain

$$\int\limits_{\Omega_0} \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx + \int\limits_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla (u_\varepsilon - g)\, dx = \int\limits_{\Omega_0} f u_\varepsilon \, dx + \int\limits_{G_\varepsilon} f (u_\varepsilon - g)\, dx + \varepsilon \int\limits_{S_\varepsilon} d(x)(u_\varepsilon - g)\, d\sigma. \tag{4}$$

Now we take any function $\varphi \in K_\varepsilon$ and multiply the equation of the problem (3) by $(\varphi - g)$. Similarly as before we get

$$\int\limits_{\Omega_0} \nabla u_\varepsilon \cdot \nabla \varphi \, dx + \int\limits_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla (\varphi - g)\, dx$$

$$= \int_{\Omega_0} f\varphi\, dx + \int_{G_\varepsilon} f(\varphi - g)\, dx + \varepsilon \int_{S_\varepsilon} d(x)(\varphi - g)\, d\sigma + \int_{S_\varepsilon} (\partial_\nu u_\varepsilon - \varepsilon d(x))(\varphi - g)\, d\sigma. \tag{5}$$

Since $\partial_\nu u_\varepsilon(x) \le \varepsilon d(x)$ and $\varphi(x) \le g(x)$ a.e. in S_ε,

$$\int_{S_\varepsilon} (\partial_\nu u_\varepsilon - \varepsilon d(x))(\varphi - g)\, d\sigma \ge 0. \tag{6}$$

Taking into account (6), it follows from equality (5) that

$$\int_{\Omega_0} \nabla u_\varepsilon \cdot \nabla\varphi\, dx + \int_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla(\varphi - g)\, dx \ge \int_{\Omega_0} f\varphi\, dx + \int_{G_\varepsilon} f(\varphi - g)\, dx + \varepsilon \int_{S_\varepsilon} d(x)(\varphi - g)\, d\sigma. \tag{7}$$

Definition 1. *A function $u_\varepsilon \in K_\varepsilon$ is called a weak solution to problem (3) if it satisfies equality (4) and inequality (7) for any arbitrary function $\varphi \in K_\varepsilon$.*

An equivalent definition reads as follows.

Definition 2. *A function $u_\varepsilon \in K_\varepsilon$ is called a weak solution to problem (3) if it satisfies the inequality*

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla(\varphi - u_\varepsilon)\, dx \ge \int_{\Omega_\varepsilon} f(\varphi - u_\varepsilon)\, dx + \varepsilon \int_{S_\varepsilon} d(x)(\varphi - u_\varepsilon)\, d\sigma \qquad \textit{for all} \quad \varphi \in K_\varepsilon. \tag{8}$$

Let us show that these definitions indeed are equivalent. Subtracting equality (4) from inequality (7), we arrive at (8). Setting $\varphi = \begin{cases} 0, & x \in \Omega_0, \\ g, & x \in G_\varepsilon, \end{cases}$ in to (8), we have

$$-\int_{\Omega_0} |\nabla u_\varepsilon|^2\, dx + \int_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla(g - u_\varepsilon)\, dx \ge -\int_{\Omega_0} f u_\varepsilon\, dx + \int_{G_\varepsilon} f(g - u_\varepsilon)\, dx + \varepsilon \int_{S_\varepsilon} d(x)(g - u_\varepsilon)\, d\sigma. \tag{9}$$

Putting $\varphi = \begin{cases} 2u_\varepsilon, & x \in \Omega_0, \\ 2u_\varepsilon - g, & x \in G_\varepsilon, \end{cases}$ in to (8), we get the reversed inequality

$$\int_{\Omega_0} |\nabla u_\varepsilon|^2\, dx + \int_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla(u_\varepsilon - g)\, dx \ge \int_{\Omega_0} f u_\varepsilon\, dx + \int_{G_\varepsilon} f(u_\varepsilon - g)\, dx + \varepsilon \int_{S_\varepsilon} d(x)(u_\varepsilon - g)\, d\sigma. \tag{10}$$

This means that (4) holds. Setting $\varphi = \begin{cases} \psi + u_\varepsilon, & x \in \Omega_0, \\ \psi + u_\varepsilon - g, & x \in G_\varepsilon, \end{cases}$ in (8), where ψ is an arbitrary function from K_ε, we get (7).

It is well known (see for instance [20]–[23]) that for any fixed value of $\varepsilon > 0$ there exists a unique solution of inequality (8).

3 Auxiliary uniform estimates

To homogenize boundary-value problems in thick multi-structures with nonhomogeneous Neumann or nonlinear Fourier conditions on the boundaries of the thin attached domains, the method of special integral identities was proposed in [7, 9]. For our problem this identity reads as follows (see [7, Chapter 2])

$$\varepsilon \int\limits_{S_\varepsilon} \frac{\varphi(x)\, d\sigma_x}{\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}} = \int\limits_{G_\varepsilon} \zeta(x_3)\, \varphi\, dx + \varepsilon \int\limits_{G_\varepsilon} \nabla_{\xi'} Y(\xi', x_3)|_{\xi'=\frac{x'}{\varepsilon}} \cdot \nabla_{x'}\varphi\, dx \tag{11}$$

for all $\varphi \in H^1(G_\varepsilon)$. Here

$$\zeta(x_3) = \frac{l_\omega(x_3)}{|\omega(x_3)|}, \qquad \omega(x_3) = \left\{\xi' \in \mathbb{R}^2 : (\xi_1 - \tfrac{1}{2})^2 + (\xi_2 - \tfrac{1}{2})^2 < \varrho^2(x_3)\right\},$$

$|\omega(x_3)|$ is the area of the circle $\omega(x_3)$, $l_\omega(x_3)$ is the length of $\partial\omega(x_3)$ for any fixed $x_3 \in [0, h]$. The auxiliary function Y is a unique solution of the following problem:

$$\Delta_{\xi'} Y = \zeta(x_3) \quad \text{in} \quad \omega(x_3), \quad \partial_{\nu'(\xi')} Y = 1 \quad \text{on} \quad \partial\omega(x_3), \quad \int\limits_{\omega(x_3)} Y(\xi', x_3)\, d\xi' = 0.$$

Then Y is 1-periodically continued with respect to ξ_1 and ξ_2; $\xi' = x'/\varepsilon$, $\nu'(\xi') = (\nu_1(\xi'), \nu_2(\xi'))$.

To obtain (11) we have to integrate by parts the last integral in (11) and take into account the boundary condition for Y and coordinates of the outward normal to the lateral surfaces of each of the cylinders $G_\varepsilon(i,j)$, $i,j = 1, ..., N-1$:

$$\overline{\nu} = \frac{1}{\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}} (\nu_1(x'/\varepsilon),\, \nu_2(x'/\varepsilon)\,, -\varepsilon\varrho'(x_3)). \tag{12}$$

Remark 2. *We do not simplify the function ζ in (11) in order to take into account also a general form of the thin curvilinear cylinders (2).*

In [7, Chapter 2] the following inequalities were proved

$$\sup_{x\in G_\varepsilon} |\nabla_{\xi'} Y(\xi', x_3)\,|_{\xi'=\frac{x'}{\varepsilon}}| \le C_0, \tag{13}$$

$$\varepsilon \int\limits_{S_\varepsilon} \varphi^2\, d\sigma_x \le C_1 \left(\varepsilon^2 \int\limits_{G_\varepsilon} |\nabla_{x'}\varphi|^2\, dx + \int\limits_{G_\varepsilon} \varphi^2\, dx \right), \tag{14}$$

$$\int\limits_{G_\varepsilon} \varphi^2\, dx \le C_2 \left(\varepsilon^2 \int\limits_{G_\varepsilon} |\nabla_{x'}\varphi|^2\, dx + \varepsilon \int\limits_{S_\varepsilon} \varphi^2\, d\sigma_x \right), \tag{15}$$

$$\|\varphi\|_{L^2(S_\varepsilon)} \le C_3 \varepsilon^{-\frac{1}{2}} \|\varphi\|_{H^1(G_\varepsilon)} \qquad \text{for all} \quad \varphi \in H^1(G_\varepsilon). \tag{16}$$

Remark 3. *Here and in what follows all constants $\{C_i\}$ and $\{c_i\}$ in inequalities are independent of the parameter ε.*

Lemma 1 ([13]). *The norm* $\|u\|_{H^1(\Omega_\varepsilon)} = \left(\int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2)\,dx\right)^{\frac{1}{2}}$ *in* $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ *and the norm* $\|\cdot\|_\varepsilon$*, which is generated by the scalar product*

$$(u, v)_\varepsilon = \int\limits_{\Omega_\varepsilon} \nabla u \cdot \nabla v\,dx, \quad u, v \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon),$$

are uniformly equivalent, i.e., there exist constants $C_1 > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and for all* $u \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ *the following estimates hold:*

$$\|u\|_\varepsilon \le \|u\|_{H^1(\Omega_\varepsilon)} \le C_4 \|u\|_\varepsilon. \tag{17}$$

Remark 4. *In fact, for the proof of Lemma 1 in [13] the following uniform Friedrichs inequality*

$$\|u\|_{L^2(\Omega_\varepsilon)} \le C_5 \|\nabla u\|_{L^2(\Omega_\varepsilon)} \quad \forall\ u \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \tag{18}$$

was employed.

Using the Cauchy–Bunyakovsky integral inequality and Cauchy's inequality $2ab \le \delta a^2 + \delta^{-1} b^2$ with $\delta > 0$ and any positive numbers a and b, then with the help of (16) and (18) we deduce from (4) that

$$\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx \le c_0(\delta_1 + \delta_2 + \delta_3)\,\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \\ + c_1(1 + \delta_1^{-1})\,\|g\|^2_{H^1(D_0)} + c_2(1 + \delta_2^{-1})\,\|f\|^2_{L^2(\Omega_\varepsilon)} + c_3(1 + \delta_3^{-1})\,\|d\|^2_{H^1(D_0)}. \tag{19}$$

Choosing $\delta_1, \delta_2, \delta_3$ so that $c_0(\delta_1 + \delta_2 + \delta_3) < \frac{1}{2}$, we have

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx \le c_4 \left(\|f\|^2_{L^2(\Omega_1)} + \|g\|^2_{H^1(D_0)} + \|d\|^2_{H^1(D_0)} \right). \tag{20}$$

By virtue of (17), we then obtain from (20) the uniform estimate

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C_3 \qquad \text{for all} \qquad \varepsilon \in (0, \varepsilon_0). \tag{21}$$

4 Formulation of the convergence theorem

In what follows, $\widetilde{u}$ denotes the zero extension of u to the parallelepiped $\Omega^+ = \Xi_0 \times (0, h)$ which is filled up by the thin curvilinear cylinders in the limit passage as $\varepsilon \to 0$, namely

$$\widetilde{u}(x) = \begin{cases} u, & x \in G_\varepsilon, \\ 0, & x \in \Omega^+ \setminus G_\varepsilon. \end{cases} \tag{22}$$

Also we introduce the characteristic function

$$\chi_{G_\varepsilon}(x) = \begin{cases} 1, & x \in G_\varepsilon, \\ 0, & x \in \Omega^+ \setminus G_\varepsilon. \end{cases}$$

Then it follows (see [7]) that

$$\chi_{G_\varepsilon} \rightharpoonup |\omega| \quad \text{weakly in} \quad L^2(\Omega^+) \quad \text{as} \quad \varepsilon \to 0. \tag{23}$$

Theorem 1. *The sequence of solutions u_ε to problem (3) satisfies the relations:*

$$\left.\begin{array}{rcll} u_\varepsilon\,|_{\Omega_0} & \rightharpoonup & u_0^- & \text{weakly in} \quad H^1(\Omega_0), \\ \widetilde{u_\varepsilon} & \rightharpoonup & |\omega(x_3)|\, u_0^+ & \text{weakly in} \quad L^2(\Omega^+), \\ \widetilde{\partial_{x_3} u_\varepsilon} & \rightharpoonup & |\omega(x_3)|\, \partial_{x_3} u_0^+ & \text{weakly in} \quad L^2(\Omega^+), \\ \widetilde{\partial_{x_i} u_\varepsilon} & \rightharpoonup & 0 & \text{weakly in} \quad L^2(\Omega^+) \quad (i = 1,\, 2) \end{array}\right\} \quad \text{as} \quad \varepsilon \to 0, \tag{24}$$

and the function $u_0(x) = \begin{cases} u_0^-, & x \in \Omega_0, \\ u_0^+, & x \in \Omega^+, \end{cases}$ *is the unique solution of the following problem*

$$\begin{cases} -\Delta_x u_0^-(x) = f(x), & x \in \Omega_0, \\ -\partial_{x_3}\big(|\omega(x_3)|\, \partial_{x_3} u_0^+(x)\big) \le |\omega(x_3)|\, f(x) + l_\omega(x_3)\, d(x), & x \in \Omega^+, \\ u_0^+(x) \le g(x), & x \in \Omega^+, \\ \big(u_0^+(x) - g(x)\big)\big(\partial_{x_3}\big(|\omega(x_3)|\, \partial_{x_3} u_0^+(x)\big) + |\omega(x_3)|\, f(x) + l_\omega(x_3)\, d(x)\big) = 0, & x \in \Omega^+, \\ \partial_\nu u_0^-(x) = 0, & x \in \partial\Omega_0 \setminus \Xi_0, \\ u_0^+(x', h) = 0, & (x', h) \in \Xi_h, \\ u_0^-(x', 0) = u_0^+(x', 0), & (x', 0) \in \Xi_0, \\ \partial_{x_3} u_0^-(x', 0) = |\omega(0)|\, \partial_{x_3} u_0^+(x', 0), & (x', 0) \in \Xi_0, \end{cases} \tag{25}$$

which is called the homogenized problem for (3).

Furthermore, the following energy convergence holds

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = E_0(u_0),$$

where

$$E_\varepsilon(u_\varepsilon) = \int_{\Omega_0} |\nabla u_\varepsilon|^2\, dx, \quad E_0(u_0) = \int_{\Omega_0} |\nabla u_0^-|^2\, dx + \int_{\Omega^+} |\omega(x_3)|\, |\partial_{x_3} u_0^+|^2\, dx.$$

Before proving Theorem 1 we investigate the homogenized problem (25).

5 Solvability of the homogenized problem

5.1 Definitions of the weak solution of the generalized problem

We see that the homogenized problem (25) is a non-standard boundary-value problem that consists of the Poisson equation in the junction body Ω_0, the variational inequalities in Ω^+ and the transmission conditions in the joint zone Ξ_0. Therefore, at first we give the definition of the weak solution to this problem and then, with the help of the theory of variational inequalities, we prove existence and uniqueness.

Let us introduce the partially anisotropic Sobolev space

$$\mathcal{H}(\Omega_1; \Xi_h) = \{u \in L^2(\Omega_1) \big| \quad \partial_{x_3} u \in L^2(\Omega_1), \quad u|_{\Omega_0} \in H^1(\Omega_0), \quad u|_{\Xi_h} = 0\}.$$

It follows from the properties of anisotropic Sobolev spaces (see [30]) that the traces of the restrictions $u^+ := u|_{\Omega^+}$ and $u^- := u|_{\Omega_0}$ on Ξ_0 are equal. In addition, since traces of functions from $\mathcal{H}(\Omega_1; \Xi_h)$ vanish on Ξ_h, there exists a constant C_0 such that

$$\int\limits_{\Omega_1} u^2\, dx \le C_0 \Big(\int\limits_{\Omega_0} |\nabla u^-|^2\, dx + \int\limits_{\Omega^+} |\partial_{x_3} u^+|^2\, dx \Big) \qquad \text{for all} \quad u \in \mathcal{H}(\Omega_1; \Xi_h).$$

In $\mathcal{H}(\Omega_1; \Xi_h)$ we introduce a norm $\|\cdot\|_{\mathcal{H}}$, which is generated by the scalar product

$$(u, v)_{\mathcal{H}} = \int\limits_{\Omega_0} \nabla u^- \cdot \nabla v^-\, dx + \int\limits_{\Omega^+} |\omega(x_3)|\, \partial_{x_3} u^+\, \partial_{x_3} v^+\, dx, \quad u, v \in \mathcal{H}(\Omega_1, \Xi_h). \tag{26}$$

We now define the subset

$$K_0 = \{\varphi \in \mathcal{H}(\Omega_1; \Xi_h) : \quad \varphi \le g \quad \text{a. e. in} \quad \Omega^+\}.$$

Obviously, K_0 is a closed and convex in $\mathcal{H}(\Omega_1; \Xi_h)$.

Let there exist a classical solution to the homogenized problem (25). Multiplying the first equation in (25) by the function u_0^-, integrating over Ω_0 and using the Gauss formula, we obtain

$$-\int\limits_{\partial\Omega_0} \partial_\nu u_0^-\, u_0^-\, d\sigma_x + \int\limits_{\Omega_0} \nabla u_0^- \cdot \nabla u_0^-\, dx = \int\limits_{\Omega_0} f u_0^-\, dx. \tag{27}$$

The integral over $\partial\Omega_0$ can also be written as

$$\int\limits_{\partial\Omega_0} \partial_\nu u_0^-\, u_0^-\, d\sigma_x = \int\limits_{\partial\Omega_0 \setminus \Xi_0} \partial_\nu u_0^-\, u_0^-\, d\sigma_x - \int\limits_{\Xi_0} \left(\partial_{x_3} u_0^-\, u_0^-\right)\big|_{x_3=0}\, dx'.$$

By taking into account the boundary condition for u_ε: $\partial_\nu u_0^- = 0$ in $\partial\Omega_0 \setminus \Xi_0$, we obtain

$$\int\limits_{\Omega_0} \nabla u_0^- \cdot \nabla u_0^-\, dx = \int\limits_{\Omega_0} f\, u_0^-\, dx - \int_{\Xi_0} \left(\partial_{x_3} u_0^-\, u_0^-\right)\big|_{x_3=0}\, dx'. \tag{28}$$

Now integrating the fourth equation of the problem (25) in Ω^+ by parts, we get

$$-\int\limits_{\Xi_0} \left(|\omega(x_3)|\, \partial_{x_3} u_0^+\, (u_0^+ - g)\right)\big|_{x_3=0}\, dx' + \int\limits_{\Xi_h} \left(|\omega(x_3)|\, \partial_{x_3} u_0^+\, (u_0^+ - g)\right)\big|_{x_3=h}\, dx'$$

$$+ \int\limits_{\Omega^+} |\omega(x_3)|\, \partial_{x_3} u_0^+\, \partial_{x_3} (u_0^+ - g)\, dx = \int\limits_{\Omega^+} |\omega(x_3)|\, f\, (u_0^+ - g)\, dx + \int\limits_{\Omega^+} l_\omega(x_3)\, d(x)\, (u_0^+ - g)\, dx. \tag{29}$$

By taking into account, that $u_0^+ = 0$ on Ξ_h, and $g = 0$ on $\Xi_0 \cup \Xi_h$ it follows that

$$-\int\limits_{\Xi_0} \left(|\omega(x_3)|\, \partial_{x_3} u_0^+\, u_0^+\right)\big|_{x_3=0}\, dx' + \int\limits_{\Omega^+} |\omega(x_3)|\, \partial_{x_3} u_0^+\, \partial_{x_3} (u_0^+ - g)\, dx$$

$$= \int\limits_{\Omega^+} |\omega(x_3)|\, f\,(u_0^+ - g)\,dx + \int\limits_{\Omega^+} l_\omega(x_3)\,d(x)\,(u_0^+ - g)\,dx. \qquad (30)$$

Let us add (28) and (30).Taking into account the transmission conditions for u_0^+ and u_0^-, we obtain

$$\int\limits_{\Omega_0} \nabla u_0^- \cdot \nabla u_0^-\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,\partial_{x_3} u_0^+\,\partial_{x_3}(u_0^+ - g)\,dx$$

$$= \int\limits_{\Omega_0} f\,u_0^-\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,f\,(u_0^+ - g)\,dx + \int\limits_{\Omega^+} l_\omega(x_3)\,d\,(u_0^+ - g)\,dx. \qquad (31)$$

Now take any function φ from K_0 and multiply the first equation of the problem (25) by φ and integrate over Ω_0. The second and forth equation multiply by the function $(\varphi - g)$ and integrate over Ω^+. Similarly as before we get

$$\int\limits_{\Omega_0} \nabla u_0^- \cdot \nabla\varphi\,dx = \int\limits_{\Omega_0} f\,\varphi\,dx - \int\limits_{\Xi_0} \left(\partial_{x_3} u_0^-\,\varphi\right)\Big|_{x_3=0}\,dx', \qquad (32)$$

$$\int\limits_{\Omega^+} |\omega(x_3)|\,\partial_{x_3} u_0^+\,\partial_{x_3}(\varphi - g)\,dx \geq \int\limits_{\Xi_0} \left(|\omega(x_3)|\,\partial_{x_3} u_0^+\,\varphi\right)\Big|_{x_3=0}\,dx'$$

$$+ \int\limits_{\Omega^+} |\omega(x_3)|\,f(x)\,(\varphi - g)\,dx + \int\limits_{\Omega^+} l_\omega(x_3)\,d(x)\,(\varphi - g)\,dx. \qquad (33)$$

Let us add (32) and (33). Then we get

$$\int\limits_{\Omega_0} \nabla u_0^- \cdot \nabla\varphi\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,\partial_{x_3} u_0^+\,\partial_{x_3}(\varphi - g)\,dx$$

$$\geq \int\limits_{\Omega_0} f\,\varphi\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,f\,(\varphi - g)\,dx + \int\limits_{\Omega^+} l_\omega(x_3)\,d\,(\varphi - g)\,dx. \qquad (34)$$

Let

$$u_0(x) = \begin{cases} u_0^-, & x \in \Omega_0, \\ u_0^+, & x \in \Omega^+. \end{cases}$$

Definition 3. *A function $u_0 \in K_0$ is called a weak solution of the problem (25) if it satisfies the equality*

$$\int\limits_{\Omega_0} \nabla u_0^- \cdot \nabla u_0^-\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,\partial_{x_3} u_0^+\,\partial_{x_3}(u_0^+ - g)\,dx$$

$$= \int\limits_{\Omega_0} f\,u_0^-\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,f\,(u_0^+ - g)\,dx + \int\limits_{\Omega^+} l_\omega(x_3)\,d\,(u_0^+ - g)\,dx \qquad (35)$$

and the inequality

$$\int\limits_{\Omega_0} \nabla u_0^- \cdot \nabla\varphi\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,\partial_{x_3} u_0^+\,\partial_{x_3}(\varphi - g)\,dx$$

$$\geq \int_{\Omega_0} f\,\varphi\,dx + \int_{\Omega^+} |\omega(x_3)|\,f\,(\varphi - g)\,dx + \int_{\Omega^+} l_\omega(x_3)\,d\,(\varphi - g)\,dx \tag{36}$$

for every function $\varphi \in K_0$.

Similarly as we proved the equivalence of Definitions 1 and 2, we can also show the equivalence of Definition 3 to the following definition.

Definition 4. *A function* $u_0 \in K_0$ *is called a weak solution of the problem (25) if it satisfies the integral inequality*

$$\int_{\Omega_0} \nabla u_0^- \cdot \nabla(\varphi - u_0^-)\,dx + \int_{\Omega^+} |\omega(x_3)|\,\partial_{x_3} u_0^+\,\partial_{x_3}(\varphi - u_0^+)\,dx$$

$$\geq \int_{\Omega_0} f\,(\varphi - u_0^-)\,dx + \int_{\Omega^+} |\omega(x_3)|\,f\,(\varphi - u_0^+)\,dx + \int_{\Omega^+} l_\omega(x_3)\,d\,(\varphi - u_0^+)\,dx \tag{37}$$

for every function $\varphi \in K_0$.

We now give the third definition of the weak solution to problem (25).

Definition 5. *A function* $u_0 \in K_0$ *is called a weak solution of the problem (25) if it satisfies the integral inequality*

$$\int_{\Omega_0} \nabla \varphi \cdot \nabla(\varphi - u_0^-)\,dx + \int_{\Omega^+} |\omega(x_3)|\,\partial_{x_3} \varphi\,\partial_{x_3}(\varphi - u_0^+)\,dx$$

$$\geq \int_{\Omega_0} f\,(\varphi - u_0^-)\,dx + \int_{\Omega^+} |\omega(x_3)|\,f(x)\,(\varphi - u_0^+)\,dx + \int_{\Omega^+} l_\omega(x_3)\,d(x)\,(\varphi - u_0^+)\,dx \tag{38}$$

for every function $\varphi \in K_0$.

Now we show that Definition 4 and Definition 5 are equivalent. Adding the inequality

$$\int_{\Omega_0} \nabla(\varphi - u_0^-) \cdot \nabla(\varphi - u_0^-)\,dx + \int_{\Omega^+} |\omega(x_3)|\,\partial_{x_3}(\varphi - u_0^+)\,\partial_{x_3}(\varphi - u_0^+)\,dx \geq 0 \quad (\varphi \in K_0)$$

to inequality (37), we get (38).

Next we take any function $\psi \in K_0$. Setting $\varphi = u_0 + t(\psi - u_0) \in K_0$ (for any $t \in [0,1]$) in inequality (38), we obtain

$$\int_{\Omega_0} \nabla(u_0^- + t(\psi - u_0^-)) \cdot \nabla(t(\psi - u_0^-))\,dx + \int_{\Omega^+} |\omega(x_3)|\,\partial_{x_3}(u_0 + t(\psi - u_0^+))\,\partial_{x_3}(t(\psi - u_0^+))\,dx$$

$$\geq \int_{\Omega_0} f\,(t(\psi - u_0^-))\,dx + \int_{\Omega^+} |\omega(x_3)|\,f\,(t(\psi - u_0^+))\,dx + \int_{\Omega^+} l_\omega(x_3)\,d\,(t(\psi - u_0^+))\,dx. \tag{39}$$

By taking into account $t \geq 0$, it follows that this inequality is equivalent to

$$\int\limits_{\Omega_0} \nabla(u_0^- + t(\psi - u_0^-)) \cdot \nabla(\psi - u_0^-)\, dx + \int\limits_{\Omega^+} |\omega(x_3)|\, \partial_{x_3}(u_0^+ + t(\psi - u_0^+))\, \partial_{x_3}(\psi - u_0^+)\, dx$$

$$\geq \int\limits_{\Omega_0} f\,(\psi - u_0^-)\, dx + \int\limits_{\Omega^+} |\omega(x_3)|\, f\,(\psi - u_0^+)\, dx + \int\limits_{\Omega^+} l_\omega(x_3)\, d\,(\psi - u_0^+)\, dx. \qquad (40)$$

Passing to the limit in (40), as $t \to 0$, we arrive at (37). Thus, all the Definitions 3, 4 and 5 are equivalent.

5.2 Existence and uniqueness of the generalized problem

We can re-write inequality (37) in the following form

$$(u, \varphi - u)_{\mathcal{H}} \geq \langle F, \varphi - u \rangle \qquad \text{for all} \quad \varphi \in K_0, \qquad (41)$$

where F is a linear continuous functional on $\mathcal{H}(\Omega_1; \Xi_h)$ defined by

$$\langle F, w \rangle = \int\limits_{\Omega_0} f\, w^-\, dx + \int\limits_{\Omega^+} |\omega(x_3)|\, f\, w^+\, dx + \int\limits_{\Omega^+} l_\omega(x_3)\, d\, w^+\, dx, \qquad \text{for} \quad w \in \mathcal{H}(\Omega_1, \Xi_h).$$

Using the theory of the variational inequalities in Hilbert spaces (see [20, Sec. 2]), we can state that there exists a unique solution of the variational inequality (41) and, consequently, of the homogenized problem (25).

6 The proof of Theorem 1

1. From (21) it follows that values

$$\|u_\varepsilon\|_{H^1(\Omega_0)}, \quad \|\widetilde{u_\varepsilon}\|_{L^2(\Omega^+)}, \quad \|\widetilde{\partial_{x_i} u_\varepsilon}\|_{L^2(\Omega^+)} \quad (i = 1, 2, 3)$$

are uniformly bounded with respect to ε. Hence, there exists a subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$, again denoted by ε, such that

$$\left.\begin{array}{rcll} u_\varepsilon|_{\Omega_0} & \rightharpoonup & u_0^- & \text{weakly in} \quad H^1(\Omega_0), \\ \widetilde{u_\varepsilon} & \rightharpoonup & |\omega(x_3)|\, (|\omega(x_3)|^{-1} u) =: |\omega|\, u_0^+ & \text{weakly in} \quad L^2(\Omega^+), \\ \widetilde{\partial_{x_i} u_\varepsilon} & \rightharpoonup & \gamma_i & \text{weakly in} \quad L^2(\Omega^+), \quad i = 1, 2, 3 \end{array}\right\} \quad \text{as } \varepsilon \to 0, \qquad (42)$$

where u_0^-, u_0^+, γ_1, γ_2, γ_3 are some functions which will be determined in what follows.

At first we determine γ_3. Take any function $\psi \in C_0^\infty(\Omega^+)$ and with the help of (11) perform the following calculations:

$$\int\limits_{\Omega^+} \widetilde{\partial_{x_3} u_\varepsilon}\, \psi\, dx = \int\limits_{G_\varepsilon} \partial_{x_3} u_\varepsilon\, \psi\, dx = -\int\limits_{G_\varepsilon} u_\varepsilon\, \partial_{x_3} \psi\, dx - \varepsilon \int\limits_{S_\varepsilon} \frac{\varrho'(x_3)\, u_\varepsilon\, \psi}{\sqrt{1 + \varepsilon^2 |\varrho'(x_3)|^2}}\, d\sigma_x$$

$$= -\int\limits_{\Omega^+} \widetilde{u_\varepsilon}\, \partial_{x_3}\psi\, dx - \int\limits_{\Omega^+} \varrho'(x_3)\, g(x_3)\, \widetilde{u_\varepsilon}\, \psi\, dx + \varepsilon \int\limits_{G_\varepsilon} \varrho'(x_3)\, \nabla_{\xi'} Y(\xi', x_3)\, |_{\xi'=\frac{x'}{\varepsilon}} \cdot \nabla_{x'}(u_\varepsilon\, \psi)\, dx. \quad (43)$$

Taking into account (13) and (21), and passing to the limit in this identity, as $\varepsilon \to 0$, we obtain

$$\int\limits_{\Omega^+} \gamma_3\, \psi\, dx = -\int\limits_{\Omega^+} (|\omega(x_3)|\, u_0^+\, \partial_{x_3}\psi\, dx + |\omega(x_3)|'\, u_0^+\, \psi)\, dx \qquad \text{for all} \quad \psi \in C_0^\infty(\Omega^+), \qquad (44)$$

whence it follows that there exists a weak derivative $\partial_{x_3} u_0^+$ and $\gamma_3 = |\omega(x_3)|\, \partial_{x_3} u_0^+$ a. e. in $x \in \Omega^+$.

Now let us find $\gamma_i, i = 1, 2$. Consider the functions

$$Y_i(\xi_i) = -\xi_i + [\xi_i], \quad i = 1, 2 \qquad (45)$$

where $[t]$ is the integer part of t. With the help of these functions we choose the following test functions:

$$\Phi_i(x) = \begin{cases} 0, & x \in \Omega_0, \\ \varepsilon Y_i\left(\frac{x_i}{\varepsilon}\right)\psi(x) + g, & x \in G_\varepsilon, \quad i = 1, 2, \end{cases} \qquad \forall \psi \in C_0^\infty(\Omega^+), \quad \psi \geq 0$$

Since $Y_i \leq 0$ and $\psi \geq 0$, $\Phi_i \in K_\varepsilon,\ i = 1, 2$. It is easy to verify that

$$\nabla(\Phi_1 - g) = \left(-\psi + \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right)\partial_{x_1}\psi,\ \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right)\partial_{x_2}\psi,\ \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right)\partial_{x_3}\psi\right), \qquad x \in G_\varepsilon,$$

$$\nabla(\Phi_2 - g) = \left(\varepsilon Y_2\left(\frac{x_2}{\varepsilon}\right)\partial_{x_1}\psi,\ -\psi + \varepsilon Y_2\left(\frac{x_2}{\varepsilon}\right)\partial_{x_2}\psi,\ \varepsilon Y_2\left(\frac{x_2}{\varepsilon}\right)\partial_{x_3}\psi\right), \qquad x \in G_\varepsilon.$$

Substituting the functions $\Phi_i - g$, i=1,2 into the inequality (7) for the solution u_ε, we get

$$\int\limits_{G_\varepsilon} \left(-\frac{\partial u_\varepsilon}{\partial x_i}\psi + \varepsilon Y_i\left(\frac{x_i}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial \psi}{\partial x_1} + \varepsilon Y_i\left(\frac{x_i}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_2}\frac{\partial \psi}{\partial x_2} + \varepsilon Y_i\left(\frac{x_i}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial \psi}{\partial x_3}\right) dx$$
$$\geq \int\limits_{G_\varepsilon} \varepsilon\, f\, Y_i\left(\frac{x_i}{\varepsilon}\right)\psi\, dx + \varepsilon^2 \int\limits_{S_\varepsilon} d\, Y_i\left(\frac{x_i}{\varepsilon}\right)\psi\, d\sigma_x, \quad i = 1, 2.$$

With the help of (14) and (21) we deduce from the previous inequality the estimate

$$\left|\int\limits_{\Omega^+} \frac{\partial u_\varepsilon}{\partial x_i}\psi\, dx\right| \leq \varepsilon \left(\int\limits_{G_\varepsilon} \left|Y_i\left(\frac{x_i}{\varepsilon}\right)(\nabla u_\varepsilon \cdot \nabla\psi - f\,\psi)\right| dx - \varepsilon \int\limits_{S_\varepsilon} |Y_i\left(\frac{x_i}{\varepsilon}\right)\, d\,\psi|\, d\sigma_x\right)$$
$$\leq \varepsilon c_1 \left(\|\nabla u_\varepsilon\|_{L^2(G_\varepsilon)}\|\nabla\psi\|_{L^2(G_\varepsilon)} + \|f\|_{L^2(G_\varepsilon)}\|\psi\|_{L^2(G_\varepsilon)}\right.$$
$$\left. + \varepsilon\, \|d\|_{L^2(S_\varepsilon)}\|\psi\|_{L^2(S_\varepsilon)}\right) \leq \varepsilon c_1\big(\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}\|\psi\|_{H^1(\Omega^+)} + \|f\|_{L^2(\Omega_1)}\|\psi\|_{L^2(\Omega^+)}$$
$$+ \|d\|_{H^1(\Omega^+)}\|\psi\|_{H^1(\Omega^+)}\big) \leq \varepsilon c_2,$$

from which, passing to the limit as $\varepsilon \to 0$, we get $\int_{\Omega^+} \gamma_i\, \psi\, dx = 0$ for all functions $\psi \in C_0^\infty(\Omega^+),\ \psi \geq 0$. This means that $\gamma_i = 0$ a. e. in $\Omega^+,\ i = 1, 2$.

2. Let us show that the traces $u_0^+|_{\Xi_0}$ and $u_0^-|_{\Xi_0}$ are equal. By virtue of the continuity of the trace operator, compact imbedding $H^{1/2}(\Xi_0) \subset L^2(\Xi_0)$ and the first relation in (42), we have

$$u_\varepsilon(x',0) \stackrel{s}{\longrightarrow} u_0^-(x',0) \quad \text{in} \quad L^2(\Xi_0) \quad \text{as} \quad \varepsilon \to 0. \tag{46}$$

Now consider the equality

$$\widetilde{u_\varepsilon}(x',0) = \chi_{\omega_0}\left(\frac{x'}{\varepsilon}\right) u_\varepsilon(x',0) \quad \text{for a. e. } (x',0) \in \Xi_0, \tag{47}$$

where $\chi_{\omega_0}(\xi')$, $\xi' \in \mathbb{R}^2$, is the 1-periodic function defined on the square Ξ_0 as

$$\chi_{\omega_0}(\xi') = \begin{cases} 1, & \xi' \in \overline{\omega(0)}, \\ 0, & [0,1]\times[0,1] \setminus \overline{\omega(0)}. \end{cases}$$

Obviously, $\chi_{\omega_0}\left(\frac{x'}{\varepsilon}\right) \rightharpoonup |\omega(0)|$ weakly in $L^2(\Xi_0)$ as $\varepsilon \to 0$. Using this fact and (46), we obtain that the right-hand side in (47) converges to $|\omega(0)|\, u_0^-$ weakly in $L^2(\Xi_0)$ as $\varepsilon \to 0$.

On the other hand, with the help of (11) we have

$$\begin{aligned} \int\limits_{\Xi_0} \widetilde{u_\varepsilon}(x',0)\, \psi(x')\, dx' = \frac{1}{h} \Bigg(& \int\limits_{\Omega^+} \widetilde{u_\varepsilon(x)}\, \psi(x')\, dx + \int\limits_{\Omega^+} (x_3 - h)\, \widetilde{\partial_{x_3} u_\varepsilon}(x)\, \psi(x')\, dx \\ & + \int\limits_{\Omega^+} g(x_3)\, \varrho'(x_3)\, (x_3 - h)\, \widetilde{u_\varepsilon}\, \psi(x')\, dx \\ & + \varepsilon \int\limits_{G_\varepsilon} \varrho'(x_3)\, (x_3 - h)\, \nabla_{\xi'} Y(\xi', x_3)\, |_{\xi' = \frac{x'}{\varepsilon}} \cdot \nabla_{x'}(u_\varepsilon \psi)\, dx \Bigg) \end{aligned} \tag{48}$$

for any function $\psi \in C_0^\infty(\Xi_0)$. Taking into account the convergence results obtained above and passing to the limit in (48) as $\varepsilon \to 0$, we obtain the following identity:

$$\begin{aligned} \int\limits_{\Xi_0} |\omega(0)|\, u_0^-(x',0)\psi(x')\, dx = \frac{1}{h} \Bigg(& \int\limits_{\Omega^+} |\omega(x_3)|\, u_0^+(x)\, \psi(x')\, dx \\ & + \int\limits_{\Omega^+} (x_3 - h)|\omega(x_3)|\, \partial_{x_3} u_0^+(x)\, \psi\, dx \\ & + \int\limits_{\Omega^+} (x_3 - h)\, g(x_3)\, \varrho'(x_3)\, |\omega(x_3)|\, u_0^+(x)\, \psi(x')\, dx \Bigg) \\ = \frac{1}{h} & \int\limits_{\Omega^+} \big(|\omega(x_3)|\, u_0^+\, \psi(x') + (x_3 - h)\, \psi(x')\, \partial_{x_3}\big(|\omega(x_3)|\, u_0^+(x)\big)\big)\, dx \\ = & \int\limits_{\Xi_0} |\omega(0)|\, u_0^+(x',0)\, \psi(x')\, dx \qquad \text{for all} \quad \psi \in C_0^\infty(\Xi_0), \end{aligned} \tag{49}$$

which implies that

$$u_0^+(x',0)=u_0^-(x',0) \quad \text{for a. e.} \quad x'\in\Xi_0.$$

3. Let us add to inequality (8) the inequality

$$\int\limits_{\Omega_0}\nabla(\varphi-u_\varepsilon)\cdot\nabla(\varphi-u_\varepsilon)\,dx+\int\limits_{G_\varepsilon}\partial_{x_3}(\varphi-u_\varepsilon)\,\partial_{x_3}(\varphi-u_\varepsilon)\,dx+\int\limits_{G_\varepsilon}\partial_{x_1}u_\varepsilon\,\partial_{x_1}u_\varepsilon\,dx+\int\limits_{G_\varepsilon}\partial_{x_2}u_\varepsilon\,\partial_{x_2}u_\varepsilon\,dx\ge 0$$

where φ is an arbitrary function from $C^1(\overline{\Omega_1})$ such that $\varphi|_{\Xi_h}=0$ and $\varphi\le g$ in Ω^+ (obviously $\varphi|_{\Omega_\varepsilon}\in K_\varepsilon$). We get

$$\int\limits_{\Omega_0}\nabla\varphi\cdot\nabla(\varphi-u_\varepsilon)\,dx+\int\limits_{G_\varepsilon}\partial_{x_1}u_\varepsilon\,\partial_{x_1}\varphi\,dx+\int\limits_{G_\varepsilon}\partial_{x_2}u_\varepsilon\,\partial_{x_2}\varphi\,dx+\int\limits_{G_\varepsilon}\partial_{x_3}\varphi\,\partial_{x_3}(\varphi-u_\varepsilon)\,dx$$

$$\ge\int\limits_{\Omega_\varepsilon}f(\varphi-u_\varepsilon)\,dx+\varepsilon\int\limits_{S_\varepsilon}d(x)(\varphi-u_\varepsilon)\,d\sigma_x \tag{50}$$

which with the help of (11) we can rewrite as

$$\int\limits_{\Omega_0}\nabla\varphi\cdot\nabla(\varphi-u_\varepsilon)\,dx+\int\limits_{\Omega^+}\widetilde{\partial_{x_1}u_\varepsilon}\,\partial_{x_1}\varphi\,dx+\int\limits_{\Omega^+}\widetilde{\partial_{x_2}u_\varepsilon}\,\partial_{x_2}\varphi\,dx+\int\limits_{\Omega^+}\chi_{G_\varepsilon}\left(\frac{x'}{\varepsilon}\right)\partial_{x_3}\varphi\,\partial_{x_3}\varphi\,dx$$

$$-\int\limits_{\Omega^+}\partial_{x_3}\varphi\,\widetilde{\partial_{x_3}u_\varepsilon}\,dx\ \ge\int\limits_{\Omega_0}f\,(\varphi-u_\varepsilon)\,dx+\int\limits_{\Omega^+}\chi_{G_\varepsilon}\left(\frac{x'}{\varepsilon}\right)f\,\varphi\,dx-\int\limits_{\Omega^+}f\widetilde{u_\varepsilon}\,dx$$

$$+\int\limits_{\Omega^+}\zeta(x_3)\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,\chi_{G_\varepsilon}d(x)\,\varphi\,dx+\varepsilon\int\limits_{G_\varepsilon}\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,\nabla_{\xi'}Y(\xi',x_3)|_{\xi'=\frac{x'}{\varepsilon}}\cdot\nabla_{x'}(d\,\varphi)\,dx$$

$$-\int\limits_{\Omega^+}\zeta(x_3)\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,d(x)\widetilde{u_\varepsilon}\,dx+\varepsilon\int\limits_{G_\varepsilon}\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}\,\nabla_{\xi'}Y(\xi',x_3)|_{\xi'=\frac{x'}{\varepsilon}}\cdot\nabla_{x'}(d\,u_\varepsilon)\,dx. \tag{51}$$

Then the fifth and seventh summands in the right-hand side of inequality (51) vanishe due to (13) and (21). Taking into account (42) and results obtained in the 1., we can pass to the limit in (51) as $\varepsilon\to 0$. As a result we obtain the identity

$$\int\limits_{\Omega_0}\nabla\varphi\cdot\nabla(\varphi-u_0^-)\,dx+\int\limits_{\Omega^+}|\omega(x_3)|\,\partial_{x_3}\varphi\,\partial_{x_3}(\varphi-u_0^+)\,dx$$

$$\ge\int\limits_{\Omega_0}f(\varphi-u_0^-)\,dx+\int\limits_{\Omega^+}|\omega(x_3)|\,f\,(\varphi-u_0^+)\,dx+\int\limits_{\Omega^+}l_\omega(x_3)\,d(x)\,(\varphi-u_0^+)\,dx. \tag{52}$$

for any function $\varphi\in K_1=\{\psi\in C^1(\overline{\Omega_1}):\ \ \psi|_{\Xi_h}=0,\ \psi\le g \text{ in } \Omega^+\}$. Since the set K_1 is dense in $K_0\subset\mathcal{H}(\Omega_1;\Xi_h)$, inequality (52) is valid for any function $\varphi\in K_0$. This means that the function u_0 is the unique solution of inequality (38) (Definition 5) and, moreover, it is the weak solution to the homogenized problem (25). Due to the uniqueness of the solution to problem

(25), the above argumentations are true for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. Thus the limits (24) hold.

4. From equalities (4) and (35) it follows that

$$E_\varepsilon(u_\varepsilon) = \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx = \int\limits_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla g\,dx + \int\limits_{\Omega_0} f\,u_\varepsilon\,dx + \int\limits_{G_\varepsilon} f\,(u_\varepsilon - g)\,dx + \varepsilon \int\limits_{S_\varepsilon} d(x)\,(u_\varepsilon - g)\,d\sigma_x, \tag{53}$$

$$E_0(u_0) = \int\limits_{\Omega_0} |\nabla u_0^-|^2\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,|\partial_{x_3} u_0^+|^2\,dx = \int\limits_{\Omega^+} |\omega(x_3)|\,\partial_{x_3} u_0^+\,\partial_{x_3} g\,dx$$

$$+ \int\limits_{\Omega_0} f\,u_0^-\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\,f\,(u_0^+ - g)\,dx + \int\limits_{\Omega^+} l_\omega(x_3)\,d(x)\,(u_0^+ - g)\,dx. \tag{54}$$

Passing to the limit in (53) similarly as for (51) and taking into account (54), we obtain $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = E_0(u_0)$. □

7 The penalty formulation and boundary integral operators

With the scalar product (26) in the Hilbert space $\mathcal{H}(\Omega_1;\Xi_h)$ let us introduce the energy functional

$$\mathrm{J}_0 = \frac{1}{2}(v,v)_{\mathcal{H}} - Lv \tag{55}$$

where the bounded functional L is given by

$$Lv := \int\limits_{\Omega_0} f(x)\,v(x)\,dx + \int\limits_{\Omega^+} |\omega(x_3)|\big(f(x) + l_\omega(x_3)\,d(x)\big)v(x)\,dx. \tag{56}$$

Then the solution $u_0 \in K_0 \subset \mathcal{H}$ is also the unique minimizer of J_0, i. e.,

$$u_0 = \arg\min_{v\in K_0} \mathrm{J}_0(v), \tag{57}$$

see [36, Remark 3.2 in Chapter I]. This solution can be approximated by a sequence of solutions $u_\rho \in \mathcal{H}$ of the penalized minimization problems

$$u_\rho = \arg\min_{v\in\mathcal{H}} \mathrm{J}_\rho(v), \tag{58}$$

where

$$\mathrm{J}_\rho(v) := \frac{1}{2}(v,v)_{\mathcal{H}} - Lv + \mathrm{j}_\rho(v) \tag{59}$$

with the penalizing functional

$$\mathrm{j}_\rho(v) := \frac{1}{\rho}\int\limits_{\Omega^+} \sup\big\{v(x) - g(x), 0\big\}\,dx = \frac{1}{\rho}\,\mathrm{j}(v).$$

Here j is convex, proper and lower semi-continuous on $\mathcal{H}$, $\mathrm{j}(v) \geq 0$ for $v \in \mathcal{H}$ and $\mathrm{j}(v) = 0$ if and only if $v \in K_0$.

Sine $\mathrm{J}_\rho(v) \to +\infty$ for $\|v\|_{\mathcal{H}} \to \infty$ uniformly for all $\rho \in [0,1]$ and, moreover

$$\mathrm{J}_\rho(u_\rho) \leq \mathrm{J}_\rho(u_0) = \mathrm{J}_0(u_0) =: c_0 < \infty,$$

there exist $c_1 < \infty$ such that

$$\|u_\rho\|_{\mathcal{H}} \leq c_1 \quad \text{uniformly for} \quad \rho \in [0,1].$$

Consequently we have

Lemma 2. (*See* [36, Theorem 7.1 in Chapter I].) *The penalty solutions u_ρ converge and satisfy*

$$\lim_{\rho \to 0} \|u_\rho - u_0\|_{\mathcal{H}} = 0 \quad \textit{and} \quad \lim_{\rho \to 0} \frac{1}{\rho} \mathrm{j}(u_\rho) = 0, \tag{60}$$

If we project u_ρ onto K_0 by taking

$$u_\rho^*(x) := \begin{cases} u_\rho(x) \quad \textit{in} \quad \Omega_0, \\ \inf\{u_\rho(x) - g(x), 0\} \quad \textit{in} \quad \Omega^+ \end{cases} \tag{61}$$

then $u_\rho^* \in K_0$ and

$$\mathrm{J}_\rho(u_\rho) \leq \mathrm{J}_0(u_0) \leq \mathrm{J}_0(u_\rho^*). \tag{62}$$

For finding $u_\rho \in \mathcal{H}$ we may solve the nonlinear corresponding Euler equation (see [35, Section 42.6]) in weak formulation,

$$\begin{aligned} 0 = &\int_{\Omega_0} \nabla u_\rho \cdot \nabla v \, dx + \int_{\Omega^+} |\omega(x_3)| \, \partial_{x_3} u_\rho \, \partial_{x_3} v \, dx \\ &- \int_{\Omega_0} f \, v \, dx - \int_{\Omega^+} |\omega(x_3)| \big(f(x) + l_\omega(x_3)\, d(x)\big)\, v(x)\, dx \\ &+ \frac{1}{\rho} \int_{\Omega^+} \max\{0, sign\big(u_\rho(x) - g(x)\big)\}\, v(x)\, dx \quad \text{for all} \quad v \in \mathcal{H}. \end{aligned} \tag{63}$$

Moreover, u_ρ can be obtained by the iteration procedure

$$(u_\rho^{n+1}, v)_{\mathcal{H}} = (u_\rho^n, v)_{\mathcal{H}} - t_n \Big((u_\rho^n, v)_{\mathcal{H}} - Lv + \frac{1}{\rho} \int_{\Omega^+} \max \Big\{0, sign\big(u_\rho^n(x) - g(x)\big)\Big\} v(x)\, dx \Big) \tag{64}$$

with $t_n > 0$ chosen properly(see [36, Section I.3]) and [35, Section 4.2.6]).

For the nonlinear system (63) as well as for the iteration procedure (64) we now can employ a boundary integral equation formulation and a corresponding fast multiple method. To this end we first apply integration by parts, and we introduce the solution of the ordinary differential equation

$$\partial_{x_3}\big(|\omega(x_3)|\, \partial_{x_3} u_\rho(x', x_3)\big) = -|\omega(x_3)|\, f(x', x_3) - l_\omega(x_3)\, d(x', x_3) + \frac{1}{\rho} \max\{0, sign\big(u_\rho(x) - g(x)\big)\}, \tag{65}$$

which is given as

$$u_\rho(x) = -\int_0^{x_3} \left(\frac{1}{|\omega(y)|} \int_0^y \big(|\omega(t)|\, f(x',t) + l_\omega(t)\, d(x',t)\big)\} \, dt \right) dy$$
$$+ \int_0^{x_3} \left(\frac{1}{|\omega(y)|} \int_0^y \frac{1}{\rho} \max\left\{0, sign\big(u_\rho(x',t) - g(x',t)\big)\right\} dt \right) dy + \Lambda_1 \int_0^{x_3} \frac{dy}{|\omega(y)|} + u_\rho(x',0), \tag{66}$$

where

$$\Lambda_1 = \widehat{F}_0(x') - \widehat{\omega} \int_0^h \left(\frac{1}{|\omega(y)|} \int_0^y \max\{0, sign\big(u_\rho(x',t) - g(x',t)\big)\} \, dt \right) dy - \widehat{\omega}\, u_\rho(x',0),$$
$$\widehat{\omega} = \left(\int_0^h \frac{dy}{|\omega(y)|} \right)^{-1} \quad \text{and} \quad \widehat{F}_0(x') = \widehat{\omega} \int_0^h \left(\frac{1}{|\omega(y)|} \int_0^y \big(|\omega(t)|\, f(x',t) + l_\omega(t)\, d(x',t)\big) \, dt \right) dy. \tag{67}$$

Then we find with integration by parts in Ω^+ the variational equation

$$\int_{\Omega^+} |\omega(x_3)|\, \partial_{x_3} u_\rho \, \partial_{x_3} v \, dx_3 \, dx' - \int_{\Omega^+} |\omega(x_3)| \big(f(x) + l_\omega(x_3)\, d(x) \big) v(x) \, dx_3 \, dx'$$
$$+ \frac{1}{\rho} \int_{\Omega^+} \big(\max\{0, sign\big(u_\rho(x) - g(x)\big)\} \big) v(x) \, dx_3 \, dx' = -|\omega(0)| \int_{\Xi_0} \partial_{x_3} u_\rho^+(x',0)\, v(x',0) \, dx'$$
$$= -\int_{\Xi_0} \Lambda_1(x')\, v(x',0) \, dx'. \tag{68}$$

In Ω_0 we represent the solution by

$$u_\rho(x) = \mathcal{V}\tau_\rho - \mathcal{W}u_\rho + \mathcal{N}f(x), \tag{69}$$

where for $x \in \Omega_0$

$$\mathcal{N}f(x) := \frac{1}{4\pi} \int_{\Omega_0} \frac{1}{|x-y|} f(y) \, dy \quad \text{is the Newton potential} \ ,$$
$$\mathcal{V}\tau_\rho(x) := \frac{1}{4\pi} \int_{\partial\Omega_0} \frac{1}{|x-y|} \tau_\rho(y) \, ds(y) \quad \text{is the simple layer boundary potential, and}$$
$$\mathcal{W}u_\rho(x) := \frac{1}{4\pi} \int_{\partial\Omega_0} \left(\frac{\partial}{\partial\nu_y} \frac{1}{|x-y|} \right) u_\rho(y) \, ds(y) \quad \text{is the double layer potential} \ .$$

With the second transmission condition on Ξ_0 and due to (68) we have

$$\tau_\rho = \left. \frac{\partial u_\rho^-}{\partial\nu} \right|_{\Xi_0} = |\omega(0)| \left. \frac{\partial u_\rho^+}{\partial x_3} \right|_{\Xi_0} = \Lambda_1(x'). \tag{70}$$

Since $\frac{\partial u_\rho}{\partial \nu} = 0$ on $\partial\Omega_0 \setminus \Xi_0$, we also have

$$\tau_\rho \in \widetilde{H}^{-\frac{1}{2}}(\Xi_0) = \left\{ \tau \in H^{-\frac{1}{2}}(\partial\Omega_0) \quad \text{with } \tau|_{\partial\Omega_0\setminus\Xi_0} = 0 \right\}.$$

Taking the normal derivative $\frac{\partial}{\partial\nu}$ on $\partial\Omega_0$ on both sides of equation (69), we obtain the nonlinear boundary integral equation for the boundary trace $\gamma_0 u_\rho$ on $\partial\Omega_0$ from

$$\int\limits_{\Omega_0} \nabla u_\rho \cdot \nabla v \, dx = \int\limits_{\partial\Omega_0} \tau_\rho \, \gamma_0 v \, dx + \int\limits_{\Omega_0} f \, v \, dx$$

and

$$\tau_\rho = \frac{1}{2} \tau_\rho + \mathcal{K}'\tau_\rho + \mathcal{D}u_\rho + \frac{\partial}{\partial\nu}\mathcal{N}f$$

with (70) as

$$\mathcal{D}u_\rho = \left(\frac{1}{2} - \mathcal{K}'\right)\left(\Lambda_1 \chi_{\Xi_0}\right) - \frac{\partial}{\partial\nu}\mathcal{N}f \quad \text{on} \quad \partial\Omega_0. \tag{71}$$

Here

$$\chi_{\Xi_0}(x) = \begin{cases} 1 & \text{for } x \in \Xi_0, \\ 0 & \text{for } x \in \partial\Omega_0 \setminus \Xi_0 \end{cases}$$

is the characteristic function of Ξ_0,

$$\mathcal{K}'\tau_\rho(x) = \frac{1}{4\pi} \int\limits_{\partial\Omega_0\setminus\{x\}} \left(\frac{\partial}{\partial\nu_x} \frac{1}{|x-y|} \right) \tau_\rho(y) \, ds(y)$$

is the adjoint to the double layer potential operator, and

$$\mathcal{D}u_\rho(x) = -\frac{\partial}{\partial\nu_x}\mathcal{W}u_\rho$$

is the hypersingular boundary integral operator on $\partial\Omega_0$. Inserting the relation (67) for Λ_1, we find the nonlinear boundary integral equation

$$\begin{aligned} \mathcal{D}u_\rho + \left(\frac{1}{2} - \mathcal{K}'\right) \widehat{\omega}\chi_{\Xi_0} u_\rho &= \left(\frac{1}{2} - \mathcal{K}'\right) \frac{\partial}{\partial\nu}\mathcal{N}f \\ + \left(\frac{1}{2} - \mathcal{K}'\right) \chi_{\Xi_0} \Big\{ \widehat{F}_0(x') &- \frac{1}{\rho} \int\limits_0^h \frac{1}{|\omega(y)|} \int\limits_0^y \max\left\{0, sign\big(u_\rho^+(x',t) - g(x',t)\big)\right\} \, dt \, dy \Big\} \end{aligned} \tag{72}$$

on $\partial\Omega_0$, where $u_\rho^+(x',t)$ is to be computed from (66) explicitly. For solving (72) we now may use the fast multipole method [33, 34] by inverting the operator

$$\widehat{\mathcal{D}} := \mathcal{D} + \left(\frac{1}{2} - \mathcal{K}'\right) \chi_{\Xi_0}\widehat{\omega} : \ \mathcal{H}^{\frac{1}{2}}(\partial\Omega_0) \mapsto \mathcal{H}^{-\frac{1}{2}}(\partial\Omega_0). \tag{73}$$

Let us assume that $\widehat{\mathcal{D}}$ is invertible which is under restrictions on the size of Ξ_0 and $\widehat{\omega}$ the case as shown in Lemma 3 below. However, we conjecture that $\widehat{\mathcal{D}}$ always is invertible. From (73) we then obtain

$$u_\rho = \widehat{\mathcal{D}}^{-1}\left(\frac{1}{2} - \mathcal{K}'\right) \tag{74}$$

$$\left(\frac{\partial}{\partial\nu}\mathcal{N}f + \chi_{\Xi_0}\left\{\widehat{F}_0 - \frac{1}{\rho}\int\limits_0^h \frac{1}{|\omega(y)|}\int\limits_0^y \max\{0, sign\big(u_\rho^+(x',t) - g(x',t)\big)\}\,dt\,dy\right\}\right) \tag{75}$$

in combination with the iteration procedure

$$u_\rho = \lim_{m\to\infty} u_\rho^m.$$

The iteration reads

$$u_\rho^{m+1} = (1-t_m)u_\rho^m - t_m\widehat{\mathcal{D}}^{-1}\left(\left(\frac{1}{2} - \mathcal{K}'\right)\frac{\partial}{\partial\nu}\mathcal{N}f \right.$$
$$\left. + \chi_{\Xi_0}\left\{\widehat{F}_0 - \frac{1}{\rho}\int\limits_0^h \frac{1}{|\omega(y)|}\int\limits_0^y \max\{0, sign\big(u_\rho^m(x',t) - g(x',t)\big)\,dt\,dy\right\}\right).$$

As initial step we take

$$u_\rho^0 := \widehat{\mathcal{D}}^{-1}\left[\left(\frac{1}{2} - \mathcal{K}'\right)\left\{\frac{\partial}{\partial\nu}\mathcal{N}f + \chi_{\Xi_0}\widehat{F}_0\right\}\right].$$

For the invertibility of $\widehat{\mathcal{D}}$ we have the following lemma.

Lemma 3. *Let $\partial\Omega_0$ be smooth. Define $\widetilde{v}_0 := \chi_{\Xi_0}v_0$ and $v_1 := v_0 - \widetilde{v}_0$ on $\partial\Omega_0$ for $v_0 \in H^{1/2}(\partial\Omega_0)$. Then*

$$\left|(\widetilde{v}_0, \mathcal{K}v_1)_{L^2(\partial\Omega_0)}\right| \le c_1|\Xi_0|^{\frac{3}{4}}\|v_0\|^2_{H^{1/2}(\partial\Omega_0)} \tag{76}$$

where c_1 depends on c in (77) but not on v_0 nor $|\Xi_0|$. In addition, if $\widehat{\omega}c_1|\Xi_0|^{\frac{3}{4}} < \gamma_1$ with

$$\gamma_1 := \inf_{v\in H^\perp}(v, \mathcal{D}v)_{L^2(\Xi_0)},$$

where $H^\perp := \{v \in H^{1/2}(\partial\Omega_0) : \ \|v\|_{H^{1/2}(\partial\Omega_0)} = 1, \ \ (v,1)_{L^2(\Xi_0)} = 0\}$, then $\widehat{\mathcal{D}}$ in (73) is invertible.

Proof. Due to the smoothness of $\partial\Omega_0$ the following inequality is satisfied:

$$\left|\frac{\nu(y)\cdot(y-x)}{|y-x|^3}\right| \le c\frac{1}{|y-x|} \tag{77}$$

for all $x \in \Xi_0$ and $y \in \partial\Omega_0 \setminus \Xi_0$.

Also we note that $\gamma_1 > 0$. Indeed, since $\mathcal{D}$ has the one-dimensional kernel spanned by the function 1 and is selfadjoint and Fredholm of index zero, it follows from $(v_0, \mathcal{D}v_0)_{L^2(\partial\Omega_0)} = 0$ that $\mathcal{D}v_0 = 0$, implying $v_0 \equiv const$ and $v_0 \equiv 0$.

Since $\sup \widetilde{v}_0 \subseteq \Xi_0$,

$$\left|(\widetilde{v}_0, \mathcal{K}v_1)_{L^2(\partial\Omega_0)}\right| \le \max_{x\in\Xi_0} |\mathcal{K}v_1(x)| \int_{\Xi_0} |\widetilde{v}_0|\, dx' \le \max_{x\in\Xi_0} |\mathcal{K}v_1(x)|\, |\Xi_0|^{\frac{3}{4}} \|v_0\|_{L^4(\partial\Omega_0)}.$$

With (77) one has

$$\max_{x\in\Xi_0} |\mathcal{K}v_1(x)| \le \max_{x\in\Xi_0} \int_{\partial\Omega_0\setminus\Xi_0} \frac{c}{|x-y|} |v_1|\, ds(y) \le$$
$$\le c \left\{ \max_{x\in\Xi_0} \int_{\partial\Omega_0\setminus\Xi_0} \frac{1}{|x-y|^{4/3}}\, ds(y) \right\}^{3/4} \left\{ \int_{\partial\Omega_0} |\chi_{(\partial\Omega_0\setminus\Xi_0)}(y) v_1(y)|^4\, ds(y) \right\}^{\frac{1}{4}} \le c_1 \|v_0\|_{L^4(\partial\Omega_0)}.$$

Hence,

$$\left|(\widetilde{v}_0, \mathcal{K}v_1)_{L^2(\partial\Omega_0)}\right| \le c_1 |\Xi_0|^{\frac{3}{4}} \|v_0\|^2_{L^4(\partial\Omega_0)}.$$

Due to Theorem 7.58 ([1]) the imbedding $H^{1/2}(\partial\Omega_0) \subset L^4(\partial\Omega_0)$ is continuous and the following estimate $\|v_0\|_{L^4(\partial\Omega_0)} \le c_2 \|v_0\|_{H^{1/2}(\partial\Omega_0)}$ holds. As a result we have (76).

The operator $\widehat{\mathcal{D}}$ is a Fredholm operator of index zero. Let us assume that $v_0 \in H^{1/2}(\partial\Omega_0)$ with $\|v_0\|_{H^{1/2}(\partial\Omega_0)} = 1$ is an eigensolution of $\widehat{\mathcal{D}}$:

$$\left\{ \mathcal{D} + \left(\frac{1}{2} - \mathcal{K}'\right) \chi_{\Xi_0} \widehat{\omega} \right\} v_0 = 0. \tag{78}$$

Since $\mathcal{D}$ has the one-dimensional kernel spanned by the function 1 and is selfadjoint and Fredholm of index zero, the compatibility condition

$$\left(\left(\frac{1}{2} - \mathcal{K}'\right) \chi_{\Xi_0} \widehat{\omega} v_0, 1 \right)_{L^2(\partial\Omega_0)} = 0$$

is satisfied which implies with $\left(\frac{1}{2} + \mathcal{K}\right) 1 = 0$ that

$$0 = \left(\chi_{\Xi_0} \widehat{\omega} v_0, \left(\frac{1}{2} - \mathcal{K}\right) 1 \right)_{L^2(\partial\Omega_0)} = \widehat{\omega}\, (\chi_{\Xi_0} v_0, 1)_{L^2(\partial\Omega_0)} = \widehat{\omega} \int_{\Xi_0} v_0\, dx'. \tag{79}$$

Then we find

$$\gamma_0 := (v_0, \mathcal{D}v_0)_{L^2(\partial\Omega_0)} \ge \gamma_1 > 0, \tag{80}$$

So, from (78) we have

$$0 < \gamma_0 = (v_0, \mathcal{D}v_0)_{L^2(\partial\Omega_0)} = -\left(v_0, \left(\tfrac{1}{2} - \mathcal{K}'\right) \chi_{\Xi_0} \widehat{\omega} v_0\right)_{L^2(\partial\Omega_0)}$$
$$= -\widehat{\omega} \left(\widetilde{v}_0, \left(\tfrac{1}{2} - \mathcal{K}'\right) \widetilde{v}_0\right)_{L^2(\partial\Omega_0)} + \widehat{\omega}\, (\widetilde{v}_1, \mathcal{K}'\widetilde{v}_0)_{L^2(\partial\Omega_0)} = -\widehat{\omega} \int_{\mathbb{R}^3\setminus\Omega_0} |\nabla U_a|^2\, dx + \widehat{\omega}\, (\mathcal{K}\widetilde{v}_1, \widetilde{v}_0)_{L^2(\partial\Omega_0)},$$

where the potential $U_a(x) = \mathcal{V}\widetilde{v}_0(x)$ for $x \in \mathbb{R}^3 \setminus \Omega_0$ decays as $\mathcal{O}(|x|^{-2})$ and ∇U_a as $\mathcal{O}(|x|^{-3})$ for $|x| \to \infty$ because of (79).

Hence,

$$0 < \gamma_1 \le \gamma_0 \le \widehat{\omega} c_1 |\Xi_0|^{3/4}$$

and $\widehat{\omega} c_1 |\Xi_0|^{3/4} < \gamma_1$ is in contradiction to the existence of $v_0 \ne 0$. This completes the proof. $\square$

8 The special case of reduction to $\mathbf{\Omega}_0$

If the given data g, d and f satisfy the additional assumptions

$$g \geq 0 \quad \text{a.e. in} \quad \Omega^+ \qquad \text{and} \qquad |w(x_3)|f(x) + l_w(x_3)d(x) \leq 0 \quad \text{in} \quad \Omega^+ \tag{81}$$

then the homogenized variational problem can be reduced to a variational inequality in Ω_0 and a postprocessing procedure in Ω^+. To this end, we employ the following lemma.

Lemma 4. *If* (81) *holds in* Ω^+ *then for* $x' \in \Xi_0$ *fixed,* u_0^+ *in* Ω^+ *has the following properties: If* $u(x', 0) \leq 0$ *then* $u_0^+(x', x_3) \leq 0$ *for all* $x_3 \in [0, h]$ *and is given by*

$$u_0^+(x', x_3) = -\int\limits_0^{x_3} \frac{1}{|\omega(y)|} \int\limits_0^{y} \big(|\omega(t)|f(x', t) + l_w(t)d(x', t)\big)dtdy + \widetilde{\Lambda}_1 \int\limits_0^{x_3} \frac{dy}{|\omega(y)|} + u_0^-(x', 0)\,, \tag{82}$$

where $\widetilde{\Lambda}_1(x') = \widehat{F}_0(x') - \widehat{\omega}u_0^-(x', 0)$ *(see (67)).*

Proof. The statement of Lemma 4 is a direct consequence of the maximum principle for the one–dimensional elliptic boundary value problem for $u_0^+(x', x_3)$ in Ω^+ for x' fixed and $0 \leq x_3 \leq h$ (see [37, p.89]). □

As a consequence of Lemma 4, it suffices for the solution in Ω^+ to know the values of $u_0^-(x', 0)$ from the solution in Ω_0 then in Ω^+ the solution $u_0^+(x', x_3)$ is given by (82).

Hence, in Ω_0 we are led to the following variational problem. Let

$$K := \{v \in H^1(\Omega_0) \quad \text{with} \quad \chi_{\Xi_0} v \leq 0\} \tag{83}$$

which is a convex and closed cone in $H^1(\Omega_0)$.

Find $v_0 \in K$ *minimizing the functional*

$$\mathcal{I}_1(v) := \frac{1}{2}\int\limits_{\Omega_0} \nabla v \cdot \nabla v dx - \int\limits_{\Omega_0} v dx \quad \text{over} \quad v \in K\,. \tag{84}$$

Since $\mathcal{I}_1$ is quadratic, there exists exactly one solution $v_0 \in K$, and we have $u_0^-(x) := v_0(x)$ in Ω_0. In order to construct v_0 we also here use the penalty method by choosing the penalty functional

$$j_1(v) := \int\limits_{\Xi_0} \sup\{v(x', 0), 0\} dx'\,. \tag{85}$$

$j_1(v)$ is convex, proper and lower semicontinuous on $H^1(\Omega_0)$ and satisfies $j_1(v) = 0$ if and only if $v \in K$ and $j_1(v) \geq 0$ for all $v \in H^1(\Omega_0)$. The penalized variational problem now reads:

Find $v_{\rho 0}$ *minimizing*

$$\mathcal{I}_{1\rho}(v) := \frac{1}{2}\int\limits_{\Omega_0} \nabla v \cdot \nabla v dx - \int\limits_{\Omega_0} f v dx + \frac{1}{\rho} j_1(v) \tag{86}$$

over $H^1(\Omega_0)$*, where* $\rho > 0$ *is the penalty parameter.*

The weak form of the corresponding Euler equation reads:

Find $v_{\rho 0} \in H^1(\Omega_0)$ as the solution of

$$\int\limits_{\Omega_0} \nabla v_{\rho 0} \cdot \nabla v dx + \frac{1}{\rho} \int\limits_{\Xi_0} \max\{0, \text{sign } v_{\rho 0}(x', 0)\} v(x', 0) dx' = \int\limits_{\Omega_0} f v dx \quad \text{for all } \ v \in H^1(\Omega_0)\,. \tag{87}$$

Then

$$\lim_{\rho \to 0} \|v_{\rho 0} - v_0\|_{H^1(\Omega_0)} = 0 \quad \text{and} \quad \lim_{\rho \to 0} \frac{1}{\rho} j_1(v_{\rho 0}) = 0 \quad \text{and} \quad v_0(x) = u_0^-(x) \quad \text{in } \ \Omega_0\,. \tag{88}$$

Moreover, $v_{\rho 0}$ can be found as a limit of the iteration procedure

$$\begin{aligned} (v_{\rho 0}^{m+1}, v)_{H^1(\Omega_0)} = (v_{\rho 0}^m, v)_{H^1(\Omega_0)} - t_m \Big((v_{\rho 0}^m, v)_{H^1(\Omega_0)} - \int\limits_{\Omega_0} f v dx \\ + \frac{1}{\rho} \int\limits_{\Xi_0} \max\{0, \text{sign } v_{\rho 0}^m(x', 0)\} v(x', 0) dx' \Big) \end{aligned} \tag{89}$$

with $t_m > 0$ chosen appropriately [36, Sec.I.3], and with $v_{\rho 0}^0(x) := 1$ for $x \in \Omega_0$.

9 A boundary integral formulation in the special case

Let us represent the solution in Ω_0 in terms of potentials as in (69), i.e.,

$$u_\rho(x) = \mathcal{V}\tau_\rho - \mathcal{W} u_\rho + \mathcal{N} f(x) \qquad \text{for} \quad x \in \Omega_0 \tag{90}$$

where $\tau_\rho = \frac{\partial u_\rho}{\partial \nu} \in \widetilde{H}^{-\frac{1}{2}}(\Xi_0)$ on $\partial\Omega_0$. Then integration by parts in (87) results in the nonlinear variational equation

$$\int\limits_{\Xi_0} \frac{\partial u_\rho}{\partial \nu} w\, dx' + \frac{1}{\rho} \int\limits_{\Xi_0} \max\{0, \text{sign } u_{\rho 0}(x', 0)\} w(x', 0) dx' = 0 \tag{91}$$

for all $w \in H^{\frac{1}{2}}(\partial\Omega_0)$ which needs to be appended by boundary integral equations associated with (90), namely the equation for $\tau_\rho \in \widetilde{H}^{-\frac{1}{2}}(\Xi_0)$:

$$\mathcal{V}\tau_\rho = \left(\frac{1}{2} + \mathcal{K}\right) u_\rho - \gamma_0 \mathcal{N} f \qquad \text{on} \quad \partial\Omega_0 \tag{92}$$

where, for brevity we write u_ρ on $\partial\Omega_0$ for the trace $\gamma_0 u_\rho$. The second equation for $u_\rho \in H^{\frac{1}{2}}(\partial\Omega_0)$ reads

$$\mathcal{D} u_\rho = \left(\frac{1}{2} - \mathcal{K}'\right) \tau_\rho - \gamma_0 \mathcal{N} f \qquad \text{on} \quad \partial\Omega_0, \tag{93}$$

which becomes with (91) the nonlinear boundary integral equation

$$\mathcal{D} u_\rho + \frac{1}{\rho} \left(\frac{1}{2} - \mathcal{K}'\right) \chi_{\Xi_0} \max\{0, \text{sign } u_{\rho 0}(x', 0)\} = -\mathcal{N} f \qquad \text{on} \quad \partial\Omega_0, \tag{94}$$

The successive iteration (89) becomes an iteration involving only the Cauchy data u_ρ on $\partial\Omega_0$ and τ_ρ. Integration by parts of (89) involving the representation of u^m and u^{m+1} as potentials (90) and using the equivalent modification

$$(u,v)_{H^1(\Omega_0)} := \int_{\Omega_0} \nabla u \cdot \nabla v \, dx + \int_{\Xi_0} uv \, dx \tag{95}$$

of the scalar product for $H^1(\Omega_0)$ we find

$$\begin{aligned}(u_\rho^{m+1}, v)_{L^2(\Xi_0)} + \int_{\partial\Omega_0} v \frac{\partial u_\rho^{m+1}}{\partial\nu} ds(x) &= (u_\rho^m, v)_{L^2(\Xi_0)} + \int_{\partial\Omega_0} v \frac{\partial u_\rho^m}{\partial\nu} ds(x) \\ - t_m \left\{ \int_{\partial\Omega_0} v \frac{\partial u_\rho^m}{\partial\nu} ds(x) + (u_\rho^m, v)_{L^2(\Xi_0)} \right. &+ \left. \frac{1}{\rho} \int_{\Xi_0} \max\{0, \text{sign } u_\rho^m(x',0)\} \, dx' \right\} \end{aligned} \tag{96}$$

for all test functions $v \in H^1(\Omega_0)$. Employing the boundary integral operators and the Calderon projector in Ω_0 then leads to the boundary integral equation iteration

$$\widehat{\mathcal{D}} u_\rho^{m+1} = \left(\frac{1}{2} - \mathcal{K}'\right) \left\{ (1-t_m)\tau_\rho^m - \frac{t_m}{\rho} \chi_{\Xi_0} \max\{0, \text{sign } u_\rho^m(x',0)\} \right\} - \frac{\partial}{\partial\nu} \mathcal{N} f, \tag{97}$$

$$\tau_\rho^{m+1} = -\chi_{\Xi_0} u_\rho^{m+1} + (1-t_m)(\tau_\rho^m + \chi_{\Xi_0} u_\rho^m) - -\frac{t_m}{\rho} \chi_{\Xi_0} \max\{0, \text{sign } u_\rho^m(x',0)\} - \frac{\partial}{\partial\nu} \mathcal{N} f, \tag{98}$$

$u_\rho^0 = 1$, $\tau_\rho^0 = 0$, $m = 0, 1, 2, \ldots$ and

$$u_\rho^{m+1}(x) = \mathcal{V}\tau_\rho^{m+1} - \mathcal{W} u_\rho^{m+1} + \mathcal{N} f(x) \qquad \text{for} \quad x \in \Omega_0.$$

For (97)-(98) we assume that the operator

$$\widehat{\mathcal{D}} = \mathcal{D} + \left(\frac{1}{2} - \mathcal{K}'\right) \chi_{\Xi_0} : \mathcal{H}^{\frac{1}{2}}(\partial\Omega_0) \mapsto \mathcal{H}^{-\frac{1}{2}}(\partial\Omega_0)$$

is invertible. Under the conditions as in Lemma 3 with $\widehat{\omega} = 1$ it follows that $\mathcal{D}$ is invertible but we have the conjecture that it always is invertible.

10 Remarks to the computational realization

For the numerical solution of the general problem one may use the successive iteration (64) in combination with a finite element discretization of Ω_0, and spline discretization of the nonlinear ordinary differential equation (65), or its solution formulae (66)-(67). More precisely, on Ω_0 one may use spatial hexagonal or tetrahedral finite elements whose element faces define boundary elements on Ξ_0. On the finite elements we chose one of the standard piecewise polynomial spaces which are globally continuous. With the trace on Ξ_0 define the tensor space of the finite element trace space with piecewise linear globally continuous splines on a partition of $0 \le x_3 \le h$ which are zero at $x_3 = h$ and take the value of the trace element on $x_3 = 0$, i.e., on Ξ_0, e.g., at its center of gravity. Then in (64) use a corresponding finite element Galerkin formulation and solve for every n the linear finite element system, e.g., with the multigrid method [38].

For the boundary integral equation formulations (74) and (97)-(98) one needs the values of the Newton potential $\frac{\partial}{\partial \nu}\mathcal{N}f$ on $\partial\Omega_0$ which can be computed in a preprocessing step by using e.g. a fast multipole method on a partition of Ω_0 [33]. Then we need the triangulation of the boundary $\partial\Omega_0$ by boundary elements and these for the approximation of u_ρ on $\partial\Omega_0$, e.g. by piecewise linear globally continuous boundary elements [34], which for (74) are coupled with the tensor product space with splines on $[0, h]$. Since for each iteration step in (74) or in (97)-(98), the Galerkin equations with $\widehat{\mathcal{D}}$ are to be solved, this should be done with the fast multipole or the ACA method and preconditioned GMRES where the simple layer potential Galerkin matrix can be used as a precondition [34]; and all matrix times vector products must be executed with fast multipole expansions.

Corresponding error analysis as well as numerical experiments are yet to be done.

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