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ASYMPTOTIC ANALYSIS OF A PARABOLIC SEMI-LINEAR PROBLEM WITH NONLINEAR BOUNDARY MULTI-PHASE INTERACTIONS IN A PERFORATED DOMAIN

T.A. Mel'nyk*, Olena A. Sivak†

Abstract

We consider a parabolic semilinear problem with rapidly oscillating coefficients in a domain Ω_ε that is $\varepsilon-$periodically perforated by small holes. The holes are divided into two $\varepsilon-$periodical sets depending on the boundary interaction at their surfaces. Two different nonlinear Robin boundary conditions $\sigma_\varepsilon(u_\varepsilon) + \varepsilon\kappa_m(u_\varepsilon) = \varepsilon g_\varepsilon^{(m)},\ m = 1, 2$, are given on the corresponding boundaries of the small holes. The asymptotic analysis of this problem is made as $\varepsilon \to 0$, namely a convergence theorem is proved without using extension operators, the asymptotic approximation for the solution is constructed and the corresponding error estimate is deduced.

AMS Subject Classification: 35B27, 35B40, 35K60.

Key Words and Phrases: Homogenization, asymptotic approximation, parabolic semi-linear problem, nonlinear boundary conditions, perforated domain.

1 Introduction and statement of the problem

The homogenization theory is at present a well-developed field of mathematics which includes a large variety of both theoretical and applied problems (see for example [1]-[8]). One class of the homogenization theory is boundary-value problems in perforated domains. In recent years a rich collection of new results on asymptotic analysis of boundary-value problems in perforated domains is appeared (see for example [9]-[42]).

The classical method proposed by E. Khruslov [43] and D. Cioranescu and J. Saint Jean Paulin [44] is based on a special bounded extension of solutions in Sobolev spaces. It was established by V. Zhikov [30], [40] that the homogenization results can be obtained without using the extension technique in Sobolev spaces in periodically perforated domains. It should be mentioned the paper [13], where the homogenization results for an elliptic problem with a nonlinear boundary condition in a perforated domain were obtained with the help of a new unfolding method that does not need any extension operators as well. Also we quote first papers [45]-[51], where boundary-value problems involving third-type boundary conditions, nonuniform Neumann and Steklov conditions at the boundary of the holes of perforated domains were studied.

The recent development of reaction diffusion systems in biology, ecology and biochemistry, and the traditional importance of these systems in physics, heat-mass transfer and engineering lead to extensive study in various aspects of nonlinear boundary-value problems in perforated domains. In

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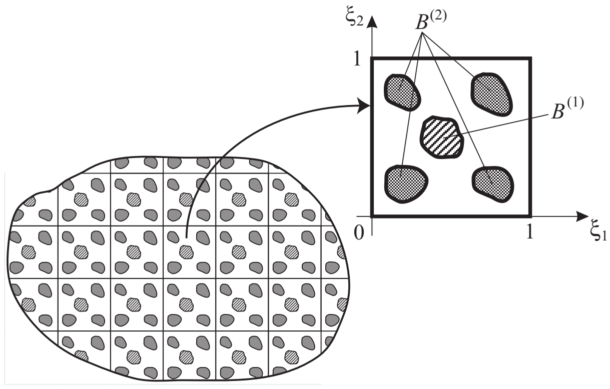

Figure 1:

the presented paper we consider a parabolic semilinear problem with rapidly oscillating coefficients in a domain Ω_ε that is $\varepsilon-$periodically perforated by small holes. The holes are divided into two $\varepsilon-$periodical sets depending on the boundary interaction at their surfaces. Two different nonlinear Robin boundary conditions $\sigma_\varepsilon(u_\varepsilon) + \varepsilon\kappa_m(u_\varepsilon) = \varepsilon g_\varepsilon^{(m)},\ m = 1, 2,$ are given on the corresponding boundaries of the small holes. Using the Zhikov's approach and the approach of the papers [10, 52], a convergence theorem is proved, the asymptotic approximation for the solution is constructed and the corresponding error estimate is deduced.

Statement of the problem. Let B be a finite union of smooth disjoint nontangent domains strictly lying in the unit square

$$\square := \{\xi \in \mathbb{R}^n : \ \ 0 < \xi_i < 1, \quad i = \overline{1,n}\}.$$

In an arbitrary way, we divide B into two sets

$$B^{(1)} := \bigcup_{k=1}^{N_1} B_k^{(1)} \quad \text{and} \quad B^{(2)} := \bigcup_{k=1}^{N_2} B_k^{(2)}.$$

Also we introduce the following notations

$$Q_0 := \square \setminus \overline{B}, \quad \mathcal{B}^{(m)} := \bigcup\nolimits_{z\in\mathbb{Z}^n} \big(z + B^{(m)}\big),$$

$$\mathcal{B}_\varepsilon^{(m)} := \varepsilon\mathcal{B}^{(m)} = \{x \in \mathbb{R}^n : \ \varepsilon^{-1}x \in \mathcal{B}^{(m)}\}, \quad m = 1, 2,$$

where ε is a small parameter. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Define the following perforated domain $\Omega_\varepsilon = \Omega\backslash\overline{\big(\mathcal{B}_\varepsilon^{(1)} \cup \mathcal{B}_\varepsilon^{(2)}\big)}$ (see Fig.1) and require the domain Ω_ε to be a domain with the Lipschitz boundary. Denote $\Gamma_\varepsilon = \partial\Omega\cap\overline{\Omega_\varepsilon}$ and $\Xi_\varepsilon^{(m)} = \Omega\cap\partial\mathcal{B}_\varepsilon^{(m)},\ \ m = 1, 2,\ \ \Xi_\varepsilon = \Xi_\varepsilon^{(1)}\cup\Xi_\varepsilon^{(2)}$.

Let $a_{ij}(\xi)$, $\xi \in \mathbb{R}^n$, $i, j = \overline{1,n}$, be smooth 1−periodic functions such that

$$\begin{aligned} &1)\ \ \forall i, j = 1, \ldots, n, \quad \forall \xi \in \mathbb{R}^n : \quad a_{ij}(\xi) = a_{ji}(\xi), \\ &2)\ \ \exists\, \varkappa_1, \varkappa_2 > 0 \quad \forall\, \xi \in \mathbb{R}^n \ \ \forall\, \eta \in \mathbb{R}^n : \quad \varkappa_1|\eta|^2 \le a_{ij}(\xi)\eta_i\eta_j \le \varkappa_2|\eta|^2. \end{aligned} \tag{1}$$

Remark 1. *Here and in the sequel we adopt the Einstein convention of summation over repeated indexes.*

Let $f_\varepsilon, f_0, g_\varepsilon^{(m)}, g_0^{(m)}$ be given functions such that $f_\varepsilon, f_0 \in L^2(0,T; L^2(\Omega))$, $g_\varepsilon^{(m)}, g_0^{(m)} \in L^2(0,T; H_0^1(\Omega))$ and

$$\begin{aligned} &f_\varepsilon \longrightarrow f_0 \quad \text{in} \quad L^2(0,T; L^2(\Omega)), \\ &g_\varepsilon^{(m)} \stackrel{w}{\longrightarrow} g_0^{(m)} \quad \text{in} \quad L^2(0,T; H_0^1(\Omega)), \quad m = 1, 2 \end{aligned} \tag{2}$$

for some fixed time $T>0$.

The given functions $h:\mathbb{R}\to\mathbb{R},\ \kappa_m:\mathbb{R}\to\mathbb{R},\ m=1,2$, are Lipschitz continuous (it is equivalent that $h,\kappa_m\in W^{1,\infty}_{loc}(\mathbb{R})$) and such that

$$\exists\, c_1>0\ \ \exists\, c_2>0:\quad c_1\le h'\le c_2,\quad c_1\le \kappa_m'\le c_2,\quad \text{a.e. in } \mathbb{R}\quad (m=1,2). \tag{3}$$

Remark 2. *In what follows we will use the following inequalities which are simply consequences of (3) (m=1,2):*

$$c_1t^2+h(0)t\le h(t)t\le c_2t^2+h(0)t,\quad c_1t^2+k_m(0)t\le k_m(t)t\le c_2t^2+k_m(0)t, \tag{4}$$

$$|h(t)|\le |h(0)|+c_5|t|,\quad |k_m(t)|\le |k_m(0)|+c_5|t|\qquad \forall t\in\mathbb{R}. \tag{5}$$

We consider the following initial/boundary-value semilinear problem with nonlinear boundary condition

$$\begin{cases} \partial_t u_\varepsilon-\mathcal{L}_\varepsilon(u_\varepsilon)+h(u_\varepsilon) &= f_\varepsilon \quad \text{in } \Omega_\varepsilon\times(0,T),\\ \sigma_\varepsilon(u_\varepsilon)+\varepsilon\kappa_1(u_\varepsilon) &= \varepsilon g_\varepsilon^{(1)} \quad \text{on } \Xi_\varepsilon^{(1)}\times(0,T),\\ \sigma_\varepsilon(u_\varepsilon)+\varepsilon\kappa_2(u_\varepsilon) &= \varepsilon g_\varepsilon^{(2)} \quad \text{on } \Xi_\varepsilon^{(2)}\times(0,T),\\ u_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon\times(0,T),\\ u_\varepsilon(x,0) &= 0 \quad \text{in } \Omega_\varepsilon, \end{cases} \tag{6}$$

where $\mathcal{L}_\varepsilon(u_\varepsilon)\equiv\partial_{x_i}\left(a_{ij}^\varepsilon(x)\partial_{x_j}u_\varepsilon(x,t)\right),\quad \sigma_\varepsilon(u_\varepsilon)\equiv a_{ij}^\varepsilon(x)\partial_{x_j}u_\varepsilon(x,t)\,\nu_i,\quad a_{ij}^\varepsilon(x)\equiv a_{ij}\left(\frac{x}{\varepsilon}\right),\ \partial_{x_i}u=\frac{\partial u}{\partial x_i},\ \ \left(\nu_1\left(\frac{x}{\varepsilon}\right),\ldots\nu_n\left(\frac{x}{\varepsilon}\right)\right)$ is the outward normal.

Recall that a function u_ε from the Sobolev space $L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon))$, where $H^1(\Omega_\varepsilon,\Gamma_\varepsilon)=\{u\in H^1(\Omega_\varepsilon):\ u|_{\Gamma_\varepsilon}=0\}$, is a weak solution to problem (6) if the following integral identity

$$\int_0^T\Big(-\int_{\Omega_\varepsilon}u_\varepsilon\partial_t\psi\,dx+\int_{\Omega_\varepsilon}a_{ij}^\varepsilon\partial_{x_j}u_\varepsilon\partial_{x_i}\psi\,dx+\int_{\Omega_\varepsilon}h(u_\varepsilon)\psi\,dx$$
$$+\varepsilon\sum_{m=1}^2\int_{\Xi_\varepsilon^{(m)}}k_m(u_\varepsilon)\psi\,ds_x\Big)\,dt=\int_0^T\Big(\int_{\Omega_\varepsilon}f_\varepsilon\psi\,dx+\varepsilon\sum_{m=1}^2\int_{\Xi_\varepsilon^{(m)}}g_\varepsilon^m\psi\,ds_x\Big)\,dt. \tag{7}$$

holds for any function $\psi\in L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon))\bigcap H^1(0,T;L^2(\Omega_\varepsilon)),\ \psi(x,T)=0$.

Our aim is to study the asymptotic behavior of the weak solution to problem (6) as $\varepsilon\to0$, i.e., when the number of the holes infinitely increases and their diameters decreases to zero. It should be noted that the limit process is accompanied by the perturbed coefficients in the nonlinear boundary conditions on the lateral sides of the holes and we study the influence of these factors on the asymptotic behavior as well.

2 Auxiliary uniform estimates

To homogenize boundary value problem (6) we use the method of integral identity which was proposed in [54], [55]. In what follows we will often use the following identity ($m=1,2$) (see [10])

$$\varepsilon\int_{\Xi_\varepsilon^{(m)}}\varphi\,ds_x=\varepsilon\int_{\Omega_\varepsilon}a_{ij}^\varepsilon(x)\,\partial_{\xi_j}\psi_0^{(m)}(\xi)|_{\xi=\frac{x}{\varepsilon}}\,\partial_{x_i}\varphi\,dx+q_m\int_{\Omega_\varepsilon}\varphi\,dx\qquad \forall\varphi\in H^1(\Omega_\varepsilon,\Gamma_\varepsilon) \tag{8}$$

where $\psi_0^{(m)}\in H^1_{\rm per}(Q_0)=\{v\in H^1(Q_0):\ v-1\text{-periodic in }\ \xi_1,\ldots,\xi_n\},\ m=1,2$, is weak solutions to the following problems

$$\begin{cases} L_{\xi\xi}\big(\psi_0^{(1)}\big)=q_1\ \text{ in }\ Q_0,\\ \sigma_\xi\big(\psi_0^{(1)}\big)=1\ \text{ on }\ S^{(1)},\\ \sigma_\xi\big(\psi_0^{(1)}\big)=0\ \text{ on }\ S^{(2)},\\ \langle\psi_0^{(1)}\rangle_{Q_0}=0, \end{cases}\qquad \begin{cases} L_{\xi\xi}\big(\psi_0^{(2)}\big)=q_2\ \text{ in }\ Q_0,\\ \sigma_\xi\big(\psi_0^{(2)}\big)=0\ \text{ on }\ S^{(1)},\\ \sigma_\xi\big(\psi_0^{(2)}\big)=1\ \text{ on }\ S^{(2)},\\ \langle\psi_0^{(2)}\rangle_{Q_0}=0, \end{cases} \tag{9}$$

$S^{(m)} = \partial B^{(m)}$, $q_m = \frac{\text{meas}\, S^{(m)}}{\text{meas}\, Q_0}$, $m = 1, 2$, $L_{\xi\xi}(\psi) \equiv \partial_{\xi_i}\left(a_{ij}(\xi)\partial_{\xi_j}\psi(\xi)\right)$, $\sigma_\xi(\psi) \equiv a_{ij}\partial_{\xi_j}\psi(\xi)\nu_i(\xi)$, $(\nu_1, \ldots, \nu_n)-$ is the outward normal to S, $\langle\psi\rangle_{Q_0} = \int_{Q_0}\psi(\xi)\,d\xi$.

Due to the regularity properties of solutions to elliptic problems we have

$$\sup_{x\in\Omega_\varepsilon}\left|\nabla_\xi\psi_0^{(m)}(\xi)|_{\xi=\frac{x}{\varepsilon}}\right| = \sup_{\xi\in Q_0}\left|\nabla_\xi\psi_0^{(m)}(\xi)\right| \le C_0 \quad (m = 1, 2). \tag{10}$$

We can deduce from (8) the following estimates (m=1, 2) (see [10])

$$\varepsilon\int_{\Xi_\varepsilon^{(m)}}\varphi^2\,ds_x \le C_1\left(\varepsilon^2\int_{\Omega_\varepsilon}|\nabla_x\varphi|^2\,dx + \int_{\Omega_\varepsilon}\varphi^2\,dx\right), \tag{11}$$

$$\int_{\Omega_\varepsilon}\varphi^2\,dx \le C_2\left(\varepsilon^2\int_{\Omega_\varepsilon}|\nabla_x\varphi|^2\,dx + \varepsilon\int_{\Xi_\varepsilon^{(m)}}\varphi^2\,ds_x\right) \qquad \forall\,\varphi\in H^1(\Omega_\varepsilon, \Gamma_\varepsilon), \tag{12}$$

where the constant C_1 and C_2 are independent of ε.

Remark 3. *In what follows all constants $\{C_i\}$ and $\{c_i\}$ in inequalities are independent of the parameter ε.*

It follows from (11) and (2) that

$$\sqrt{\varepsilon}\sum\nolimits_{m=1}^{2}\|g_\varepsilon^{(m)}\|_{L^2(0,T;L^2(\Xi_\varepsilon^{(m)}))} \le C_5. \tag{13}$$

Also with the help of (11) and (12) it is easy to prove that the usual norm $\|\cdot\|_{H^1(\Omega_\varepsilon)}$ is uniformly equivalent with respect to ε to a new norm

$$\|u\|_\varepsilon := \left(\int_{\Omega_\varepsilon}|\nabla u|^2\,dx + \varepsilon\int_{\Xi_\varepsilon}u^2\,ds_x\right)^{1/2}$$

in the space $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$, i.e., there exist constants $C_3 > 0$, $C_4 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $u \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ the following relations hold

$$C_3\|u\|_{H^1(\Omega_\varepsilon)} \le \|u\|_\varepsilon \le C_4\|u\|_{H^1(\Omega_\varepsilon)}. \tag{14}$$

3 Existence and uniqueness of the solution to problem (6)

Using the operator method, developed for instance in [56], we can prove existence and uniqueness of the solution of problem (6).

Theorem 1. *At each fixed value of ε problem (6) has exactly one solution $u_\varepsilon \in L^2(0, T; H^1(\Omega_\varepsilon, \Gamma_\varepsilon))$ for which the following estimate*

$$\max_{0\le t\le T}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$$
$$\le C_1\Big(1 + \|f_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \sqrt{\varepsilon}\sum\nolimits_{m=1}^{2}\|g_\varepsilon^{(m)}\|_{L^2(0,T;L^2(\Xi_\varepsilon^{(m)}))}\Big) \le C_2 \tag{15}$$

holds, where the constants C_1 and C_2 are independent of ε, $f_\varepsilon, g_\varepsilon^{(m)}$ and u_ε.

Proof. Boundary value problem (6) can be formulated as an operator equation. To this end, we define operator $\mathcal{A}(t) : H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \to (H^1(\Omega_\varepsilon, \Gamma_\varepsilon))^*$ and functional $F(t) \in (H^1(\Omega_\varepsilon, \Gamma_\varepsilon))^*$ for a.e. $t \in [0, T]$ as follows

$$\mathcal{A}(t)u(v) = \int_{\Omega_\varepsilon}a_{ij}^\varepsilon\partial_{x_j}u\,\partial_{x_j}v\,dx + \int_{\Omega_\varepsilon}h(u)v\,dx + \varepsilon\sum_{m=1}^{2}\int_{\Xi_\varepsilon^{(m)}}k_m(u)v\,ds_x,$$
$$u, v \in L^2(0, T; H^1(\Omega_\varepsilon, \Gamma_\varepsilon)),$$

$$F(t)v = \int_{\Omega_\varepsilon} f_\varepsilon v\, dx + \varepsilon \sum_{m=1}^{2} \int_{\Xi_\varepsilon^{(m)}} g_\varepsilon^{(m)} v\, ds_x \quad v \in L^2(0,T; H^1(\Omega_\varepsilon, \Gamma_\varepsilon).$$

Now using these notations we rewrite equality (7) in the form of the following identity

$$-\int_0^T (u_\varepsilon(t,\cdot), v'(t,\cdot))\, dt + \int_0^T \mathcal{A}(t) u_\varepsilon(v)\, dt = \int_0^T F(t)(v(t,\cdot))\, dt \tag{16}$$

for any function $v \in L^2(0,T; H^1(\Omega_\varepsilon, \Gamma_\varepsilon)) \bigcap H^1(0,T; L^2(\Omega_\varepsilon))$, $v(T) = 0$; here $(\cdot\,,\cdot)$ is the inner product in $L^2(\Omega_\varepsilon)$, $v' = \frac{dv}{dt}$.

Similar as in theorem 2.1 (p.108 [56]) one can prove that a function $u_\varepsilon \in L^2(0,T; H^1(\Omega_\varepsilon, \Gamma_\varepsilon))$ is the weak solution to problem (6) if and only if u_ε is a solution to the following abstract Cauchy problem (see [56, Theorem 1.5])

$$u_\varepsilon' + \mathcal{A} u_\varepsilon = F \quad \text{in} \quad L^2\big(0,T; (H^1(\Omega_\varepsilon, \Gamma_\varepsilon))^*\big), \quad u_\varepsilon(0) = 0, \tag{17}$$

Since $\big(L^2(0,T; H^1(\Omega_\varepsilon, \Gamma_\varepsilon))\big)^* \cong L^2\big(0,T; (H^1(\Omega_\varepsilon, \Gamma_\varepsilon))^*\big)$, here and what follows we use $\mathcal{A}$ to denote its realization

$$\mathcal{A} : L^2\big(0,T; H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\big) \mapsto L^2\big(0,T; (H^1(\Omega_\varepsilon, \Gamma_\varepsilon))^*\big),$$

and the same for the functional F.

By virtue of Lemma 4.2 [56], from properties of the operator

$$\mathcal{A}(t) : H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \to (H^1(\Omega_\varepsilon, \Gamma_\varepsilon))^*$$

it follows the corresponding properties of its realization.

Now let's prove some properties of the operator $\mathcal{A}(t) : H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \to (H^1(\Omega_\varepsilon, \Gamma_\varepsilon))^*$.

(1) Operator $\mathcal{A}$ is bounded. Using (1), (4), (5) we can prove the following inequality

$$\|\mathcal{A}u\|_{(H^1(\Omega_\varepsilon, \Gamma_\varepsilon))^*} \le c_1 \left(\|u\|_{H^1(\Omega_\varepsilon)} + 1\right), \quad u \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon).$$

(2) Operator $\mathcal{A}$ is monotone. With the help of (1), (3) we obtain

$$\begin{aligned} &< \mathcal{A}u - \mathcal{A}v, u - v > \\ &\quad = \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_j}(u - v) \partial_{x_i}(u - v)\, dx + \int_{\Omega_\varepsilon} (h(u) - h(v))(u - v)\, dx \\ &\qquad + \varepsilon \sum_{m=1}^{2} \int_{\Xi_\varepsilon^{(m)}} (k_m(u) - k_m(v))(u - v)\, ds_x \\ &\quad \ge \varkappa_1 \int_{\Omega_\varepsilon} |\nabla(u - v)|^2\, dx + c_1 \int_{\Omega_\varepsilon} (u - v)^2\, dx + \varepsilon c_1 \sum_{m=1}^{2} \int_{\Xi_\varepsilon^{(m)}} (u - v)^2\, ds_x \ge 0, \\ &\qquad u, v \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon). \end{aligned} \tag{18}$$

(3) Operator $\mathcal{A}$ is hemicontinuous. Indeed, for each $u, v \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ the real valued function

$$z \to \mathcal{A}(u + zv)(v) = \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_j}(u + zv)\, \partial_{x_i} v\, dx + \int_{\Omega_\varepsilon} h(u + zv) v\, dx$$

$$+\varepsilon \sum_{m=1}^{2} \int_{\Xi_\varepsilon^{(m)}} k_m(u + zv) v\, ds_x$$

is continuous.

Thus the realization $\mathcal{A} : L^2\big(0,T; H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\big) \mapsto L^2\big(0,T; (H^1(\Omega_\varepsilon, \Gamma_\varepsilon))^*\big)$ is bounded, monotone and hemicontinuous. It means the $\mathcal{A}$ is type M (see Lema 2.1 [56]).

(4) Operator $\mathcal{A}$ is coercive. Using (1), (4), (5) for each $v \in L(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon))$ we have

$$\int_0^T Av(v)\,dt = \int_0^T \Bigg[\int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_j} v \partial_{x_i} v\,dx + \int_{\Omega_\varepsilon} h(v)v\,dx$$

$$+\varepsilon \sum_{m=1}^2 \int_{\Xi_\varepsilon^{(m)}} k_m(v)v\,ds_x\Bigg]\,dt \geq \varkappa_1 \int_0^T \int_{\Omega_\varepsilon} |\nabla v|^2\,dx\,dt$$

$$+\int_0^T \int_{\Omega_\varepsilon} (h(0)v + c_1 v^2)\,dx\,dt + \varepsilon \int_0^T \sum_{m=1}^2 \int_{\Xi_\varepsilon^{(m)}} (k_m(0)v + c_1 v^2)\,ds_x\,dt$$

$$\geq \varkappa_1 \int_0^T \int_{\Omega_\varepsilon} |\nabla v|^2\,dx\,dt - |h(0)| \int_0^T \int_{\Omega_\varepsilon} |v|\,dx\,dt + c_1 \int_0^T \int_{\Omega_\varepsilon} v^2\,dx\,dt$$

$$-\varepsilon \int_0^T \sum_{m=1}^2 |k_m(0)| \int_{\Xi_\varepsilon^{(m)}} |v|\,ds_x\,dt + \varepsilon c_1 \int_0^T \sum_{m=1}^2 \int_{\Xi_\varepsilon^{(m)}} v^2\,ds_x\,dt.$$

Using Cauchy's inequality with δ ($ab \leq \delta a^2 + \frac{b^2}{4\delta}$, $\ a,b,\delta > 0$) for second and forth integrals we deduce

$$\int_0^T Av(v)\,dt \geq \varkappa_1 \int_0^T \int_{\Omega_\varepsilon} |\nabla v|^2\,dx\,dt - \delta \int_0^T \int_{\Omega_\varepsilon} v^2\,dx\,dt$$

$$-\int_0^T \int_{\Omega_\varepsilon} \frac{|h(0)|^2}{4\delta}\,dx\,dt + c_1 \int_0^T \int_{\Omega_\varepsilon} v^2\,dx\,dt - \varepsilon \int_0^T \sum_{m=1}^2 \delta_m |k_m(0)| \int_{\Xi_\varepsilon^{(m)}} v^2\,ds_x\,dt$$

$$-\varepsilon \int_0^T \sum_{m=1}^2 \frac{|k_m(0)|}{4\delta_m} \int_{\Xi_\varepsilon^{(m)}} ds_x\,dt + \varepsilon c_1 \int_0^T \sum_{m=1}^2 \int_{\Xi_\varepsilon^{(m)}} v^2\,ds_x\,dt. \tag{19}$$

Set $\delta = c_1$ and δ_m such that $c_1 - \delta_m |k_m(0)| > 0$. Then taking (14) and $\operatorname{meas} \Xi_\varepsilon^{(m)} \simeq \mathcal{O}(\varepsilon^{-1})$ into account, we deduce from (19) that

$$\int_0^T Av(v)\,dt \geq c_2 \int_0^T \left(\int_{\Omega_\varepsilon} |\nabla v|^2\,dx + \varepsilon \int_{\Xi_\varepsilon} v^2\,ds_x\right)\,dt - c_3$$

$$= c_4 \int_0^T \|v\|_\varepsilon^2\,dt - c_5 \geq c_6 \int_0^T \|v\|^2_{H^1(\Omega_\varepsilon)}\,dt - c_5 = c_6 \|v\|^2_{L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon))} - c_5.$$

Now it follows from Corollary 4.1 [56] that problem (17) has a unique solution.

It remains to prove inequality (15). For this we multiply the first equation from (6) by u_ε and integrate over $\Omega_\varepsilon \times (0,t)$. Then integrating by parts and taking into account $\int_0^t u_\varepsilon \partial_t u_\varepsilon\,dt = \frac{1}{2} u_\varepsilon^2(x,t)$, we obtain

$$\frac{1}{2} \int_{\Omega_\varepsilon} u_\varepsilon^2(x,t)\,dx + \int_0^t \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_j} u_\varepsilon \partial_{x_i} u_\varepsilon\,dx\,dt + \int_0^t \int_{\Omega_\varepsilon} h(u_\varepsilon) u_\varepsilon\,dx\,dt$$

$$+\varepsilon \sum_{m=1}^2 \int_0^t \int_{\Xi_\varepsilon^{(m)}} k_m(u_\varepsilon) u_\varepsilon\,ds_x dt = \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon\,dx\,dt$$

$$+\varepsilon \sum_{m=1}^2 \int_0^t \int_{\Xi_\varepsilon^{(m)}} g_\varepsilon^{(m)} u_\varepsilon\,ds_x\,dt.$$

Due to (1), (4) we get

$$\frac{1}{2} \int_{\Omega_\varepsilon} u_\varepsilon^2(x,t)\,dx + c_1 \int_0^t \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \varepsilon \int_{\Xi_\varepsilon} u_\varepsilon^2\,ds_x\right)\,dt + \varepsilon c_2 \int_0^t \int_{\Xi_\varepsilon} u_\varepsilon\,ds_x\,dt$$

$$+c_3\int_0^t\int_{\Omega_\varepsilon} u_\varepsilon\,dx\,dt \le \int_0^t\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon\,dx\,dt + \varepsilon\sum_{m=1}^{2}\int_0^t\int_{\Xi_\varepsilon} g_\varepsilon^{(m)} u_\varepsilon\,ds_x\,dt.$$

Taking into account (14) the previous inequality we rewrite as follows

$$\frac{1}{2}\int_{\Omega_\varepsilon} u_\varepsilon^2(x,t)\,dx + c_4\int_0^t \|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}\,dt \le c_2\varepsilon\int_0^t\int_{\Xi_\varepsilon}|u_\varepsilon|\,ds_x\,dt$$
$$+\,c_3\int_0^t\int_{\Omega_\varepsilon}|u_\varepsilon|\,dx\,dt + \int_0^t\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon\,dx\,dt + \varepsilon\sum_{m=1}^{2}\int_0^t\int_{\Xi_\varepsilon} g_\varepsilon^{(m)} u_\varepsilon\,ds_x\,dt. \quad (20)$$

Using Hölder's inequality and (14), we deduce from (20) the following inequality

$$\frac{1}{2}\int_{\Omega_\varepsilon} u_\varepsilon^2(x,t)\,dx + c_4\|u_\varepsilon\|^2_{L^2(0,t;H^1(\Omega_\varepsilon))} \le c_5\|u_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}$$
$$+\,\|f_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))}\|u_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}$$
$$+\,c_6\sqrt{\varepsilon}\sum_{m=1}^{2}\|g_\varepsilon^{(m)}\|_{L^2(0,t;L^2(\Xi_\varepsilon^{(m)}))}\|u_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))} \quad (21)$$

From (21) we get two inequalities

$$\max_{0\le\tau\le t}\|u_\varepsilon(x,\tau)\|^2_{L^2(\Omega_\varepsilon)}$$
$$\le c_7\Big(1+\|f_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))}+\sqrt{\varepsilon}\sum_{m=1}^{2}\|g_\varepsilon^{(m)}\|_{L^2(0,t;L^2(\Xi_\varepsilon^{(m)}))}\Big)\|u_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))} \quad (22)$$

and

$$\|u_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))} \le c_8\Big(1+\|f_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))}+\sqrt{\varepsilon}\sum_{m=1}^{2}\|g_\varepsilon^{(m)}\|_{L^2(0,t;L^2(\Xi_\varepsilon^{(m)}))}\Big), \quad (23)$$

from which we discover the inequality (15). □

4 Convergence theorem

In the sequel, $\widetilde{y}$ denotes the zero-extension of a function y defined on Ω_ε into the domain Ω. Also we introduce the following characteristic function

$$\chi_{Q_0}(\xi) = \begin{cases} 1, & x\in Q_0, \\ 0, & x\in \square\setminus Q_0. \end{cases} \quad (24)$$

It is known that $\chi^\varepsilon_{Q_0}(x) := \chi_{Q_0}(\frac{x}{\varepsilon}) \xrightarrow{w} |Q_0|$ in $L^2(\Omega)$ as $\varepsilon\to 0$.

Lemma 1. *Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a sequence in $L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon))$ uniformly bounded in ε in $L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon))$ and such that*

$$\widetilde{\kappa_m(v_\varepsilon)} \xrightarrow{w} \zeta_m \quad in \quad L^2(\Omega\times(0,T)) \quad as \quad \varepsilon\to 0 \qquad (m=1,2).$$

Then for any function $\psi\in L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon))$

$$\varepsilon\int_{\Xi_\varepsilon^{(m)}\times(0,T)}\kappa_m(v_\varepsilon)\,\psi\,ds_x\,dt \to q_m\int_{\Omega\times(0,T)}\zeta_m\,\psi\,dx\,dt \quad as \quad \varepsilon\to 0$$
$$(m=1,2). \quad (25)$$

The proof for lemma is similar to the proof of Lemma 2 in [10]. By the same way, using (2), we can prove that for any function $\psi \in L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon))$

$$\varepsilon \int_{\Xi_\varepsilon^{(m)}\times(0,T)} g_\varepsilon^{(m)} \psi \, ds_x \, dt \; \rightarrow \; |S^{(m)}| \int_{\Omega\times(0,T)} g_0^{(m)} \, \psi \, dx \, dt \quad \text{as} \quad \varepsilon \to 0$$

$$(m = 1,2). \tag{26}$$

Consider $1-$periodic solutions $T_l, \ l = 1,\ldots,n$, to the following cell-type problems

$$\begin{cases} \mathcal{L}_{\xi\xi}\big(T_l\big) = -\partial_{\xi_i} a_{il} \ \text{ in } \ Q_0, \\ \sigma_\xi\big(T_l\big) = -a_{il}\,\nu_i \ \text{ on } \ S, \quad \langle T_l\rangle_{Q_0} = 0. \end{cases} \tag{27}$$

It is easy to prove the existence and uniqueness of the solutions to these problems (see for instance [7],[1],[10]).

With the help of $T_l, \ l = 1,\ldots,n$, we define the coefficients of the homogenized matrix $\{\widehat{a}_{ij}\}$ by the formula

$$\widehat{a}_{ij} = \langle a_{ij} + a_{ik}\partial_{\xi_k} T_j\rangle_{Q_0}, \quad i,j \in \{1,2,\ldots,n\}. \tag{28}$$

It is easy to see that

$$\widehat{a}_{ij} = \langle a_{kl}\,\partial_{\xi_k}(\xi_i + T_i)\,\partial_{\xi_l}(\xi_j + T_j)\rangle_{Q_0} \tag{29}$$

i.e., the matrix $\{\widehat{a}_{ij}\}$ is symmetric and it is well known that it is elliptic (see [7],[1]).

Theorem 2. *For the solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to problem (6) the following convergences*

$$\left.\begin{aligned} \widetilde{u_\varepsilon} &\overset{w}{\longrightarrow} |Q_0| v_0 &&\text{in} \quad L^2(0,T;L^2(\Omega)), \\ \widetilde{a_{ij}^\varepsilon\,\partial_{x_j} u_\varepsilon} &\overset{w}{\longrightarrow} \widehat{a}_{ij}\partial_{x_j} v_0 &&\text{in} \quad L^2(0,T;L^2(\Omega)), \quad i = 1,\ldots,n, \end{aligned}\right\} \quad \text{as } \varepsilon \to 0, \tag{30}$$

hold, where v_0 is a unique weak solution to the problem

$$\begin{cases} |Q_0|\partial_t v_0(x,t) - \widehat{a}_{ij}\,\partial^2_{x_i x_j} v_0(x,t) + |Q_0| h(v_0) + \sum\limits_{m=1}^{2} |S^{(m)}|\,\kappa_m(v_0(x,t)) \\ = \sum\limits_{m=1}^{2} |S^{(m)}|\, g_0^{(m)}(x,t) + |Q_0|\, f_0(x,t), \quad (x,t) \in \Omega\times(0,T), \\ v_0(x,t) = 0, \quad (x,t) \in \partial\Omega\times(0,T), \\ v_0(x,0) = 0, \quad x \in \Omega, \end{cases} \tag{31}$$

which is called homogenized problem for (6).

Proof. **1.** It follows from (15) and (3) that the values

$$\|\widetilde{u_\varepsilon}(\cdot\,,t)\|_{L^2(\Omega)} \ \ (\text{for all } t\in[0,T]), \qquad \|\widetilde{u_\varepsilon}\|_{L^2(\Omega\times(0,T))}, \qquad \|\widetilde{h(u_\varepsilon)}\|_{L^2(0,T;L^2(\Omega))},$$

$$\|\widetilde{a_{ij}^\varepsilon\,\partial_{x_j} u_\varepsilon}\|_{L^2(0,T;L^2(\Omega))}, \quad i = 1,\ldots,n, \quad \|\widetilde{\kappa_m(u_\varepsilon)}\|_{L^2(0,T;L^2(\Omega))}, \quad m = 1,2,$$

are uniformly bounded with respect to ε. Hence there exists a subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$, again denoted by $\{\varepsilon\}$, such that

$$\left.\begin{aligned} \forall\, t\in[0,T] \quad \widetilde{u_\varepsilon(\cdot,t)} &\overset{w}{\longrightarrow} |Q_0|\, v_0(\cdot,t) \ \text{ in } L^2(\Omega), \\ \widetilde{u_\varepsilon} &\overset{w}{\longrightarrow} v_1 \qquad\qquad \text{in } L^2(0,T;L^2(\Omega)), \\ \widetilde{h(u_\varepsilon)} &\overset{w}{\longrightarrow} \mu \qquad\qquad\ \text{in } L^2(0,T;L^2(\Omega)), \\ \widetilde{a_{ij}^\varepsilon\,\partial_{x_j} u_\varepsilon} &\overset{w}{\longrightarrow} \gamma_i \qquad\qquad \text{in } L^2(0,T;L^2(\Omega)), \ i = 1,\ldots,n, \\ \widetilde{\kappa_m(u_\varepsilon)} &\overset{w}{\longrightarrow} \zeta_m \qquad\qquad \text{in } L^2(0,T;L^2(\Omega)), \ m = 1,2, \end{aligned}\right\} \tag{32}$$

as $\varepsilon \to 0$ where $v_0, v_1, \mu \quad \gamma_i,\ i = 1,\dots,n,\quad \zeta_m,\ m = 1,2$, are some functions which will be determined in what follows.

According to Fubini's theorem we have $v_1(\cdot,t) \in L^2(\Omega)$ for a.e. $t \in (0,T)$. We conclude from (32) that $v_1(x,t) = |Q_0| v_0(x,t)$ for a.e. $(x,t) \in \Omega \times (0,T)$ and consequently $v_0 \in L^2(\Omega \times (0,T))$.

2. Obviously the ε-periodic functions $T_l\left(\frac{\cdot}{\varepsilon}\right),\ l = 1,\dots,n$, defined in (27) satisfy the following relations

$$\begin{cases} \partial_{x_i}\left(a_{ij}(\xi)\partial_{\xi_j}T_l(\xi)|_{\xi=\frac{x}{\varepsilon}}\right) + \partial_{x_i}a^{\varepsilon}_{il}(x) = 0 \quad \forall\, x \in \Omega_\varepsilon, \\ \left(a_{ij}(\xi)\,\partial_{\xi_j}T_l(\xi)\,\nu_i(\xi) + a_{il}(\xi)\nu_i(\xi)\right)\Big|_{\xi=\frac{x}{\varepsilon}} = 0 \quad \forall\, x \in \Xi_\varepsilon. \end{cases}$$

Multiplying the first relation by $u_\varepsilon(x,t)\,\phi(x)\eta(t)$, where $\phi \in C_0^\infty(\Omega)$, $\eta \in C^1([0,T])$, $\eta(T) = 0$ and integrating over $\Omega_\varepsilon \times (0,T)$, we obtain

$$\int_0^T \int_{\Omega_\varepsilon} \left(a_{ij}(\xi)\,\partial_{\xi_j}T_l(\xi) + a_{il}(\xi)\right)|_{\xi=\frac{x}{\varepsilon}} \left(u_\varepsilon\,\partial_{x_i}\phi + \phi\,\partial_{x_i}u_\varepsilon\right)\eta\,dx\,dt = 0, \quad l = \overline{1,n}. \tag{33}$$

Put the following test-function $\varphi(x,t) = \varepsilon T_l(\frac{x}{\varepsilon})\phi(x)\eta(t),\ (x,t) \in \Omega_\varepsilon \times (0,T)$ into the integral identity (7). The result is as follows

$$\begin{aligned} -\,\varepsilon \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon T_l\left(\frac{x}{\varepsilon}\right)\phi(x)\partial_t\eta(t)\,dx\,dt \\ + \int_0^T \int_{\Omega_\varepsilon} a^{\varepsilon}_{ij}(x)\partial_{x_j}u_\varepsilon \partial_{\xi_i}T_l(\xi)|_{\xi=\frac{x}{\varepsilon}}\,\phi(x)\eta(t)\,dx\,dt \\ + \varepsilon \int_0^T \int_{\Omega_\varepsilon} a^{\varepsilon}_{ij}(x)\partial_{x_j}u_\varepsilon\, T_l\left(\frac{x}{\varepsilon}\right)\partial_{x_i}\phi(x)\eta(t)\,dx\,dt \\ + \varepsilon \int_0^T \int_{\Omega_\varepsilon} h(u_\varepsilon)T_l\left(\frac{x}{\varepsilon}\right)\phi(x)\eta(t)\,dx\,dt \\ + \varepsilon^2 \sum_{m=1}^{2} \int_0^T \int_{\Xi^{(m)}_\varepsilon} \kappa_m(u_\varepsilon)\,T_l\left(\frac{x}{\varepsilon}\right)\phi(x)\eta(t)\,ds_x\,dt \\ = \varepsilon \int_0^T \int_{\Omega_\varepsilon} f_\varepsilon\,T_l\left(\frac{x}{\varepsilon}\right)\phi(x)\eta(t)\,dx + \varepsilon^2 \sum_{m=1}^{2} \int_0^T \int_{\Xi^{(m)}_\varepsilon} g^{(m)}_\varepsilon\,T_l\,\phi(x)\eta(t)\,ds_x\,dt. \end{aligned} \tag{34}$$

Using (2), (3) and the identities (8), it follows from (34) that

$$\int_0^T \int_{\Omega_\varepsilon} a^{\varepsilon}_{ij}(x)\partial_{x_j}u_\varepsilon \partial_{\xi_i}T_l(\xi)|_{\xi=\frac{x}{\varepsilon}}\,\phi(x)\eta(t)\,dx\,dt = \mathcal{O}(\varepsilon) \quad \text{as} \quad \varepsilon \to 0, \quad l = \overline{1,n}. \tag{35}$$

Subtracting (34) from (33), we get

$$\begin{aligned} \int_0^T \int_{\Omega} \left(a_{ij}(\xi)\,\partial_{\xi_j}T_l(\xi) + a_{il}(\xi)\right)|_{\xi=\frac{x}{\varepsilon}}\,\widetilde{u_\varepsilon}\,\eta\,\partial_{x_i}\phi\,dx\,dt \\ + \int_0^T \int_{\Omega} \widetilde{a^{\varepsilon}_{il}\,\partial_{x_i}u_\varepsilon}\,\phi\,\eta\,dx\,dt = \mathcal{O}(\varepsilon), \qquad l = \overline{1,n}. \end{aligned} \tag{36}$$

In (36) we regard that the functions $a_{ij}\,\partial_{\xi_j}T_l + a_{il},\ l = 1,\dots,n$, are equal to zero on B.

Let us find the limit of the first summand in the left-hand side of (36). At first we note that the limit function $v_0(\cdot,t)$ in (32) belongs to $H_0^1(\Omega)$ for a.e. $t \in (0,T)$ because of the conectedness of the domain $\mathbb{R}^n \setminus \overline{\left(\mathcal{B}^{(1)} \cup \mathcal{B}^{(2)}\right)}$ (see [40]-[29]). Since $\left(a_{ij}(\xi)\,\partial_{\xi_j}T_l(\xi) + a_{il}(\xi)\right)\nu_i(\xi) = 0$ at $\xi \in S$ and the vector-functions

$$\mathbf{F}_l = \left(a_{1j}(\xi)\,\partial_{\xi_j}T_l(\xi) + a_{1l}(\xi),\dots,\ a_{nj}(\xi)\,\partial_{\xi_j}T_l(\xi) + a_{nl}(\xi)\right), \quad l = 1,\dots,n, \tag{37}$$

are solenoidal in Q_0 (see (27)), their zero-extensions into $\square \setminus Q_0$ are also solenoidal in the weak sense, i.e.,

$$\int_{Q_0} \mathbf{F}_l(\xi) \cdot \nabla\psi(\xi)\, d\xi = \int_{\square} \mathbf{F}_l(\xi) \cdot \nabla\psi(\xi)\, d\xi = 0 \quad \forall \psi \in C^\infty_{\rm per}(\square), \quad l = 1, \dots, n.$$

Then using results by V.V. Zhikov (see [40, Lemma 2.3]), we get that

$$\lim_{\varepsilon\to 0} \int_0^T\!\!\int_\Omega \big(a_{ij}(\xi)\, \partial_{\xi_j} T_l(\xi) + a_{il}(\xi)\big)|_{\xi=\frac{x}{\varepsilon}}\, \widetilde{u_\varepsilon}\, \partial_{x_i}\phi\, \eta\, dx\, dt = \int_0^T\!\!\int_\Omega \widehat{a}_{il}\, v_0\, \partial_{x_i}\phi\, \eta\, dx\, dt.$$

As a results, it follows from (36) in the limit passage as $\varepsilon \to 0$ that

$$\int_0^T \int_\Omega \widehat{a}_{il}\, v_0\, \partial_{x_i}\phi\, \eta\, dx\, dt + \int_0^T \int_\Omega \gamma_l\, \phi\, \eta dx\, dt = 0$$

$$\forall \phi \in C_0^\infty(\Omega),\ \eta \in C^1([0,T]),\ \eta(T) = 0 \quad (l = 1, \dots, n),$$

i.e.,

$$\gamma_l(x,t) = \widehat{a}_{il}\, \partial_{x_i} v_0(x,t) \quad \text{for a.e. } (x,t) \in \Omega \times (0,T) \quad (l = 1, \dots, n). \tag{38}$$

3. Using the extension by zero and the identities (8), we rewrite the integral identity (7) with test-function $\varphi(x)\eta(t)$, where $\varphi \in C_0^\infty(\Omega)$, $\eta \in C^1([0,T])$, $\eta(T) = 0$ in the following way

$$\begin{aligned}
&- \int_0^T \int_\Omega \widetilde{u_\varepsilon} \varphi(x) \partial_t \eta(t)\, dx\, dt + \int_0^T \int_\Omega \widetilde{a^\varepsilon_{ij}\, \partial_{x_j} u_\varepsilon}\, \partial_{x_i}\varphi(x)\eta(t)\, dx\, dt \\
&\qquad + \int_0^T \int_{\Omega_\varepsilon} \widetilde{h(u_\varepsilon)} \varphi(x)\, \eta(t)\, dx\, dt \\
&+ \sum_{m=1}^{2} \Big(\underbrace{\varepsilon \int_0^T\!\!\int_{\Omega_\varepsilon} a^\varepsilon_{ij}(x)\, \partial_{\xi_j} \psi_0^{(m)}(\xi)|_{\xi=\frac{x}{\varepsilon}} \big(\kappa'_m(u_\varepsilon)\, \partial_{x_i} u_\varepsilon\, \varphi(x) + \kappa_m(u_\varepsilon)\, \partial_{x_i}\varphi(x)\big)\eta(t)\, dx\, dt} \\
&\qquad + q_m \int_0^T \int_\Omega \widetilde{\kappa_m(u_\varepsilon)}\, \varphi(x)\eta(t)\, dx \Big) = \int_0^T \int_\Omega \chi^\varepsilon_{Q_0}\, f_\varepsilon \varphi(x)\eta(t)\, dx\, dt \\
&+ \sum_{m=1}^{2} \Big(\underbrace{\varepsilon \int_0^T \int_{\Omega_\varepsilon} a^\varepsilon_{ij}(x)\, \partial_{\xi_j} \psi_0^{(m)}(\xi)|_{\xi=\frac{x}{\varepsilon}}\, \big(\partial_{x_i} g^{(m)}_\varepsilon\, \varphi(x) + g^{(m)}_\varepsilon\, \partial_{x_i}\varphi(x)\big)\eta(t)\, dx\, dt} \\
&\qquad + q_m \int_0^T \int_\Omega \chi^\varepsilon_{Q_0}\, g^{(m)}_\varepsilon\, \varphi(x)\eta(t)\, dx\, dt \Big) \\
&\qquad\qquad \forall\, \varphi \in C_0^\infty(\Omega)\ \forall\, \eta \in C^1([0,T]), \eta(T) = 0.
\end{aligned} \tag{39}$$

It is easy to see that the underbraced summands in (39) vanish as $\varepsilon \to 0$; the first one is due to (3), (10) and (15), the second one is due to (10) and (2).

Taking into account (32), (38) and (2), we pass to the limit in (39) as $\varepsilon \to 0$. As a result we get the identity

$$\begin{aligned}
&- |Q_0| \int_0^T \int_\Omega v_0\, \varphi(x) \partial_t \eta(t)\, dx\, dt + \int_0^T \int_\Omega \widehat{a}_{ij}\, \partial_{x_j} v_0\, \partial_{x_i}\varphi(x)\eta(t)\, dx\, dt \\
&\qquad + \int_0^T \int_{\Omega_\varepsilon} \mu\, \varphi(x)\eta(t)\, dx\, dt + \sum_{m=1}^{2} q_m \int_0^T \int_\Omega \zeta_m\, \varphi(x)\eta(t)\, dx\, dt \\
&\qquad\qquad = |Q_0| \int_0^T \int_\Omega f_0 \varphi(x)\eta(t)\, dx\, dt + \sum_{m=1}^{2} |S^{(m)}| \int_0^T \int_\Omega g_0^{(m)}\, \varphi(x)\eta(t)\, dx\, dt
\end{aligned} \tag{40}$$

for any function $\varphi \in C_0^\infty(\Omega), \eta \in C^1([0,T]), \eta(T)=0$. Since the space of functions

$$\{\varphi(x)\,\eta(t): \ \varphi \in C_0^\infty(\Omega), \quad \eta \in C^1([0,T]), \quad \eta(T)=0\}$$

is dense in the space $\big\{\psi \in L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon)) \bigcap H^1(0,T;L^2(\Omega_\varepsilon)) : \psi|_{t=T}=0\big\}$ (see [57, p.301]), identity (40) is valid for any function $\psi \in L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon)) \bigcap H^1(0,T;L^2(\Omega_\varepsilon))$ such that $\psi|_{t=T}=0$.

4. Multiplying the differential equation in problem (6) by u_ε and integrating over $\Omega_\varepsilon \times (0,T)$, we obtain

$$\frac{1}{2}\int_{\Omega_\varepsilon} u_\varepsilon^2(x,T)\,dx + \int_0^T \int_{\Omega_\varepsilon} a_{ij}^\varepsilon\,\partial_{x_j}u_\varepsilon\,\partial_{x_i}u_\varepsilon\,dx\,dt + \int_0^T\int_{\Omega_\varepsilon} h(u_\varepsilon)u_\varepsilon\,dx\,dt$$
$$+\varepsilon\sum_{m=1}^{2}\int_0^T\int_{\Xi_\varepsilon^{(m)}} \kappa_m(u_\varepsilon)\,u_\varepsilon\,ds_x\,dx$$
$$=\int_0^T\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon\,dx\,dt + \varepsilon\sum_{m=1}^{2}\int_0^T\int_{\Xi_\varepsilon^{(m)}} g_\varepsilon^{(m)} u_\varepsilon\,ds_x\,dt.$$

With the help of (2), (8), (25), (26) and (40) we can find

$$\lim_{\varepsilon\to 0}\left(\frac{1}{2}\int_{\Omega_\varepsilon} u_\varepsilon^2(x,T)\,dx + \int_0^T\int_{\Omega_\varepsilon} a_{ij}^\varepsilon\,\partial_{x_j}u_\varepsilon\,\partial_{x_i}u_\varepsilon\,dx\,dt + \int_0^T\int_{\Omega_\varepsilon} h(u_\varepsilon)u_\varepsilon\,dx\,dt\right.$$
$$\left. +\,\varepsilon\sum_{m=1}^{2}\int_0^T\int_{\Xi_\varepsilon^{(m)}} \kappa_m(u_\varepsilon)\,u_\varepsilon\,ds_x\,dx\right) = \lim_{\varepsilon\to 0}\left(\int_0^T\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon\,dx\,dt\right.$$
$$\left. +\varepsilon\sum_{m=1}^{2}\int_0^T\int_{\Xi_\varepsilon^{(m)}} g_\varepsilon^{(m)}u_\varepsilon\,ds_x\,dt\right) = \lim_{\varepsilon\to 0}\left(\int_0^T\int_\Omega f_\varepsilon \widetilde{u_\varepsilon}\,dx\,dt\right.$$
$$+\sum_{m=1}^{2}\left(\varepsilon\int_0^T\int_{\Omega_\varepsilon} a_{ij}^\varepsilon\,\partial_{\xi_j}\psi_0^{(m)}(\xi)\Big|_{\xi=\frac{x}{\varepsilon}}\left(\partial_{x_i}g_\varepsilon^{(m)}u_\varepsilon + g_\varepsilon^{(m)}\partial_{x_i}u_\varepsilon\right)dx\,dt\right.$$
$$\left.\left. +q_m\int_0^T\int_\Omega g_\varepsilon^{(m)}\widetilde{u_\varepsilon}\,dx\,dt\right)\right) = |Q_0|\int_0^T\int_\Omega f_0 v_0\,dx\,dt$$
$$+\sum_{m=1}^{2}|S^{(m)}|\int_0^T\int_\Omega g_0^{(m)}v_0\,dx\,dt = \frac{|Q_0|}{2}\int_\Omega v_0^2(x,T)\,dx$$

$$+\int_0^T\int_\Omega \widehat{a}_{ij}\partial_{x_j}v_0\partial_{x_i}v_0\,dx\,dt \quad + \int_0^T\int_{\Omega_\varepsilon}\mu v_0\,dx\,dt \quad + \sum_{m=1}^{2}q_m\int_0^T\int_\Omega \zeta_m v_0\,dx\,dt. \qquad (41)$$

5. Now it remains to determine the functions $\mu,\ \zeta_1$ and ζ_2. For this we will use the method of Browder and Minty, a remarkable technique which somehow applies to the corresponding inequality of monotonicity to justify passing to a weak limit within a nonlinearity.

Thanks to (1) and (3), the inequality of monotonicity in our case reads as follows

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\varepsilon}(u_\varepsilon(x,T)-\psi(x,T))^2\,dx \\
&\qquad+\int_0^T\int_{\Omega_\varepsilon} a_{ij}^\varepsilon\,\partial_{x_j}\big(u_\varepsilon-\psi-\varepsilon T_p\,\partial_{x_p}\psi\big)\,\partial_{x_i}\big(u_\varepsilon-\psi-\varepsilon T_q\,\partial_{x_q}\psi\big)\,dx\,dt \\
&\qquad\qquad+\int_0^T\int_{\Omega_\varepsilon}\big(h(u_\varepsilon)-h(\psi)\big)(u_\varepsilon-\psi)\,dx\,dt \\
&\qquad+\varepsilon\sum_{m=1}^{2}\int_0^T\int_{\Xi_\varepsilon^{(m)}}\big(\kappa_m(u_\varepsilon)-\kappa_m(\psi)\big)\big(u_\varepsilon-\psi\big)\,ds_x\,dt\ge 0 \\
&\qquad\qquad\qquad\forall\,\psi=\varphi(x)\,\eta(t),\quad \varphi\in C_0^\infty(\Omega),\quad \eta\in C([0,T]), \qquad (42)
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\varepsilon}u_\varepsilon^2(x,T)\,dx+\int_0^T\int_{\Omega_\varepsilon}a_{ij}^\varepsilon\,\partial_{x_j}u_\varepsilon\,\partial_{x_i}u_\varepsilon\,dx\,dt \\
&\qquad+\int_0^T\int_{\Omega_\varepsilon}h(u_\varepsilon)u_\varepsilon\,dx\,dt+\varepsilon\sum_{m=1}^{2}\int_0^T\int_{\Xi_\varepsilon^{(m)}}\kappa_m(u_\varepsilon)\,u_\varepsilon\,ds_x\,dx \\
&\qquad+\int_0^T\int_{\Omega_\varepsilon}a_{ij}^\varepsilon\,\big(\partial_{x_j}\psi+\partial_{\xi_j}T_p\,\partial_{x_p}\psi\big)\,\big(\partial_{x_i}\psi+\partial_{\xi_i}T_q\,\partial_{x_q}\psi\big)\,dx\,dt \\
&-2\int_0^T\int_{\Omega}\widetilde{a_{ij}^\varepsilon\,\partial_{x_j}}u_\varepsilon\,\partial_{x_i}\psi\,dx\,dt-2\int_0^T\int_{\Omega_\varepsilon}a_{ij}^\varepsilon\,\partial_{x_j}u_\varepsilon\,\partial_{\xi_i}T_q\,\partial_{x_q}\psi\,dx\,dt \\
&\qquad-2\varepsilon\int_0^T\int_{\Omega_\varepsilon}a_{ij}^\varepsilon\,\big(\partial_{x_j}u_\varepsilon-\partial_{x_j}\psi-\partial_{\xi_j}T_p\,\partial_{x_p}\psi\big)\,T_q\,\partial^2_{x_ix_q}\psi\,dx\,dt \\
&\qquad\qquad+\varepsilon^2\int_0^T\int_{\Omega_\varepsilon}a_{ij}^\varepsilon\,T_p\,T_q\,\partial^2_{x_jx_p}\psi\,\partial^2_{x_ix_q}\psi\,dx\,dt \\
&\qquad\qquad-\int_{\Omega_\varepsilon}u_\varepsilon(x,T)\psi(x,T)\,dx+\frac{1}{2}\int_{\Omega_\varepsilon}\psi^2(x,T)\,dx \\
&\qquad\qquad-\int_0^T\int_{\Omega_\varepsilon}\big(h(\psi)u_\varepsilon+h(u_\varepsilon)\psi-h(\psi)\psi\big)\,dx\,dt \\
&-\varepsilon\sum_{m=1}^{2}\int_0^T\int_{\Xi_\varepsilon^{(m)}}\big(\kappa_m(\psi)\,u_\varepsilon+\kappa_m(u_\varepsilon)\,\psi-\kappa_m(\psi)\,\psi\big)\,ds_x\,dt\ge 0 \\
&\qquad\qquad\qquad\forall\,\psi=\varphi(x)\eta(t),\ \varphi\in C_0^\infty(\Omega),\ \eta\in C([0,T]). \qquad (43)
\end{aligned}
$$

The limit of the first two lines in (43) is equal to the right-hand side in (41). The integral in the third line can be rewritten in the form

$$
\int_0^T\left(\int_{\Omega_\varepsilon}\Big(a_{ij}(\xi)\,\partial_{\xi_j}\big(\xi_p+T_p\big)\partial_{\xi_i}\big(\xi_q+T_q\big)\Big)|_{\xi=\frac{x}{\varepsilon}}\,\partial_{x_p}\varphi(x)\,\partial_{x_q}\varphi(x)\,dx\right)\eta^2(t)\,dt. \qquad (44)
$$

It follows from [40] that its limit equals

$$
\int_0^T\int_{\Omega}\widehat{a}_{pq}\,\partial_{x_p}\varphi(x)\,\partial_{x_q}\varphi(x)\eta^2(t)\,dx\,dt=\int_0^T\int_{\Omega}\widehat{a}_{pq}\,\partial_{x_p}\psi(x,t)\,\partial_{x_q}\psi(x,t)\,dx\,dt.
$$

Due to (35) the second integral in the forth line vanishes. Obviously, the limits of the summands in the fifth and the sixth lines are equal to zero and the limit in the seventh line equals

$$
-|Q_0|\int_{\Omega}v_0(x,T)\psi(x,T)\,dx+\frac{|Q_0|}{2}\int_{\Omega}\psi^2(x,T)\,dx.
$$

It follows easily that the limit in the eighth line is equal

$$\int_0^T \int_\Omega \big(|Q_0| h(\psi) v_0 + \mu\psi - |Q_0| h(\psi)\psi\big)\; dx\, dt.$$

The limits of the integrals in the last line can be found with the help of Lemma 1. Thus we have

$$\begin{aligned}\frac{|Q_0|}{2}\int_\Omega (v_0(x,T)-\psi(x,T))^2\,dx + \int_0^T\int_\Omega \widehat{a}_{ij}\,\partial_{x_j}(v_0-\psi)\,\partial_{x_i}(v_0-\psi)\,dx \\ + \int_0^T\int_\Omega (\mu - |Q_0|h(\psi))\,(v_0-\psi)\,dx\,dt \\ + \sum_{m=1}^{2} q_m \int_0^T\int_\Omega \big(\zeta_m - |Q_0|\kappa_m(\psi)\big)\,\big(v_0-\psi\big)\,dx\,dt \ge 0. \quad (45)\end{aligned}$$

Evidently, this inequality holds for any function $\psi \in L^2(0,T;H^1(\Omega))$.

Fix any $\tau(x,t)=\varphi(x)\eta(t),\ \varphi\in C_0^\infty(\Omega),\ \eta\in C([0,T])$ and set $\psi := v_0-\lambda\tau$ ($\lambda>0$) in (45). We get

$$\begin{aligned}\lambda\frac{|Q_0|}{2}\int_{\Omega_\varepsilon} \tau^2\,dx + \lambda\int_0^T\int_\Omega \widehat{a}_{ij}\,\partial_{x_j}\tau\,\partial_{x_i}\tau\,dx\,dt \\ + \int_0^T\int_\Omega (\mu - |Q_0|h(v_0-\lambda\tau))\,\tau\,dx\,dt \\ + \sum_{m=1}^{2} q_m \int_0^T\int_\Omega \big(\zeta_m - |Q_0|\kappa_m(v_0-\lambda\tau)\big)\,\tau\,dx\,dt \ge 0.\end{aligned}$$

In the limit (as $\lambda\to 0$) we obtain

$$\int_0^T\int_\Omega (\mu - |Q_0|h(v_0))\,\tau\,dx\,dt + \int_\Omega \sum_{m=1}^{2} q_m\big(\zeta_m - |Q_0|\kappa_m(v_0)\big)\,\psi\,dx \ge 0.$$

Replacing τ by $-\tau$, we deduce that in fact quality holds above. Thus

$$\mu(x,t) + \sum_{m=1}^{2} q_m\zeta_m(x,t) = |Q_0|h(v_0) + \sum_{m=1}^{2} |S^{(m)}|\,\kappa_m(v_0(x,t)) \quad \text{for a.e.} \quad (x,t)\in\Omega\times(0,T). \quad (46)$$

6. Returning to (40), we see that the function v_0 satisfies the following integral identity

$$\begin{aligned}-|Q_0|\int_0^T\int_\Omega v_0\partial_t\psi\,dx\,dt + \int_0^T\int_\Omega \widehat{a}_{ij}\,\partial_{x_j}v_0\,\partial_{x_i}\psi\,dx\,dt \\ + |Q_0|\int_0^T\int_\Omega h(v_0)\psi\,dx\,dt + \sum_{m=1}^{2}|S^{(m)}|\int_0^T\int_\Omega \kappa_m(v_0)\,\psi\,dx\,dt \\ = |Q_0|\int_0^T\int_\Omega f_0\psi\,dx\,dt + \sum_{m=1}^{2}|S^{(m)}|\int_0^T\int_\Omega g_0^{(m)}\,\psi\,dx\,dt \quad (47)\end{aligned}$$

for any function $\psi \in L^2(0,T;H^1(\Omega_\varepsilon,\Gamma_\varepsilon))\bigcap H^1(0,T;L^2(\Omega_\varepsilon)), \psi(x,T)=0$. Hence v_0 is a weak solution to the limit problem (31). Thanks to (3) this solution is unique.

Due to the uniqueness of the solution to problem (31), the above argumentations hold for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. □

5 Asymptotic approximation to the solution

In this section we assume that functions $f_0, g_0^{(1)}$ and $g_0^{(2)}$ belong to the space $C^1(\overline{\Omega} \times [0,T])$ and the following fitting condition

$$|Q_0|\, h(0) + \sum_{m=1}^{2} |S^{(m)}|\, \kappa_m(0) = \sum_{m=1}^{2} |S^{(m)}|\, g_0^{(m)}(x,0) + |Q_0|\, f_0(x,0)$$

is satisfied for all $x \in \partial\Omega$. Then by virtue of Theorem 6.1 ([53, Section V]) these assumptions together with the condition (3) provide the existence of the unique solution v_0 to the homogenized problem (31) from the space $C^{2,1}(\overline{\Omega} \times [0,T])$ and, in addition, this solution has the derivatives $\partial^2_{tx_i} v_0$, $i = 1, \ldots, n$ from the space $L^2(\Omega \times (0,T))$.

We take the following approximation

$$\overline{u}_\varepsilon := v_0(x,t) + \varepsilon T_k\left(\tfrac{x}{\varepsilon}\right) \partial_{x_k} v_0(x,t) \tag{48}$$

to the solution u_ε to problem (6). Here v_0 is the solution to problem (31) and $T_1, \ldots, T_n$ are the solutions to the cell-type problems (27). Substituting the difference $u_\varepsilon - \overline{u}_\varepsilon$ in problem (6), we find the residuals both in the differential equation and boundary conditions. Straightforward calculations show that

$$\begin{aligned}
\partial_t(u_\varepsilon - \overline{u}_\varepsilon) - \mathcal{L}_\varepsilon\,(u_\varepsilon - \overline{u}_\varepsilon) &= f_\varepsilon - f_0 - h(u_\varepsilon) + h(v_0) - \sum_{m=1}^{2} q_m \left(g_0^{(m)} - \kappa_m(v_0)\right) \\
&+ \Big(a_{ij}(\xi) + a_{ik}(\xi)\partial_{\xi_k} T_j(\xi) - \tfrac{1}{|Q_0|}\, \widehat{a}_{ij}\Big)\Big|_{\xi = \frac{x}{\varepsilon}}\ \partial^2_{x_i x_j} v_0 \\
&- \varepsilon T_k\left(\frac{x}{\varepsilon}\right) \partial^2_{t\,x_k} v_0 + \varepsilon \partial_{x_i}\left(F_i^\varepsilon(x,t)\right), \quad (x,t) \in \Omega_\varepsilon \times (0,T);
\end{aligned} \tag{49}$$

$$\sigma_\varepsilon\,(u_\varepsilon - \overline{u}_\varepsilon) = -\varepsilon \kappa_m(u_\varepsilon) + \varepsilon g_\varepsilon^{(m)} - F_i^\varepsilon(x,t)\, \nu_i, \quad x \in \Xi_\varepsilon^{(m)} \quad (m = 1, 2), \tag{50}$$

where $F_i^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}) T_k(\frac{x}{\varepsilon})\, \partial^2_{x_j x_k} v_0(x,t)$, $i = 1, \ldots, n$, and

$$(u_\varepsilon - \overline{u}_\varepsilon)\big|_{\Gamma_\varepsilon} = -\varepsilon T_k\left(\tfrac{x}{\varepsilon}\right) \partial_{x_k} v_0(x,t), \tag{51}$$

$$(u_\varepsilon - \overline{u}_\varepsilon)\big|_{t=0} = 0. \tag{52}$$

Let φ_ε be a smooth function in $\overline{\Omega}$ such that $0 \le \varphi_\varepsilon \le 1$, $\varphi_\varepsilon(x) = 1$ if $dist(x, \partial\Omega) \le \varepsilon$, and $\varphi_\varepsilon(x) = 0$ if $dist(x, \partial\Omega) \ge 2\varepsilon$. Obviously,

$$|\nabla_x \varphi_\varepsilon| \le c\, \varepsilon^{-1} \quad \text{in } \overline{\Omega}. \tag{53}$$

With the help of φ_ε we define the following functions

$$\psi_\varepsilon(x,t) = -\varepsilon \varphi_\varepsilon(x)\, T_k\left(\tfrac{x}{\varepsilon}\right)\, \partial_{x_k} v_0(x,t)$$

and

$$w_\varepsilon(x,t) = u_\varepsilon(x,t) - \overline{u}_\varepsilon(x,t) - \psi_\varepsilon(x,t), \quad (x,t) \in \overline{\Omega}_\varepsilon \times (0,T).$$

It is easy to verify that for each $t \in (0,T)$ $\text{supp}\,(\psi_\varepsilon) \subset \mathcal{U}_{2\varepsilon} = \{x \in \overline{\Omega}_\varepsilon : \ \text{dist}(x, \partial\Omega) \le 2\varepsilon\}$,

$$\max_{0 \le t \le T} \|\psi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le c\varepsilon^{\frac{3}{2}} \quad \text{and} \quad \|\psi_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \le c\varepsilon^{\frac{1}{2}}. \tag{54}$$

In addition, w_ε is a solution to the following problem

$$\begin{cases}
\partial_t w_\varepsilon - \mathcal{L}_\varepsilon(w_\varepsilon) = f_\varepsilon - f_0 + h(v_0) - h(u_\varepsilon) - \sum_{m=1}^{2} q_m \left(g_0^{(m)}(x) - \kappa_m(v_0)\right) \\
+ \varepsilon \partial_{x_i}\left(F_i^\varepsilon(x)\right) - \varepsilon T_k\left(\frac{x}{\varepsilon}\right) \partial^2_{t\,x_k} v_0 + \mathcal{L}_\varepsilon(\psi_\varepsilon) \\
+ \left(a_{ij}(\xi) + a_{ik}(\xi)\partial_{\xi_k} T_j(\xi) - |Q_0|^{-1} \hat{a}_{ij}\right)\big|_{\xi = \frac{x}{\varepsilon}}\ \partial^2_{x_i x_j} v_0 \quad \text{in } \Omega_\varepsilon \times (0,T); \\
\sigma_\varepsilon(w_\varepsilon) = -\varepsilon \kappa_m(u_\varepsilon) + \varepsilon g_\varepsilon^{(m)}(x) - \varepsilon F_i^\varepsilon(x)\, \nu_i - \sigma_\varepsilon(\psi_\varepsilon) \ \text{on } \Xi_\varepsilon^{(m)} \times (0,T) \ (m = 1, 2); \\
w_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon \times (0,T); \\
w_\varepsilon = 0 \quad \text{on } \Omega_\varepsilon \times \{t = 0\}.
\end{cases}$$

Multiplying the equation of this problem by w_ε, then integrating by parts and subtracting identities (8) for $\varphi_m = \kappa_m(v_0)\, w_\varepsilon,\ m = 1, 2$, we get

$$
\begin{aligned}
&\int_0^t \int_{\Omega_\varepsilon} w_\varepsilon \partial_t w_\varepsilon \, dx\, dt + \int_0^t \int_{\Omega_\varepsilon} a_{ij}^\varepsilon\, \partial_{x_j} w_\varepsilon\, \partial_{x_i} w_\varepsilon \, dx\, dt \\
&\quad + \int_0^T \int_{\Omega_\varepsilon} (h(u_\varepsilon) - h(v_0)) w_\varepsilon \, dx\, dt + \varepsilon \sum_{m=1}^2 \int_0^t \int_{\Xi_\varepsilon^{(m)}} \big(\kappa_m(u_\varepsilon) - \kappa_m(v_0)\big)\, w_\varepsilon \, ds_x\, dt \\
&\qquad = \int_0^t \int_{\Omega_\varepsilon} (f_\varepsilon - f_0) w_\varepsilon \, dx\, dt + \sum_{m=1}^2 \Big(\varepsilon \int_0^t \int_{\Xi_\varepsilon^{(m)}} g_\varepsilon^{(m)}\, w_\varepsilon \, ds_x\, dt \\
&\quad - q_m \int_0^t \int_{\Omega_\varepsilon} g_0^{(m)}\, w_\varepsilon \, dx\, dt \Big) - \varepsilon \sum_{m=1}^2 \int_0^t \int_{\Omega_\varepsilon} a_{ij}^\varepsilon(x)\, \partial_{\xi_j} \psi_0|_{\xi = \frac{x}{\varepsilon}} \partial_{x_i} \left(\kappa_m(v_0)\, w_\varepsilon \right) \, dx\, dt \\
&\quad + \int_0^t \int_{\Omega_\varepsilon} \big(a_{ij}(\xi) + a_{ik}(\xi) \partial_{\xi_k} T_j(\xi) - |Q_0|^{-1} \widehat{a}_{ij} \big)\, |_{\xi = \frac{x}{\varepsilon}}\ \partial^2_{x_i x_j} v_0\, w_\varepsilon \, dx\, dt \\
&\qquad - \varepsilon \int_0^t \int_{\Omega_\varepsilon} T_k(\xi)|_{\xi = \frac{x}{\varepsilon}}\, \partial^2_{t\, x_k} v_0\ w_\varepsilon \, dx\, dt \\
&\qquad + \varepsilon \int_0^t \int_{\Omega_\varepsilon} F_i^\varepsilon \partial_{x_i} w_\varepsilon \, dx\, dt - \int_0^t \int_{\Omega_\varepsilon} a_{ij}^\varepsilon\, \partial_{x_j} \psi_\varepsilon\, \partial_{x_i} w_\varepsilon \, dx\, dt. \qquad (55)
\end{aligned}
$$

Due to (1), (3) and (14) the left-hand side of (55) is estimated by the following way

$$
\begin{aligned}
&\int_0^t \int_{\Omega_\varepsilon} w_\varepsilon \partial_t w_\varepsilon \, dx\, dt + \int_0^t \int_{\Omega_\varepsilon} a_{ij}^\varepsilon\, \partial_{x_j} w_\varepsilon\, \partial_{x_i} w_\varepsilon \, dx\, dt \\
&\quad + \int_0^t \int_{\Omega_\varepsilon} (h(u_\varepsilon) - h(v_0)) w_\varepsilon \, dx\, dt + \varepsilon \sum_{m=1}^2 \int_0^t \int_{\Xi_\varepsilon^{(m)}} (\kappa_m(u_\varepsilon) - \kappa_m(v_0))\, w_\varepsilon \, ds_x\, dt \\
&\qquad \ge \frac{1}{2} \int_{\Omega_\varepsilon} w_\varepsilon^2(x,t)\, dx + c_1 \|w_\varepsilon\|^2_{L^2(0,t;H^1(\Omega_\varepsilon))} \\
&\quad - c_2 \left| \int_0^t \int_{\Omega_\varepsilon} (\varepsilon T_k \partial_{x_k} v_0 + \psi_\varepsilon) w_\varepsilon \, dx\, dt \right| - c_3 \varepsilon \left| \int_0^t \int_{\Xi_\varepsilon} \big(\varepsilon T_k\, \partial_{x_k} v_0 + \psi_\varepsilon\big) w_\varepsilon \, ds_x\, dt \right|. \qquad (56)
\end{aligned}
$$

Now estimate the summands in the right-hand side of (55). Evidently,

$$
\left| \int_{\Omega_\varepsilon \times (0,t)} (f_\varepsilon - f_0)\, w_\varepsilon \, dx\, dt \right| \le \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon \times (0,t))} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}.
$$

With the help of (8), (2) and (10) we bound the second and third terms:

$$
\begin{aligned}
&\int_0^t \left| \sum_{m=1}^2 \Big(\varepsilon \int_{\Xi_\varepsilon^{(m)}} g_\varepsilon^{(m)}\, w_\varepsilon \, ds_x - q_m \int_{\Omega_\varepsilon} g_0^{(m)}\, w_\varepsilon \, dx \Big) \right| dt \\
&\quad \le \int_0^t \sum_{m=1}^2 \Big(\varepsilon \Big| \int_{\Omega_\varepsilon} a_{ij}^\varepsilon\ \partial_{\xi_j} \psi_0(\xi) \big|_{\xi = \frac{x}{\varepsilon}}\ \partial_{x_i} (g_\varepsilon^{(m)} w_\varepsilon)\, dx \Big| \\
&\qquad + q_m \Big| \int_{\Omega_\varepsilon} g_\varepsilon^{(m)} w_\varepsilon \, dx - \int_{\Omega_\varepsilon} g_0^{(m)} w_\varepsilon \, dx \Big| \Big) dt \\
&\le c_1 \varepsilon \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))} + c_2 \sum_{m=1}^2 \|g_\varepsilon^{(m)} - g_0^{(m)}\|_{L^2(\Omega_\varepsilon \times (0,T))} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}
\end{aligned}
$$

and

$$\varepsilon \int_0^t \sum_{m=1}^2 \left| \int_{\Omega_\varepsilon} a_{ij}^\varepsilon(x)\, \partial_{\xi_j}\psi_0\big|_{\xi=\frac{x}{\varepsilon}}\, \partial_{x_i}\left(\kappa_m(v_0)w_\varepsilon\right)\, dx\right| dt$$

$$\leq \varepsilon c_3 \int_{\Omega_\varepsilon\times(0,t)} |\nabla v_0|\, |w_\varepsilon|\, dx\, dt + \varepsilon c_4 \sum_{m=1}^2 \int_{\Omega_\varepsilon\times(0,t)} |\kappa_m(v_0)|\, |\nabla w_\varepsilon|\, dx\, dt$$

$$\leq \varepsilon c_5\, \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}.$$

Thanks to (28) and the fact that the vector-functions (37) are weak solenoidal in the square $\square$, it follows from Lemma 16.4 ([1]) that

$$\left| \int_{\Omega_\varepsilon\times(0,t)} \Big(a_{ij}(\xi) + a_{ik}(\xi)\partial_{\xi_k}T_j(\xi) - \tfrac{1}{|Q_0|}\, \widehat{a}_{ij} \Big)\, |_{\xi=\frac{x}{\varepsilon}}\;\; \partial^2_{x_ix_j} v_0\, w_\varepsilon\, dx\, dt \right|$$

$$\leq \varepsilon c_6\, \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}.$$

It is easy to see that

$$\varepsilon \left| \int_0^t \int_{\Omega_\varepsilon} T_k(\xi)|_{\xi=\frac{x}{\varepsilon}}\, \partial^2_{tx_k} v_0\;\; w_\varepsilon\, dx\, dt\right| \leq \varepsilon c_7 \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}$$

and

$$\varepsilon \left| \int_{\Omega_\varepsilon\times(0,t)} F_i^\varepsilon \partial_{x_i} w_\varepsilon\, dx\, dt \right| \leq \varepsilon c_8\, \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}.$$

The last summand in (55) is estimated with the help of Lemma 1.5 ([7]) and (53):

$$\left| \int_{\Omega_\varepsilon\times(0,t)} a_{ij}^\varepsilon\, \partial_{x_j}\psi_\varepsilon\, \partial_{x_i} w_\varepsilon\, dx\, dt \right| = \left| \int_{\mathcal{U}_{2\varepsilon}\times(0,t)} a_{ij}^\varepsilon\, \partial_{x_j}\psi_\varepsilon\, \partial_{x_i} w_\varepsilon\, dx\, dt \right|$$

$$\leq c_9 \int_{\mathcal{U}_{2\varepsilon}\times(0,t)} |\nabla v_0|\, |\nabla w_\varepsilon|\, dx\, dt + \varepsilon c_{10} \int_{\mathcal{U}_{2\varepsilon}\times(0,t)} |D^2 v_0|\, |\nabla w_\varepsilon|\, dx\, dt$$

$$\leq c_9 \|v_0\|_{L^2(0,t;H^1(\mathcal{U}_{2\varepsilon}))} \|w_\varepsilon\|_{L^2(0,t;H^1(\mathcal{U}_{2\varepsilon}))} + \varepsilon c_{10} \|v_0\|_{L^2(0,t;H^2(\Omega)} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon)}$$

$$\leq c_{11} \varepsilon^{\frac{1}{2}} \|v_0\|_{L^2(0,t;H^2(\Omega))} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}. \tag{57}$$

It is remain to bound last two terms in (56). Evidently that

$$\varepsilon \left| \int_0^t \int_{\Omega_\varepsilon} T_k \partial_{x_k} v_0\, w_\varepsilon\, dx\, dt \right| \leq \varepsilon c_{12} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}$$

and

$$\left| \int_0^t \int_{\Omega_\varepsilon} \psi_\varepsilon w_\varepsilon\, dx\, dt \right| \leq \varepsilon^{\frac{3}{2}} c_{13} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}.$$

According to (8) and (10) we have

$$\varepsilon^2 \left| \int_0^t \int_{\Xi_\varepsilon} T_k\, \partial_{x_k} v_0\, w_\varepsilon\, ds_x\, dt \right|$$

$$\leq 2\varepsilon^2 \int_{\Omega_\varepsilon\times(0,t)} \left| a_{ij}^\varepsilon\;\partial_{\xi_j}\psi_0(\xi)|_{\xi=\frac{x}{\varepsilon}}\, \partial_{x_i}\big(T_k\, \partial_{x_k} v_0\, w_\varepsilon\big) \right| dx\, dt$$

$$+ c_{14}\, \varepsilon \int_{\Omega_\varepsilon\times(0,t)} \left| T_k\, \partial_{x_k} v_0\, w_\varepsilon \right| dx\, dt \;\; \leq c_{15}\varepsilon \|v_0\|_{L^2(0,t;H^2(\Omega))} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}. \tag{58}$$

Similarly we have

$$\varepsilon \left| \int_0^t \int_{\Xi_\varepsilon} \psi_\varepsilon \, w_\varepsilon \, ds_x \, dt \right| \le c_{16} \varepsilon^{\frac{1}{2}} \|v_0\|_{L^2(0,t;H^2(\Omega))} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}.$$

Thus, taking into account estimates obtained above, we deduce from (55) that

$$\frac{1}{2} \int_{\Omega_\varepsilon} w_\varepsilon^2(x,t)\, dx + c_1 \|w_\varepsilon\|^2_{L^2(0,t;H^1(\Omega_\varepsilon))} \le c_{17} \varepsilon^{\frac{1}{2}} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}$$

$$+ \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon \times (0,t))} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}$$

$$+ c_2 \sum_{m=1}^{2} \|g_\varepsilon^{(m)} - g_0^{(m)}\|_{L^2(\Omega_\varepsilon \times (0,t))} \|w_\varepsilon\|_{L^2(0,t;H^1(\Omega_\varepsilon))}.$$

We can now proceed analogously to the proof of (15) and obtain

$$\max_{0\le t\le T} \|w_\varepsilon(x,t)\|_{L^2(\Omega_\varepsilon)} + \|w_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$$
$$\le c_{18} \left(\varepsilon^{\frac{1}{2}} + \|f_\varepsilon - f_0\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \sum_{m=1}^{2} \|g_\varepsilon^{(m)} - g_0^{(m)}\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \right). \quad (59)$$

It follows from (54) and (59) that

$$\max_{0\le t\le T} \|u_\varepsilon - \overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|u_\varepsilon - \overline{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))}$$
$$\le C \left(\varepsilon^{\frac{1}{2}} + \|f_\varepsilon - f_0\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \sum_{m=1}^{2} \|g_\varepsilon^{(m)} - g_0^{(m)}\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \right). \quad (60)$$

where the constant C is independent of ε. Thus, we have proved the following result.

Theorem 3. *Let the assumption made at the beginning of this section and conditions for functions h, κ_1, κ_2 (see (3)) are satisfied. Let $f_\varepsilon \in L^2(0,T;L^2(\Omega))$ and $g_\varepsilon^{(m)} \in L^2(0,T;H_0^1(\Omega)), m = 1,2$. Then between the solution u_ε to problem (6) and the approximation function (48) the estimate (60) holds.*

6 Conclusion

An important problem for existing multiscale methods is their stability and accuracy. The proof of the error estimate between the constructed approximation and the exact solution is a general principle that has been applied to the analysis of a multiscale method efficiency (see [58]). We have proved a such estimate in Theorem 3. It follows from results proved in the paper that for applied problems in perforated domains we can use the corresponding homogenized problem, which are more simple, instead of the initial problem with the sufficient plausibility.

Now it is easy to understand how to conduct investigation of boundary-value problems in perforated domains in the case of the $p-$multiphase boundary interactions. In this case we should use solutions to special problems (as (9)) that correspond to each surface interaction and then deduce the respective integral identities, with the help of them it will be possible to perform the asymptotic analysis as before. Similarly we can make the asymptotic investigation of the initial/boundary-value problems for the reaction-diffusion semi-linear systems with the $p-$multiphase nonlinear boundary interactions in perforated domains.

It should be stressed here that we do not know asymptotic behavior for the solution to similar problem in the case when together with nonuniform Neumann or Robin conditions there are also the Dirichlet conditions on some boundaries of the halls. This question is still open.

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