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A LOCAL AND LOW-ORDER NAVIER-STOKES-KORTEWEG SYSTEM

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Abstract. We consider the dynamics of a compressible fluid that can occur in two phases and allows for phase transition, say of liquid↔vapour type. Various Diffuse-Interface (DI) model approaches with Van-der-Waals like constitutive relations have been suggested in the last two decades, among them generalized Navier-Stokes-Korteweg systems. The latter ones typically involve higher-order differential or complex non-local operators to describe capillarity effects. This makes them in particular numerically very expansive.
We introduce in this contribution a new alternative approach within the Navier-Stokes-Korteweg class. Capillarity is modeled by only first-order differential operators and an additional simple elliptic equation for an unknown, called order parameter. We show that the order-parameter model is thermodynamically consistent and present a first local well-posedness result for classical solutions. It is a remarkable feature of the new model that the advective part can be chosen hyperbolic even for nonmonotone Van-der-Waals pressure isotherms if the parameter that controls the coupling between density and order parameter is large enough. It is conjectures that solutions of the classical Navier-Stokes-Korteweg system appear as limits of the solutions of the order-parameter model for strong coupling.

Key words. Phase Transition in Compressible Fluids, Diffuse Interface Modelling, Short-Time Existence.

1. Introduction. For the mathematical description of a homogeneous compressible fluid with liquid↔vapour phase transitions one uses either models which display the phase boundary as a sharp front (sharp interface, SI) or as a steep transition, smeared out over a small-scale distance (diffuse interface, DI). Here we focus on the DI ansatz. The DI ansatz is often favoured because also topological changes in the phase distribution as the merger or separation of bubbles/drops are covered. Moreover, and may be even more important from the numerical point of view, only one set of equations has to be solved in a single spatial domain. Recall that the SI model requires the solution of a free-boundary value problem, i.e., some kind of front tracking is necessary.
The DI approach for compressible fluids requires a subtle interaction modelling of the capillary forces close to phase boundaries and the hydrodynamical motion of the fluid. Motivated by original work of Korteweg, in [6] Dunn&Serrin have introduced the most frequently used instance of a DI model: the local Navier-Stokes-Korteweg equations (NSK). Static equilibrium solutions of this system satisfy the Euler-Lagrange equations of the Van-der-Waals functional [16]. We recall that the Van-der-Waals functional is one of the well-accepted models for the description of two-phase equilibria. For penalizing density variations it contains the gradient of the density. Actually this derivative is finally responsible for third-order differential operators in the classical local NSK model [1, 6]. Alternatively one can start from models which penalize density variations by zero-order but non-local terms (e.g. [5]). Let us note in passing that this is the functional which Van-der Waals originally suggested. In fact the local model associated to his name nowadays is just a kind of approximation. The non-local approach has been used to derive nonlocal NSK models by means of the Herivel-Lin principle [13]. Again the dynamic model contains as equilibrium solutions minimizers of the above-mentioned nonlocal functional. This model does not introduce additional orders of differentiation but a nonlocal integral operator which leads to comparable difficulties in numerics.
We will start in this contribution from an alternative functional. This functional does not depend on density alone but on a second unknown which can be seen as a kind of order parameter. Differential operators are only applied to the order parameter but not to density. This access to equilibrium density distributions has been suggested –up to the knowledge of the author– by Brandon&Lin&Rogers in [2]. We note that the models in [15] for phase transition in solid media have similar structure. Based on the construction by Brandon&Rogers&Li we present a new model –called order-parameter model– which contains only first order terms (neglecting the viscous part) in the evolution equations. Additionally

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a linear elliptic equation for the order parameter has to be solved. The order-parameter model will be introduced in Sect. 2 together with a review of the thermodynamical setting and the two above-mentioned NSK models. We will derive a natural energy balance for the new model and justify it in this way. Besides these issues it is a remarkable property of the order-parameter model that it contains a scaling number which controls the coupling between the density field and the order parameter. If this parameter is large enough the first order part of the order-parameter model is hyperbolic (eventhough the underlying Euler system is of mixed elliptic-hyperbolic type). This cannot be achieved for the local NSK model. We conjecture furthermore that solutions of the order-parameter model converge to those of the classical local NSK model if the coupling parameter tends to infinity.

In Sect. 3 a rigorous result for the order-parameter NSK model is derived: we prove that the initial-value problem for the order parameter NSK model admits unique classical solutions. The proof uses the standard method of successive approximations. However, it should be noted that the regularity requirements for the classical solutions are less strict than for the third-order local NSK problem. We also present a first convergence result for the large coupling limit.

2. Equilibrium Functionals and Navier-Stokes-Korteweg Modelling. We describe our hydro-thermodynamical setting in Sect. 2.1, the equilibrium problems are reviewed in Sec. 2.2. Main part is Sect. 2.3 on the dynamical models. In particular the new order-parameter model is introduced and discussed.

2.1. Hydro- and Thermomechanical Background. We consider here the isothermal setting and start from a given pressure function $p : (0, b) \to (0, \infty)$, $b > 0$. We suppose that the reference temperature is below the critical temperature and consider a Van-der-Waals type pressure, i.e.,

$$p(\rho) = \frac{RT_*\rho}{b-\rho} - a\rho^2, \tag{2.1}$$

where a, R, T_* are positive constants, R being the specific gas constant. Associated with the pressure is the energy function $W : (0, b) \to (0, \infty)$ given through

$$p(\rho) = \rho W'(\rho) - W(\rho). \tag{2.2}$$

As noted above, for fixed values a, b, we choose the constant reference temperature T_* so small such that p is monotone decreasing in some non-empty interval. The graph of p has then a shape as indicated in Fig. 2.1. From (2.2) we observe $p'(\rho) = \rho W''(\rho)$ so that W has non-convex double-well structure. With the notations from Fig. 2.1 we can define phases. If the density ρ lies in the interval $(0, \alpha_1]$, $((\alpha_1, \alpha_2))$, $\{[\alpha_2, b)\}$ the corresponding fluid state is called liquid (spinodal) {vapour}.

2.2. Equilibrium functionals. Liquid-vapour equilibrium configurations can be associated with minimizers of free energy functionals. We consider a bounded domain $\Omega \subset \mathbb{R}^d$ in $d \in \{1, 2, 3\}$ space dimensions with smooth boundary $\partial\Omega$. Let us define

$$\mathcal{A}(m) = \left\{ \rho : \Omega \to (0, b) \,\middle|\, \int_\Omega \rho(\mathbf{x})\, d\mathbf{x} = m \right\}, \tag{2.3}$$

for $m \in (0, \text{meas}(\Omega)b)$. If m is prescribed in $(\alpha_1 \text{meas}(\Omega), \alpha_2 \text{meas}(\Omega))$ we expect a two-phase equilibrium due to the non-convex shape of W. We recall that there is no unique minimizer of the SI functional

$$F^0[\rho] = \int_\Omega W(\rho(\mathbf{x}))\, d\mathbf{x}. \tag{2.4}$$

This is the major mathematical reason to consider extended functionals of the form

$$F^\varepsilon[\rho, c] = F^0[\rho] + G^\varepsilon[\rho, c]. \tag{2.5}$$

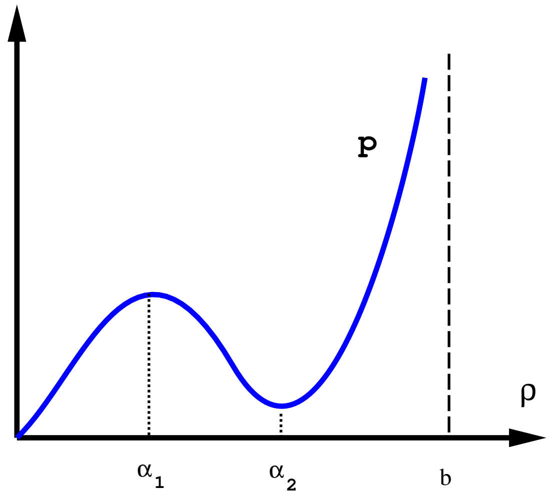

FIG. 2.1. *The figure shows the graph of the Van-der-Waals pressure function $p = p(\rho)$ for sufficiently low temperature. The sign of the derivative of p changes for $\rho = \alpha_{1/2}$.*

Here $\varepsilon > 0$ is a small parameter. The meaning of the additional unknown $c : \Omega \to \mathbb{R}$ becomes clear below in (2.11).
The classical local Van-der-Waals functional [16] results from the choice

$$G^\varepsilon[\rho, c] = G^\varepsilon_{\text{local}}[\rho] = \int_\Omega \frac{\varepsilon^2}{2} |\nabla\rho|^2 \, d\mathbf{x}. \tag{2.6}$$

The minimizer is searched for in $\mathcal{A}(m) \cap H^1(\Omega)$. In [12] it has been proven that minimizers of (2.5) with $G^\varepsilon_{\text{local}}$ given by (2.6) –if they exist– converge for $\varepsilon \to 0$ to a minimizer of (2.4). In this way a reasonable solution to the variational problem for (2.4) is uniquely selected. Let us note for later use that the Euler-Lagrange equation for the functional F^ε with $G^\varepsilon_{\text{local}}$ from (2.6) is

$$-C^\varepsilon[\rho, c] + W(\rho) = c, \quad C^\varepsilon[\rho, c] = C^\varepsilon_{\text{local}}[\rho] = \varepsilon^2 \Delta\rho, \tag{2.7}$$

where the constant $c \in \mathbb{R}$ is due to the integral constraint (2.3). Obviously (2.7) is a (nonlinear) elliptic equation such that existing minimizers of (2.5), (2.6) can be expected to be smooth functions. Actually already in [16] another non-local choice has been suggested which is defined through

$$\begin{aligned} G^\varepsilon[\rho, c] &= G^\varepsilon_{\text{global}}[\rho] = \frac{1}{4} \int_\Omega \int_\Omega k_\varepsilon(\mathbf{x}, \mathbf{y}) \Big(\rho(\mathbf{x}) - \rho(\mathbf{y})\Big)^2 \, d\mathbf{y} d\mathbf{x}, \\ k_\varepsilon(\mathbf{x}, \mathbf{y}) &= \frac{1}{\varepsilon^d} k\left(\frac{\mathbf{x}}{\varepsilon}, \frac{\mathbf{y}}{\varepsilon}\right). \end{aligned} \tag{2.8}$$

We suppose the kernel function $k : \Omega^2 \to \mathbb{R}$ in (2.8) to be nonnegative such that $G^\varepsilon_{\text{global}}$ penalizes variations of the density field. Moreover we require the symmetry condition

$$k_\varepsilon(\mathbf{x}, \mathbf{y}) = k_\varepsilon(\mathbf{y}, \mathbf{x}). \tag{2.9}$$

The minimizer ρ is searched for in $\mathcal{A}(m)$, no further (differential) regularity is required. Also in this case a sharp-interface result for the sharp interface limit $\varepsilon \to 0$, analoguous to the one in [12], has been established. We refer to [4]. In this case the Euler-Lagrange equation associated to (2.5), (2.8) is provided by

$$-C^{\varepsilon}[\rho,c]+W'(\rho)=c,\quad C^{\varepsilon}[\rho,c]=C^{\varepsilon}_{\text{global}}[\rho]=\int_{\Omega}k_{\varepsilon}\left(\cdot,\mathbf{y}\right)\Big(\rho(\mathbf{y})-\rho(\cdot)\Big)\,d\mathbf{y}. \tag{2.10}$$

Thus we obtain a nonlinear integral equation which in principle can admit discontinuous solutions. Yet another idea can be taken from [2] where it is searched for the density $\rho\in\mathcal{A}(m)$ and a kind of order parameter $c\in H^1(\Omega)$ which minimize (2.5) with $G^{\varepsilon}_{\text{order}}$ given by

$$G^{\varepsilon}_{\text{order}}[\rho,c]=\int_{\Omega}\frac{\alpha^2}{2}(\rho(\mathbf{x})-c(\mathbf{x}))^2+\frac{\varepsilon^2}{2}|\nabla c(\mathbf{x})|^2\,d\mathbf{x}. \tag{2.11}$$

Note that no derivative of ρ appears in (2.11), differentiation is only applied to the additional unknown c. The coupling between density and order parameter is controlled by $\alpha>0$. Moreover no integral convolution is used. Nevertheless a sharp interface result as for the models above is available, due to [14]. Our new dnamical model will rely on the functional in (2.11). Since $G^{\varepsilon}_{\text{order}}$ in (2.11) depends on two unknowns we obtain the Euler-Lagrange system

$$-C^{\varepsilon}[\rho,c]+W'(\rho)=c,\quad C^{\varepsilon}[\rho,c]=C^{\varepsilon}_{\text{order}}[\rho,c]=\alpha^2(c-\rho), \tag{2.12}$$

$$0=\varepsilon^2\Delta c+\alpha^2(\rho-c). \tag{2.13}$$

2.3. Dynamical Modelling with the Navier-Stokes-Korteweg Approach. In this section we turn to dynamics. Analogous to the SI functional 2.4 we recall the hyperbolic-elliptic Euler system with Van-der-Waals pressure isotherm. We assume that then that the DI dynamics is governed by systems of the Navier-Stokes-Korteweg structure. This is justified afterwards by showing for the three choices from the preceding section that the respective systems are thermodynamically consistent.

2.3.1. The Euler and the Navier-Stokes-Korteweg System. Let $\Omega_T:=\mathbb{R}^d\times(0,T)$, $T>0$ and assume $d=2$ for simplicity. The isothermal Euler system for the unknowns density $\rho=\rho(\mathbf{x},t):\Omega_T\to(0,b)$ and velocity $\mathbf{v}=\mathbf{v}(\mathbf{x},t)=(v_1(\mathbf{x},t),v_2(\mathbf{x},t))^T:\Omega_T\to\mathbb{R}^2$ is given by

$$\begin{array}{rcccl}\rho_t & + & \operatorname{div}(\rho\mathbf{v}) & = & 0,\\ (\rho\mathbf{v})_t & + & \operatorname{div}(\rho\mathbf{v}\mathbf{v}^T+p(\rho)\mathbf{I}) & = & 0\end{array} \tag{2.14}$$

in Ω_T. Here $\mathbf{I}\in\mathbb{R}^{d\times d}$ denotes the unity matrix.
The Euler system (2.14) for the conservative unknown $\mathbf{u}=(\rho,\rho v_1,\rho v_2)^T$ can be written as

$$\mathbf{u}+\mathbf{f}_1(\mathbf{u})_{x_1}+\mathbf{f}_2(\mathbf{u})_{x_2}=0,$$
$$\mathbf{f}_1(\mathbf{u})=(\rho v_1,\rho v_1^2+p(\rho),\rho v_1v_2)^T,\qquad \mathbf{f}_2(\mathbf{u})=(\rho v_2,\rho v_1v_2,\rho v_2^2+p(\rho))^T.$$

The eigenvalues of the Jacobian $\xi_1Df_1(\mathbf{u})+\xi_2D\mathbf{f}_2(u)\in\mathbb{R}^{3\times3}$ for $\boldsymbol{\xi}=(\xi_1,\xi_2)^T\in\mathcal{S}^1$ are

$$\lambda_1(\rho,\mathbf{v};\boldsymbol{\xi})=\mathbf{v}\cdot\boldsymbol{\xi}-\sqrt{p'(\rho)},\ \lambda_2(\rho,\mathbf{v};\boldsymbol{\xi})=\lambda_3(\rho,\mathbf{v};\boldsymbol{\xi})=\mathbf{v}\cdot\boldsymbol{\xi},\ \lambda_4(\rho,\mathbf{v};\boldsymbol{\xi})=\mathbf{v}\cdot\boldsymbol{\xi}+\sqrt{p'(\rho)}. \tag{2.15}$$

We observe that (2.14) is a mixed hyperbolic-elliptic type system. It is elliptic for $\rho\in(\alpha_1,\alpha_2)$. This failure of hyperbolicity is one of the major reasons that standard modern numerical discretization methods cannot be applied. These rely on the hyperbolicity of the system.
Next we present the general class of NSK systems. For the unknowns density $\rho=\rho(\mathbf{x},t):\Omega_T\to(0,b)$,

velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) : \Omega_T \to \mathbb{R}^2$ and possibly an order-parameter $c = c(\mathbf{x}, t) : \Omega_T \to \mathbb{R}$ the NSK system is given by

$$\begin{array}{rcccl} \rho_t & + & \operatorname{div}(\rho\mathbf{v}) & = & 0, \\ (\rho\mathbf{v})_t & + & \operatorname{div}(\rho\mathbf{v}\mathbf{v}^T + p(\rho)\mathbf{I}) & = & \operatorname{div}(\mathbf{T}) + \rho\nabla C^\varepsilon[\rho, c] \end{array} \tag{2.16}$$

in Ω_T. The matrix $\mathbf{T} = \mathbf{T}(\mathbf{x}, t) \in \mathbb{R}^{d\times d}$ in (2.16) stands for the viscous part of the stress tensor which is given for $\lambda, \mu \in \mathbb{R}$ with $\mu \geq 0$ and $3\lambda + 2\mu > 0$ by

$$\mathbf{T}_{ij} := \lambda \operatorname{div}(\mathbf{v})\delta_{ij} + 2\mu\mathbf{D}_{ij}, \quad \mathbf{D}_{ij} := \frac{1}{2}\Big(v_{j,x_i} + v_{i,x_j}\Big) \qquad (i, j \in \{1, 2\}). \tag{2.17}$$

The essential term to model phase transition phenomena is the operator C^ε. Several choices for C^ε have been presented in Sect. 2.2 and will be discussed in the sequel.

2.3.2. The Classical third-order local NSK System. The classical choice that corresponds to (2.5), (2.6) with $C^\varepsilon_{\text{local}}$ from (2.7) makes the momentum equations in (2.16) third-order evolution equations. Adding a viscous stress tensor to (2.16) in this case one obtains the classical Navier-Stokes-Korteweg system that has been widely analyzed in literature and used for numerical simulations. The third order terms in (2.16) might look strange for the first view (and in fact cause lots of trouble for numerics) but do not contradict thermodynamics. Rather we have

PROPOSITION 2.1. *Let $(\rho, \mathbf{v})$ be a classical solution of (2.16) with C^ε given by (2.7). Assume that there is a constant $\bar\rho > 0$ such that $\rho(\mathbf{x}, t) - \bar\rho, |\mathbf{v}(\mathbf{x}, t)|, |\nabla\mathbf{v}(\mathbf{x}, t)|$ tend to 0 for all $t \in (0, T)$ if $|\mathbf{x}| \to \infty$. Assume that all spatial derivatives of ρ up to order two are bounded.*
Then we have for $t \in (0, T)$

$$\begin{aligned} &\frac{d}{dt}\Big(\int_{\mathbb{R}^2} \frac{1}{2}\rho(\mathbf{x}, t)|\mathbf{v}(\mathbf{x}, t)|^2 + W(\rho(\mathbf{x}, t))\, d\mathbf{x} + G^\varepsilon_{local}[\rho(., t)]\Big) \\ &+ \int_{\mathbb{R}^2} 2\mu\mathbf{D}(\mathbf{v}(\mathbf{x}, t)) : \mathbf{D}(\mathbf{v}(\mathbf{x}, t)) + \lambda(\operatorname{div}(\mathbf{v}(\mathbf{x}, t)))^2\, d\mathbf{x} = 0. \end{aligned} \tag{2.18}$$

Proof. cf. [1, 13]. □

The assumptions on the classical solution in Propsition 2.1 have been selected such that boundary integral due to partial integration vanish. We remark that we tacitly supposed the classical solution here (and in the sequel) such that all integrals in the energy balances exist. Under the assumptions of the lemma this requires in particular $W(\tilde\rho) = 0$.

NOTE 2.2. *We presented the NSK-system (2.16) for $\Omega = \mathbb{R}^2$. For a bounded domain Ω reasonable boundary conditions are*

$$\mathbf{v} = 0, \quad \frac{\partial}{\partial\mathbf{n}}\rho = 0 \ \text{in } \partial\Omega. \tag{2.19}$$

Here $\mathbf{n}$ denotes the normal vector associated with $\partial\Omega$. In this case the energy balance (2.18) is also valid.
The third-order term in (2.16) forces us to impose the second artificial boundary condition in (2.19). Note that this choice leads to a 90^o degree contact angle between the phases at the boundary.

Before we turn to the next case let us stress that the local capillarity term C^ε does not change the advective first order part in (2.16): it remains to be of mixed type.

2.3.3. The Nonlocal NSK System. The choice $C^{\varepsilon}_{\text{global}}$ from (2.10) in (2.16) has been introduced in [13]. As a counterpart to Proposition 2.1 we can formulate

PROPOSITION 2.3. *Let $(\rho, \mathbf{v})$ be a classical solution of (2.16) with C^{ε}_{global} given by (2.10). Assume that there is a constant $\bar{\rho} > 0$ such that $\rho(\mathbf{x}, t) - \bar{\rho}, |\mathbf{v}(\mathbf{x}, t)|, |\nabla \mathbf{v}(\mathbf{x}, t)|$ tend to 0 for all $t \in (0, T)$ if $|\mathbf{x}| \to \infty$.*
Then we have for $t \in (0, T)$

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^2} \frac{1}{2}\rho(\mathbf{x}, t)|\mathbf{v}(\mathbf{x}, t)|^2 + W(\rho(\mathbf{x}, t))\,d\mathbf{x} + G^{\varepsilon}_{global}[\rho(., t)]\Big) \\
&\quad + \int_{\mathbb{R}^2} 2\mu \mathbf{D}(\mathbf{v}(\mathbf{x}, t)) : \mathbf{D}(\mathbf{v}(\mathbf{x}, t)) + \lambda(\operatorname{div}(\mathbf{v}(\mathbf{x}, t)))^2\,d\mathbf{x} = 0.
\end{aligned} \tag{2.20}$$

Proof. Let

$$\eta = \eta(\rho, \mathbf{m}) = \frac{|\mathbf{m}|^2}{2\rho} + W(\rho), \quad \mathbf{m} = \rho\mathbf{v}.$$

We multiply each equation in the system (2.16) with $C^{\varepsilon} = C^{\varepsilon}_{\text{global}}$ with the corresponding components of the gradient of η, that are

$$-\frac{1}{2}|\mathbf{v}|^2 + W'(\rho), v_1, v_2.$$

We add up all equations and integrate with respect to space. Using the required bounds on th classical solution all terms in (2.20) are standard except the energy contribution $G^{\varepsilon}_{\text{global}}$. This one appears from the capillarity term $\rho \nabla C^{\varepsilon}_{\text{global}}[\rho]$, precisely we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2} \frac{1}{2}\rho(\mathbf{x}, t)|\mathbf{v}(\mathbf{x}, t)|^2 + W(\rho(\mathbf{x}, t))\,d\mathbf{x} \\
&= \int_{\mathbb{R}^2}\Big(\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)\cdot \nabla \int_{\Omega} k_{\varepsilon}(\mathbf{x}, \mathbf{y})\Big(\rho(\mathbf{y}, t) - \rho(\mathbf{x}, t)\Big)\,d\mathbf{y}\Big)\,d\mathbf{x}. \\
&= -\int_{\mathbb{R}^2}\Big(\operatorname{div}(\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)) \int_{\mathbb{R}^2} k_{\varepsilon}(\mathbf{x}, \mathbf{y})\Big(\rho(\mathbf{y}, t) - \rho(\mathbf{x}, t)\Big)\,d\mathbf{y}\Big)\,d\mathbf{x} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_t(\mathbf{x}, t) k_{\varepsilon}(\mathbf{x}, \mathbf{y})\Big(\rho(\mathbf{y}, t) - \rho(\mathbf{x}, t)\Big)\,d\mathbf{y}\,d\mathbf{x} \\
&=: \mathcal{T}_{\text{global}}.
\end{aligned}$$

The result is now a consequence of the following computation where we use the symmetry of k_{ε} as required in (2.9).

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k_{\varepsilon}(\mathbf{x}, \mathbf{y})(\rho(\mathbf{y}, t) - \rho(\mathbf{x}, t))^2\,d\mathbf{y}d\mathbf{x} \\
&= 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k_{\varepsilon}(\mathbf{x}, \mathbf{y})(\rho_t(\mathbf{y}, t) - \rho_t(\mathbf{x}, t))(\rho(\mathbf{y}, t) - \rho(\mathbf{x}, t))\,d\mathbf{y}d\mathbf{x} \\
&= 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k_{\varepsilon}(\mathbf{y}, \mathbf{x})(\rho_t(\mathbf{y}, t)(\rho(\mathbf{y}, t) - \rho(\mathbf{x}, t))\,d\mathbf{y}d\mathbf{x} \\
&\quad + 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k_{\varepsilon}(\mathbf{x}, \mathbf{y})(\rho_t(\mathbf{x}, t)(\rho(\mathbf{x}, t) - \rho(\mathbf{y}, t))\,d\mathbf{y}d\mathbf{x} \\
&= -4\mathcal{T}_{\text{global}}.
\end{aligned}$$

□

NOTE 2.4. *The same relation as in (2.20) can be obtained for (2.16) with* $C^\varepsilon := C^\varepsilon_{global}$ *the initial-boundary-value problem in some bounded domain* Ω *if the single condition*

$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ in } \partial\Omega \tag{2.21}$$

holds. Nevertheless the symmetry condition (2.9) leads to a 90^o *contact angle with the boundary* $\partial\Omega$*, i.e., it acts like a boundary condition.*

2.3.4. A Low-Order Order-Parameter NSK System. Finally, a possible NSK system comes from the choice $C^\varepsilon_{\text{order}}$ in (2.12). This system has to be complemented by an equation for the order parameter c. If we select the Euler-Lagrange equation (2.13) the complete system reads

$$\begin{aligned} \rho_t \quad + \quad \operatorname{div}(\rho\mathbf{v}) \quad &= \quad 0, \\ (\rho\mathbf{v})_t \quad + \quad \operatorname{div}(\rho\mathbf{v}\mathbf{v}^T + p(\rho)\mathbf{I}) \quad &= \quad \alpha^2\rho\nabla(c-\rho), \\ 0 \quad &= \quad \varepsilon^2\Delta c + \alpha^2(\rho - c). \end{aligned} \tag{2.22}$$

We obtain a first-order system coupled to an additional linear elliptic equation. Henceforth we will call it the *order-parameter NSK system.* Up to the knowledge of the author this system has not been suggested in the literature before. We start again with the associated result on energy dissipation.

PROPOSITION 2.5. *Let* $(\rho, \mathbf{v}, c)$ *be a classical solution of (2.22). Assume that there is a constant* $\bar\rho > 0$ *such that* $\rho(\mathbf{x},t) - \bar\rho, c(\mathbf{x},t) - \bar\rho, |\mathbf{v}(\mathbf{x},t)|, |\nabla\mathbf{v}(\mathbf{x},t)|$ *tend to* 0 *for all* $t \in (0,T)$ *if* $|\mathbf{x}| \to \infty$.
Then we have for $t \in (0,T)$

$$\begin{aligned} &\frac{d}{dt}\Big(\int_{\mathbb{R}^2} \frac{1}{2}\rho(\mathbf{x},t)|\mathbf{v}(\mathbf{x},t)|^2 + W(\rho(\mathbf{x},t))\,d\mathbf{x} + G^\varepsilon_{order}[\rho(.,t)]\Big) \\ &+ \int_{\mathbb{R}^2} 2\mu\mathbf{D}(\mathbf{v}(\mathbf{x},t)) : \mathbf{D}(\mathbf{v}(\mathbf{x},t)) + \lambda(\operatorname{div}(\mathbf{v}(\mathbf{x},t)))^2\,d\mathbf{x} = 0. \end{aligned} \tag{2.23}$$

Proof. With the same approach as in the proof of Proposition 2.3 we multiply the first three equations in (2.22) with $-\frac{1}{2}|\mathbf{v}|^2 + W'(\rho)$, v_1, v_2, respectively, add up and integrate with respect to $\mathbb{R}^2$. Using the first boundary condition in (2.24) it is standard to derive

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^2} \frac{1}{2}\rho(\mathbf{x},t)|\mathbf{v}(\mathbf{x},t)|^2 + W(\rho(\mathbf{x},t))\,d\mathbf{x} \\ &= \quad \alpha^2 \int_{\mathbb{R}^2} \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\cdot\nabla(c(\mathbf{x},t) - \rho(\mathbf{x},t))\,d\mathbf{x}. \\ &= \quad -\alpha^2 \int_{\mathbb{R}^2} \operatorname{div}(\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t))(c(\mathbf{x},t) - \rho(\mathbf{x},t))\,d\mathbf{x} \\ &= \quad \alpha^2 \int_{\mathbb{R}^2} \rho_t(\mathbf{x},t)(c(\mathbf{x},t) - \rho(\mathbf{x},t))\,d\mathbf{x} \\ &=: \quad \mathcal{T}_{\text{order}}. \end{aligned}$$

For the last line we used the continuity equation. The elliptic equation in (2.22) and the second condition in (2.24) yield

$$\begin{aligned} 0 \quad &= \quad \int_{\mathbb{R}^2} \varepsilon^2 c_t(\mathbf{x},t)\Delta c(\mathbf{x},t) + \alpha^2 c_t(\mathbf{x},t)(\rho(\mathbf{x},t) - c(\mathbf{x},t))\,d\mathbf{x} \\ &= \quad -\varepsilon^2\frac{d}{dt}\int_{\mathbb{R}^2}\frac{1}{2}|\nabla c|^2(\mathbf{x},t)\,d\mathbf{x} + \int_{\mathbb{R}^2}\alpha^2 c_t(\mathbf{x},t)(\rho(\mathbf{x},t) - c(\mathbf{x},t))\,d\mathbf{x}. \end{aligned}$$

Thus we obtain

$$\begin{aligned}\mathcal{T}_{\text{order}} &= -\frac{d}{dt}\int_{\mathbb{R}^2}\frac{\varepsilon^2}{2}c^2(\mathbf{x},t)\,d\mathbf{x} + \alpha^2\int_{\mathbb{R}^2}\rho_t(\mathbf{x},t)(c(\mathbf{x},t)-\rho(\mathbf{x},t)) + c_t(\mathbf{x},t)(\rho(\mathbf{x},t)-c(\mathbf{x},t))\,d\mathbf{x}\\ &= -\frac{d}{dt}\int_{\mathbb{R}^2}\frac{\varepsilon^2}{2}c^2(\mathbf{x},t)\frac{\alpha^2}{2}(\rho(\mathbf{x},t)-c(\mathbf{x},t))^2\,d\mathbf{x},\end{aligned}$$

which implies the statement of the proposition.

□

The same kind of remark as for theother two models applies to (2.22).

NOTE 2.6. *If for some bounded domain Ω the boundary conditions*

$$\mathbf{v}\cdot\mathbf{n} = 0, \qquad \frac{\partial}{\partial \mathbf{n}}c = 0 \ \text{in } \partial\Omega \tag{2.24}$$

hold, the estiamte (2.23) is also valid. We conjecture that the chosen Neumann conditions for c imply a 90^o contact angle with the boundary $\partial\Omega$.

The system (2.22) contains only local differentiation operators. The price to pay is an additional equation for the order parameter. But this equation is a simple linear elliptic equation which can be solved extremely fast numerically, at least if a fixed mesh is used. But there is another issue which makes (2.22) attractive from the numerical point of view. One can rewrite the momentum equations in (2.22) as follows

$$(\rho\mathbf{v})_t + \operatorname{div}(\rho\mathbf{v}\mathbf{v}^T + \tilde{p}(\rho)\mathbf{I}) = \alpha^2\rho\nabla c, \qquad \tilde{p}(\rho) := p(\rho) + \frac{\alpha^2}{2}\rho^2.$$

For α large enough we have

$$\tilde{p}'(\rho) > 0.$$

In view of the formulas in (2.15) the advective part in (2.22) is now *hyperbolic*. This is a major advantage compared to the local NSK system (2.16) which remains to be of elliptic type. In the case here we observe directly how the capillarity term regularizes the non-monotone pressure. Note that the remaining inhomogenity $\alpha^2\rho\nabla c$ appears to be harmless as c is the solution of an elliptic equation with a source which is expected to be bounded in $L^\infty(\Omega_T)$.

We conclude this section with some notes on possible further extensions of the order parameter Euler-Korteweg system.

NOTE 2.7.

(i) *From the motivation of the Euler-Korteweg models via equilibrium functionals it is consequential to determine c in (2.22) from an elliptic constraint. However one might also want to evolve c directly. For an advective evolution with some relaxation parameter $\delta > 0$ the last line in (2.22) would become*

$$\frac{\delta}{\rho}\Big((\rho c)_t + \operatorname{div}(\rho c\mathbf{v})\Big) = \varepsilon^2\Delta c + \alpha^2(\rho - c).$$

(ii) *One could argue that a choice of the coupling parameter α that makes the advective part in (2.22) hyperbolic is far from being physical. However we conjecture that solutions of (2.22) tend for $\alpha\to\infty$ (and $\varepsilon > 0$ fixed!) to a solution of the accepted local NSK system (2.16). We cannot prove this conjecture but will give a consistency result in Theorem 3.1 (ii) below. Numerical material that supports this conjecture can be found in [10].*

(iii) Also for the nonlocal model one can split the capillarity term to obtain the new pressure

$$\tilde{p}(\rho, \mathbf{x}) = p(\rho) + \frac{1}{2} K_\varepsilon(\mathbf{x}) \rho^2.$$

Here $K_\varepsilon(\mathbf{x}) = \int_\Omega k_\varepsilon(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$. Note that now $\tilde{p}$ depends additionally on space which makes it complicated to control the monotonicity of $\tilde{p}$. In fact for usual choices of the kernel function k and $\Omega = \mathbb{R}^d$ the primitive K_ε does depend neither on ε nor on $\mathbf{x}$. In this case one can obtain an uniformly monotone new pressure We refer to the forthcoming PhD-thesis of J. Haink (see [10] for a short version).

3. Short-Time Existence of Smooth Solutions for the Order-Parameter Model. For the classical local NSK-model and the nonlocal NSK-model a number of wellposedness results are available. Short-time existence of classical solutions for initial-value problems has been established in [9] and [13]. The existence proof of weak solutions for the initial-value case in the sense of Lions-Feireisl can be found in [3] and in [8]. It is remarkable that the latter work on the nonlocal system deals with general viscous stress tensors while work on the local NSK-system requires restrictions on the viscosity modelling. The paper [11] deals with strong solutions in a bounded domain for the local NSK-system. In the sequel we address the question of short-time existence for smooth solutions for the new order-parameter model.

3.1. Preliminaries. To simplify notations we restrict ourselves in this section to the spatially two-dimensional case $d = 2$, consider a simplified viscosity mechanism and set $\varepsilon = 1$. We define for an arbitrary number $R > 0$ the open space-time set $\Omega_R := \mathbb{R}^2 \times (0, R)$.
The simplified version of the order-parameter model is given by

$$\begin{aligned} \rho^\alpha_t + (v^\alpha_1 \rho^\alpha)_{x_1} + (v^\alpha_2 \rho^\alpha)_{x_2} &= 0, \\ (\rho^\alpha v^\alpha_1)_t + (\rho^\alpha (v^\alpha_1)^2 + p(\rho^\alpha))_{x_1} + (\rho v^\alpha_1 v^\alpha_2)_{x_2} - \alpha^2 \rho^\alpha (c^\alpha - \rho^\alpha)_{x_1} - \Delta v^\alpha_1 &= 0, \\ (\rho^\alpha v^\alpha_2)_t + (\rho^\alpha v^\alpha_1 v^\alpha_2)_{x_1} + (\rho^\alpha (v^\alpha_2)^2 + p(\rho^\alpha))_{x_2} - \alpha^2 \rho^\alpha (c^\alpha - \rho^\alpha)_{x_2} - \Delta v^\alpha_2 &= 0, \\ - \Delta c^\alpha &= \alpha^2 (\rho^\alpha - c^\alpha). \end{aligned} \tag{3.1}$$

For the pressure function $p = p(\rho^\alpha)$ we assume $p \in C^4(\mathbb{R})$, the specific form from (2.1) is not used in this section. By $\alpha > 0$ we denote the coupling constant.
We consider the initial-value problem for (3.1) with the initial condition

$$\rho^\alpha(.,0) = \rho_0, \quad \mathbf{v}^\alpha(.,0) = \mathbf{v}_0 \text{ in } \mathbb{R}^2. \tag{3.2}$$

Thereby we assume throughout the section that there is a constant $\bar{\rho} > 0$ such that the components of the initial function $\mathbf{w}_0 := (\rho_0, v_{01}, v_{02})^T$ satisfy

$$\rho_0 - \bar{\rho} \in H^4(\mathbb{R}^2),\ v_{01},\ v_{02} \in H^4(\mathbb{R}^2),\ \rho_0 > 0. \tag{3.3}$$

Actually it is not necessary to put $\rho_0 - \bar{\rho} \in H^4(\mathbb{R}^2)$ but the necessary sharper regularity makes the proof more complex without gaining so much.
The continuous functions $\mathbf{w}^\alpha := (\rho^\alpha, v^\alpha_1, v^\alpha_2)^T : \bar{\Omega}_T \to \mathbb{R}^3$, $c^\alpha : \bar{\Omega}_T \to \mathbb{R}$ are called a **classical solution of (3.1), (3.2)** if all functions in (3.1) exist in the classical sense as continuous functions, (3.1) holds pointwise in Ω_T, and (3.2) holds pointwise in $\mathbb{R}^2$.
In what follows we use the notations

$$\tilde{\rho}^\alpha := \rho^\alpha - \bar{\rho},\ \tilde{\mathbf{w}}^\alpha := (\rho^\alpha - \bar{\rho}, v^\alpha_1, v^\alpha_2)^T,\ \tilde{\rho}_0 := \rho_0 - \bar{\rho},\ \tilde{\mathbf{w}}_0 := (\rho_0 - \bar{\rho}, v_{01}, v_{02})^T.$$

By $\mathcal{C}, \mathcal{D} : [0,\infty) \to [0,\infty)$ we denote continuous generic functions which might change from line to line. They can depend on $\mathbf{w}_0$, α, and the pressure p.
Note that for the elliptic equation we cannot impose an independent initial datum. But $c^\alpha(.,0) : \mathbb{R}^2 \to \mathbb{R}$ is chosen compatible, i.e. we suppose $c^\alpha(.,0) = \tilde{c}^\alpha(.,0) + \bar{\rho}$ sucht that $\tilde{c}^\alpha(.,0)$ is the the weak solution of the elliptic problem

$$-\Delta \tilde{c}^\alpha(.,0) + \alpha^2 \tilde{c}^\alpha(.,0) = \alpha^2 \tilde{\rho}_0 \text{ in } \mathbb{R}^2. \tag{3.4}$$

3.2. The Main Theorem for the Order-Parameter NSK-System. Let us state the key result of this section.
Theorem 3.1. *Let $\mathbf{w}_0$ satisfy (3.3).*

(i) There exists a constant $T_* > 0$ such that the initial-value problem (3.1), (3.2), (3.4) has a classical solution $(\mathbf{w}^\alpha, c^\alpha)$ in $\bar{\Omega}_{T_*}$. The classical solution $(\mathbf{w} = (\rho^\alpha, v_1^\alpha, v_2^\alpha)^T, c^\alpha)$ satisfies

$$\begin{aligned} \tilde{\rho}^\alpha &\in L^\infty(0,T_*; H^4(\mathbb{R}^2)),\ \rho^\alpha > 0, \\ v_1^\alpha, v_2^\alpha &\in L^\infty(0,T_*; H^4(\mathbb{R}^2)), \\ c^\alpha - \bar{\rho} &\in L^\infty(0,T_*; H^5(\mathbb{R}^2)). \end{aligned}$$

The solution $(\mathbf{w}^\alpha, c^\alpha)$ is unique in the class of classical solutions.

(ii) Let W from (2.2) such that $W(\tilde{\rho}) = W'(\tilde{\rho}) = 0$. For all $t \in [0, T_*)$ we have

$$\lim_{\alpha\to\infty} \|\rho^\alpha(.,t) - c^\alpha(.,t)\|_{L^2(\mathbb{R}^2)} = 0.$$

As discussed above the L^2-convergence from Theorem 3.1 does not suffice to prove the convergence of solutions of the low-order NSK system to a solution of the local NSK system as the coupling parameter tends to infinity. It is just a consistency result.
We divide the proof of Theorem 3.1 in three steps. In Step I we derive a-priori estimates for a linearization of (3.1) that are used in Step II to perform the existence part of the proof by successive approximations. In Step III we consider the limit statement (ii). Both steps are close to the method of proof in [13]. For Steps I/II we skip the index α. The assumptions of Theorem 3.1 are supposed to hold for the section's rest.
Step I: The Linearized Problem. To prove Theorem 3.1 we introduce a linearization of (3.1). Define for $k \in \mathbb{N}_0$ and $R > 0$ the function sets

$$H^k_{\bar{\rho}}(0,R) = \Big\{ \mathbf{w} = (\rho, v_1, v_2)^T : \Omega_R \to \mathbb{R}^3 \,\Big|\, \tilde{\mathbf{w}}, \in L^\infty(0,R; H^k(\mathbb{R}^2)) \Big\},$$

$$\mathcal{W}(0,R) =$$

$$\Big\{ (\mathbf{w}, c) \in H^4_{\bar{\rho}}(0,R) \,\Big|\, \rho > 0,\ \mathbf{w}_t \text{ exists and is in } L^\infty(0,R; H^2(\mathbb{R}^2)),\ \mathbf{v} \in L^2(0,R; H^5(\mathbb{R}^2)),$$
$$\mathbf{v}_t \text{ exists and is in } L^2(0,R; H^3(\mathbb{R}^2)) \Big\}.$$

Let $\hat{\mathbf{w}} \in \mathcal{W}(0,R)$. Embedding theorems for Sobolev and Bochner spaces show that we have the relations

$$D^{\boldsymbol{\alpha}} \hat{\mathbf{w}} \in C([0,R]; C(\mathbb{R}^2)) \quad \boldsymbol{\alpha} \in \mathbb{N}_0^2,\ |\boldsymbol{\alpha}| \le 1, \tag{3.5}$$

$$D^{\boldsymbol{\alpha}} \hat{\mathbf{v}} \in C([0,R]; C(\mathbb{R}^2)) \quad \boldsymbol{\alpha} \in \mathbb{N}_0^2,\ |\boldsymbol{\alpha}| \le 2. \tag{3.6}$$

The differential operator $D^{\boldsymbol{\alpha}}$ (for arbitrary $\boldsymbol{\alpha} \in \mathbb{N}_0^2$) in the last two equations is defined through $D^{\boldsymbol{\alpha}} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}}$. Let us define the constant $C_0 = C_0(\hat{\mathbf{w}}, R)$ as a positive number such that

$$\left\|\frac{1}{\hat{\rho}}\right\|_{L^\infty(\Omega_R)} + \|\hat{\rho}_t\|_{L^\infty(\Omega_R)} + \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^2,\, |\boldsymbol{\alpha}|\le 2} \|D^{\boldsymbol{\alpha}}\hat{\mathbf{w}}\|_{L^\infty(\Omega_R)} < C_0(\hat{\mathbf{w}}, R). \tag{3.7}$$

The inverse of $\hat{\rho}$ is bounded in the L^∞-norm since $\hat{\rho}$ is bounded from below by a positive constant due to continuity. The time derivative of $\hat{\rho}$ and the space derivatives of $\hat{\mathbf{w}}$ are bounded in $L^\infty(\Omega_R)$ due to $\hat{\rho}_t \in L^\infty(0, T, H^2(\mathbb{R}^2))$, $\hat{\mathbf{w}} \in H^4_{\hat{\rho}}(0, T)$, and Sobolev embedding. Thus the number $C_0(\hat{\mathbf{w}}, R)$ in (3.7) exists.
Now let $T > 0$ and choose aome $\mathbf{F} = (F_1, F_2, F_3)^T \in L^\infty(0, T; L^2(\mathbb{R}^2))$. We consider the following linear problem for $\mathbf{w} : \bar{\Omega}_T \to \mathbb{R}^3$ and $\tilde{c} : \bar{\Omega}_T \to \mathbb{R}$

$$\begin{aligned} L(\hat{\mathbf{w}})[\mathbf{w}, c] &= \mathbf{F} && \text{in } \Omega_T, \\ -\Delta\tilde{c} + \alpha^2\tilde{c} &= \alpha^2\tilde{\rho} && \text{in } \mathbb{R}^2 \times [0, T), \\ \mathbf{w}(., 0) &= \mathbf{w}_0 && \text{in } \mathbb{R}^2. \end{aligned} \tag{3.8}$$

It remains to specify the operator $L(\hat{\mathbf{w}})$ in (3.8) which is given by

$$L(\hat{\mathbf{w}})[\mathbf{w}, c] = \begin{pmatrix} L_1(\hat{\mathbf{w}})[\mathbf{w}, \tilde{c}] \\ L_2(\hat{\mathbf{w}})[\mathbf{w}, \tilde{c}] \\ L_3(\hat{\mathbf{w}})[\mathbf{w}, \tilde{c}] \end{pmatrix} = \mathbf{w}_t + A_1(\hat{\mathbf{w}})\mathbf{w}_{x_1} + A_2(\hat{\mathbf{w}})\mathbf{w}_{x_2} + \begin{pmatrix} 0 \\ -\alpha^2\big(\tilde{c}_{x_1} - \rho_{x_1}\big) - \hat{\rho}^{-1}\Delta v_1 \\ -\alpha^2\big(\tilde{c}_{x_2} - \rho_{x_2}\big) - \hat{\rho}^{-1}\Delta v_2 \end{pmatrix}.$$

Here the matrices A_1, A_2 are defined through

$$A_1(\hat{\mathbf{w}}) = \begin{pmatrix} \hat{v}_1 & \hat{\rho} & 0 \\ p(\hat{\rho})\hat{\rho}^{-1} & \hat{v}_1 & 0 \\ 0 & 0 & \hat{v}_1 \end{pmatrix}, \quad A_2(\hat{\mathbf{w}}) = \begin{pmatrix} \hat{v}_2 & 0 & \hat{\rho} \\ 0 & \hat{v}_2 & 0 \\ p(\hat{\rho})\hat{\rho}^{-1} & 0 & \hat{v}_2 \end{pmatrix}.$$

Note that a classical solution $(\mathbf{w}, c)$ of (3.1), (3.2), (3.4) satisfies the linearized problem (3.8) with $\mathbf{F} = 0$ and $\hat{\mathbf{w}} \equiv \mathbf{w}$.
We look now for a solution $(\mathbf{w}, \tilde{c})$ of (3.8) in the set of weak solutions, i.e., we suppose that $\tilde{\mathbf{w}}$ and $\tilde{c}$ are in $L^2(0, T; H^1(\mathbb{R}^2))$ and
satisfy

(i) $\mathbf{w}_t$ exists in $L^2(0, T; H^{-1}(\mathbb{R}^2))$,

(ii) the weak formulation of (3.8) (see below) holds for almost all $t \in [0, T]$ and all test functions ψ in $H^1(\mathbb{R}^2)$,

(iii) $\mathbf{w}(., 0) = \mathbf{w}_0$ a.e. in $\mathbb{R}^2$.

Note that condition (iii) makes sense since we have in particular $\tilde{\mathbf{w}} \in C([0, T]; L^2(\mathbb{R}^2))$ by $\tilde{\mathbf{w}} \in C([0, T]; H^1(\mathbb{R}^2))$ and (i). The weak form of the equations in (3.8) is given for ψ in $H^1(\mathbb{R}^2)$ and $t \in [0, T]$ by

$$\begin{aligned} \int_\Omega \mathbf{w}_t(\mathbf{x}, t)\psi(\mathbf{x}, t)\, d\mathbf{x} &= \mathbf{R}[\hat{\mathbf{w}}, \mathbf{w}, \tilde{c}, \psi](t) + \int_\Omega \mathbf{F}(\mathbf{x}, t)\psi(\mathbf{x}, t)\, d\mathbf{x}, \\ \int_\Omega \nabla\tilde{c}(\mathbf{x}, t) \cdot \nabla\psi(\mathbf{x}, t) + \alpha^2\tilde{c}(\mathbf{x}, t)\psi(\mathbf{x}, t)\, d\mathbf{x} &= \int_\Omega \alpha^2\tilde{\rho}(\mathbf{x}, t)\psi(\mathbf{x}, t)\, d\mathbf{x}. \end{aligned} \tag{3.9}$$

The term $\mathbf{R}$ above is given by

$$\begin{aligned}\mathbf{R}[\hat{\mathbf{w}},\mathbf{w},\tilde{c},\psi](t) \quad := \quad & \int_{\mathbb{R}^2} -\big(A_1(\hat{\mathbf{w}}(\mathbf{x},t))\mathbf{w}_{x_1}(\mathbf{x},t) + A_2(\hat{\mathbf{w}}(\mathbf{x},t))\mathbf{w}_{x_2}(\mathbf{x},t)\big)\cdot\psi(\mathbf{x},t)\,d\mathbf{x} \\ & + \int_{\mathbb{R}^2} \frac{1}{(\hat{\rho}(\mathbf{x},t))^2}\begin{pmatrix} 0 \\ \nabla\hat{\rho}(\mathbf{x},t)\cdot\nabla v_1(\mathbf{x},t) \\ \nabla\hat{\rho}(\mathbf{x},t)\cdot\nabla v_2(\mathbf{x},t)\end{pmatrix}\psi(\mathbf{x},t)\,d\mathbf{x} \\ & + \int_{\mathbb{R}^2} \frac{-1}{\hat{\rho}(\mathbf{x},t)}\begin{pmatrix} 0 \\ \nabla v_1(\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t) \\ \nabla v_2(\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t)\end{pmatrix} d\mathbf{x} \\ & + \int_{\mathbb{R}^2} \begin{pmatrix} 0 \\ \alpha^2\big(\tilde{c}_{x_1}(\mathbf{x},t) - \rho_{x_1}(\mathbf{x},t)\big) \\ \alpha^2\big(\tilde{c}_{x_2}(\mathbf{x},t) - \rho_{x_2}(\mathbf{x},t)\big)\end{pmatrix}\psi(\mathbf{x},t)\,d\mathbf{x}.\end{aligned}$$

The problem (3.8) is a linear problem with variable coefficients consisting of a transport equation for ρ and linear advection-diffusion equations for the components of $\mathbf{v}$ and an elliptic equation for c. The existence of a unique weak solution can be established by Galerkin methods as e.g. in textbooks like [7]. Note that in the course of this proof the transport equation is first regularized by an artificial diffusion term. This term can then be eliminated by a straightforward vanishing viscosity process which is uniform in appropriate energy estimates.

Next we present the basic a-priori energy estimate for weak solutions of (3.8).

LEMMA 3.2 (L^2-estimates). *Let $\hat{\mathbf{w}} \in \mathcal{W}(0,T)$. Suppose that $\mathbf{w}_0$ satisfies (3.3) and that we have* $\mathbf{F} \in L^\infty(0,T;L^2(\mathbb{R}^2))$.
The weak solution $(\mathbf{w},\tilde{c})$ of (3.8) satisfies for almost all $t \in [0,T]$ the estimate

$$\begin{aligned}& \|\tilde{\mathbf{w}}(.,t)\|^2_{L^2(\mathbb{R}^2))} + G_{order}[\rho(.,t),\tilde{c}(.,t)](\mathbf{x}) + \|\nabla\mathbf{v}\|^2_{L^2(0,t;L^2(\mathbb{R}^2))} \\ & \leq \exp\big(\mathcal{C}(C_0(\hat{\mathbf{w}},t))t\big)\left(\|\tilde{\mathbf{w}}_0\|^2_{L^2(\mathbb{R}^2)} + \mathcal{C}(C_0(\hat{\mathbf{w}},t))\int_0^t \|\mathbf{F}(.,s)\|^2_{L^2(\mathbb{R}^2)}\,ds\right).\end{aligned} \tag{3.10}$$

The term $G_{order} = G^1_{order}$ is defined as in (2.11) with $d = 2$.

Proof. We assume first that $\mathbf{F}$ and $\hat{\mathbf{w}}$ are regular enough such that the weak solution of (3.8) is actually a classical solution of (3.8). We define the matrix $A_0(\hat{\mathbf{w}}) := \mathrm{diag}(1,\hat{\rho},\hat{\rho})$, multiply the system (3.8) with $A_0(\hat{\mathbf{w}}(.,t))\tilde{\mathbf{w}}(.,t)$ for $t \in [0,T]$, and integrate over $\mathbb{R}^2$. The terms we get are estimated as follows:

$$\begin{aligned}& \int_{\mathbb{R}^2} \mathbf{w}_t(\mathbf{x},t)\cdot A_0(\hat{\mathbf{w}}(\mathbf{x},t))\tilde{\mathbf{w}}(\mathbf{x},t)\,d\mathbf{x} \\ & = \frac{1}{2}\frac{d}{dt}\|\sqrt{A_0(\hat{\mathbf{w}}(.,t))}\tilde{\mathbf{w}}(.,t)\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{2}\int_{\mathbb{R}^2}\hat{\rho}_t(\mathbf{x},t)\big((v_1(\mathbf{x},t))^2 + (v_2(\mathbf{x},t))^2\big)\,d\mathbf{x} \\ & \geq \frac{1}{2}\frac{d}{dt}\|\sqrt{A_0(\hat{\mathbf{w}}(.,t))}\tilde{\mathbf{w}}(.,t)\|^2_{L^2(\mathbb{R}^2)} - \mathcal{C}(C_0)\big(\|v_1(.,t)\|^2_{L^2(\mathbb{R}^2)} + \|v_2(.,t)\|^2_{L^2(\mathbb{R}^2)}\big).\end{aligned} \tag{3.11}$$

In the last line and for the rest of the proof we skip the arguments of C_0. We proceed with the advection terms and get for all $t \in (0,T)$ from (3.7)

$$\begin{aligned}
&\int_{\mathbb{R}^2} A_1(\hat{\mathbf{w}}(\mathbf{x},t))\mathbf{w}_{x_1}(\mathbf{x},t)\cdot A_0(\hat{\mathbf{w}}(\mathbf{x},t))\tilde{\mathbf{w}}(\mathbf{x},t)\,d\mathbf{x}\\
&= \int_{\mathbb{R}^2} -\frac{1}{2}\hat{v}_{1,x_1}(\mathbf{x},t)(\tilde{\rho}(\mathbf{x},t))^2 + \hat{\rho}(\mathbf{x},t)\tilde{\rho}(\mathbf{x},t)v_{1,x_1}(\mathbf{x},t) - (p(\hat{\rho}(\mathbf{x},t))v_1(\mathbf{x},t))_{x_1}\tilde{\rho}(\mathbf{x},t)\\
&\qquad + \hat{v}_1(\mathbf{x},t)\hat{\rho}(\mathbf{x},t)(v_1(\mathbf{x},t)v_{1,x_1}(\mathbf{x},t) + v_2(\mathbf{x}.t)v_{2,x_1}(\mathbf{x},t))\,d\mathbf{x}\\
&\leq \delta\big(\|v_{1,x_1}(.,t)\|^2_{L^2(\mathbb{R}^2)} + \|v_{2,x_1}(.,t)\|^2_{L^2(\mathbb{R}^2)}\big) + \mathcal{C}(C_0+\delta^{-1})\|\tilde{\mathbf{w}}(.,t)\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{3.12}$$

Note that integration by parts produce no trace terms by means of $\hat{\mathbf{w}} \in \mathcal{W}(0,T)$. To derive the last line in (3.12) we used Young's inequality with $\delta \in (0,1)$.
The same arguments as above imply

$$\begin{aligned}
&\int_{\mathbb{R}^2} A_2(\hat{\mathbf{w}}(\mathbf{x},t))\mathbf{w}_{x_2}(\mathbf{x},t)A_0(\hat{\mathbf{w}}(\mathbf{x},t))\tilde{\mathbf{w}}(\mathbf{x},t)\,d\mathbf{x}\\
&\quad\leq \delta\big(\|v_{1,x_2}(.,t)\|^2_{L^2(\mathbb{R}^2)} + \|v_{2,x_2}(.,t)\|^2_{L^2(\mathbb{R}^2)}\big) + \mathcal{C}(C_0+\delta^{-1})\|\tilde{\mathbf{w}}(.,t)\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{3.13}$$

For the last term in the definition of $\mathbf{R}$ in (3.9) we proceed as in the proof of Proposition 2.5. In particular we use the elliptic equation for $\tilde{c}$ in (3.8). Precisely we obtain

$$\begin{aligned}
&-\alpha^2\int_{\mathbb{R}^2} \hat{\rho}\nabla\big(\tilde{c}(\mathbf{x},t)-\rho(\mathbf{x},t)\big)\cdot\mathbf{v}(\mathbf{x},t)\,d\mathbf{x}\\
&=\alpha^2\int_{\mathbb{R}^2} \big(\tilde{c}(\mathbf{x},t)-\rho(\mathbf{x},t)\big)\big(\hat{\rho}(\mathbf{x},t)\mathrm{div}(\mathbf{v}(\mathbf{x},t)) + \mathbf{v}(\mathbf{x},t)\cdot\nabla\hat{\rho}(\mathbf{x},t)\big)\,d\mathbf{x}\\
&=\alpha^2\int_{\mathbb{R}^2} \big(\tilde{\rho}(\mathbf{x},t)-\tilde{c}(\mathbf{x},t)\big)\big(\tilde{\rho}_t(\mathbf{x},t) + \hat{v}_1(\mathbf{x},t)\tilde{\rho}_{x_1}(\mathbf{x},t) + \hat{v}_2(\mathbf{x},t)\tilde{\rho}_{x_2}(\mathbf{x},t)\\
&\qquad -F_1(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\nabla\hat{\rho}(\mathbf{x},t)\big)\,d\mathbf{x}\\
&\geq \frac{d}{dt}G_{\text{order}}[\rho(.,t),\tilde{c}(.,t)](\mathbf{x})\\
&\quad + \int_{\mathbb{R}^2} \nabla\tilde{c}(\mathbf{x},t)\cdot\big(\tilde{\rho}(\mathbf{x},t)\hat{\mathbf{v}}(\mathbf{x},t)\big) + \tilde{c}(\mathbf{x},t)\tilde{\rho}(\mathbf{x},t)\mathrm{div}(\hat{\mathbf{v}}(\mathbf{x},t))\,d\mathbf{x}\\
&\quad - \mathcal{C}(C_0)\left(\|\tilde{\mathbf{w}}\|^2_{L^2(\mathbb{R}^2)} + \|\mathbf{F}(.,t)\|^2_{L^2(\mathbb{R}^2)}\right)\\
&\geq \frac{d}{dt}G_{\text{order}}[\rho(.,t),\tilde{c}(.,t)](\mathbf{x}) - \mathcal{C}(C_0)\left(\|\tilde{\mathbf{w}}\|^2_{L^2(\mathbb{R}^2)} + \|\mathbf{F}(.,t)\|^2_{L^2(\mathbb{R}^2)}\right).
\end{aligned} \tag{3.14}$$

For the last estimate we needed the elliptic regularity estimates of the type

$$\|\tilde{c}(.,t)\|_{L^2(\mathbb{R}^2)} + \|\nabla\tilde{c}(.,t)\|_{L^2(\mathbb{R}^2)} < C\|\tilde{\rho}(.,t)\|_{L^2(\mathbb{R}^2)}.$$

hold for $t \in [0,T]$ and a constant $C > 0$ not depending on $\mathbf{w}$.
To treat the viscous part in (3.8) consider for $i = 1,2$

$$-\int_{\mathbb{R}^2} \frac{1}{\hat{\rho}(\mathbf{x},t)}\Delta v_i(\mathbf{x},t)\hat{\rho}(\mathbf{x},t)v_i(\mathbf{x},t)\,d\mathbf{x} = \|\nabla v_i(.,t)\|^2_{L^2(\mathbb{R}^2)}. \tag{3.15}$$

If we collect the estimates (3.11), (3.12), (3.13), (3.14), (3.15), choose δ sufficiently small, and use

$$\int_{\mathbb{R}^2} \mathbf{F}(\mathbf{x},t)\cdot A_0(\hat{\mathbf{w}}(\mathbf{x},t))\tilde{\mathbf{w}}(\mathbf{x},t)\,d\mathbf{x} \le \mathcal{C}(C_0)\left(\|\tilde{\mathbf{w}}(.,t)\|^2_{L^2(\mathbb{R}^2)} + \|\mathbf{F}(.,t)\|^2_{L^2(\mathbb{R}^2)}\right)$$

for all $t \in [0,T]$ we arrive at

$$\begin{aligned}(3.16)\quad &\frac{d}{dt}\left(\frac{1}{2}\|\sqrt{A_0(\hat{\mathbf{w}}(.,t))}\tilde{\mathbf{w}}(.,t)\|^2_{L^2(\mathbb{R}^2)} + G_{\text{order}}[\rho(.,t),\tilde{c}(.,t)](\mathbf{x})\right) + \|\nabla\mathbf{v}(.,t)\|^2_{L^2(\mathbb{R}^2)}\\ &\qquad \le \mathcal{C}(C_0)\left(\|\tilde{\mathbf{w}}(.,t)\|^2_{L^2(\mathbb{R}^2)} + \|\mathbf{F}(.,t)\|^2_{L^2(\mathbb{R}^2)}\right).\end{aligned}$$

As a consequence of Gronwall's inequality, the estimate (3.7), and $\int_0^t \mathcal{C}(C_0(\hat{\mathbf{w}},s))\,ds \le \mathcal{C}(C_0(\hat{\mathbf{w}},t))t$, ($C_0(\hat{\mathbf{w}},.)$ is an increasing function!), we get

$$\begin{aligned}&\|\tilde{\mathbf{w}}(.,t)\|^2_{L^2(\mathbb{R}^2)} + G_{\text{order}}[\rho(.,t),\tilde{c}(.,t)](\mathbf{x})\\ &\quad\le \exp\left(\mathcal{C}(C_0(\hat{\mathbf{w}},t))t\right)\left(\|\tilde{\mathbf{w}}_0\|^2_{L^2(\mathbb{R}^2)} + \mathcal{C}(C_0(\hat{\mathbf{w}},t))\int_0^t \|\mathbf{F}(.,s)\|^2_{L^2(\mathbb{R}^2)}\,ds\right).\end{aligned}$$

Using this result again in (3.16) and integration with respect to time leads to the statement of the lemma for smooth functions $\mathbf{F}$ and $\hat{\mathbf{w}}$. For $\mathbf{F} \in L^\infty(0,T;L^2(\mathbb{R}^2))$ and $\hat{\mathbf{w}} \in \mathcal{W}(0,T)$ the statement follows by a density argument. □

Let us note that the term in (3.14) could have been directly estimated in terms of C_0, $\|\tilde{\mathbf{w}}\|_{L^2(\mathbb{R}^2)}$, and $\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}$. In the way we proceeded we have been able to recover the natural energy estimate for the (linearized) order-parameter NSK-system in (3.10).

Next we prove estimates on higher-order derivatives of $\mathbf{w}$ and $\tilde{c}$. In order to do this let a number $k \ge 2, k \in \mathbb{N}$ and $\hat{\mathbf{w}} \in \mathcal{W}(0,T) \cap H^k_{\bar{\rho}}(0,T)$ be given. We define for $t \in [0,T]$ the number $C_k = C_k(\hat{\mathbf{w}},t)$ to be a positive constant such that

$$(3.17)\quad \left\|\frac{1}{\hat{\rho}}\right\|_{L^\infty(\Omega_t)} + \|\hat{\rho}_t\|_{L^\infty(\Omega_t)} + \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^2,\,|\boldsymbol{\alpha}|\le 2} \|D^{\boldsymbol{\alpha}}\hat{\mathbf{w}}\|_{L^\infty(\Omega_t)} + \|\hat{\mathbf{w}} - (\bar{\rho},0,0)^T\|_{L^\infty(0,t;H^k(\mathbb{R}^2))} \le C_k(\hat{\mathbf{w}},t)$$

holds.

Let us consider the following problem for $\mathbf{w} : \Omega_T \to \mathbb{R}^3$ which we obtain if we apply (formally) the operator $D^{\boldsymbol{\alpha}}$ to (3.8):

$$(3.18)\quad \begin{aligned} L(\hat{\mathbf{w}})[D^{\boldsymbol{\alpha}}\mathbf{w}, D^{\boldsymbol{\alpha}}\tilde{c}] &= D^{\boldsymbol{\alpha}}\mathbf{F} - \mathcal{L}^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})[\mathbf{w}] && \text{in } \Omega_T,\\ -\Delta D^{\boldsymbol{\alpha}}\tilde{c} + \alpha^2 D^{\boldsymbol{\alpha}}\tilde{c} &= \alpha D^{\boldsymbol{\alpha}}\rho && \text{in } \mathbb{R}^2\times[0,T),\\ D^{\boldsymbol{\alpha}}\mathbf{w}(.,0) &= D^{\boldsymbol{\alpha}}\mathbf{w}_0 && \text{in } \mathbb{R}^2.\end{aligned}$$

Here we used the operator

$$\mathcal{L}^{\boldsymbol{\alpha}}(\hat{\mathbf{w}}) = (\mathcal{L}_1^{\boldsymbol{\alpha}}(\hat{\mathbf{w}}), \mathcal{L}_2^{\boldsymbol{\alpha}}(\hat{\mathbf{w}}), \mathcal{L}_3^{\boldsymbol{\alpha}}(\hat{\mathbf{w}}))^T,$$

defined for $\boldsymbol{\alpha} = (0,0)^T$ by $\mathcal{L}_1^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})][\mathbf{w}] = (0,0,0)^T$ and otherwise by

$$\begin{aligned}
\mathcal{L}_1^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})][\mathbf{w}] &= \sum_{\boldsymbol{\beta}\in\mathcal{I}(\boldsymbol{\alpha})} \Big(D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\hat{v}_1 D^{\boldsymbol{\beta}}\rho_{x_1} + D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\hat{v}_2 D^{\boldsymbol{\beta}}\rho_{x_2} + D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\hat{\rho} D^{\boldsymbol{\beta}} v_{x_1} + D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\hat{\rho} D^{\boldsymbol{\beta}} v_{x_2}\Big),\\
\mathcal{L}_2^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})][\mathbf{w}] &= \sum_{\boldsymbol{\beta}\in\mathcal{I}(\boldsymbol{\alpha})} \Big(D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\hat{v}_1 D^{\boldsymbol{\beta}} v_{1,x_1} + D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\hat{v}_2 D^{\boldsymbol{\beta}} v_{1,x_2} + D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\left(\frac{p'(\hat{\rho})}{\hat{\rho}}\right) D^{\boldsymbol{\beta}}\rho_{x_1}\\
&\qquad - D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\left(\frac{1}{\hat{\rho}}\right) D^{\boldsymbol{\beta}}\Delta v_1\Big),\\
\mathcal{L}_3^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})][\mathbf{w}] &= \sum_{\boldsymbol{\beta}\in\mathcal{I}(\boldsymbol{\alpha})} \Big(D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\hat{v}_1 D^{\boldsymbol{\beta}} v_{2,x_1} + D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\hat{v}_2 D^{\boldsymbol{\beta}} v_{2,x_2} + D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\left(\frac{p'(\hat{\rho})}{\hat{\rho}}\right) D^{\boldsymbol{\beta}}\rho_{x_2}\\
&\qquad - D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\left(\frac{1}{\hat{\rho}}\right) D^{\boldsymbol{\beta}}\Delta v_2\Big).
\end{aligned}$$

The index set $\mathcal{I}(\boldsymbol{\alpha})$ is given by

$$\mathcal{I}(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} = (\beta_1,\beta_2)^T \in \mathbb{N}_0^2 \,|\, 0 \le \beta_1 \le \alpha_1,\ 0 \le \beta_2 \le \alpha_2,\ |\boldsymbol{\beta}| < |\alpha|\}. \tag{3.19}$$

The order parameter c does not show up in the definition of $\mathcal{L}^{\boldsymbol{\alpha}}$ since only linear operators are applied to c. The subsequent lemma gives typical estimates on the operators $\mathcal{L}^{\boldsymbol{\alpha}}$ applied to sufficiently regular functions. For a proof we refer to Lemma 4.3 in [13].

LEMMA 3.3. *For $k \in \{2,4\}$ suppose that $(\rho_0 - \bar{\rho}, v_{01}, v_{02})^T \in H^k(\mathbb{R}^2)$ and $\mathbf{F} \in L^\infty(0,T;H^k(\mathbb{R}^2))$ holds. Furthermore assume that the function $\hat{\mathbf{w}}$ satisfies (3.17) for $t \in [0,T]$ with $C_k(\hat{\mathbf{w}},t) > 0$ and that $\mathbf{w}$ is in $H^k_{\bar{\rho}}(0,T)$ with $\mathbf{v}$ in $L^2(0,T;H^{k+1}(\mathbb{R}^2))$.*
Then we have for $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ with $|\boldsymbol{\alpha}| \le k$, almost all $t \in (0,T)$, and $i = 2,3$ the estimates

$$\|\mathcal{L}_1^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})[\mathbf{w}](.,t)\|_{L^2(\mathbb{R}^2)} \le \mathcal{C}(C_k(\hat{\mathbf{w}},t))\|\mathbf{w}(.,t)\|_{H^k(\mathbb{R}^2)}, \tag{3.20}$$

$$\|\mathcal{L}_i^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})[\mathbf{w}](.,t)\|_{L^2(\mathbb{R}^2)} \le \mathcal{C}(C_k(\hat{\mathbf{w}},t))\Big(\|\mathbf{v}(.,t)\|_{H^{k+1}(\mathbb{R}^2)} + \|\mathbf{w}(.,t)\|_{H^k(\mathbb{R}^2)}\Big). \tag{3.21}$$

With Lemma 3.3 we can prove

LEMMA 3.4 (Higher-order estimates). *Let $k \in \{2,4\}$ and $\hat{\mathbf{w}} \in \mathcal{W}(0,T)$. Suppose that $\mathbf{w}_0$ satisfies the conditions (3.3) and that $\mathbf{F} \in L^\infty(0,T,H^k(\mathbb{R}^2))$ holds.*
The weak solution $(\mathbf{w},\tilde{c})$ of (3.8) with $\tilde{\mathbf{w}} \in L^\infty(0,T;H^k(\mathbb{R}^2))$ and $\tilde{c} \in L^\infty(0,T;H^{k+1}(\mathbb{R}^2))$ satisfies for almost all $t \in [0,T]$ the estimate

$$\begin{aligned}
&\|\tilde{\mathbf{w}}(.,t)\|^2_{H^k(\mathbb{R}^2)} + \|\tilde{c}(.,t)\|^2_{H^{k+1}(\mathbb{R}^2)} + \|\mathbf{v}\|^2_{L^2(0,t;H^{k+1}(\mathbb{R}^2))}\\
&\le \exp\big(\mathcal{C}(C_k(\hat{\mathbf{w}},t))t\big)\left(\|\tilde{\mathbf{w}}_0\|^2_{H^k(\mathbb{R}^2)} + \mathcal{C}(C_k(\hat{\mathbf{w}},t))\int_0^t \|\mathbf{F}(.,s)\|^2_{H^k(\mathbb{R}^2)}\,ds\right).
\end{aligned} \tag{3.22}$$

Moreover we have for $k = 4$

$$\begin{aligned}
&t^{-1}\|\rho(.,t) - \rho_0\|_{L^\infty(\mathbb{R}^2)} + \|\rho_t\|_{L^\infty(\Omega_t)}\\
&\le \mathcal{C}(C_0(\hat{\mathbf{w}},t))\left(\operatorname*{esssup}_{s\in[0,t]}\Big\{\|\tilde{\mathbf{w}}(.,s)\|_{H^4(\mathbb{R}^2)}\Big\} + \|\mathbf{F}\|_{L^\infty(0,t;H^2(\mathbb{R}^2))}\right).
\end{aligned} \tag{3.23}$$

Proof. We consider the system of differential equations in (3.18). Let us first assume that $\hat{\mathbf{w}}$, $\tilde{\mathbf{w}}_0$ and $\mathbf{F}$ are sufficiently regular functions such that their spatial derivatives up to order 4 exist as continuous

functions decaying to zero for $|\mathbf{x}| \to \infty$, and such that the same holds for the solutions $\tilde{\mathbf{w}}, \tilde{c}$ of (3.8). We skip again the arguments of $C_0 = C_0(\hat{\mathbf{w}}, t)$ and $C_k = C_k(\hat{\mathbf{w}}, t)$.
Let $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ with $|\boldsymbol{\alpha}| \in \{0, \dots, k\}$ be given. We multiply the first set of equations in (3.18) by $A_0(\hat{\mathbf{w}}) D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}$ and integrate with respect to $\mathbb{R}^2$. As in Lemma 3.2 we estimate all arising terms and get for the time derivative

$$\begin{aligned}(3.24)\qquad &\int_{\mathbb{R}^2} D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}_t(\mathbf{x}, t) \cdot A_0(\hat{\mathbf{w}}(\mathbf{x}, t)) D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}(\mathbf{x}, t)\, d\mathbf{x} \\ &\geq \frac{1}{2} \frac{d}{dt} \|\sqrt{A_0(\hat{\mathbf{w}}(., t))} D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}(., t)\|^2_{L^2(\mathbb{R}^2)} - \mathcal{C}(C_0) \|D^{\boldsymbol{\alpha}} \mathbf{v}(., t)\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Also in the same way we compute for $i = 1, 2$ and some constant $\delta \in (0, 1)$

$$\begin{aligned}(3.25)\qquad &\int_{\mathbb{R}^2} A_i(\hat{\mathbf{w}}(\mathbf{x}, t)) D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}(\mathbf{x}, t) \cdot A_0(\hat{\mathbf{w}}(\mathbf{x}, t)) D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}(\mathbf{x}, t)\, d\mathbf{x} \\ &\leq \delta\big(\|D^{\boldsymbol{\alpha}} v_{1,x_i}(., t)\|^2_{L^2(\mathbb{R}^2)} + \|D^{\boldsymbol{\alpha}} v_{2,x_i}(., t)\|^2_{L^2(\mathbb{R}^2)}\big) + \mathcal{C}(C_0 + \delta^{-1}) \|\tilde{\mathbf{w}}(., t)\|^2_{H^k(\mathbb{R}^2)}.\end{aligned}$$

Furthermore we obtain from $k \geq 2$ and Lemma 3.3.

$$\begin{aligned}(3.26)\qquad &\alpha^2 \left| \int_{\mathbb{R}^2} \hat{\rho}(\mathbf{x}, t) \nabla\big(D^{\boldsymbol{\alpha}} \tilde{c}(., t)(\mathbf{x}) - D^{\boldsymbol{\alpha}} \rho(\mathbf{x}, t)\big) \cdot D^{\boldsymbol{\alpha}} \mathbf{v}(\mathbf{x}, t)\, d\mathbf{x} \right| \\ &\leq \mathcal{C}(C_k + \delta^{-1}) \left(\|\tilde{\mathbf{w}}(., t)\|^2_{H^k(\mathbb{R}^2)} + + \|\tilde{c}(., t)\|^2_{H^{k+1}(\mathbb{R}^2)} \right) + \delta \|\mathbf{v}(., t)\|^2_{H^{k+1}(\mathbb{R}^2)} \\ &\leq \mathcal{C}(C_k + \delta^{-1}) \|\tilde{\mathbf{w}}(., t)\|^2_{H^k(\mathbb{R}^2)} + \delta \|\mathbf{v}(., t)\|^2_{H^{k+1}(\mathbb{R}^2)}.\end{aligned}$$

For the last line we used the energy estimate for $\tilde{c}$ obtained from the elliptic equation in (3.18).
To treat the viscous part in (3.18) consider for $i = 1, 2$

$$(3.27)\qquad -\int_{\mathbb{R}^2} \frac{1}{\hat{\rho}(\mathbf{x}, t)} \Delta D^{\boldsymbol{\alpha}} v_i(\mathbf{x}, t) \hat{\rho}(\mathbf{x}, t) D^{\boldsymbol{\alpha}} v_i(\mathbf{x}, t)\, d\mathbf{x} = \|\nabla D^{\boldsymbol{\alpha}} v_i(., t)\|^2_{L^2(\mathbb{R}^2)}.$$

We collect the estimates (3.24), (3.25), (3.26), (3.27). Then, for $t \in (0, T)$ and sufficiently small $\delta \in (0, 1)$, we deduce the inequality

$$\begin{aligned}(3.28)\qquad &\frac{d}{dt}\left(\frac{1}{2} \|\sqrt{A_0(\hat{\mathbf{w}}(\mathbf{x}, t))} D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}(., t)\|^2_{L^2(\mathbb{R}^2)} \right) + \|\nabla D^{\boldsymbol{\alpha}} \mathbf{v}(., t)\|^2_{L^2(\mathbb{R}^2)} \\ &\leq \mathcal{C}(C_0) \|\tilde{\mathbf{w}}(., t)\|^2_{H^k(\mathbb{R}^2)} + \|\mathbf{F}(., t)\|^2_{H^k(\mathbb{R}^2)} + \left| \int_{\mathbb{R}^2} \mathcal{L}^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})[\mathbf{w}](\mathbf{x}, t) \cdot A_0(\hat{\mathbf{w}}(\mathbf{x}, t)) D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}(\mathbf{x}, t)\, d\mathbf{x} \right|.\end{aligned}$$

It remains to estimate the last term. We apply Lemma 3.3 and, for $\delta \in (0, 1)$, we get

$$\begin{aligned}&\left| \int_{\mathbb{R}^2} \mathcal{L}^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})[\mathbf{w}](\mathbf{x}, t) \cdot A_0(\hat{\mathbf{w}}(\mathbf{x}, t)) D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}(\mathbf{x}, t)\, d\mathbf{x} \right| \\ &\leq \mathcal{C}(C_0) \|\mathcal{L}^{\boldsymbol{\alpha}}(\hat{\mathbf{w}})[\mathbf{w}](., t)\|_{L^2(\mathbb{R}^2)} \|\tilde{\mathbf{w}}(., t)\|_{H^k(\mathbb{R}^2)} \\ &\leq \mathcal{C}(C_k)\Big(\|\nabla v_1(., t)\|_{H^k(\mathbb{R}^2)} + \|\nabla v_2(., t)\|_{H^k(\mathbb{R}^2)} + \|\tilde{\mathbf{w}}(., t)\|_{H^k(\mathbb{R}^2)} \Big) \|\tilde{\mathbf{w}}(., t)\|_{H^k(\mathbb{R}^2)} \\ &\leq \mathcal{C}(C_k)\Big(\delta(\|\nabla v_1(., t)\|^2_{H^k(\mathbb{R}^2)} + \|\nabla v_2(., t)\|_{H^k(\mathbb{R}^2)})^2 + \delta^{-1} \|\tilde{\mathbf{w}}(., t)\|^2_{H^k(\mathbb{R}^2)} \Big).\end{aligned}$$

We use the last estimate in (3.28). Since (3.28) holds for each $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ with $|\boldsymbol{\alpha}| \leq k$ we can sum up over all such $\boldsymbol{\alpha}$ and obtain for $\delta \in (0, 1)$ sufficiently small

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2} \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^2, |\boldsymbol{\alpha}| \leq k} \|\sqrt{A_0(\hat{\mathbf{w}}(\mathbf{x}, t))} D^{\boldsymbol{\alpha}} \tilde{\mathbf{w}}(., t)\|^2_{L^2(\mathbb{R}^2)} \right) + \|\mathbf{v}(., t)\|^2_{H^{k+1}(\mathbb{R}^2)} \\ &\qquad \leq \mathcal{C}(C_k) \left(\|\tilde{\mathbf{w}}(., t)\|^2_{H^k(\mathbb{R}^2)} + \|\mathbf{F}(., t)\|_{H^k(\mathbb{R}^2)} \right).\end{aligned}$$

As in the proof of Lemma 3.2 the application of Gronwall's inequality implies (3.22) for sufficiently regular coefficients and initial function The estimate (3.22) itself does only rely on H^4-norms. Therefore a density argument gives the complete statement.
The second and third statement follow directly from the differentiated continuity equation in (3.18) and Sobolev embedding. □

Step II: Proof of Theorem 3.1 (i). Using the a-priori estimates from Step I, in the second step we make use of the method of successive approximations to derive Theorem 3.1.

Proof of Theorem 3.1(i). An induction proof shows that there is a $T_{**} \in (0,T)$ and a constant $C_{**} > 0$ such that for $n \in \mathbb{N}$ there is an unique weak solution $(\mathbf{w}_n, \tilde{c}_n) \in \mathcal{W}(0,T_{**})$ of the linear problem

$$(3.29)\qquad \begin{aligned} L(\mathbf{w}_{n-1})[\mathbf{w}_n, c_n] &= \mathbf{0} && \text{in } \Omega_{T_{**}} \\ -\Delta\tilde{c}_n + \alpha^2\tilde{c}_n &= \alpha^2\tilde{\rho}_n && \text{in } \mathbb{R}^2 \times [0,T_{**}) \\ \mathbf{w}_n(.,0) &= \mathbf{w}_0 && \text{in } \mathbb{R}^2, \end{aligned}$$

and such that the uniform estimate

$$(3.30)\qquad \begin{aligned} C_4(\mathbf{w}_n, T_{**}) &= \operatorname*{ess\,sup}_{t\in[0,T_{**}]} \left\{ \|\tilde{\mathbf{w}}_n(.,t)\|_{H^4(\mathbb{R}^2)} \right\} + \|\mathbf{v}_n\|_{L^2(0,T_{**};H^5(\mathbb{R}^2))} \\ &\quad + \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^2,\, |\boldsymbol{\alpha}|\le 2} \|D^{\boldsymbol{\alpha}}\mathbf{w}_n\|_{L^\infty(\Omega_T)} \\ &\quad + \|\rho_n^{-1}\|_{L^\infty(\Omega_{T_{**}})} + \|\rho_{n,t}\|_{L^\infty(\Omega_{T_{**}})} \\ &\le C_{**} \end{aligned}$$

holds. The numbers T_{**} and C_{**} are independent of n. The induction proof is straightforward by using Lemma 3.2, Lemma 3.4 and $\tilde{\mathbf{w}}_0 \in H^4(\mathbb{R}^2)$. The required time-regularity in the definition of $\mathcal{W}(0,T_{**})$ is a consequence of $\mathbf{w}_n$ being a weak solution of $(3.29)_1$.
The functions $\mathbf{W}_n \in L^\infty(0,T_{**};H^2(\mathbb{R}^2))$ and $\mathbf{c}_n \in L^\infty(0,T_{**};H^2(\mathbb{R}^2))$ defined by

$$\mathbf{W}_n = \mathbf{w}_n - \mathbf{w}_{n-1},\ \tilde{\mathbf{c}}_n = \tilde{c}_n - \tilde{c}_{n-1} \quad n = 1,2,\ldots$$

build the unique weak solution of the problem

$$(3.31)\qquad \begin{aligned} L(\mathbf{w}_{n-1})[\mathbf{W}_n, \tilde{\mathbf{c}}_n] &= \mathbf{F}_n && \text{in } \Omega_{T_{**}} \\ -\Delta\tilde{\mathbf{c}}_n + \alpha^2\tilde{\mathbf{c}}_n &= \alpha^2(\rho_n - \rho_{n-1}) && \text{in } \mathbb{R}^2 \times [0,T_{**}), \\ \mathbf{F}_n &:= -L(\mathbf{w}_{n-1})[\mathbf{w}_{n-1}, \tilde{c}_{n-1}], \\ \mathbf{W}_n(.,0) &= \mathbf{0} && \text{in } \mathbb{R}^2. \end{aligned}$$

Note that we have $\mathbf{F}_n \in L^\infty(0,T_{**};H^2(\mathbb{R}^2))$ due to the regularity of $\mathbf{w}_{n-1}$ and Sobolev embeddings. Thus we can apply the inequality (3.22) in Lemma 3.4 with $k = 2$ and $\mathbf{F} = \mathbf{F}_n$. We obtain for almost all $t \in (0, T_{**}]$ the estimate

$$
\begin{aligned}
&\|\mathbf{W}_n(.,t)\|^2_{H^2(\mathbb{R}^2)} \\
&\leq \mathcal{C}(C_2(\mathbf{w}_{n-1}, T_{**})) \int_0^t \|\mathbf{F}_n(.,s)\|^2_{H^2(\mathbb{R}^2)}\, ds \\
&= \mathcal{C}(C_2(\mathbf{w}_{n-1}, T_{**})) \int_0^t \|L(\mathbf{w}_{n-2})[\mathbf{w}_{n-1}, \tilde{c}_{n-1}](.,s) - L(\mathbf{w}_{n-1})[\mathbf{w}_{n-1}, \tilde{c}_{n-1}](.,s)\|^2_{H^2(\mathbb{R}^2)}\, ds \\
&\leq \mathcal{C}(C_2(\mathbf{w}_{n-1}, T_{**})) \\
&\quad \times \mathcal{D}\Big(\big\|\rho^{-1}_{n-1}\big\|_{L^\infty(\Omega_{T_{**}})} + \|\tilde{\mathbf{w}}_{n-1}\|_{L^\infty(0,T_{**};H^3(\mathbb{R}^2))} \\
&\qquad\qquad + \big\|\rho^{-1}_{n-2}\big\|_{L^\infty(\Omega_{T_{**}})} + \|\tilde{\mathbf{w}}_{n-2}\|_{L^\infty(0,T_{**};H^4(\mathbb{R}^2))}\Big) \\
&\quad \times \int_0^t \|\mathbf{W}_{n-1}(.,s)\|^2_{H^2(\mathbb{R}^2)}\, ds. \\
&\leq \mathcal{C}\left(C_{**}\right) \operatorname*{ess\,sup}_{s\in[0,T_{**}]} \big\{ \|\mathbf{W}_{n-1}(.,s)\|^2_{H^2(\mathbb{R}^2)} \big\} t.
\end{aligned}
\tag{3.32}
$$

For the last line we used the Sobolev embedding and the uniform estimate (3.30). Let us point out that we tacitly used again the elliptic estimates for the iterates $\tilde{c}_n$ in terms of lower-order Sobolev-norms for the density $\rho_n - \bar{\rho}$. Thus $\tilde{c}_n$ does not show up in the estimates.
We choose now $T_* \in (0, T_{**}]$ such that we have with $\mathcal{C}$ as in the last line of (3.32) the estimate

$$
\mathcal{C}\left(C_{**}\right) T_* < \frac{1}{2}. \tag{3.33}
$$

Thus we observe from (3.32) that $\{\tilde{\mathbf{w}}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $L^\infty(0, T_*; H^2(\mathbb{R}^2))$. There exists a function $\tilde{\mathbf{w}} \in L^\infty(0, T_*; H^2(\mathbb{R}^2))$ with $\tilde{\mathbf{w}}_n \to \tilde{\mathbf{w}}$ in this space. Elliptic energy estimates and the second relation in (3.31) show that $\{\tilde{c}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(0, T_*; H^4(\mathbb{R}^2)$. We denote the limit by $\tilde{c}$.
Furthermore from (3.30) (and energy estimate for $\tilde{c}_n$) we conclude (by taking subsequences)

$$
\tilde{\mathbf{w}}_n \stackrel{*}{\rightharpoonup} \tilde{\mathbf{w}} \text{ in } L^\infty(0, T_*; H^4(\mathbb{R}^2)),\ \mathbf{v}_n \rightharpoonup \mathbf{v} \text{ in } L^2(0, T_*; H^5(\mathbb{R}^2)) \text{ and } \tilde{c}_n \rightharpoonup \tilde{c} \text{ in } L^2(0, T_*; H^5(\mathbb{R}^2)).
$$

Thus we have

$$
\tilde{\mathbf{w}} \in L^\infty(0, T_*; H^4(\mathbb{R}^2)) \text{ and } \mathbf{v}, \tilde{c} \in L^2(0, T_*; H^5(\mathbb{R}^2)). \tag{3.34}
$$

Now, from the notion of a weak solution for (3.8), in particular from the equation (3.9), and the convergence of the sequence $\{(\tilde{\mathbf{w}}_n, c_n)\}_{n\in\mathbb{N}}$ in particular in the space $L^\infty(0, T_*; H^2(\mathbb{R}^2))$ we conclude that the limit functions $\mathbf{w}, \tilde{c}$ satisfies

$$
\begin{aligned}
-\int_{\Omega_{T_*}} \mathbf{w}\psi_t(\mathbf{x},t)\, d\mathbf{x}dt &= \int_0^{T_*} \mathbf{R}[\mathbf{w}, \mathbf{w}, \tilde{c}, \psi](t)\, dt, \\
-\int_{\Omega_{T_*}} \tilde{c}(\mathbf{x},t)\Delta\psi(\mathbf{x},t)\, d\mathbf{x}dt &= \alpha^2 \int_{\Omega_{T_*}} (\rho(\mathbf{x},t) - \bar{\rho} - \tilde{c}(\mathbf{x},t))\psi(\mathbf{x},t)\, d\mathbf{x}dt
\end{aligned}
\tag{3.35}
$$

for all $\psi \in C_0^\infty(\Omega_{T_*})$ (now a function of $\mathbf{x}$ and t!).
Due to integration by parts we can shift all spatial derivatives in the integrand $\mathbf{R}$ of the weak formulation for $\mathbf{w}$ of (3.35) from the test function since $\tilde{\mathbf{w}}, \tilde{c} \in L^\infty(0, T_*; H^4(\mathbb{R}^2))$. Then (3.35) and (3.34) show

that the time derivative of $\tilde{\mathbf{w}}$ exists and is in $L^\infty(0,T_*;H^2(\mathbb{R}^2))$. Moreover we have for the velocity components $\mathbf{v}_t \in L^2(0,T_*;H^3(\mathbb{R}^2))$ with the same argument and $\mathbf{v} \in L^2(0,T_*;H^5(\mathbb{R}^2))$ from (3.34). Since the ρ_n-component of $\mathbf{w}_n \in \mathcal{W}(0,T_{**})$ is uniformly bounded from below by a positive constant ρ is positive. From $\rho_t \in L^\infty(0,T_*;H^2(\mathbb{R}^2))$ we conclude (by differenciation of the elliptic equation for c with respect to time and once more elliptic regularity) that $\tilde{c}_t$ exists and is in $L^2(0,T_*;H^3(\mathbb{R}^2)$. This implies that $\mathbf{w}$ and all its derivatives that appear in (3.1) are continuous functions in Ω_{T_*} (see (3.6), (3.6)). We can now define the function $c := \tilde{c} - \bar{\rho}$ which is that regular such that Δc exists as continuous function. The functions $\mathbf{w}, c$ satisfy the equations in (3.1) as classical solutions. For the function c we have $c \in C(\mathbb{R}^2 \times [0,T_*])$.
The uniqueness of classical solution $(\mathbf{w}, c)$ follows in the same spirit as above. □

Proof of Theorem 3.1(ii). In this part the index α is used again. Since we have $W(\tilde{\rho}) = W'(\tilde{\rho}) = 0$ and $\tilde{\rho}(.,t) \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R})$ for $t \in [0,T_*]$ we can apply Proposition 2.5 to the classical solution $(\mathbf{w}^\alpha, c^\alpha)$ from (i) and get

$$\begin{aligned}&\int_{\mathbb{R}^2} \frac{1}{2}\rho^\alpha(\mathbf{x},t)|\mathbf{v}^\alpha(\mathbf{x},t)|^2 + W(\rho^\alpha(\mathbf{x},t)) + \frac{\alpha^2}{2}(\rho^\alpha(\mathbf{x},t) - c^\alpha(\mathbf{x},t))^2 + \frac{1}{2}|\nabla c^\alpha(\mathbf{x},t)^2|\,d\mathbf{x} \\ &\leq \int_{\mathbb{R}^2} \frac{1}{2}\rho_0(\mathbf{x})|\mathbf{v}_0(\mathbf{x})|^2 + W(\rho_0(\mathbf{x})) + \frac{\alpha^2}{2}(\rho_0(\mathbf{x}) - c^\alpha(\mathbf{x},0))^2 + \frac{1}{2}|\nabla c^\alpha(\mathbf{x},0)^2|\,d\mathbf{x}.\end{aligned}$$

This leads in particular for some $C > 0$ independent of α to

$$\begin{aligned}&\|\rho^\alpha - c^\alpha\|_{L^\infty(0,T_*;L^2(\mathbb{R}^2))} \\ &\leq \frac{2}{\alpha^2}\int_{\mathbb{R}^2} \frac{1}{2}\rho_0(\mathbf{x})|\mathbf{v}_0(\mathbf{x})|^2 + W(\rho_0(\mathbf{x}))\,d\mathbf{x} + \|\rho_0 - c^\alpha(.,0)\|^2_{L^2(\mathbb{R}^2))} + \frac{1}{\alpha^2}\|\nabla c^\alpha(.,0)\|^2_{L^2(\mathbb{R}^2)} \\ &\leq \frac{2}{\alpha^2}\int_{\mathbb{R}^2} \frac{1}{2}\rho_0(\mathbf{x})|\mathbf{v}_0(\mathbf{x})|^2 + W(\rho_0(\mathbf{x}))\,d\mathbf{x} + C\|\rho_0 - c^\alpha(.,0)\|^2_{L^2(\mathbb{R}^2))} \\ &=: Z_1^\alpha + Z_2^\alpha.\end{aligned}$$

To get the last estimate we used the weak formulation $(3.9)_2$ with $t = 0$. Obviously we have $Z_1^\alpha \to 0$ for $\alpha \to \infty$. Moreover since $\tilde{\rho}_0 \in H^4(\mathbb{R}^2)$ we have in particular for some constant $D > 0$ independent of α the inequality

$$\|\tilde{c}^\alpha(.,0)\|_{H^2(\mathbb{R}^2)} \leq D\|\tilde{\rho}_0\|_{H^4(\mathbb{R}^2)}.$$

Thus we observe again directly from the weak formulation $(3.9)_2$ with $t = 0$ that Z_2^α tends to 0 for $\alpha \to \infty$. This proves (ii). □

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