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Abstract

In this paper we consider the problem of two-dimensional steady flow of a viscous incompressible fluid at low Reynolds number past a porous body of arbitrary shape whose boundary is a closed Lipschitz curve. Assuming that the flow inside the porous body is governed by the Brinkman equation, we consider indirect layer potential representations corresponding to Brinkman, Stokes and Oseen systems. We show that the boundary integral equations of the Oseen-Brinkman coupling turn out to be a regular perturbation of those of the Stokes-Brinkman coupling. This allows us to prove that the difference between the first terms of the matched asymptotic expansions of the Stokes system and of the Oseen system is of the order $|\ln Re|^{-3}$ uniformly in any compact region of $\mathbb{R}^2$. Since to us is not known whether the coupled Brinkman-Navier-Stokes two-dimensional interface problem has a solution and whether it can be approximated (if exists) by the Brinkman-Oseen transmission problem, these estimates can be considered as a first step in the complete analysis of 2D low Renolds number flow past a porous body.

Keywords: Low Reynolds number flow, Brinkman, Stokes and Oseen equations, Lipschitz domain, boundary integral equations, matched asymptotic expansions, asymptotic estimates.

AMS Subject classification: 76D, 76M.

1 Introduction

The problem of viscous incompressible flow past porous particles has been an interesting subject due to its importance in many engineering, chemical, biological, or geological applications such as the processes involving chemical reactions and modeling of polymer molecules as porous particles, the cell or enzyme immobilization process, or the flow of rivers through porous banks (see e.g. [1]). The two-dimensional viscous flow problems past slightly deformed circular and arbitrarily shaped voids have been treated in [2, 3].

The potential theory for Stokes and Brinkman operators provides efficient methods to treat boundary value problems in Lyapunov or Lipschitz domains in $\mathbb{R}^n$ (n = 2, 3) and in Hölder or Sobolev spaces (see e. g. [4]-[6]). Mitrea and Wright [7] have recently presented the corresponding analysis for the Stokes system in Lipschitz domains in $\mathbb{R}^n$ (see also [8, 9]). In [4], we studied boundary value problems that describe flows past porous particles based on the assumption of creeping flow, which implies that the corresponding Reynolds number is equal to zero. The same assumption has been also the basis of several studies concerning viscous incompressible flows past solid bodies (see e.g. [10]-[12]). Whereas the exterior Dirichlet problem for the Stokes system with prescribed velocity at infinity is uniquely solvable in three dimensions and, therefore, provides a reasonable first approximation to Navier - Stokes flows, the situation for two - dimensional problems is drastically different due to the Stokes paradox.

In two dimensions, the problem of low Reynolds number viscous incompressible flow past a stationary solid body, which is uniform at infinity has a long history (see e.g.[13]-[26]). The Stokes approximation of the Navier-Stokes equation, which neglects the effects of inertial forces, leads to the Stokes paradox, in view of which the exterior Dirichlet problem for the Stokes system in an exterior domain in $\mathbb{R}^2$, with a prescribed velocity on the boundary and given uniform behaviour at infinity, does not have any solution. On the other hand, the Oseen approximation of the Navier-Stokes equation takes into account the uniform flow at infinity and leads to the Oseen equation (see e.g. [17]). The low Reynolds number flow problem is a singular perturbation problem, in which the Stokes approximation is valid in a compact inner (Stokes) region containing the body, and the Oseen approximation in the complementary region, i. e., the exterior (Oseen) region. Kaplun [13] and Proudman and Pearson [14] employed the method of matched asymptotic expansions for the problem of low Reynolds number flow past a circular cylinder or a solid sphere. They were able to determine analytically the first two terms in a formal asymptotic expansion with respect to low Reynolds number, of the drag force exerted on the body. The method of formal matched asymptotic expansions has then been extended to solid cylinders with arbitrary smoothly bounded cross sections by Hsiao and McCamy [17] who used the stream function formulation and boundary integral equations of the first kind. Formally matched asymptotics based on boundary integral equations of the second kind can be found in ([21, 24, 26]).

In [19, 20] Hsiao and McCamy finally proved rigorous estimates for the difference between the Stokes and the Oseen expansions with respect to powers of $(\ln Re)^{-1}$. In particular they have shown that the boundary integral operators corresponding to the Oseen flow can be seen as regular perturbations of those to the Stokes flow in the compact vicinity of the obstacle. Then, in addition, they have employed the famous result by Finn and Smith [16] on the approximation of the exterior Navier - Stokes flow by the first terms of the Oseen expansion, and so, rigorously justified the Stokes expansion for exterior Navier-Stokes flows. More recently, Sazonov in [25] also obtained for the two dimensional Stokes problem the Oseen expansions and the corresponding rigorous results for Navier-Stokes solutions in appropriate Slobodetski spaces $W^{\ell,p}$, including an existence proof for the $2D$ Navier - Stokes flow for small Reynolds numbers and a C^∞ cross section.

In this paper, the $2D$ obstacle is assumed to be porous. In [5], the authors developed a matched asymptotic analysis for the two-dimensional steady flow of a viscous incompressible fluid at low Reynolds number past a porous body of arbitrary shape whose boundary is a closed Lyapunov or a Lipschitz curve. The flow region (i.e. $\mathbb{R}^2$) is divided into three distinct regions. One of them is the region inside the porous body, where the flow is modeled by the Brinkman equation. The other two overlapping regions of clear fluid are the inner (Stokes) and outer (Oseen) regions governed by the Stokes equation and the Oseen equation, respectively. The flow velocity and boundary traction fields across the fluid - porous interface are (weakly) continuous. Using indirect boundary integral representations, the problems corresponding to the inner region are reduced to uniquely solvable systems of Fredholm integral equations of the second kind in Hölder or Sobolev spaces, respectively, while the outer problems corresponding to the outer region are solved by using the singularity method. The inner problems are constituted as transmission problems based on Brinkman potentials in the porous body coupled with Stokes, respectively Oseen potentials in its exterior. The corresponding transmission conditions result in boundary integral equations of the second kind on the transmission boundary where those of the Oseen - Brinkman coupling turn out to be a regular perturbation of those of the Stokes-Brinkman coupling. This allows us to show in the same manner as Hsiao and McCamy in [19, 20] that the difference between the first terms of the matched asymptotic expansions of the Stokes system as obtained in our work [5] and of the Oseen system is of the order $|\ln Re|^{-3}$ uniformly in any compact domain of $\mathbb{R}^2$. Since to us is not known whether the coupled Brinkman Navier-Stokes two dimensional interface problem has a solution and whether it can be approximated (if exists) by the Brinkman-Oseen transmission problem, these estimates can be considered as a first step in the complete analysis of 2D low Reynolds number flow past a porous body.

1.1 Preliminaries

Next, we collect the basic Sobolev spaces that are used in this paper.

The porous obstacle is described by $D_0 \subset \mathbb{R}^2$ which is a bounded Lipschitz domain whose boundary is denoted by Γ.

- We shall use the following spaces, which are defined on the domain D_0[1]:

$$H^1_{\text{div}}(D_0) := \big\{\mathbf{w} \in H^1(D_0) : \nabla\cdot\mathbf{w} = 0 \ \text{ in } \ D_0\big\},$$

$$\widetilde{H}^{-1}(D_0) := \{f \in H^{-1}(\mathbb{R}^2) : \text{supp}\, f \subseteq \overline{D}_0\},$$

$$H^1(D_0, \mathrm{P}_{\mathrm{St}}) = \big\{(\mathbf{u}, p) \in H^1_{\text{div}}(D_0) \times L^2(D_0) : \mathrm{P}_{\mathrm{St}}(\mathbf{u}, p) = \mathbf{0} \text{ in } D_0\big\},$$

where the operator $\mathrm{P}_{\mathrm{St}} : \mathrm{H}^1_{\text{div}}(\mathrm{D}_0) \times \mathrm{L}^2(\mathrm{D}_0) \to \mathbb{R}^2$ is given by $\mathrm{P}_{\mathrm{St}}(\mathbf{u}, p) = -\nabla^2\mathbf{u} + \nabla p$.

The boundary traction or conormal derivative operator $\mathbf{t} : H^1(D_0, \mathrm{P}_{\mathrm{St}}) \to H^{-1/2}(\Gamma)$ is given by

$$\langle \mathbf{t}(\mathbf{u}, p), \phi\rangle := 2\int_{D_0} E_{jk}(\mathbf{u})E_{jk}(\mathcal{Z}\phi)d\mathbf{x} - \int_{D_0} p\nabla\cdot\mathcal{Z}\phi d\mathbf{x} \ \text{ for all } \ \phi \in H^{1/2}(\Gamma), \tag{1.1}$$

where $E_{jk}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)$, and $E_{jk}(\mathcal{Z}\phi)$, defined similarly. For given $(\mathbf{u}, p) \in H^1(D_0, \mathrm{P}_{\mathrm{St}})$, (1.1) defines a bounded linear functional $\mathbf{t}(\mathbf{u}, p) \in H^{-1/2}(\Gamma)$, hence, it is also

[1] We use the notation $H^s(D_0)$ instead of $\big(H^s(D_0)\big)^2$, and similarly for all other product Sobolev spaces.

a linear bounded operator. In addition, for every $(\mathbf{u}, p) \in H^1(D_0, \mathrm{P_{St}})$ and $\mathbf{w} \in H^1(D_0)$, the following Green formula holds:

$$2\int_{D_0} E_{jk}(\mathbf{u})E_{jk}(\mathbf{w})d\mathbf{x} = \int_{D_0} p\nabla\cdot\mathbf{w}d\mathbf{x} + \langle \mathbf{t}(\mathbf{u}, p), \gamma_0\mathbf{w}\rangle,$$

where the trace mapping $\gamma_0 : H^1(D_0) \to H^{1/2}(\Gamma)$ is an extension of $C^0(\overline{D}_0) \ni v \longmapsto v|_\Gamma$, and and $\mathcal{Z} : H^{1/2}(\Gamma) \to H^1(D_0)$ is a right inverse of γ_0.

• In the exterior domain D_e we define:

$$H^1_{\mathrm{div,loc}}(D_e) = \{\mathbf{w} \in H^1_{\mathrm{loc}}(D_e) : \nabla\cdot\mathbf{w} = 0 \ \text{ in } \ D_e\},$$

$$H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}}) = \{(\mathbf{u}, p) \in H^1_{\mathrm{div,loc}}(D_e) \times L^2_{\mathrm{loc}}(D_e) : \mathrm{P_{St}}(\mathbf{u}, p) = \mathbf{0} \text{ in } D_e\}.$$

The linear bounded operator $\mathbf{t} : H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}}) \to H^{-1/2}(\Gamma)$ is now given by

$$\langle \mathbf{t}(\mathbf{u}, p), \phi\rangle := -2\int_{D_e} E_{jk}(\mathbf{u})E_{jk}(\mathcal{Z}_c\phi)d\mathbf{x} - \int_{D_e} p\nabla\cdot(\mathcal{Z}_c\phi)d\mathbf{x} \ \text{ for all } \ \phi \in H^{1/2}(\Gamma). \quad (1.2)$$

In addition, for every $(\mathbf{u}, p) \in H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}})$ and $\mathbf{w} \in H^1_{\mathrm{comp}}(\overline{D}_e)$ one has the following Green's formula:

$$2\int_{D_e} E_{jk}(\mathbf{u})E_{jk}(\mathbf{w})d\mathbf{x} = \int_{D_e} p\nabla\cdot\mathbf{w}d\mathbf{x} - \langle \mathbf{t}(\mathbf{u}, p), \gamma_{c0}\mathbf{w}\rangle,$$

where $\gamma_{c0} : H^1_{\mathrm{loc}}(D_e) \to H^{1/2}(\Gamma)$ is the trace operator for the exterior domain D_e, and $\mathcal{Z}_c : H^{1/2}(\Gamma) \to H^1_{\mathrm{comp}}(\overline{D}_e)$ is a right inverse of γ_{c0} (for more details see e.g. [7]).

Hereafter we use the notation $\mathbf{t}(\mathbf{u})$ instead of $\mathbf{t}(\mathbf{u}, p)$, but we understand that the boundary traction field $\mathbf{t}(\mathbf{u})$ depends on both $\mathbf{u}$ and p. Also, note that the above concepts can be extended to the Brinkman equation, by using the spaces $H^1(D_0, \mathrm{P_{Br}})$ and $H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}})$ instead of $H^1(D_0, \mathrm{P_{Br}})$ and $H^1_{\mathrm{loc}}(D_e, \mathrm{P_{St}})$, where $\mathrm{P_{Br}}(\mathbf{u}, p) = \nabla^2\mathbf{u} - \chi^2\mathbf{u} - \nabla p$.

• The Sobolev trace spaces $H^r(\Gamma)$ for $|r| \le 1$, are given by

$$H^r(\Gamma) = \left\{f|_\Gamma : f \in H^{\frac{1}{2}+r}_{\mathrm{loc}}(\mathbb{R}^2)\right\} \ \text{ if } \ 0 < r < 1; \ \ H^0(\Gamma) = L^2(\Gamma) \ \text{ for } \ r = 0, \ \text{ and}$$

$$H^1(\Gamma) = \{f \in L^2(\Gamma) : |\nabla_\tau f| \in L^2(\Gamma)\} \ \text{ for } \ r = 1;$$

here ∇_τ is the tangential derivative on Γ; for $r \in [0, 1]$, the space $H^{-r}(\Gamma)$ is the dual space of $H^r(\Gamma)$ with respect to the $\langle\cdot,\cdot\rangle_{L^2(\Gamma)}$ - duality.

2 Formulation of the problem

We consider the problem of determining the steady two-dimensional low Reynolds number flow of a viscous incompressible fluid of infinite expanse past a stationary porous body with permeability k_0. The porous body occupies the bounded, simply connected Lipschitz domain $D_0 \subset \mathbb{R}^2$, i. e., its boundary is a Lipschitz curve $\Gamma \in C^{0,1}$. Let $D_e = \mathbb{R}^2 \setminus \overline{D}_0$ be the exterior domain with boundary Γ. We assume that far away from the porous body, the flow is uniform with the velocity field U_∞ in the x_1-direction. Moreover, we assume that the resulting flow in the exterior domain D_e, due to the presence of the porous body, is a steady low Reynolds number flow, with the non-dimensional velocity and pressure fields $\mathbf{v}^e$ and p^e, respectively, and that the flow inside the porous body with the velocity and pressure fields $\mathbf{v}^i$ and p^i is governed by the continuity and Brinkman equations. The non - dimensionalization is done using $\mathbf{x} = \mathbf{x}^{'}/a, \mathbf{v} = \mathbf{v}^{'}/U_\infty, p = p^{'}/\frac{\mu U_\infty}{a}$ where a is a characteristic length related to the size of the porous body D_0, the dynamic viscosity of the fluid is μ, and the prime superscript refers to dimensional variables. In addition, the velocity and boundary traction fields are supposed to be continuous across Γ, and the perturbation due to the presence of the porous body vanishes at infinity. In view of these assumptions, the flow fields have to satisfy the continuity and Navier-Stokes equations in the exterior domain D_e:

$$\nabla \cdot \mathbf{v}^e = 0 \ \text{ in } D_e, \tag{2.1}$$

$$-\nabla p^e + \nabla^2 \mathbf{v}^e - Re\ \mathbf{v}^e \cdot \nabla \mathbf{v}^e = \mathbf{0} \ \text{ in } D_e; \tag{2.2}$$

the continuity and Brinkman equations in the bounded domain D_0:

$$\nabla \cdot \mathbf{v}^i = 0 \ \text{ in } D_0, \tag{2.3}$$

$$-\nabla p^i + (\nabla^2 - \chi^2)\mathbf{v}^i = \mathbf{0} \ \text{ in } D_0; \tag{2.4}$$

the transmission conditions:

$$\mathbf{v}^i = \mathbf{v}^e \quad \mathbf{t}(\mathbf{v}^i) = \mathbf{t}(\mathbf{v}^e) \ \text{ on } \Gamma, \tag{2.5}$$

and the conditions at infinity:

$$\mathbf{v}^e(\mathbf{x}) \to \mathbf{i}, \quad p^e(\mathbf{x}) \to 0 \ \text{ as } |\mathbf{x}| \to \infty. \tag{2.6}$$

Here $Re = \rho a U_\infty/\mu$ is the Reynolds number, ρ is the constant fluid density and χ is the positive Brinkman constant given by $\chi = a/\sqrt{k_0}$. In addition, $\mathbf{i}$ is the unit vector corresponding to the x_1-axis, and $\mathbf{t}(\mathbf{v}^e)$ and $\mathbf{t}(\mathbf{v}^i)$ are the boundary traction fields corresponding to $(\mathbf{v}^e, p^e)$ and $(\mathbf{v}^i, p^i)$, respectively (see (1.1) and (1.2)). For classical solutions $\mathbf{v} \in C^1(\overline{D}), p \in C^0(\overline{D})$, the traction field $\mathbf{t}(\mathbf{v}) = (t_1(\mathbf{v}), t_2(\mathbf{v}))$ at the boundary Γ can be written as

$$t_j(\mathbf{v})(\mathbf{x}) = T_{j\ell}(\mathbf{v})(\mathbf{x}) n_\ell(\mathbf{x})\,, \tag{2.7}$$

where

$$T_{j\ell}(\mathbf{v})(\mathbf{x}) = -p(\mathbf{x})\delta_{j\ell} + \left(\frac{\partial v_j(\mathbf{x})}{\partial x_\ell} + \frac{\partial v_\ell(\mathbf{x})}{\partial x_j}\right), \quad j, \ell = 1, 2, \tag{2.8}$$

are the components of the stress tensor field $\mathbf{T}(\mathbf{v})$ corresponding to the fields $\mathbf{v}$ and p. By $\mathbf{n}$ we denote the outward unit normal to Γ defined at almost all points $\mathbf{x} \in \Gamma$. Hereafter we use Einstein's repeated-index summation convention.

For the weak solutions $(\mathbf{u}, p) \in H^1(D_0, P_{st})$ as considered here, the boundary traction is defined by (1.1), which is an extension of (2.7), (2.8) from classical to weak solutions.

2.1 The Stokes and Oseen approximation

As mentioned in the introduction, it may be noted that the Stokes approximation of the Navier-Stokes equation given by

$$\nabla \cdot \mathbf{v}_0 = 0 \ \text{ in } D_e, \tag{2.9}$$

$$-\nabla p_0 + \nabla^2 \mathbf{v}_0 = \mathbf{0} \ \text{ in } D_e, \tag{2.10}$$

with a given boundary datum $\mathbf{f}$ and the condition at infinity (2.6) has not any solution (Hsiao [20]). However, if the far field condition is replaced by

$$\mathbf{v}_0(\mathbf{x}) \to \mathbf{A} \ln |\mathbf{x}| + O(1), \quad p_0(\mathbf{x}) \to 0 \ \text{ as } |\mathbf{x}| \to \infty. \tag{2.11}$$

for an arbitrarily given constant vector $\mathbf{A}$, one can obtain the solution to the corresponding problem, which may be called the Stokes problem. Hsiao and MacCamy [17] used a matching procedure in order to determine the constant vector $\mathbf{A}$ that is related to the force $\mathbf{F}$ on the solid cylinder. Since the above linearization of the Navier-Stokes equation is not a uniformly valid approximation, Hsiao [20] introduced the linearization $\mathbf{v}^e = \mathbf{i} + \mathbf{Q}$ that reduces the Navier-Stokes equations (2.1) and (2.2) to

$$\nabla \cdot \mathbf{Q} = 0 \ \text{ in } D_e, \tag{2.12}$$

$$-\nabla P + \nabla^2 \mathbf{Q} - Re(\mathbf{i} \cdot \nabla)\mathbf{Q} = Re(\mathbf{Q} \cdot \nabla)\mathbf{Q} \quad \text{in} \quad D_e. \tag{2.13}$$

Neglecting the non-linear inertia term, one obtains the Oseen equations

$$\nabla \cdot \mathbf{Q}_0 = 0 \ \text{ in } D_e, \tag{2.14}$$

$$-\nabla P_0 + \nabla^2 \mathbf{Q}_0 - Re(\mathbf{i} \cdot \nabla)\mathbf{Q}_0 = \mathbf{0} \ \text{ in } D_e, \tag{2.15}$$

and the corresponding boundary conditions being

$$\mathbf{Q}_0 \to -\mathbf{i} + \mathbf{f} \quad \text{on } \Gamma, \quad \text{and} \quad \mathbf{Q}_0 \to \mathbf{0}, \quad P_0 \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{2.16}$$

The above problem is referred to as the Oseen problem. Hsiao [20] constructed asymptotic expansions using the fundamental solution to the Oseen equation and the stream function approach. Then, Hsiao and MacCamy [19] have shown that the formal expansions developed by them based on the matching procedure for the Navier-Stokes equations are identical to the ones given by the corresponding expansions of the integral equations. Here the boundary integral equations for the Oseen equation turn out to be regular perturbations of those for the Stokes equation if Re tends to zero. They also have presented rigorous estimates corresponding to these asymptotic expansions.

It seems to us that there is no corresponding result for the case of the Navier-Stokes together with the Brinkman system. In order to make an attempt, as a first step, we have developed a matched asymptotic analysis for this problem (see [5]). The flow region is divided into three distinct but overlapping regions. One of them is the region inside the porous body, where the flow is described by the Brinkman model, and the other two regions of clear fluid are the inner (Stokes) and outer (Oseen) regions governed by the Stokes equation and the Oseen equation, respectively. By using some indirect boundary integral representations, the inner problems corresponding to the inner region have been reduced to uniquely solvable systems of second kind Fredholm integral equations in some Hölder or Sobolev spaces, while the outer problems corresponding to the outer region are solved by using the method of fundamental solutions. In fact, we have obtained the following asymptotic expansions with respect to small Reynolds number Re:

$$\begin{aligned} \mathbf{v}^i &= 4\pi(\ln Re)^{-1}\mathbf{v}^{(0)} + 4\pi(\ln Re)^{-2}\mathbf{v}^{(1)} + O((\ln Re)^{-3}), \\ p^i &= 4\pi(\ln Re)^{-1}p^{(0)} + 4\pi(\ln Re)^{-2}p^{(1)} + O((\ln Re)^{-3}), \end{aligned} \tag{2.17}$$

and

$$\mathbf{v}^e = 4\pi(\ln Re)^{-1}\mathbf{v}^0 + 4\pi(\ln Re)^{-2}\mathbf{v}^1 + O((\ln Re)^{-3}),$$
$$p^e = 4\pi(\ln Re)^{-1}p^0 + 4\pi(\ln Re)^{-2}p^1 + O((\ln Re)^{-3}), \tag{2.18}$$

respectively. Also, the corresponding asymptotic expansion of the force is

$$\mathbf{F} = 4\pi(\ln Re)^{-1}\mathbf{F}^0 + 4\pi(\ln Re)^{-2}\mathbf{F}^1 + O((\ln Re)^{-3}). \tag{2.19}$$

The constant vectors $\mathbf{F^0}$ and $\mathbf{F^1}$ have been obtained by using the matching procedure, i. e., by matching the solution of the zero and first order Stokes problems in the inner region with the solution of the corresponding Oseen problems in the outer region.

3 Boundary integral formulation for the Stokes-Brinkman coupled system

Next, we refer to the transmission problem for the Stokes-Brinkman coupled system. Let us consider the following layer potential representations for the Brinkman system

$$\mathbf{v}^B(\mathbf{x}) = \mathbf{W}_{\chi^2}(\mathbf{x}, \mathbf{\Phi}) + \mathbf{V}_{\chi^2}(\mathbf{x}, \mathbf{h}), \tag{3.1}$$

$$p^B(\mathbf{x}) = P^d_{\chi^2}(\mathbf{x}, \mathbf{\Phi}) + P^s_{\chi^2}(\mathbf{x}, \mathbf{h}) \ \text{ for } \ \mathbf{x} \in D_0, \tag{3.2}$$

as well as for the Stokes system in the inner region

$$\mathbf{v}^{ST}(\mathbf{x}) = \mathbf{W}(\mathbf{x}, \mathbf{\Phi}) + \mathbf{V}(\mathbf{x}, \mathbf{h}) + \frac{1}{4\pi}\boldsymbol{\omega}, \tag{3.3}$$

$$p^{ST}(\mathbf{x}) = P^d(\mathbf{x}, \mathbf{\Phi}) + P^s(\mathbf{x}, \mathbf{h}) \ \text{ for } \ \mathbf{x} \in D_e, \tag{3.4}$$

where $(\mathbf{V}_{\chi^2}(\cdot, \mathbf{h})), P^s_{\chi^2}(\cdot, \mathbf{h}))$ and $(\mathbf{W}_{\chi^2}(\cdot, \mathbf{\Phi}), P^d_{\chi^2}(\cdot, \mathbf{\Phi}))$ are single- and double-layer potentials together with their corresponding pressure potentials for the Brinkman system. These potentials have the densities $\mathbf{h}$ and $\mathbf{\Phi}$, respectively. Similarly, the layer potentials for the Stokes system are denoted by $(\mathbf{V}(\cdot, \mathbf{h}), P^s(\cdot, \mathbf{h}))$ and $(\mathbf{W}(\cdot, \mathbf{\Phi}), P^d(\cdot, \mathbf{\Phi}))$. In addition, $(\mathbf{\Phi}, \mathbf{h})$ are the unknown densities and $\boldsymbol{\omega}$ the unknown vector that will be obtained in the Sobolev spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ and $\mathbb{R}^2$, respectively. It may be noted that the above boundary integral representations are equivalent to those of the corresponding ones used in [5] for the zero and first order terms of the Brinkman and Stokes expansions. Note that the additional constant $\boldsymbol{\omega}$ is added here in order to control the far-field behaviour of the Stokes velocity field, and replaces a similar quantity used in [5]. This constant will be obtained from the matching conditions.

The velocity and stress satisfy the transmission conditions

$$\gamma_{c0}\mathbf{v}^{ST} = \gamma_0\mathbf{v}^B \ \text{ and } \ \mathbf{t}(\mathbf{v}^{ST}) = \mathbf{t}(\mathbf{v}^B) \ \text{ on } \ \Gamma, \tag{3.5}$$

where γ_{c0} is the exterior trace operator, γ_0 is the inner trace operator and $\mathbf{t}(\mathbf{v}^{ST})$ and $\mathbf{t}(\mathbf{v}^B)$ are defined as in (1.1) and (1.2).

By applying (3.5) on (3.1) - (3.4), we obtain the following system of integral equations of the second kind for the unknowns $\mathbf{\Phi} \in H^{1/2}(\Gamma), \mathbf{h} \in H^{-1/2}(\Gamma)$ and $\boldsymbol{\omega} \in \mathbb{R}^2$ on Γ:

$$\big(\mathbf{I} - \mathbf{K}_{\chi^2,0}\big)\mathbf{\Phi} - \mathcal{V}_{\chi^2,0}\mathbf{h} + \frac{1}{4\pi}\boldsymbol{\omega} = \mathbf{0}, \tag{3.6}$$

$$\left(\mathbf{I}+\mathbf{K}^*_{\chi^2,0}\right)\mathbf{h}-\mathbf{D}_{\chi^2,0}\mathbf{\Phi}=\mathbf{0}, \tag{3.7}$$

appended by the additional condition

$$\int_\Gamma \mathbf{h}\; d\Gamma=\mathbf{A}, \tag{3.8}$$

where $\mathbf{I}$ is the identity operator, and the constant vector $\mathbf{A}$ expresses the logarithmic behaviour of the Stokes velocity field at infinity (see the condition (2.11)) and will be a given quantity later on. Also, $\mathcal{V}_{\chi^2,0}$, $\mathbf{K}_{\chi^2,0}$, $\mathbf{K}^*_{\chi^2,0}$ and $\mathbf{D}_{\chi^2,0}$ are the complementary single-layer, double-layer, adjoint of the double-layer boundary integral operators and the complementary hypersingular operator for the Stokes-Brinkman coupled system. In fact, if $\mathbf{A}=\mathbf{0}$ is given then the behaviour of $\mathbf{v}^{ST}(\infty)=\frac{1}{4\pi}\boldsymbol{\omega}$ is part of the solution and can not also be described. This explains the Stokes paradox.

If we now consider the following expressions

$$\mathbf{\Phi}=\mathbf{\Phi}_1(\ln Re)^{-1}+\mathbf{\Phi}_2(\ln Re)^{-2}, \quad \mathbf{h}=\mathbf{h}_1(\ln Re)^{-1}+\mathbf{h}_2(\ln Re)^{-2}$$

$$\mathbf{A}=\mathbf{A}_1(\ln Re)^{-1}+\mathbf{A}_2(\ln Re)^{-2}, \quad \boldsymbol{\omega}=\boldsymbol{\omega}_1(\ln Re)^{-1}+\boldsymbol{\omega}_2(\ln Re)^{-2},$$

which correspond to the first two terms in (2.17) and (2.18), then we can use similar arguments to those in [5] to show that the corresponding system of equations (3.6) and (3.7) has a unique solution $(\mathbf{\Phi}^\ell,\mathbf{h}^\ell,\boldsymbol{\omega}^\ell)\in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)\times\mathbb{R}^2$, for $\ell=1,2$. Consequently, we get the following existence and uniqueness result.

Theorem 3.1 *If D_0 is a bounded Lipschitz domain in $\mathbb{R}^2$, then the system of equations given in* (3.6) – (3.8) *expressed in the form*

$$\mathfrak{A}\begin{pmatrix}\mathbf{\Phi}\\ \mathbf{h}\\ \boldsymbol{\omega}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{A}\end{pmatrix} \tag{3.9}$$

where

$$\mathfrak{A}=\begin{pmatrix}\mathbf{I}-\mathbf{K}_{\chi^2,0} & -\mathcal{V}_{\chi^2,0} & \frac{1}{4\pi}\mathbb{I}\\ -\mathbf{D}_{\chi^2,0} & \mathbf{I}+\mathbf{K}^*_{\chi^2,0} & \mathbf{0}\\ \mathbf{0} & \int_\Gamma \cdot d\Gamma & \mathbf{0}\end{pmatrix}, \tag{3.10}$$

has a unique solution $(\mathbf{\Phi},\mathbf{h},\boldsymbol{\omega})\in H^{\frac{1}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)\times\mathbb{R}^2$ if $\Gamma\in C^{0,1}$.

Proof. The proof of this theorem is based on the compactness of the four complementary layer potential operators in (3.10) which was proved in [5, Thoerem 5.2], as well as on potential theory for Stokes and Brinkman operators. Moreover, for any eigensolution we have $\int_\Gamma \mathbf{h}^0 d\Gamma=0$ and therefore [5, Thoerem 4.1] gives that the kernel of $\mathfrak{A}$ is trivial and hence Fredholm's alternative applies. □

3.1 The matching procedure

In order to determine the constant $\mathbf{A}$ that gives meaningful results, we follow the arguments in Hsiao and MacCamy [17] and Hsiao [20]. First, we briefly review the matching procedure using the Stokes and Oseen expansions. The formal expansion in the inner region, called the Stokes expansion is

$$\mathbf{v}^{ST} \sim \sum_{k=1}^{\infty} \mathbf{v}^{ST}(\mathbf{x}, \mathbf{A}_k)(\ln Re)^{-k}, \tag{3.11}$$

where $\mathbf{v}^{ST}(\mathbf{x}, \mathbf{A}_k)$ are the Stokes velocity fields, that correspond to the constants $\mathbf{A} = \mathbf{A}_k$. Since the single-layer potential for the Stokes system has the following behaviour at infinity,

$$\mathbf{Vh}(\mathbf{x}) \sim \frac{1}{4\pi}\mathbf{A} \ln \frac{1}{|\mathbf{x}|} + \frac{1}{4\pi} \frac{\mathbf{A} \cdot \mathbf{x}}{|\mathbf{x}|} \frac{\mathbf{x}}{|\mathbf{x}|} + O(|\mathbf{x}|^{-1}) \quad \text{as} \quad \mathbf{x} \to \infty, \tag{3.12}$$

which in terms of the outer variables $\mathbf{x} = \xi/Re$ takes the form

$$\mathbf{Vh}(\xi/Re) \sim \frac{1}{4\pi}\mathbf{A} \ln Re + \frac{1}{4\pi}\mathbf{A} \ln \frac{1}{|\boldsymbol{\xi}|} + \frac{1}{4\pi} \frac{\mathbf{A} \cdot \boldsymbol{\xi}}{|\boldsymbol{\xi}|} \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + O(Re|\boldsymbol{\xi}|), \tag{3.13}$$

we get

$$\mathbf{v}^{ST} \sim \frac{1}{4\pi}\mathbf{A}_1 + (\ln Re)^{-1} \frac{1}{4\pi}\Big\{\mathbf{A}_1 \ln \frac{1}{|\boldsymbol{\xi}|} + \frac{1}{4\pi} \frac{\mathbf{A} \cdot \boldsymbol{\xi}}{|\boldsymbol{\xi}|} \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + \boldsymbol{\omega}_1 + \mathbf{A}_2\Big\} + O\big((\ln Re)^{-2}\big), \tag{3.14}$$

as $Re \to 0$, with $\boldsymbol{\xi}$ fixed. On the other hand, in terms of the outer variables one has

$$\mathbf{Q}(\boldsymbol{\xi}) := \mathbf{v}^O(\boldsymbol{\xi}/Re) - \mathbf{i}, P(\boldsymbol{\xi}) = p^O(\boldsymbol{\xi}/Re)/Re, \tag{3.15}$$

and neglecting the term $(\mathbf{Q} \cdot \nabla_\xi)\mathbf{Q}$, the corresponding problem for the Navier - Stokes equations reduces to the following Oseen problem

$$\begin{array}{lll} \Delta_\xi \mathbf{Q} - \nabla_\xi P = \frac{\partial}{\partial \xi_1}\mathbf{Q} & \text{and} \;\; \nabla_\xi \cdot \mathbf{Q} = 0, & \text{in} \;\; D_e^\xi, \\ \mathbf{Q} = -\mathbf{i} \;\; \text{for} \;\; \boldsymbol{\xi} \to \mathbf{0} & \text{and} \;\; \mathbf{Q}(\infty) = \mathbf{0}, & \end{array} \tag{3.16}$$

where $D_e^\xi := \{\boldsymbol{\xi} \in \mathbb{R}^2 \wedge \boldsymbol{\xi}/Re \in D_e\}$. Note that the region D_e^ξ reduces to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ as $Re \to 0$. The Oseenlet located at the origin and acting in the x_1 - direction,

$$\begin{array}{l} \mathbf{Q}^*(\boldsymbol{\xi}) = -2e^{\xi_1/2}K_0\big(\frac{1}{2}|\boldsymbol{\xi}|\big)\mathbf{i} + 2\nabla_\xi\big\{e^{\xi_1/2}K_0\big(\frac{1}{2}|\boldsymbol{\xi}|\big) + \ln(|\boldsymbol{\xi}|)\big\}, \\ P^*(\boldsymbol{\xi}) = -2\xi_1|\boldsymbol{\xi}|^{-2}, \end{array} \tag{3.17}$$

solves the above Oseen problem. Now, let us consider the outer expansion corresponding to the Oseen flow

$$\begin{array}{l} \mathbf{v}^O(\boldsymbol{\xi}/Re) \sim \mathbf{i} + \sum_{k=1}^{\infty} \mathbf{Q}_k(\boldsymbol{\xi})(\ln Re)^{-k}, \\ P^O(\boldsymbol{\xi}) \sim \sum_{k=1}^{\infty} P_k(\boldsymbol{\xi})(\ln Re)^{-k}. \end{array} \tag{3.18}$$

These expansions should satisfy the Oseen equation everywhere except at $\boldsymbol{\xi} = 0$ and vanish at infinity. Thus, their terms will be solutions for the following sequence of problems:

$$\begin{array}{l} \Delta_\xi \mathbf{Q}_k - \nabla_\xi P_k = \frac{\partial}{\partial \xi_1}\mathbf{Q}_k + \mathbf{R}_k \quad \text{for} \quad \boldsymbol{\xi} \neq 0, \\ \nabla_\xi \cdot \mathbf{Q}_k = \mathbf{0}, \\ \mathbf{Q}_k \to \mathbf{0}, \;\; P_k \to 0 \quad \text{for} \quad |\boldsymbol{\xi}| \to \infty, \\ \mathbf{R}_1 = \mathbf{0}, \;\; \mathbf{R}_k = \sum_{m=1}^{k-1} \mathbf{Q}_m \cdot \nabla_\xi \mathbf{Q}_{k-m}, \quad \text{for} \;\; k \geq 2. \end{array} \tag{3.19}$$

Since none of the pairs $(\mathbf{Q}_n, P_n)$ is unique, we choose $\mathbf{Q}_1(\boldsymbol{\xi}) = a_1\mathbf{Q}_1^*(\boldsymbol{\xi})$, where a_1 is an appropriate constant that will be obtained from the matching procedure. Then, we have

$$\mathbf{Q}_1(\boldsymbol{\xi}) = a_1\big\{\mathbf{i}[\ln|\boldsymbol{\xi}| + c_E - \ln 4] - \tfrac{\xi_1}{|\boldsymbol{\xi}|}\tfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\big\} \quad \text{as} \quad \boldsymbol{\xi} \to \mathbf{0}, \tag{3.20}$$

and hence, the Oseen velocity behaves like

$$\mathbf{v}^O(\boldsymbol{\xi}) = \mathbf{i} + a_1\big\{\mathbf{i}[\ln|\boldsymbol{\xi}| + c_E - \ln 4]\frac{\xi_1}{|\boldsymbol{\xi}|}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\big\}(\ln Re)^{-1} + O\big((\ln Re)^{-2}\big) + O(|\boldsymbol{\xi}|\ln|\boldsymbol{\xi}|), \tag{3.21}$$

where $c_E = 0,5772\dots$ is Euler's constant. Now, using the matching principle for (3.14) and (3.21), we get

$$\begin{aligned}&\tfrac{1}{4\pi}\mathbf{A}_1 + \tfrac{1}{4\pi}(\ln Re)^{-1}\big\{\mathbf{A}_1\ln\tfrac{1}{|\boldsymbol{\xi}|} + \tfrac{\mathbf{A}_1\cdot\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\tfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + \boldsymbol{\omega}_1 + \mathbf{A}_2\big\} + O\big((\ln Re)^{-2}\big)\\ &\sim \mathbf{i} + a_1(\ln Re)^{-1}\big\{\mathbf{i}[\ln\tfrac{1}{|\boldsymbol{\xi}|} + c_E - \ln 4] - \tfrac{\boldsymbol{\xi}_1}{|\boldsymbol{\xi}|}\tfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\big\} + O(|\boldsymbol{\xi}|\ln|\boldsymbol{\xi}|).\end{aligned} \tag{3.22}$$

Equating the coefficients of like powers of $\ln Re$, we obtain

$$\begin{aligned}&\mathbf{A}_1 = 4\pi\mathbf{i},\\ &\mathbf{i}\ln\tfrac{1}{|\boldsymbol{\xi}|} + \tfrac{\boldsymbol{\xi}_1}{|\boldsymbol{\xi}|}\tfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + \boldsymbol{\omega}_1 + \mathbf{A}_2 = a_1\big\{\mathbf{i}[-\ln\tfrac{1}{|\boldsymbol{\xi}|} + c_E - \ln 4] - \tfrac{\boldsymbol{\xi}_1}{|\boldsymbol{\xi}|}\tfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\big\}.\end{aligned} \tag{3.23}$$

and hence

$$\mathbf{A}_1 = 4\pi\mathbf{i}, \quad a_1 = -1, \quad \mathbf{A}_2 = (\ln 4 - c_E)4\pi\mathbf{i} - \boldsymbol{\omega}_1. \tag{3.24}$$

Note that the constants $\mathbf{A}_1$ and $\mathbf{A}_2$ have similar expressions to those obtained in [5], [17], [20]. Consequently, for $k = 1$, we have

$$\begin{aligned}&\boldsymbol{\phi}_1 - \mathbf{K}_{\chi^2,0}\boldsymbol{\phi}_1 - \mathcal{V}_{\chi^2,0}\mathbf{h}_1 + \boldsymbol{\omega}_1 = \mathbf{0},\\ &\mathbf{h}_1 + \mathbf{K}^*_{\chi^2,0}\mathbf{h}_1 - \mathbf{D}_{\chi^2,0}\boldsymbol{\phi}_1 = \mathbf{0},\\ &\textstyle\int_\Gamma \mathbf{h}_1 d\Gamma = \mathbf{A}_1 = 4\pi\mathbf{i},\end{aligned} \tag{3.25}$$

and, for $k = 2$:

$$\begin{aligned}&\boldsymbol{\phi}_2 - \mathbf{K}_{\chi^2,0}\boldsymbol{\phi}_2 - \mathcal{V}_{\chi^2,0}\mathbf{h}_2 + \boldsymbol{\omega}_2 = \mathbf{0},\\ &\mathbf{h}_2 + \mathbf{K}^*_{\chi^2,0}\mathbf{h}_2 - \mathbf{D}_{\chi^2,0}\boldsymbol{\phi}_2 = \mathbf{0},\\ &\textstyle\int_\Gamma \mathbf{h}_2 d\Gamma = \mathbf{A}_2 = 4\pi(\ln 4 - c_E)\mathbf{i} - \boldsymbol{\omega}_1.\end{aligned} \tag{3.26}$$

In view of Theorem 3.1, each of the above systems of equations has a unique solution in the space $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times \mathbb{R}^2$.

4 Oseen-Brinkman coupled system

Next, we consider the transmission problem for the Oseen-Brinkman coupled system near the porous body. There the Oseen and continuity equations are given by

$$\Delta_\xi\mathbf{v}^O - \nabla_\xi P^O = \tfrac{\partial}{\partial\xi_1}\mathbf{v}^O \quad \text{and} \quad \nabla_\xi\cdot\mathbf{v}^O = 0 \quad \text{in} \quad D_e^\xi. \tag{4.1}$$

We have to add the transmission conditions

$$\mathbf{v}^O = -\mathbf{i} + \mathbf{v}^i \quad \text{and} \quad \mathbf{t}(\mathbf{v}^O) = \mathbf{t}(\mathbf{v}^i) \quad \text{on} \quad \Gamma. \tag{4.2}$$

Now, in order to define the corresponding boundary layer potentials, let us consider the fundamental solution of the Oseen system, also called the Oseenlet [25]

$$\begin{aligned}
&G^O_{1k}(Re;\mathbf{z}) = \tfrac{1}{2\pi}\Bigg[-e^{Re\frac{z_1}{2}}K_0(Re\tfrac{|\mathbf{z}|}{2})\delta_{1k} + \tfrac{1}{Re}\tfrac{\partial}{\partial z_k}\Bigg(\ln|\mathbf{z}| + e^{Re\frac{z_1}{2}}K_0(Re\tfrac{|\mathbf{z}|}{2})\Bigg)\Bigg],\\
&\text{for}\quad k=1,2;\\
&G^O_{22}(Re;\mathbf{z}) = -\tfrac{1}{2\pi Re}\tfrac{\partial}{\partial z_1}\Bigg(\ln|\mathbf{z}| + e^{Re\frac{z_1}{2}}K_0(Re\tfrac{|\mathbf{z}|}{2})\Bigg),\\
&\Pi^O_j(\mathbf{z}) = \tfrac{1}{2\pi}\tfrac{z_j}{|\mathbf{z}|^2},
\end{aligned} \tag{4.3}$$

where

$$K_0(z) = \sum_{k=0}^{\infty}\frac{z^{2k}}{2^{2k}(k!)^2}\Bigg(\psi(k+1) - \ln\frac{z}{2}\Bigg), \tag{4.4}$$

$$\psi(1) = -c_E, \quad \psi(k+1) = -c_E + \sum_{j=1}^{k}\frac{1}{j}.$$

Note that the function series in (4.4) for $k \geq 1$ converges for all $z \in \mathbb{C}$. The derivation of the Oseenlet in 2D case can be obtained by using the method of Fourier transform.

4.1 Expansion of the 2D Oseenlet

An elementary but lengthy computation shows that the 2D Oseenlet can be expressed in terms of the 2D Stokeslet (i. e., the fundamental solution of the Stokes system in $\mathbb{R}^2$), as a perturbation of it (see [17, 25]), namely

$$G^O_{jk}(Re;\mathbf{z}) = G_{jk}(\mathbf{z}) + \frac{1}{4\pi}(\ln Re\ \mathbf{I} - \mathbf{J}) + Re\ln Re\ G^{R1}_{jk}(Re;\mathbf{z}) + Re\ G^{R2}_{jk}(Re;\mathbf{z}), \tag{4.5}$$

where

$$\mathbf{J} = \begin{pmatrix} \ln 4 - C_E & 0 \\ 0 & \ln 4 - C_E + 1 \end{pmatrix}.$$

In addition, in view of (4.4) in (4.3) the first remainder kernel G^{R1}_{jk} can be written as the series

$$G^{R1}_{jk}(\epsilon;\mathbf{z}) = \sum_{\ell=0}^{\infty}\epsilon^\ell p^{\ell+1}_{jk}(\mathbf{z}), \tag{4.6}$$

where $p^{\ell+1}_{jk}(\mathbf{z})$ are homogeneous polynomials of degree $(\ell+1)$ and $\epsilon := Re$, and this series converges for any $R > 0$ uniformly with respect to $\epsilon \in [0,1]$ and for $|\mathbf{z}| \leq R$. Similarly, the second remainder kernel G^{R2}_{jk} can be written as the series

$$G^{R2}_{jk}(\epsilon;\mathbf{z}) = \sum_{\ell=0}^{\infty}\epsilon^\ell \mathcal{H}^{\ell+1}_{jk}(\mathbf{z}), \tag{4.7}$$

where $\mathcal{H}^{\ell+1}_{jk}(\mathbf{z})$ are pseudohomogeneous kernels of degree $(\ell+1)$ (see [27, Chapter 7, p.354]), and this series also converges uniformly with respect to $\epsilon \in [0,1]$, $|\mathbf{z}| \leq R$.

By using the 2D Oseenlet, we are now in the position to define the desired layer potential operators for the Oseen system in $\mathbb{R}^2$.

4.2 Layer potential operators for the Oseen system

The layer potential operators for the Oseen system can be constructed similalry to the layer potential operators for the Stokes system (see e. g. [12, Chapter 3]; [5]). In addition, taking into account (4.5), we obtain the following relation between the single-layer potentials for Stokes and Oseen systems:

$$(\mathbf{V}^O_{Re}\mathbf{g}^{Re})_j(\mathbf{x}) := (\mathbf{V}\mathbf{g}^{Re})_j(\mathbf{x}) + \frac{1}{4\pi}\left(\ln Re\ \mathbf{I} - \mathbf{J}\right)\int_\Gamma \mathbf{g}d\mathbf{\Gamma} \\ + Re(\ln Re)\int_\Gamma G^{R1}_{jk}(Re;\mathbf{x}-\mathbf{y})g^{Re}_k(\mathbf{y})d\Gamma + Re\int_\Gamma G^{R2}_{jk}(Re;\mathbf{x}-\mathbf{y})g^{Re}_k(\mathbf{y})d\Gamma, \tag{4.8}$$

where $\mathbf{g}^{Re} \in H^{-\frac{1}{2}}(\Gamma)$. By using the notation $\mathbf{V}^R_{Re}$ for the sum of the last two operators in (4.8), we have

$$(\mathbf{V}^O_{Re}\mathbf{g}^{Re})_j(\mathbf{x}) = (\mathbf{V}\mathbf{g}^{Re})_j(\mathbf{x}) + (\mathbf{V}^R_{Re}\mathbf{g}^{Re})_j(\mathbf{x}) + \frac{1}{4\pi}\left(\ln Re\ \mathbf{I} - \mathbf{J}\right)\int_\Gamma \mathbf{g}d\mathbf{\Gamma}. \tag{4.9}$$

The pressure operator P^O_{Re} associated with the single-layer operator $\mathbf{V}^O_{Re}$ is given by

$$P^O\mathbf{g}^{Re}(\mathbf{x}) = P^S\mathbf{g}^{Re}(\mathbf{x}) = \frac{1}{2\pi}\int_\Gamma \Pi^O_j(\mathbf{x}-\mathbf{y})g^{Re}_j(\mathbf{y})d\Gamma, \tag{4.10}$$

where its kernel has the form

$$\Pi^O_j(\mathbf{z}) = \Pi^S_j(\mathbf{z}) = \frac{1}{2\pi}\frac{z_j}{|\mathbf{z}|^2}. \tag{4.11}$$

The corresponding relation between the double layer potentials is

$$(\mathbf{W}^O_{Re}\mathbf{\Psi}^{Re})_j(\mathbf{x}) := (\mathbf{W}\mathbf{\Psi}^{Re})_j(\mathbf{x}) + Re(\ln Re)\int_\Gamma K^{R1}_{jk}(Re;\mathbf{x}-\mathbf{y})\Psi^{Re}_k(\mathbf{y})d\Gamma \\ + Re\int_\Gamma K^{R2}_{jk}(Re;\mathbf{x}-\mathbf{y})\Psi^{Re}_k(\mathbf{y})d\Gamma, \tag{4.12}$$

where $\mathbf{\Psi} \in H^{\frac{1}{2}}(\Gamma)$. Also, using the notation $\mathbf{W}^R_{Re}$ for the complementary operator $(\mathbf{W}^O_{Re} - \mathbf{W})$, we have

$$(\mathbf{W}^O_{Re}\mathbf{\Psi}^{Re})_j(\mathbf{x}) = (W^S\mathbf{\Psi}^{Re})_j(\mathbf{x}) + (W^R_{Re}\mathbf{\Psi}^{Re})_j(\mathbf{x}). \tag{4.13}$$

The pressure operator $P^{d,O}$ associapted with the above double layer operator for the Oseen system admits the decomposition

$$P^{d,O}\mathbf{\Psi}^{Re}(\mathbf{x}) = P^d\mathbf{\Psi}^{Re}(\mathbf{x}) + Re\int_\Gamma \Lambda^R_{jk}(Re;\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\Psi^{Re}_j(\mathbf{y})d\Gamma, \\ = P^d\mathbf{\Psi}^{Re}(\mathbf{x}) + P^R_{Re}\mathbf{\Psi}^{Re}(\mathbf{x}). \tag{4.14}$$

4.3 Representation of the desired solution for the transmission problem corresponding to the Oseen-Brinkman coupled system

Now we represent the Oseen velocity and pressure fields in terms of the following boundary layer potentials

$$\begin{aligned} \mathbf{v}^O(\mathbf{x}) &= \mathbf{W}^O_{Re}(\mathbf{x}, \Phi^{Re}) + \mathbf{V}^O_{Re}(\mathbf{x}, \mathbf{h}^{Re}), \\ P^O(\mathbf{x}) &= P^{d,O}(\mathbf{x}, \Phi^{Re}) + P^S(\mathbf{x}, \mathbf{h}^{Re}), \end{aligned} \tag{4.15}$$

which in view of the expansions (4.8) and (4.12), reduce to

$$\begin{aligned} \mathbf{v}^O(\mathbf{x}) \ &= \ \mathbf{W}(\mathbf{x}, \mathbf{\Phi}^{Re}) + \mathbf{V}(\mathbf{x}, \mathbf{h}^{Re}) + \frac{1}{4\pi}\left(\ln Re\ \mathbf{I} - \mathbf{J}\right)\int_\Gamma \mathbf{h}^{Re} d\Gamma \\ & \quad + \mathbf{W}^R_{Re}(\mathbf{x}, \mathbf{\Phi}^{Re}) + \mathbf{V}^R_{Re}(\mathbf{x}, \mathbf{h}^{Re}), \end{aligned} \tag{4.16}$$

$$P^O(\mathbf{x}) = P^d(\mathbf{x}, \mathbf{\Phi}^{Re}) + P^S(\mathbf{x}, \mathbf{h}^{Re}) + P^R_{Re}(\mathbf{x}, \mathbf{\Phi}^{Re}). \tag{4.17}$$

By imposing the transmission condition (4.2) to our layer potential representations (4.16) and (4.17), we obtain the following system of Fredholm integral equations of the second kind with the unknown $(\mathbf{\Phi}^{Re}, \mathbf{h}^{Re})^T \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$:

$$\begin{aligned} & \big(\mathbf{I} - \mathbf{K}_{\chi^2,0}\big)\mathbf{\Phi}^{Re} - \big(\mathcal{V}_{\chi^2,0}\big)\mathbf{h}^{Re} \\ & + Re \ln Re \left\{\mathbf{K}^R_{Re}\mathbf{\Phi}^{Re} + \mathcal{V}^R_{Re}\mathbf{h}^{Re}\right\} + \frac{1}{4\pi}\left(\ln Re\ \mathbf{I} - \mathbf{J}\right)\int_\Gamma \mathbf{h}^{Re}\ d\Gamma + \mathbf{i} = \mathbf{0}, \end{aligned} \tag{4.18}$$

$$\Big(\mathbf{I} + \mathbf{K}^*_{\chi^2,0}\Big)\mathbf{h}^{Re} - \mathbf{D}_{\chi^2,0}\mathbf{\Phi}^{Re} + Re \ln Re \left\{\mathbf{D}^R_{Re}\mathbf{\Phi}^{Re} - \big(\mathbf{K}^R_{Re}\big)^*\mathbf{h}^{Re}\right\} = \mathbf{0}, \tag{4.19}$$

where $\mathcal{V}_{\chi^2,0}$, $\mathbf{K}_{\chi^2,0}$, $\mathbf{K}^*_{\chi^2,0}$ and $\mathbf{D}_{\chi^2,0}$ are the complementary single-layer, double-layer, adjoint of the double-layer potential operators and the complementary hypersingular operator for the Stokes-Brinkman-coupled system. More details concerning these operators together with the corresponding operators $\mathbf{K}^R_{Re}$, $\mathcal{V}^R_{Re}$, $\big(\mathbf{K}^R_{Re}\big)^*$ and $\mathbf{D}^R_{Re}$ are given in the Appendix.

The system of equations (4.18) - (4.19) can be expressed in the matrix form as

$$\mathfrak{A}_{Re}\begin{pmatrix} \mathbf{\Phi}^{Re} \\ \mathbf{h}^{Re} \\ \boldsymbol{\omega}^{Re} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{A}^{Re} \end{pmatrix}, \ \mathfrak{A}_{Re} = \mathfrak{A} + Re(\ln Re)\mathfrak{B}_{Re}, \tag{4.20}$$

provided

$$\frac{1}{4\pi}\boldsymbol{\omega}^{Re} = \tfrac{1}{4\pi}(\ln Re)\big(\mathbf{I} - (\ln Re)^{-1}\mathbf{J}\big)\mathbf{A}^{Re} + \mathbf{i}. \tag{4.21}$$

Here the operator $\mathfrak{A}$ is as given in (3.10) and

$$\mathfrak{B}_{Re} = \begin{pmatrix} \mathbf{K}^R_{Re} & \mathcal{V}^R_{Re} & \mathbf{0} \\ \mathbf{D}^R_{Re} & -\big(\mathbf{K}^R_{Re}\big)^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{4.22}$$

Lemma 4.1 *The operator $\mathfrak{A}_{Re}$ is a regular perturbation of $\mathfrak{A}$. In particular, there exists $Re_0 > 0$, such that ${\mathfrak{A}_{Re}}^{-1} : H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \times \mathbb{R}^2 \to H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \times \mathbb{R}^2$ is given by*

$$\begin{aligned} {\mathfrak{A}_{Re}}^{-1} \ &= \ \big(\mathbf{I} - Re(\ln Re)\mathfrak{A}^{-1}\mathfrak{B}_{Re}\big)^{-1}\mathfrak{A}^{-1} = \sum_{k=0}^{\infty}(Re \ln Re)^k(-\mathfrak{A}^{-1}\mathfrak{B}_{Re})^k\mathfrak{A}^{-1}, \\ &=: \ \mathfrak{A}^{-1} + (\ln Re)^{-5}\mathfrak{C}_{Re}\mathfrak{A}^{-1}. \end{aligned} \tag{4.23}$$

for $0 \le Re \le Re_0$. Note that $\mathfrak{C}_{Re} = \sum_{k=1}^{\infty}(Re \ln Re)^{k+5}(-\mathfrak{A}^{-1}\mathfrak{B}_{Re})^k\mathfrak{A}^{-1}$.

Proof. The operator in (4.22) although depending on Re, is uniformly bounded for $0 \leq Re \leq 1$, because of (4.3) and (4.4). Hence, with $\mathfrak{A}^{-1}$ available, $\mathfrak{A}_{Re}$ can be inverted in terms of the Neumann series provided

$$\|Re \ln Re \; \mathfrak{A}^{-1}\mathfrak{B}_{Re}\| \leq |Re \ln Re| \; \|\mathfrak{A}^{-1}\| \; \|\mathfrak{B}_{Re}\| < 1.$$

Indeed, there exists a constant Re_0 such that for $0 \leq Re \leq Re_0$ the previous inequality is satisfied. In addition, we take into account the property that the operator $\mathfrak{B}^{Re}$ is bounded on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ since each of its entries has the boundedness property. Hence, the lemma is proved. □

Now, in order to solve the system (4.20) we consider the following expressions:

$$\begin{aligned}
\mathbf{\Phi}^{Re} &= \mathbf{\Phi}_0 + (\ln Re)^{-1}\mathbf{\Phi}_1 + (\ln Re)^{-2}\mathbf{\Phi}_2 + (\ln Re)^{-3}\mathbf{\Phi}_R, \\
\mathbf{h}^{Re} &= \mathbf{h}_0 + (\ln Re)^{-1}\mathbf{h}_1 + (\ln Re)^{-2}\mathbf{h}_2 + (\ln Re)^{-3}\mathbf{h}_R, \\
\boldsymbol{\omega}^{Re} &= \boldsymbol{\omega}_0 + (\ln Re)^{-1}\boldsymbol{\omega}_1 + (\ln Re)^{-2}\boldsymbol{\omega}_2 + (\ln Re)^{3}\boldsymbol{\omega}_R, \\
\mathbf{A}^{Re} &= \mathbf{A}_0 + (\ln Re)^{-1}\mathbf{A}_1 + (\ln Re)^{-2}\mathbf{A}_1 + (\ln Re)^{-3}\mathbf{A}_R.
\end{aligned} \tag{4.24}$$

In addition, taking into account the relations (3.8) and (4.21) we find that

$$\mathbf{A}^{Re} = \int_\Gamma \mathbf{h}^{Re} d\Gamma = (\ln Re)^{-1}\Big(\mathbf{I} + \sum_{k=1}^{\infty}(\ln Re)^{-k}\mathbf{J}^k\Big)(4\pi(\ln 4 - c_E)\mathbf{i} - \boldsymbol{\omega}^{Re}). \tag{4.25}$$

This constant controls the behaviour at infinity of the Stokes flow, and hence it should be finite, i.e., $|\mathbf{A}^{Re}| < \infty$. Equating the like powers of $\ln Re$ in (4.25), for the leading order term we get $\mathbf{A}_0 = \mathbf{0}$. Then taking into account the corresponding leading order system in (4.20) and the uniqueness result of the solution of this system, we get $(\boldsymbol{\phi}_0, \mathbf{h}_0, \boldsymbol{\omega}_0) = \mathbf{0}$. Similarly, we get $\mathbf{A}_1 = 4\pi\mathbf{i}$ and from the corresponding system in (4.20) we get the solution $(\boldsymbol{\phi}_1, \mathbf{h}_1, \boldsymbol{\omega}_1)$. The next order term gives $\mathbf{A}_2 = 4\pi(\ln 4 - c_E)\mathbf{i} - \boldsymbol{\omega}_1$, and correspondingly, $(\boldsymbol{\phi}_2, \mathbf{h}_2, \boldsymbol{\omega}_2)$. Finally, the remainder term of (4.25) satisfies the equation

$$\begin{aligned}
\mathbf{A}_R \;=\; & 4\pi\sum_{k=0}^{\infty}(\ln Re)^{-k}\mathbf{J}^{k+2}\mathbf{i} - \sum_{k=0}^{\infty}(\ln Re)^{-k}\mathbf{J}^{k+1}\big(\mathbf{I} + (\ln Re)^{-1}\mathbf{J}\big)\big\}\boldsymbol{\omega}_1 \\
& -\big\{\mathbf{I} + \sum_{k=0}^{\infty}(\ln Re)^{-k}\mathbf{J}^k\big(\mathbf{I} + (\ln Re)^{-1}\mathbf{J}\big)\big\}\boldsymbol{\omega}_2 \\
& +(\ln Re)^{-1}\big\{\mathbf{I} - \sum_{k=0}^{\infty}(\ln Re)^{-k}\mathbf{J}^k\big(\mathbf{I} + (\ln Re)^{-1}\mathbf{J}\big)\big\}\boldsymbol{\omega}_R.
\end{aligned} \tag{4.26}$$

Note that the above procedure provides the Stokes-Brinkman expansion from the Oseen-Brinkman expansion in a rigorous manner which reveals the validity of the formal matching procedure developed in [5].

As a consequence of (4.26) we have the following lemma.

Lemma 4.2 *If $|\mathbf{A}^{Re}| < \infty$ for $0 \leq Re \leq 1$, then there exist c_1 and c_2 such that*

$$|\mathbf{A}_R| \leq c_1 + |\ln Re|^{-1}c_2|\boldsymbol{\omega}_R|.$$

Proof. The proof of this result is a direct consequence of (4.26) and of the boundedness assumption about $\mathbf{A}^{Re}$. □

Now we are in the position to provide our main result:

Theorem 4.3 *There exists* $Re_0 > 0$ *such that*

$$\|\mathbf{v}^O - \mathbf{v}^{ST}_{Re}\|_{H^1(D^R_e)} + \|\mathbf{v}^{OB} - \mathbf{v}^B_{Re}\|_{H^1(D_0)} \\ + \|p^O - p^{ST}_{Re}\|_{L^2(D^R_e)} + \|p^{OB} - p^B_{Re}\|_{L^2(D_0)} \le c(R)|\ln Re|^{-3}, \quad (4.27)$$

for $0 \le Re < Re_0$*, where* $D^R_e = \{\mathbf{x} \in D_e : |\mathbf{x}| \le R\}$.

Proof. It may be noted that corresponding to the system (4.20), $(\boldsymbol{\phi}_\ell, \mathbf{h}_\ell, \boldsymbol{\omega}_\ell)^T$ for $\ell = 0, 1, 2$ are determined giving $\mathbf{A}_\ell, \ell = 0, 1, 2$. Also, we have the estimate for $\mathbf{A}_R$ available. Hence, in order to estimate the remainder terms, from (4.23) we have

$$(\boldsymbol{\phi}_R, \mathbf{h}_R, \boldsymbol{\omega}_R)^T = \mathfrak{A}^{-1}(\mathbf{0}, \mathbf{0}, \mathbf{A}_R)^T + (\ln Re)^{-5}\mathfrak{C}_{Re}\mathfrak{A}^{-1}\left\{(\ln Re)^{-1}\mathbf{A}_1 + (\ln Re)^{-2}\mathbf{A}_2\right\}. \quad (4.28)$$

In view of (4.28) and Lemma 4.2, there exist the constants $c_3, c_4 > 0$ such that we have the following estimate for $\boldsymbol{\omega}_R$:

$$|\boldsymbol{\omega}_R| \le (c_3|\mathbf{A}_R| + c_4) \le (c_1c_3 + c_4) + c_2c_3|\ln Re|^{-1}|\boldsymbol{\omega}_R|. \quad (4.29)$$

Now, one can find $Re_1 \in (0, Re_0]$ and $c_5 > 0$ such that for all $Re \in [0, Re_1]$, we have the uniform estimates $|\boldsymbol{\omega}_R| \le c_5$, $\|\boldsymbol{\phi}_R\|_{H^{1/2}(\Gamma)} \le c_5$ and $\|\mathbf{h}_R\|_{H^{-1/2}(\Gamma)} \le c_5$. In addition, taking into account the equality

$$\begin{pmatrix} \boldsymbol{\Phi}_{Re} \\ \mathbf{h}_{Re} \\ \boldsymbol{\omega}_{Re} \end{pmatrix} - (\ln Re)^{-1}\begin{pmatrix} \boldsymbol{\Phi}_1 \\ \mathbf{h}_1 \\ \boldsymbol{\omega}_1 \end{pmatrix} - (\ln Re)^{-2}\begin{pmatrix} \boldsymbol{\Phi}_2 \\ \mathbf{h}_2 \\ \boldsymbol{\omega}_2 \end{pmatrix} = (\ln Re)^{-3}\begin{pmatrix} \boldsymbol{\Phi}_R \\ \mathbf{h}_R \\ \boldsymbol{\omega}_R \end{pmatrix}, \quad (4.30)$$

we get just the desired estimates (4.27). This completes the proof of Theorem 4.3. □

The above estimates corresponding to the Brinkman-Oseen transmission can be thought of as a first step in the complete analysis of 2D low Reynolds number flow past a porous body including the Navier-Stokes equations.

5 Appendix : The layer potential operators for Stokes, Brinkman and Oseen equations

5.1 Fundamental solutions of the Brinkman and Stokes equations in $\mathbb{R}^2$

• The fundamental solutions of the Brinkman equations in $\mathbb{R}^2$ is given by (see e.g. [12, p. 81]):

$$\mathcal{G}^{\chi^2}_{jk}(\mathbf{x} - \mathbf{y}) = \delta_{jk}A_1(\chi|\mathbf{x} - \mathbf{y}|) + \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x} - \mathbf{y}|^2}A_2(\chi|\mathbf{x} - \mathbf{y}|) \quad \text{and} \\ \Pi^{\chi^2}_j(\mathbf{x} - \mathbf{y}) = 2\frac{x_j - y_j}{|\mathbf{x} - \mathbf{y}|^2}, \quad (5.1)$$

where

$$A_1(z) = 2\{K_0(z) + z^{-1}K_1(z) - z^{-2}\},$$
$$A_2(z) = 2\{-K_0(z) - 2z^{-1}K_1(z) + 2z^{-2}\}, \tag{5.2}$$

and K_0 and K_1 are the modified Bessel functions of the second kind of order 0 and 1, respectively. The corresponding stress and pressure tensors $\mathbf{S}^{\chi^2}$ and $\mathbf{\Lambda}^{\chi^2}$ associated with the simple layer potential have the following components (see e.g. [12, p. 82, 196]):

$$S_{ijk}^{\chi^2}(\mathbf{x}-\mathbf{y}) = -\Pi_j^{\chi^2}(\mathbf{x}-\mathbf{y})\delta_{ik} + \frac{\partial \mathcal{G}_{ij}^{\chi^2}(\mathbf{x}-\mathbf{y})}{\partial x_k} + \frac{\partial \mathcal{G}_{kj}^{\chi^2}(\mathbf{x}-\mathbf{y})}{\partial x_i}$$

$$= -2\left\{\delta_{ik}\frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^2}D_1(\chi|\mathbf{x}-\mathbf{y}|) + \left(\delta_{kj}\frac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|^2} + \delta_{ij}\frac{x_k - y_k}{|\mathbf{x}-\mathbf{y}|^2}\right)D_2(\chi|\mathbf{x}-\mathbf{y}|)\right.$$
$$\left. + \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^4}D_3(\chi|\mathbf{x}-\mathbf{y}|)\right\}, \tag{5.3}$$

$$\Lambda_{ik}^{\chi^2}(\mathbf{x}-\mathbf{y}) = 2\frac{\delta_{ik}}{|\mathbf{x}-\mathbf{y}|^2}\big(-\chi^2|\mathbf{x}-\mathbf{y}|^2\ln|\mathbf{x}-\mathbf{y}| - 2\big) + 8\frac{(x_i - y_i)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^4}, \tag{5.4}$$

where

$$D_1(z) = 2K_2(z) + 1 - 4z^{-2},$$
$$D_2(z) = 2K_2(z) + zK_1(z) - 4z^{-2}, \tag{5.5}$$
$$D_3(z) = -8K_2(z) - 2zK_1(z) + 16z^{-2},$$

and K_2 is the modified Bessel function of the second kind of order 2. The series representation of $K_0(z)$ is given in (4.4), and the corresponding representations of $K_1(z)$ and $K_2(z)$ are given by

$$K_1(z) = \sum_{k=0}^{\infty}\frac{z^{2k-1}}{2^{2k}(k!)^2}\left\{1 - 2k\left(\psi(k+1) - \ln\frac{z}{2}\right)\right\}, \tag{5.6}$$

$$K_2(z) = -\frac{1}{2} + \frac{2}{z^2} + \sum_{k=1}^{\infty}\frac{z^{2k}}{2^{2k}(k!)^2}\left(\frac{k}{k+1}\psi(k+1) - \frac{1}{2(k+1)^2} - \frac{k}{k+1}\ln\frac{z}{2}\right). \tag{5.7}$$

- The fundamental solutions of the Stokes equations in $\mathbb{R}^2$ are (see e.g. [12, p. 38])

$$\mathcal{G}_{jk}(\mathbf{x}-\mathbf{y}) = -\delta_{jk}\ln|\mathbf{x}-\mathbf{y}| + \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^2}, \quad \Pi_j(\mathbf{x}-\mathbf{y}) = 2\frac{x_j - y_j}{|\mathbf{x}-\mathbf{y}|^2}. \tag{5.8}$$

The corresponding stress and pressure tensors $\mathbf{S}$ and $\mathbf{\Lambda}$ have the components (see e.g. [12, p. 39, 132])

$$S_{ijk}(\mathbf{x}-\mathbf{y}) = -4\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^4},$$
$$\Lambda_{ik}(\mathbf{x}-\mathbf{y}) = 4\left(-\frac{\delta_{ik}}{|\mathbf{x}-\mathbf{y}|^2} + 2\frac{(x_i - y_i)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^4}\right). \tag{5.9}$$

5.2 The layer potential operators for the Stokes and Brinkman equations

The *single* and *double-layer potential operators*, $\mathbf{V}_{\chi^2}(\cdot,\mathbf{g})$ and $\mathbf{W}_{\chi^2}(\cdot,\mathbf{h})$, associated with the Brinkman system operating on the densities $\mathbf{g}$ and $\mathbf{h}$, respectively, are given for $\mathbf{x}\in\mathbb{R}^2\setminus\Gamma$ by (see e.g. [27]):

$$\mathbf{V}_{\chi^2}(\mathbf{x},\mathbf{g})=\frac{1}{4\pi}\int_\Gamma \mathcal{G}^{\chi^2}(\mathbf{x}-\mathbf{y})\cdot\mathbf{g}(\mathbf{y})d\Gamma(\mathbf{y}), \mathbf{V}_{\chi^2}: H^{-1/2+\sigma}(\Gamma)\to H^{1+\sigma}(D_0) \quad \text{and} \tag{5.10}$$

$$\left(\mathbf{W}_{\chi^2}\right)_k(\mathbf{x},\mathbf{h})=\frac{1}{4\pi}\int_\Gamma S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})h_j(\mathbf{y})d\Gamma(\mathbf{y}),\quad \mathbf{W}_{\chi^2}: H^{1/2+\sigma}(\Gamma)\to H^{1+\sigma}(D_0), \tag{5.11}$$

where $\mathbf{x}\in\mathbb{R}^2\setminus\Gamma$ and $|\sigma|\le\frac{1}{2}$. Also let $P^s_{\chi^2}(\cdot,\mathbf{g})$ and $P^d_{\chi^2}(\cdot,\mathbf{h})$ be the corresponding pressure operators given by

$$P^s_{\chi^2}(\mathbf{x},\mathbf{g})=\frac{1}{4\pi}\int_\Gamma \Pi^{\chi^2}_j(\mathbf{x}-\mathbf{y})g_j(\mathbf{y})d\Gamma(\mathbf{y}),\ P^s_{\chi^2}: H^{-1/2+\sigma}(\Gamma)\to H^{\sigma}(D_0)\quad \text{and} \tag{5.12}$$

$$P^d_{\chi^2}(\mathbf{x},\mathbf{h})=\frac{1}{4\pi}\int_\Gamma \Lambda^{\chi^2}_{j\ell}(\mathbf{x}-\mathbf{y})n_\ell(\mathbf{y})h_j(\mathbf{y})d\Gamma(\mathbf{y}),\quad P^d_{\chi^2}: H^{1/2+\sigma}(\Gamma)\to H^{\sigma}(D_0), \tag{5.13}$$

for $\mathbf{x}\in\mathbb{R}^2\setminus\Gamma$.

The *single* and *double-layer potentials*, $\mathbf{V}(\cdot,\mathbf{g})$ and $\mathbf{W}(\cdot,\mathbf{h})$, associated with the Stokes system, with the densities $\mathbf{g}$ and $\mathbf{h}$, respectively, can be obtained as in (5.10) and (5.11), but with the terms $\mathcal{G}$ and $S_{jk\ell}$ instead of $\mathcal{G}^{\chi^2}$ and $S^{\chi^2}_{jk\ell}$. Similarly, the corresponding pressure terms $P^s(\cdot,\mathbf{g})$ and $P^d(\cdot,\mathbf{h})$ can be obtained as in (5.12) and (5.13), but with Π_j and $\Lambda_{j\ell}$ instead of ${\Pi_j}^{\chi^2}$ and $\Lambda^{\chi^2}_{j\ell}$. The mapping properties correspond to those of (5.10) - (5.13) with D_0 replaced by $D_e\cup B_R$ where $B_R:=\{\mathbf{x}\in\mathbb{R}^2:|\mathbf{x}|<R\}$ and $R>0$ large enough so that $\overline{D}_0\subset B_R$.

5.3 Complementary single and double-layer integral operators

Now, for $\sigma\in\mathbb{R}$, $|\sigma|\le 1/2$, we consider the single- and double-layer integral operators on Γ for the Brinkman system $\mathcal{V}_{\chi^2}: H^{-1/2+\sigma}(\Gamma)\to H^{1/2+\sigma}(\Gamma)$ and $\mathbf{K}_{\chi^2}: H^{1/2+\sigma}(\Gamma)\to H^{1/2+\sigma}(\Gamma)$. Similarly, let $\mathcal{V}: H^{-1/2+\sigma}(\Gamma)\to H^{1/2+\sigma}(\Gamma)$ and $\mathbf{K}: H^{1/2+\sigma}(\Gamma)\to H^{1/2+\sigma}(\Gamma)$ be the layer potential operators corresponding to the Stokes system. The integral operators $\mathcal{V}_{\chi^2}$ and $\mathbf{K}_{\chi^2}$ are given by

$$\begin{aligned}\left(\mathcal{V}_{\chi^2}\mathbf{g}\right)_j(\mathbf{x}) &= \frac{1}{4\pi}\int_\Gamma \mathcal{G}^{\chi^2}(\mathbf{x}-\mathbf{y})\cdot\mathbf{g}(\mathbf{y})d\Gamma(\mathbf{y}),\\ \left(\mathbf{K}_{\chi^2}\mathbf{h}\right)_j(\mathbf{x}) &= p.v.\ \frac{1}{4\pi}\int_\Gamma S^{\chi^2}_{jk\ell}(\mathbf{y}-\mathbf{x})n_\ell(\mathbf{y})h_j(\mathbf{y})d\Gamma(\mathbf{y}),\qquad \text{where } \mathbf{x}\in\Gamma,\end{aligned}$$

for all $\mathbf{g}\in H^{-1/2+\sigma}(\Gamma)$, $\mathbf{h}\in H^{1/2+\sigma}(\Gamma)$, and similarly for the integral operators $\mathcal{V}$ and $\mathbf{K}$. Here *p.v.* refers to the principal value of a Cauchy-singular integral on Γ.

Also, let $\mathbf{D}_{\chi^2}: H^{1/2+\sigma}(\Gamma)\to H^{-1/2+\sigma}(\Gamma)$ be the operator

$$\left(\mathbf{D}_{\chi^2}\mathbf{h}\right)_j(\mathbf{x})=\text{p.f.}\int_\Gamma D^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y})h_\ell(\mathbf{y})d\Gamma(\mathbf{y}), \tag{5.14}$$

where

$$D^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y}) = -\Lambda^{\chi^2}_{\ell k}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})n_j(\mathbf{x}) + \Big(\frac{\partial}{\partial x_j}S^{\chi^2}_{\ell ik}(\mathbf{y}-\mathbf{x}) + \frac{\partial}{\partial x_i}S^{\chi^2}_{\ell jk}(\mathbf{y}-\mathbf{x})\Big)n_i(\mathbf{x})n_k(\mathbf{y}).$$

Here $p.f.$ denotes the Hadamard finite part integral. The corresponding operator for the Stokes system is denoted by $\mathbf{D}_0$. The operators $\mathbf{D}_{\chi^2}$ and $\mathbf{D}_0$ belong to the class of hypersingular operators and each of them is self - adjoint.

The adjoint to the operator $\mathbf{K}_{\chi^2}$ with respect to the $L^2(\Gamma)$ duality is given by

$$\mathbf{K}^*_{\chi^2} : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \qquad (\mathbf{K}^*_{\chi^2}\mathbf{g})_j(\mathbf{x}) = p.v. \int_\Gamma K^{\chi^2}_{j\ell}(\mathbf{x},\mathbf{y})g_\ell(\mathbf{y})d\Gamma(\mathbf{y}). \tag{5.15}$$

Similarly the corresponding operator for the Stokes system is denoted by $\mathbf{K}^*$.

The Stokes - Brinkman coupling involves the complementary single and double layer integral operators

$$\mathcal{V}_{\chi^2,0} := \mathcal{V}_{\chi^2} - \mathcal{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$
$$\mathbf{K}_{\chi^2,0} := \mathbf{K}_{\chi^2} - \mathbf{K} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma),$$

and the adjoint to the complementary double-layer integral operator

$$\mathbf{K}^*_{\chi^2,0} := \mathbf{K}^*_{\chi^2} - \mathbf{K}^* : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

In addition, the complementary hypersingular boundary integral operator

$$\mathbf{D}_{\chi^2,0} := \mathbf{D}_{\chi^2} - \mathbf{D} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

All of these complementary layer potential operators are linear, compact mappings (see [5, Theorem 5.2]).

5.4 The layer potential operators for the Oseen equation

In order to determine the expressions of the layer potential operators corresponding to Oseen - Brinkman coupled system, let us observe that on the boundary Γ

$$\begin{aligned}
\gamma_{c0}\mathbf{v}^O &= \Big(\frac{1}{2}\mathbf{I} + \mathbf{K}_O\Big)\Phi^{Re} + \mathcal{V}\mathbf{h}^{Re} + \mathbf{K}^R_{Re}\mathbf{\Phi}^{Re} + \mathcal{V}^R_{Re}\mathbf{h}^{Re}\\
\gamma_0\mathbf{v}^i &= \Big(-\frac{1}{2}\mathbf{I} + \mathbf{K}_{\chi^2}\Big)\Phi^{Re} + \mathcal{V}_{\chi^2}\mathbf{h}^{Re}\\
t_j(\mathbf{v}^O) &= \gamma_{c0}\left\{-P^{d,O}_{Re}(\cdot,\Phi^{Re})\delta_{j\ell} + \big(\partial_\ell W^O_{j,Re}(\cdot,\Phi^{Re}) + \partial_j W^O_{\ell,Re}(\cdot,\Phi^{Re})\big)\right\}n_\ell\\
&\quad + \gamma_{c0}\left\{-P^O(\cdot,\mathbf{h}^{Re})\delta_{j\ell} + \big(\partial_\ell V^O_{j,Re}(\cdot,\mathbf{h}^{Re}) + \partial_j V^O_{\ell,Re}(\cdot,\mathbf{h}^{Re})\big)\right\}n_\ell\\
t_j(\mathbf{v}^i) &= \gamma_0\left\{-P^d_{\chi^2}(\cdot,\Phi^{Re}),\delta_{j\ell} + \big(\partial_\ell W^{\chi^2}_j(\cdot,\Phi^{Re}) + \partial_j W^{\chi^2}_\ell(\cdot,\Phi^{Re})\big)\right\}n_\ell\\
&\quad + \gamma_0\left\{-P_{\chi^2}(\cdot,\mathbf{h}^{Re})\delta_{j\ell} + \big(\partial_\ell V^{\chi^2}_j(\cdot,\mathbf{h}^{Re}) + \partial_j V^{\chi^2}_\ell(\cdot,\mathbf{h}^{Re})\big)\right\}n_\ell
\end{aligned}$$

As follows from (4.8) and (4.12), as well as the equations (4.18) and (4.19), the coupling between the Oseen and Brinkman equations introduces not only the above mentioned operators, but also some additional operators corresponding to the perturbation of the Oseen kernels. They are given by

$$(\mathcal{V}^R_{Re}\mathbf{h}^{Re})_j(\mathbf{x}) = Re\ln Re\int_\Gamma G^{R1}_{jk}(Re;\mathbf{x}-\mathbf{y})h^{Re}_k(\mathbf{y})d\Gamma(\mathbf{y}) + Re\int_\Gamma G^{R2}_{jk}(Re;\mathbf{x}-\mathbf{y})h^{Re}_k(\mathbf{y})d\Gamma(\mathbf{y}),$$

$$\left(\mathbf{K}^R_{Re}\mathbf{\Phi}^{Re}\right)_j(\mathbf{x}) = Re\ln Re\int_\Gamma K^{R1}_{jk}(Re;\mathbf{x}-\mathbf{y})\Phi^{Re}_k(\mathbf{y})d\Gamma(\mathbf{y}) + Re\int_\Gamma K^{R2}_{jk}(Re;\mathbf{x}-\mathbf{y})\Phi^{Re}_k(\mathbf{y})d\Gamma(\mathbf{y}),$$

$$\begin{aligned}
\left(\left(\mathbf{K}^R_{Re}\right)^*\mathbf{h}^{Re}\right)_j(\mathbf{x}) &:= n_\ell(\mathbf{x})\left(\partial_{x_\ell}V^R_{j,Re} + \partial_{x_j}V^R_{\ell,Re}\right)(\mathbf{x},\mathbf{h}^{Re}) \\
&= \Bigg\{Re\ln Re\int_\Gamma \left\{\partial_{x_\ell}G^{R1}_{jk}(Re;\mathbf{x}-\mathbf{y}) + \partial_{x_j}G^{R1}_{\ell k}(Re;\mathbf{x}-\mathbf{y})\right\}h^{Re}_k(\mathbf{y})d\Gamma(\mathbf{y}) \\
&+ Re\int_\Gamma \left\{\partial_{x_\ell}G^{R2}_{jk}(Re;\mathbf{x}-\mathbf{y}) + \partial_{x_j}G^{R2}_{\ell k}(Re;\mathbf{x}-\mathbf{y})\right\}h^{Re}_k(\mathbf{y})d\Gamma(\mathbf{y})\Bigg\}n_\ell(\mathbf{x}) \\
\left(\mathbf{D}^R_{Re}\mathbf{\Phi}^{Re}\right)_j(\mathbf{x}) &:= n_\ell(\mathbf{x})\left(-\delta_{j\ell}P^R_{Re} + \left\{\partial_{x_\ell}W^R_{j,Re} + \partial_{x_j}W^R_{\ell,Re}\right\}\right)(\mathbf{x},\mathbf{\Phi}^{Re}) \\
&= n_\ell(\mathbf{x})\Bigg\{Re\ln Re\int_\Gamma \left\{\partial_{x_\ell}K^{R1}_{jk}(Re;\mathbf{x}-\mathbf{y}) + \partial_{x_j}K^{R1}_{\ell k}(Re;\mathbf{x}-\mathbf{y})\right\}\Phi^{Re}_k(\mathbf{y})d\Gamma(\mathbf{y}) \\
&+ Re\left\{-\int_\Gamma \Lambda^R_{ik}\delta_{j\ell}(Re;\mathbf{x}-\mathbf{y})n_k(\mathbf{y})\Phi^{Re}_i(\mathbf{y})d\Gamma(\mathbf{y})\right\} \\
&+ Re\int_\Gamma \left\{\partial_{x_\ell}K^{R2}_{jk}(Re;\mathbf{x}-\mathbf{y}) + \partial_{x_j}K^{R2}_{\ell k}(Re;\mathbf{x}-\mathbf{y})\right\}\Phi^{Re}_k(\mathbf{y})d\Gamma(\mathbf{y})\Bigg\}.
\end{aligned}$$

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References

[1] Cheng AH-D, Ouazar D. *Groundwater*, Chapter 8 in Boundary Element Techniques in Geomechanics, eds. G.D. Manolis and T.G. Davies. CMP/Elsevier, 1993: 243-294.

[2] Raja Sekhar GP, Sano O. Two - dimensional viscous flow past a slightly deformed circular cavity in a porous medium. *Fluid. Dyn. Res.* 2001; **28**:281-293.

[3] Raja Sekhar GP, Sano O. Two-dimensional viscous flow in a granular material with a void of arbitrary shape. *Physics of Fluids.* 2003; **15**:554-567.

[4] Kohr M, Raja Sekhar GP, Wendland WL. Boundary integral method for Stokes flow past a porous body. *Mathematical Methods in the Applied Sciences.* 2008; **31**: 1065-1097.

[5] Kohr M, Wendland WL, Raja Sekhar GP. Boundary integral equations for two - dimensional low Reynolds number flow past a porous body, *Mathematical Methods in the Applied Sciences*, 2009; **32**: 922-962.

[6] Kohr M, Raja Sekhar GP. Existence and uniqueness result for two-dimensional porous media flows with porous inclusions based on Brinkman equation. *Engr. Analysis with Boundary Elements.* 2007; **31**:604-613.

[7] Mitrea M, Wright M. Boundary value problems for the Stokes system in arbitrary Lipschitz domains. Preprint, 2008.

[8] Escauriaza L, Mitrea M. Transmission problems and spectral theory for singular integral operators on Lipschitz domains. *J. Funct. Anal.* 2004; **216**:141–171.

[9] Brown R, Mitrea I, Mitrea M, Wright M. Mixed boundary value problems for the Stokes system. *Transactions of the American Mathematical Society.* (2009, in press).

[10] Van Dyke M. *Perturbation Methods in Fluid Mechanics.* Academic Press: New York, 1975.

[11] Galdi GP. *An introduction to the Mathematical Theory of the Navier-Stokes equations.* Vol. I and II. Springer-Verlag; New York, 1994.

[12] Kohr M, Pop I. *Viscous Incompressible Flow for Low Reynolds Numbers.* WIT Press: Southampton (UK), 2004.

[13] Kaplun S. Low Reynolds number flow past a circular cylinder. *J. Fluid Mech.* 1957; **6**:595-603.

[14] Proudman I, Pearson JRA. Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. *J. Fluid Mech.* 1957; **2**:237-262.

[15] Finn R, On the exterior stationary problem for the Navier - Stokes equations, and associated perturbation problems. *Arch. Rational Mech. Anal.* 1965: **19**: 363-406.

[16] Finn R, Smith DR. On the stationary solutions of the Navier-Stokes equations in two dimensions. *Arch. Rational Mech. Anal.* 1967; **25**:26-39.

[17] Hsiao GC, MacCamy RC. Solutions of boundary value problems by integral equations of the first kind. *SIAM Rev.* 1973; **15**:687-705.

[18] Hsiao GC, Wendland WL, A finite element method for an integral equation of the first kind, *J. Math. Anal. Appl.*, 1977; **58**: 449-481.

[19] Hsiao GC, MacCamy RC. Singular perturbations for the two-dimensional viscous flow problem. *Lecture Notes in Math.*, Springer: Berlin. 1982; **942**:229-244.

[20] Hsiao GC. Integral representations of solutions for two dimensional viscous flow problem. *Integral Equations and Operator Theory.* 1982; **5**:533-547.

[21] Shintani K, Umemura A, Takano A. Low-Reynolds-number flow past an elliptic cylinder. *J. Fluid Mech.* 1983; **136**:227-289.

[22] Hsiao GC, Kopp P, Wendland WL, A Galerkin-collocation method for boundary integral equations of the first kind, *Mathematical Methods in the Applied Sciences*, 1984; **6**: 280-325.

[23] Lee SH, Leal LG. Low-Reynolds-number flow past cylindrical bodies of arbitrary cross-sectional shape. *J. Fluid Mech.* 1986; **164**: 401-427.

[24] Power H. Matched-asymptotic analysis of low Reynolds number flow past a cylinder of arbitrary cross-sectional shape. *Math. Applic. Comp.* 1990; **9**:111-122.

[25] Sazonov LI. The justification of an asymptotic expansion for the solution of the two - dimensional flow problem at small Reynolds numbers. *Izvestiya R A N : Ser. Mat.* 2003; **67**: 125-154.

[26] Kohr M. An application of the method of matched asymptoic expansions for low Reynolds number flow past a cylinder of arbitrary cross-section. *Int. J. Math. Math. Sci.* 2004; **45-48**:2525-2535.

[27] Hsiao GC, Wendland WL. *Boundary Integral Equations: Variational Methods.* Springer–Verlag: Heidelberg, 2008.

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