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Contents

Graded Mesh Refinement and Error Estimates of Higher Order for DGFE-solutions of Elliptic Boundary Value Problems in Polygons*

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Abstract

Error estimates for DGFE-solutions are well investigated if one assumes that the exact solution is sufficiently regular. In this paper we consider a Dirichlet and a mixed boundary value problem for a linear elliptic equation in a polygon. It is well known, that the first derivatives of the solutions develop singularities near reentrant corner points or points where the boundary conditions change. Based on regularity results formulated in Sobolev-Slobodetskii spaces and weighted spaces of Kondratjev type we prove error estimates of higher order for DGFE-solutions using a suitable graded mesh refinement near boundary singular points. The main tools are: regularity investigations for the exact solution relying on general results for elliptic boundary value problems, error analysis for the interpolation in Sobolev-Slobodetskii spaces and error estimates for DGFE-solutions on special graded refined meshes combined with estimates in weighted Sobolev spaces. Our main result is that there exist a local grading of the mesh and a piecewise interpolation by polynoms of higher degree, such that we will get the same order $O(h^{\alpha})$ of approximation as in the smooth case.

Keywords: elliptic boundary value problems, discontinuous Galerkin method, weighted Sobolev spaces, Sobolev-Slobodetskii spaces, graded mesh refinement

AMS Subject Classification: 35J50, 65M60, 65M15, 65M12

Introduction

In a number of complex problems from science and technology we meet the requirement to apply efficient, robust, reliable and accurate numerical methods for the solution of partial differential equations. In computational fluid dynamics, the finite volume method, using piecewise constant (and hence, discontinuous) approximations of a sought solution, is very popular. On the other hand, in structural mechanics or electro-magnetic fields one usually applies conforming Galerkin finite element techniques, using continuous piecewise polynomial approximations on a finite element mesh. The combination of the finite element and finite volume approaches leads to the discontinuous Galerkin finite element method (DGFEM), which uses piecewise polynomial approximations of the sought solution on a finite element mesh without any requirement on the continuity between neighbouring elements. This causes that the DGFEM leads to very flexible schemes for the solution of complicated problems.

The DGFEM was first used in [45] for the solution of a neutron linear transport equation and analyzed theoretically in [44] and later in [38]. Nearly simultaneously the discontinuous Galerkin techniques were developed for the numerical solution of elliptic or parabolic problems ([55], [5]). Further, the DGFEM

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was applied to nonlinear conservation laws ([15]), compressible flows ([7], [8], [9], [18], [20], [33], [54]), simulation of compressible low Mach number flows at incompressible limits ([28], [29]), solution of incompressible viscous flows ([49], reactive transport in porous media flows ([51]), shallow water flows ([17]), the Hamilton-Jacobi equations ([37]), biharmonic equation ([50]) and the Maxwell equations ([34]).

The literature devoted to theoretical analysis of the DGFEM is rapidly growing. It is possible to mention only a limited number of works dealing with the theory of the DGFEM applied to elliptic or parabolic problems and nonlinear convection-diffusion problems as, e.g. [5], [6], [12], [19], [21], [22], [24], [25], [35], [36], [43], [46], [47], [52], [53].

In works dealing with error estimates of the DGFEM, the analyzed problems are considered in polygonal domains (or polyhedral domains in 3D case) and the error analysis is carried out under the assumption that the exact solutions are sufficiently regular, namely that they are elements of some Sobolev space H^m, where $m \geq 2$. However, this assumption is not always realistic. It is well-known, see [39], [32], [16], [40], [41], that boundary corners, edges and points, where different boundary conditions meet, cause a singular behaviour of the solution in the vicinity of these points. Then the global regularity of the exact solutions is lower. These situations were analyzed in the framework of conforming Galerkin finite element methods. In this case it is possible to apply the theory of interpolation in the Sobolev-Slobodetskii spaces of functions with the so-called noninteger (or fractional) derivatives. See, e.g. [11] or [26]. The low regularity of the exact solutions causes a decrease of order of accuracy in error estimates. Another possibility is to use a graded mesh refinement in a neighbourhood of singular boundary points, where the grading parameter depends on the local behaviour of the solution which is given by the exponent of the distance to the singular boundary points, see e.g. [48], [1], [4], [2]. Due to the local grading of the mesh, one gets the same order $O(h^\alpha)$ of approximation as in the smooth case, where h is the maximal size of the mesh.

In this paper we transfer the method of graded mesh refinements to DGFEM solutions of elliptic boundary value problems of second order in polygons. To our knowledge, in the framework of the DGFEM, this is a new method and have not yet been analyzed. It demands a generalization of the FEM-techniques based on estimates in Sobolev-Slobodetskii and weighted Sobolev spaces combined with the graded mesh refinement to the DGFEM. The main tools are the detailed regularity investigation for the exact solution of the boundary value problem with Dirichlet or mixed boundary conditions relying on general results for elliptic boundary value problems, error analysis for the interpolation in Sobolev-Slobodetskii spaces and error estimates for DGFE-solutions on special graded refined meshes combined with estimates in weighted Sobolev spaces. Our main result is that there exist a local grading and a piecewise interpolation by polynoms of higher degree, such that we shall get the same order of approximation for DGFEM solutions as in the smooth case.

The structure of the paper is as follows: In Section 1, the continuous problem is formulated. Section 2 is concerned with properties of Sobolev-Slobodetskii and weighted Sobolev spaces and the analysis of the behaviour of the exact solution near singular boundary points in case of Dirichlet and mixed boundary conditions. A detailed description of the leading singular terms of weak solutions is derived. In Section 3, the discrete discontinuous Galerkin is introduced. Section 4 contains some auxiliary results from the analysis of the discontinuous Galerkin method and their generalization to the Sobolev-Slobodetskii spaces. In Section 5, an abstract error estimate is proved. It is used in Section 6 for obtaining the global error estimate in terms of the mesh size. Section 7 is devoted to the proof of the main result of the paper, which is the optimal higher-order error estimate in the maximal size of the mesh, obtained with the aid of the graded mesh refinement and weighted Sobolev spaces. In Conclusion some open problems and subjects for future work are formulated.

1 Continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where $\Gamma_D \cap \Gamma_N = \emptyset$, Γ_D and Γ_N are open sets in $\partial\Omega$ and the one-dimensional measure of Γ_D is positive. We consider the following linear elliptic boundary value problem

$$L(x,D_x)u = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + cu = f \quad \text{in } \Omega, \tag{1.1}$$

$$B_1(x,D_x)u = u = 0 \quad \text{on } \Gamma_D, \tag{1.2}$$

$$B_2(x,D_x)u = \sum_{i,j=1}^{2} a_{ij}(x)\,\frac{\partial u}{\partial x_j}\,n_i = q \quad \text{on } \Gamma_N. \tag{1.3}$$

Because of the derivation of regularity properties of the solution, for simplicity we consider the homogeneous Dirichlet boundary condition. The nonhomogeneous Dirichlet condition can be treated in such a way that it is prolonged inside the domain Ω, the exact solution is expressed as the sum of this prolongation and a function satisfying equation (1.1) and the boundary condition (1.3) with right-hand sides modified due to the prolongation of the Dirichlet condition and the homogeneous Dirichlet boundary condition (1.2).

If $a_{ij}, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $q \in L^2(\Gamma_N)$, we can introduce the weak formulation of problem (1.1) – (1.3). We denote $V = \{v \in H^1(\Omega); v|_{\Gamma_D} = 0\}$ (where the restriction $v|_{\Gamma_D}$ is considered in the sense of traces). The *weak solution* of problem (1.1) – (1.3) is defined as a function $u \in V$ such that

$$a(u,v) = (f,v) + (q,v)_{\Gamma_N} \quad \forall\, v \in V, \tag{1.4}$$

where

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij}\,\frac{\partial u}{\partial x_j}\,\frac{\partial v}{\partial x_i} + cuv\right)\mathrm{d}x, \tag{1.5}$$

$$(f,v) = \int_\Omega fv\,\mathrm{d}x, \quad (q,v)_{\Gamma_N} = \int_{\Gamma_N} qv\,\mathrm{d}v. \tag{1.6}$$

If, moreover,

$$c \geq 0 \quad \text{and} \quad \mu_1|\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\,\xi_i\,\xi_j \leq \mu_2|\xi|^2 \tag{1.7}$$

for all $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$ and all $x \in \Omega$, then the assumptions of the Lax–Milgram theorem hold:

$$|a(w,v)| \leq \alpha_1\|w\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \quad \forall\, w,\, v \in V, \tag{1.8}$$

$$a(v,v) \geq \alpha_2\|v\|^2_{H^1(\Omega)} \quad \forall\, v \in V, \tag{1.9}$$

with constants $\alpha_1,\ \alpha_2 > 0$. Hence, problem (1.1) – (1.3) has a unique weak solution $u \in V$.

2 Regularity properties of the solution

2.1 Sobolev–Slobodetskii spaces and weighted Sobolev spaces

Let us assume that $\hat\Omega \subset \mathbb{R}^2$ is a bounded domain with a Lipschitz boundary $\partial\hat\Omega$, $k \geq 0$ is an integer and $\gamma \in (0,1)$. In what follows we shall work with the Sobolev–Slobodovskii space $H^{k+\gamma}(\hat\Omega) = W^{k+\gamma,2}(\hat\Omega)$, defined as the subspace of $H^k(\hat\Omega)$ formed by all functions v for which

$$I_{k,\gamma,\hat\Omega}(v) = \left(\sum_{|\alpha|=k}\int_{\hat\Omega}\int_{\hat\Omega}\frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x-y|^{2+2\gamma}}\,\mathrm{d}x\mathrm{d}y\right)^{1/2} < +\infty. \tag{2.1}$$

The space $H^{k+\gamma}(\hat\Omega)$ equipped with the norm

$$\|v\|_{H^{k+\gamma}(\hat\Omega)} = \left(\|v\|_{H^k(\hat\Omega)} + I_{k,\gamma,\hat\Omega}(v)^2\right)^{1/2}, \quad v \in H^{k+\gamma}(\hat\Omega), \tag{2.2}$$

is a reflexive Banach space. The functional

$$|v|_{H^{k+\gamma}(\hat{\Omega})} = I_{k,\gamma,\hat{\Omega}}(v), \quad v \in H^{k+\gamma}(\hat{\Omega}), \tag{2.3}$$

is a seminorm in $H^{k+\gamma}(\hat{\Omega})$. (For details, see, e. g., [30]).

Over the boundary $\partial\hat{\Omega}$ of $\hat{\Omega} \subset \mathbb{R}^2$ we can also define the Sobolev–Slobodetskii space $H^{\tilde{\gamma}}(\partial\hat{\Omega})$ for $\tilde{\gamma} \in (0,1)$ formed by functions $v \in L^2(\partial\hat{\Omega})$ such that

$$I_{0,\tilde{\gamma},\partial\hat{\Omega}}(v) = \left(\int_{\partial\hat{\Omega}} \int_{\partial\hat{\Omega}} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2\tilde{\gamma}}} \,\mathrm{d}x\mathrm{d}y \right)^{1/2} < +\infty. \tag{2.4}$$

$H^{\tilde{\gamma}}(\partial\hat{\Omega})$ equipped with the norm

$$\|v\|_{H^{\tilde{\gamma}}(\partial\hat{\Omega})} = \left(\|v\|^2_{L^2(\partial\hat{\Omega})} + I_{0,\tilde{\gamma},\partial\hat{\Omega}}(v)^2 \right)^{1/2} \tag{2.5}$$

is a Banach space.

Now let us consider the case $k=0,\ \gamma \in (\frac{1}{2},1]$. By [31] or [30], the space $H^{\gamma-1/2}(\partial\hat{\Omega}))$ is formed by the traces on $\partial\hat{\Omega}$ of all functions $v \in H^{\gamma}(\hat{\Omega})$. In $H^{\gamma-1/2}(\partial\hat{\Omega})$ we can introduce the norm defined by

$$\|v\|_{H^{\gamma-1/2}(\partial\hat{\Omega})} = \inf_{\substack{\tilde{v}\in H^{\gamma}(\hat{\Omega}) \\ \tilde{v}|_{\partial\hat{\Omega}}=v}} \|\tilde{v}\|_{H^{\gamma}(\hat{\Omega})} \quad \forall\, v \in H^{\gamma-1/2}(\partial\hat{\Omega}). \tag{2.6}$$

This norm is equivalent to the norm defined by (2.5) with $\tilde{\gamma} = \gamma - 1/2$ (see, e. g., [31]). In what follows, we shall use the norm (2.6). Then for any $v \in H^{\gamma}(\hat{\Omega})$ we can write

$$\|v\|_{L^2(\partial\hat{\Omega})} \le \hat{C}\, \|v\|_{H^{\gamma-1/2}(\partial\hat{\Omega})} \le \hat{C}\, \|v\|_{H^{\gamma}(\hat{\Omega})}, \tag{2.7}$$

where the constant depends on $\hat{\Omega}$, but is independent of $v \in H^{\gamma}(\hat{\Omega})$.

The regularity theory for elliptic boundary value problems in non-smooth domains with corners and edges is well developed, especially in the framework of weighted Sobolev spaces. We formulate here regularity results for solutions of the general weak problem (1.1) – ((1.3) in the following weighted Sobolev spaces of Kondrat'ev type. We denote by M the set of singular boundary points, which consists of corner points and boundary points, where the type of the boundary condition changes. We assume that the set $M = \{O_i, i=1,\dots,m\}$ is finite. We introduce the function space $C^\infty_M(\Omega) := \{u \in C^\infty(\overline{\Omega}) : \operatorname{supp} u \cap M = \emptyset\}$. For $1 \le p < \infty$ the space $V^{k,p}(\Omega,\vec{\beta})$ is the closure of the set $C^\infty_M(\Omega)$ with respect to the norm

$$\|u\|_{V^{k,p}(\Omega,\vec{\beta})} = \left(\sum_{|\alpha|\le k} \int_\Omega \Pi_{i=1}^m r_i^{p(\beta_i-k+|\alpha|)} |D^\alpha u|^p \mathrm{d}x \right)^{1/p}, \tag{2.8}$$

where $r_i = r_i(x) = \operatorname{dist}(x, O_i)$ and $\vec{\beta} = (\beta_1, \cdots, \beta_m)$ is a vector of real numbers. The product in (2.8) means that the weights work locally in neighorborhoods of the points O_i. For simplicity we will consider later the case that only one corner point O occurs and the norm (2.8) reads in this case

$$\|u\|_{V^{k,p}(\Omega,\beta)} = \left(\sum_{|\alpha|\le k} \int_\Omega r^{p(\beta-k+|\alpha|)} |D^\alpha u|^p \mathrm{d}x \right)^{1/p}, \tag{2.9}$$

where $r = r(x) = \operatorname{dist}(x, O)$ and β is a real number.

We remark that

$$V^{k,p}(\Omega,\vec{\beta}) \subset V^{k-1,p}(\Omega,\vec{\beta}-1) \tag{2.10}$$

and

$$\|u\|_{V^{k-1,p}(\Omega,\vec{\beta}-1)} \le \|u\|_{V^{k,p}(\Omega,\vec{\beta})}. \tag{2.11}$$

Analogously to (2.6) we introduce the trace space $V^{k-\frac{1}{p},p}(\partial\Omega,\vec{\beta})$ for an integer $k > 0$: The space $V^{k-\frac{1}{p},p}(\partial\Omega,\vec{\beta})$ consists of traces on $\partial\Omega$ of functions in $V^{k,p}(\Omega,\vec{\beta})$ and is equipped with the norm

$$\|u\|_{V^{k-\frac{1}{p},p}(\partial\Omega,\vec{\beta})} = \inf \|v\|_{V^{k,p}(\Omega,\vec{\beta})},$$

where the infimum is taken over the set of all functions $v \in V^{k,p}(\Omega,\vec{\beta})$ such that $v = u$ on $\partial\Omega$. For more details, see [42].

2.2 Behaviour of the solutions near singular boundary points

Let us assume that u is the weak solution of (1.1) – (1.3) for $f \in L_2(\Omega)$ and for smooth coefficients a_{ij}, c and boundary datum q. As we already mentioned, we assume that the set $\overline{\Gamma}_D \cap \overline{\Gamma}_N$ is formed by a finite number of points.

In order to apply the regularity theory in weighted Sobolev spaces we demand that

$$u \in V \cap V^{1,2}(\Omega, 0). \tag{2.12}$$

In [48, Property (R)], it is shown that (2.12) is satisfied for a large class of problems including Dirichlet problems and mixed boundary value problems. Note that pure Neumann problems are excluded. In [48], the regularity problem was considered, namely, there was discussed for which β the solution u is contained in the space $V^{2+k,2}(\Omega,\beta)$, provided $f \in W^{k,2}(\Omega)$ and $q \in W^{k+\frac{1}{2},2}(\partial\Omega)$. The regularity is given by the distribution of the eigenvalues of a parameter dependent boundary value problem. One can get this parameter dependent boundary value problem by considering the principal parts of the differential operators L, B_1 and B_2 of the boundary value problem (1.1) – (1.3) with frozen coefficients at points of M, using spherical coordinates, followed by a Mellin transform with respect to r. Let us illustrate this approach for a polygonal domain $\Omega \subset I\!R^2$ which has only one conical point O on its boundary. For simplicity we assume that there is a ball-neighbourhood of O, for which Ω coincides with the cone $\mathcal{C} = \{(r,\omega) : 0 < r < \infty, 0 < \omega < \omega_0\}$. Here (r,ω) are the standard polar coordinates. We consider a special boundary value problem in $\mathcal{C}$, which is generated by the principal parts of $-L$, B_1, and B_2 with frozen coefficients in O:

$$\begin{array}{rclcll} L_0(O,D_x)u(x) & := & -\sum\limits_{i,j=1}^{2} a_{i,j}(O)\frac{\partial^2}{\partial x_i \partial x_j}u(x) & = & f(x) & \text{in } \mathcal{C}, \\ B_{0,1}(O,D_x)u(x) & := & u(x) & = & g_1(x) & \text{on } (\partial\mathcal{C})_D, \\ B_{0,2}(O,D_x)u(x) & := & \sum\limits_{i,j=1}^{2} a_{i,j}(O)\frac{\partial u(x)}{\partial x_j}n_i & = & g_2(x) & \text{on } (\partial\mathcal{C})_N. \end{array}$$

Introducing polar coordinates (r,ω) and using the Mellin transform

$$\hat{u}(\alpha,\omega) = \frac{1}{\sqrt{2\pi}}\int_0^\infty r^{-\alpha-1}u(r,\omega)dr$$

we obtain a boundary value problem with the parameter α:

$$\begin{array}{rclll} \hat{L}(\omega,D_\omega,\alpha)\hat{u}(\alpha,\omega) & = & \hat{F}(\alpha,\omega) & \text{for } \omega \in G = (0,\omega_0), \\ \hat{B}_j(\omega,D_\omega,\alpha)\hat{u}(\alpha,\omega) & = & \hat{G}_j(\alpha,\omega) & \text{for } \omega \in \partial G,\ j = 1,2, \end{array} \tag{2.13}$$

where $\hat{F} = r^{2m}\hat{f}$ and $\hat{G}_j = r^{m_j}\hat{g}_j$. We denote the operator belonging to the boundary value problem (2.13) by

$$\mathcal{A}(\alpha) = (\hat{L}(\omega,D_\omega,\alpha), \hat{B}_j(\omega,D_\omega,\alpha), j = 1,2) \tag{2.14}$$

The distribution of the eigenvalues α (those complex numbers α_0 for which non-trivial solutions $\hat{u}$ of (2.13) for $\hat{F} = 0$ and $\hat{G}_j = 0$, $j = 1,2$, exist) in a certain strip in the complex plane determines the regularity. The following theorem was proved in [39] and can also be found in [42, 48].

Theorem 1. *Let Ω be a bounded domain with one angular corner point O and smooth boundary else. The weak solution u of problem (1.1) – (1.3) with the right hand side $f \in V^{k,2}(\Omega, 1-H_0+\varepsilon+k)$ and the Neumann datum $q \in V^{k+\frac{1}{2},2}(\Gamma_N, 1-H_0+\varepsilon+k)$ is contained in $V^{2+k,2}(\Omega, 1-H_0+\varepsilon+k)$, and*

$$\|u\|_{V^{2+k,2}(\Omega,1-H_0+\varepsilon+k)} \leq C(\|f\|_{V^{k,2}(\Omega,1-H_0+\varepsilon+k)} + \|q\|_{V^{k+\frac{1}{2},2}(\partial\Omega,1-H_0+\varepsilon+k)}), \tag{2.15}$$

where $H_0 = \Re(\alpha_0)$. Here, α_0 is an eigenvalue of problem (2.13) with maximal real part, such that the strip $0 < \Re(\alpha) < \Re(\alpha_0)$ is free of eigenvalues; $\varepsilon > 0$ is an arbitrarily small real number.

Remark 1. *If $H_0 \leq 1$, then $W^{k,2}(\Omega) \subset V^{k,2}(\Omega, 1-H_0+\varepsilon+k)$ and $W^{k+\frac{1}{2},2}(\partial\Omega) \subset V^{k+\frac{1}{2},2}(\partial\Omega, 1-H_0+\varepsilon+k)$ and the estimate (2.15) implies that*

$$\|u\|_{V^{2+k,2}(\Omega,1-H_0+\varepsilon+k)} \leq C\|f\|_{W^{k,2}(\Omega)} + \|q\|_{W^{k+\frac{1}{2},2}(\partial\Omega)}. \tag{2.16}$$

Furthermore, there is an asymptotic expansion of the solution u with respect to the distance to the singular boundary point O, see e.g. [39], [41].

Theorem 2. *Let Ω be a bounded domain with one angular corner point O and smooth boundary else. Assume that the volume force densities $f \in V^{k,2}(\Omega, 1-H_0+\varepsilon+k)$ and that the Neumann datum $q \in V^{k+\frac{1}{2},2}(\Gamma_N, 1-H_0+\varepsilon+k)$. Suppose that $\mathcal{A}(\alpha)$ is invertible on the line $Re\alpha = k+1-\tilde{\beta}$, where $0 < \beta - \tilde{\beta} < 1$. Then the weak solution $u \in V$ of the boundary problem admits the following decomposition:*

$$u = u_{reg} + \sum_{\gamma\in\Lambda_{\tilde{\beta}}} \eta c_\gamma v_\gamma(r,\omega) \ . \tag{2.17}$$

Here, $u_{reg}\big|_\Omega \in V^{k+2,2}(\Omega, \tilde{\beta})$ and

$$\Lambda_{\tilde{\beta}} = \{\gamma = (\alpha,\mu,\kappa) : \ \alpha \text{ is an eigenvalue of } \mathcal{A}(\alpha) \text{ in the strip} \\ 0 < Re\alpha < k+1-\tilde{\beta};\ \mu = 1,\ldots,I_\alpha;\ \kappa = 0,\ldots,M_{\alpha,\mu}\}\ , \tag{2.18}$$

I_α denotes the geometrical multiplicity of α, $\{\Phi_{\alpha,\mu,\kappa},\ \mu = 1,\ldots,I_\alpha; \kappa = 0,\ldots,M_{\alpha,\mu}\}$ is a canonical system of Jordan chains of $\mathcal{A}(\alpha)$ with respect to eigenvalue α, $M_{\alpha,\mu}+1$ are the lengths of the Jordan chains, η is a cut-off function which isequal to 1 near O and the singular functions v_γ are of the form

$$v_\gamma(r,\varphi) = r^\alpha \sum_{q=0}^{\kappa} \frac{(\ln r)^q}{q!} \Phi_{\alpha,\mu,\kappa-q}(\omega),\ . \tag{2.19}$$

The coefficients c_γ are constants.

Theorem1 and Theorem2 can be formulated for polygons, where more than one singular boundary point occur. In this case we have to consider instead of H_0 the vector $\vec{H}_0 = (H_{0,1},\cdots,H_{0,m})$ and the corresponding weighted spaces $V^{k,p}(\Omega,\vec{\beta})$ with the norm (2.8). The decomposition (2.17) reads then

$$u = u_{\text{reg}} + \sum_{i=1}^{m} \sum_{\gamma_i\in\Lambda_{\tilde{\beta}_i}} \eta_i c_{\gamma_i} v_{\gamma_i}(r_i,\omega_i)\ .$$

Let us discuss the Dirchlet and the mixed boundary value problem for the Laplacian as an example. Let O_i be one of the corner points. If Dirichlet conditions are prescribed on both sides of O_i, then we say shortly $D-D$ boundary conditions are given, if a Dirichlet condition on one side and Neumann condition on the other side are prescribed, then we say shortly $D-N$ boundary conditions occur.

Example 1. *Let Ω be a polygon and O_i a singular boundary point from M with the angle $\omega_{0,i}$. We consider the mixed boundary value problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= q \quad \text{on } \Gamma_N, \end{aligned} \tag{2.20}$$

where the right hand sides f and q satisfy the assumptions of Theorem 1 where H_0 is replaced by $\vec{H}_0$. It holds in the $D-D$-case that $H_{0_i} = \frac{\pi}{\omega_{0_i}}$, and in the $D-N$ case that $H_{0_i} = \frac{\pi}{2\omega_{0_i}}$. The asymptotic expansion (2.17) with the ith term for the pure Dirchlet conditions $D-D$ reads: If we choose $k=0,\ \tilde{\beta} = 0,\ \omega_0 > \pi$, then

$$u = u_{reg} + \cdots + \eta_i c_i r_i^{\frac{\pi}{\omega_{0_i}}} \sin(\frac{\pi}{\omega_{0_i}}\omega_i) + \cdots \tag{2.21}$$

where $u_{reg} \in V^{2,2}(\Omega, \tilde{\beta})$.

For the mixed boundary conditions we get the singular expansion choosing again $k = 0, \tilde{\beta} = 0$:

$$u = u_{reg} + \cdots + \eta_i c_i r_i^{\frac{\pi}{2\omega_{0_i}}} \cos(\frac{\pi}{2\omega_{0_i}}\omega_i) + \cdots , \quad \textit{for } \frac{\pi}{2} < \omega_{0_i} < \frac{3\pi}{2}, \tag{2.22}$$

$$u = u_{reg} + \cdots + \eta_i c_{i,1} r_i^{\frac{\pi}{2\omega_{0_i}}} \cos(\frac{\pi}{2\omega_{0_i}}\omega_i) + \eta_i c_{i,2} r_i^{\frac{3\pi}{2\omega_{0_i}}} \cos(\frac{3\pi}{2\omega_{0_i}}\omega_i) + \cdots , \quad \textit{for } \frac{3\pi}{2} < \omega_0 < 2\pi. \tag{2.23}$$

In what follows we need a Green's formula in weighted Sobolev spaces. Let us denote

$$Lu = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu, \quad A = (a_{ij})_{i,j=1,2}\,. \tag{2.24}$$

Theorem 3. *Let Ω be a bounded polygon with the singular boundary points $M = \{O_i, i = 1\cdots m\}$, $u \in V^{2,2}(\Omega, 1 - \vec{H}_0 + \varepsilon)$. Then*

$$\int_\Omega Luv\,dx = \int_\Omega A\nabla u \cdot \nabla v dx + \int_\Omega cuv\,dx + \int_{\partial\Omega} A\nabla u \cdot nv\,ds \quad \forall v \in V^{1,2}(\Omega, \vec{H}_0 - \varepsilon). \tag{2.25}$$

Proof. See [41], p.317. □

□

Now we discuss the behaviour of solutions near a singular boundary point O_i of the more general boundary value problem (1.1) – (1.3). For simplicity we omit the index i. The calculation of H_0 can be done by different methods, e.g. one can make an local ansatz $u_{\text{sing}} = r^\alpha \Phi(\alpha, \omega)$ and calculate the corresponding eigenvalues and eigenfunctions of $\mathcal{A}(\alpha)$. Another possibility is to consider the principal parts with frozen coefficients of the boundary value problem (1.1) – (1.3) at O and transform it to a mixed boundary value problem for the Laplacian. The latter method leads to a general estimate of H_0:

Lemma 1. *Let Ω be a polygon. Then the leading singular term which corresponds to the singular point O of the weak solution is characterized by H_0, where*

$$H_0 > \frac{1}{2}, \quad \textit{in the D-D case}, \tag{2.26}$$

$$H_0 > \frac{1}{4}, \quad \textit{in the D-N case}, \tag{2.27}$$

$$H_0 > \frac{1}{2}, \quad \textit{in the D-N case for } \omega_0 < \pi, \tag{2.28}$$

$$H_0 = \frac{1}{2}, \quad \textit{in the D-N case for } \omega_0 = \pi. \tag{2.29}$$

Proof. We shall proceed in several steps.

1st STEP– *Transformation to a mixed problem for the anisotropic Laplacian*
We start from the original problem (1.1) – (1.3) and consider the special problem:

$$-\text{div}(A(O)\nabla u) = F \quad \text{in } \Omega, \tag{2.30}$$
$$u = 0 \quad \text{on } \Gamma_D, \tag{2.31}$$
$$A(O)\nabla u \cdot n = Q \quad \text{on } \Gamma_N. \tag{2.32}$$

Let be $\lambda_1, \lambda_2 > 0$ be the eigenvalues of the matrix $A(O)$ with $\lambda_1 \leq \lambda_2$. We consider a corresponding orthonormal system of eigenvectors $\vec{e_1}, \vec{e_2}$, such that for the matrix H, whose columns are $\vec{e_1}, \vec{e_2}$, holds

$$\det H = 1.$$

The linear principal axes transformation

$$x = Hy, \; y = H^\top x$$

is a rotation around O and preserves the opening angle ω_0 as well the length of vectors. We denote by Ω_y the rotated domain and use the index y for all transformed quantities. Due to Green's formula (2.25), the boundary problem (2.30), (2.31), (2.32) will be transformed to

$$-(\lambda_1 \frac{\partial^2 u_y}{\partial y_1^2} + \lambda_2 \frac{\partial^2 u_y}{\partial y_2^2}) = F_y \quad \text{in } \Omega_y, \tag{2.33}$$
$$u_y = 0 \quad \text{on } \Gamma_{y,D}, \tag{2.34}$$
$$D(\lambda_1, \lambda_2)\nabla u_y \cdot n_y = Q_y \quad \text{on } \Gamma_{y,N}, \tag{2.35}$$

where $D(\lambda_1, \lambda_2)$ is the diagonal matrix.

2nd STEP– *Transformation to a mixed problem for the isotropic Laplacian*
We transform the boundary value problem (2.33), (2.34), (2.35) to a mixed boundary value problem for the isotropic Laplacian setting

$$z_1 = \frac{y_1}{\sqrt{\lambda_1}}, \quad z_2 = \frac{y_2}{\sqrt{\lambda_2}}. \tag{2.36}$$

We mark all quantities with the index z in the $z_1 - z_2$ coordinates. The transformed mixed boundary value problem reads:

$$-\Delta u_z = F_z \quad \text{in } \Omega_z, \tag{2.37}$$
$$u_z = 0 \quad \text{on } \Gamma_{z,D}, \tag{2.38}$$
$$\nabla u_z \cdot n_z = Q_z \quad \text{on } \Gamma_{z,N}. \tag{2.39}$$

In particular, we are interested in the opening angle of the transformed cone K_z with the tip O. Since the transformation (2.36) maps straight lines to straight lines, we get

$$\tan \omega_{z,0} = \sqrt{\frac{\lambda_1}{\lambda_2}} \tan \omega_0 \tag{2.40}$$

It follows

$$\omega_{z,0} \leq \omega_0 \quad \text{for } 0 < \omega_0 < \frac{\pi}{2}, \quad \frac{\pi}{2} < \omega_0 < \frac{3\pi}{2} \text{ and } \frac{3\pi}{2} < \omega_0 < 2\pi. \tag{2.41}$$

Furthermore,

$$\text{if } \omega_0 = \frac{\pi}{2} \text{ or } \omega_0 = \frac{3\pi}{2}, \text{ then } \omega_{z,0} = \omega_0. \tag{2.42}$$

3rd STEP– *Backward transformation of the leading singular term*
Due to example 1, the leading singular terms of solutions of the boundary value problem (2.37), (2.38), (2.39) have the following form for pure Dirichlet conditions

$$u_{z,\text{sing}} = r_z^{\frac{\pi}{\omega_{z,0}}} \sin(\frac{\pi}{\omega_{z,0}} \omega_z),$$

for mixed boundary conditions

$$u_{z,\text{sing}} = r_z^{\frac{\pi}{2\omega_{z,0}}} \cos(\frac{\pi}{2\omega_{z,0}}\omega_z).$$

We retransform $u_{z,\text{sing}}$ to the $x_1 - x_2-$coordinates. We have

$$r_z^2 = z_1^2 + z_2^2 = \frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} = r_y^2 \left(\frac{\cos\omega_y^2}{\lambda_1} + \frac{\sin\omega_y^2}{\lambda_2} \right)$$

and, as before,

$$\tan\omega_z = \frac{z_2}{z_1} = \sqrt{\frac{\lambda_1}{\lambda_2}} \tan\omega_y,$$

which leads to $\omega_z = \arctan(\sqrt{\frac{\lambda_1}{\lambda_2}}\tan\omega_y)$ piecewise in the intervals $0 < \omega_0 < \frac{\pi}{2}$, $\frac{\pi}{2} < \omega_0 < \frac{3\pi}{2}$ and $\frac{3\pi}{2} < \omega_0 < 2\pi$. Note that for the exceptional angles (2.42) holds. Having in mind that for the backward transformation from the y-coordinates into the x-coordinates, the relations $r_y = r_x$ and $\omega_y = \omega_x$ hold, we finally get the following results:

For pure Dirichlet conditions we have

$$u_{x,\text{sing}} = r_x^{\frac{\pi}{\omega_{z,0}}} \sqrt{\frac{\cos\omega_x^2}{\lambda_1} + \frac{\sin\omega_x^2}{\lambda_2}} \sin\left(\frac{\pi}{\omega_{z,0}} \arctan(\sqrt{\frac{\lambda_1}{\lambda_2}}\tan\omega_x) \right) = r_x^{\frac{\pi}{\omega_{z,0}}} \Phi(\frac{\pi}{\omega_{z,0}}, \omega_x), \qquad (2.43)$$

where $\omega_{z,0} = \arctan(\sqrt{\frac{\lambda_1}{\lambda_2}}\tan\omega_0)$ is defined piecewise as above.

For mixed boundary conditions we have

$$u_{x,\text{sing}} = r_x^{\frac{\pi}{2\omega_{z,0}}} \sqrt{\frac{\cos\omega_x^2}{\lambda_1} + \frac{\cos\omega_x^2}{\lambda_2}} \sin\left(\frac{\pi}{2\omega_{z,0}} \arctan(\sqrt{\frac{\lambda_1}{\lambda_2}}\tan\omega_x) \right) = r_x^{\frac{\pi}{2\omega_{z,0}}} \Psi(\frac{\pi}{2\omega_{z,0}}, \omega_x). \qquad (2.44)$$

Note, that (2.40), (2.41) and (2.42) determine $\omega_{z,0}$. Setting $H_0 = \frac{\pi}{\omega_{z,0}}$ for the Dirichlet problem and $H_0 = \frac{\pi}{2\omega_{z,0}}$ for the mixed problem, we have proved the lemma. □

□

In what follows we consider polygons, that means more than one singular boundary point occur. Theorem1 can be modified and we get:

Corollary 1. *The weak solution u of problem (1.1) – (1.3) is an element of the space $V^{2+k,2}(\Omega, 1 - \vec{H}_0 + \varepsilon + k)$, provided $f \in V^{k,2}(\Omega, 1 - \vec{H}_0 + \varepsilon + k)$ and $q \in V^{k+\frac{1}{2},2}(\Gamma_N, 1 - \vec{H}_0 + \varepsilon + k)$. Thus, if $k = 0$, we have $u \in V^{2,2}(\Omega, 1 - \vec{H}_0 + \varepsilon)$, where $\varepsilon > 0$ is arbitrarily small.*

We have two possibilities for simplification with respect to the number of singular boundary points:

- We use a localozation and consider different graded mesh refinements separately for individual singular points.
- We set $H_0 = \min\{H_{0,i}, i = 1, \cdots, m\}$ and consider the same (finest) graded mesh refinement for every singular point.

I what follows we restrict our theoretical considerations to the last case. If one is interested in numerical experiments then different gradings should be investigated. It follows from Lemma 1 and the above corollary that for every singular boundary point the leading singularity of the solution is characterized by the opening angle $\omega_{z_i,0_i}$. We set $H_0 = \min\{H_{0,i}, i = 1, \cdots, m\}$ and consider Corollary 1 with the modified weighted spaces endowed with the norm

$$\|u\|_{V^{k,2}(\Omega,\beta)} = \left(\sum_{|\alpha| \le k} \int_\Omega r^{2(\beta-k+|\alpha|)} |D^\alpha u|^2 \mathrm{d}x \right)^{1/2}, \qquad (2.45)$$

where $r = r(x) = \mathrm{dist}(x, M)$ and $\beta = 1 - H_0 + \varepsilon + k$. Moreover, since we are interested in the treatment of singularities, we shall consider the case when $H_0 \le 1$.

It is possible to show, compare [32], Theorem 1.4.5.3, that the solution u of the boundary value problem (1.1) – (1.3) belongs to $H^{1+H_0-\varepsilon}(\Omega)$ for an arbitrarily small positive ε. The idea is going back to Babuŝka, who proposed to check for which p the leading singularity is from $W^{2,p}(\Omega)$ and then to use imbedding theorems. Choosing an appropriate fixed ε and setting

$$\gamma = H_0 - \varepsilon, \tag{2.46}$$

we shall consider

$$\gamma \in \left(\frac{1}{2}, 1\right) \tag{2.47}$$

such that the solution of problem (1.1) - (1.3) satisfies the condition

$$u \in H^{1+\gamma}(\Omega). \tag{2.48}$$

This means that for the Dirichlet problem, condition (2.47) is satisfied, for the mixed boundary conditions, we have to assume that $\omega_0 < \pi$.

3 Discontinuous Galerkin discretization

Let $\mathcal{T}_h$ be a triangulation of the domain Ω with standard properties. This means that $\mathcal{T}_h$ is formed by a finite number of closed triangles with mutually disjoint interiors. If $K, K' \in \mathcal{T}_h$ are different elements, then $K \cap K' = \emptyset$ or $K \cap K'$ is a common side of K and K' or $K \cap K'$ is a common vertex of K and K'. Moreover, we assume that the points of $\overline{\Gamma}_D \cap \overline{\Gamma}_N$ are vertices of elements $K \in \mathcal{T}_h$, adjacent to $\partial\Omega$. The sides of $K \in \mathcal{T}_h$ will be called faces.

In our further considerations we shall use the following notation. For an element $K \in \mathcal{T}_h$ we set $h_K = \mathrm{diam}(K),\ h = \max_{K\in\mathcal{T}_h} h_K$. By ρ_K we denote the radius of the largest circle inscribed into K and by $|K|$ we denote the two-dimensional Lebesgue measure of K.

By $\mathcal{F}_h$ we denote the system of all faces of all elements $K \in \mathcal{T}_h$. Further, we define the set of all innner faces by

$$\mathcal{F}_h^I = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \Omega\}, \tag{3.1}$$

and the set of all boundary faces by

$$\mathcal{F}_h^B = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \partial\Omega\}. \tag{3.2}$$

Further, we distinguish the sets of the Dirichlet and Neumann faces:

$$\mathcal{F}_h^D = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \Gamma_D\}, \quad \mathcal{F}_h^N = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \Gamma_N\}. \tag{3.3}$$

Obviously, $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$, $\mathcal{F}_h^B = \mathcal{F}_h^D \cup \mathcal{F}_h^N$.

For each $\Gamma \in \mathcal{F}_h$ we choose a unit vector $\boldsymbol{n}_\Gamma$ orthogonal to Γ. We assume that for $\Gamma \in \mathcal{F}_h^B$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$. For each face $\Gamma \in \mathcal{F}_h^I$ the orientation of $\boldsymbol{n}_\Gamma$ is arbitrary but fixed. Finally, by $d(\Gamma)$ we denote the diameter of $\Gamma \in \mathcal{F}_h$.

Over a triangulation $\mathcal{T}_h$, for any reals number $s > 0$ we define the *broken Sobolev (- Slobodetskii) space*

$$H^s(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^s(K)\ \forall\, K \in \mathcal{T}_h\} \tag{3.4}$$

equipped with the seminorm

$$|v|_{H^s(\Omega,\mathcal{T}_h)} = \left(\sum_{K\in\mathcal{T}_h} |v|^2_{H^s(K)}\right)^{1/2}. \tag{3.5}$$

If $\Gamma \in \mathcal{F}_h^I$, then there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial K_\Gamma^{(R)}$. We use the convention that $\boldsymbol{n}_\Gamma$ is the outer normal to the element $K_\Gamma^{(L)}$ and the inner normal to the element $K_\Gamma^{(R)}$. For $v \in H^1(\Omega, \mathcal{T}_h)$ and $\Gamma \in \mathcal{F}_h^I$ we introduce the following notation:

$$
\begin{aligned}
& v|_\Gamma^{(L)} = \text{the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \quad v|_\Gamma^{(R)} = \text{the trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma, \\
& \langle v \rangle_\Gamma = \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)} \right), \quad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}.
\end{aligned} \tag{3.6}
$$

The value $[v]_\Gamma$ depends on the orientation of $\boldsymbol{n}_\Gamma$, but the value $[v]_\Gamma \boldsymbol{n}_\Gamma$ is independent of this orientation.

If $[\cdot]_\Gamma, \langle \cdot \rangle_\Gamma$ and $\boldsymbol{n}_\Gamma$ appear in an integral $\int_\Gamma \ldots \, \mathrm{d}S$, where $\Gamma \in \mathcal{F}_h^I$, we omit the subscript Γ and write simply $[\cdot], \langle \cdot \rangle$ and $\boldsymbol{n}$.

Let $p \geq 1$ be an integer. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions

$$
S_{hp} = \{ v \in L^2(\Omega); v|_K \in P^p(K), \ \forall K \in \mathcal{T}_h \}, \tag{3.7}
$$

where $P^p(K)$ denotes the space of all polynomials on K of degree $\leq p$.

In view of (2.48) and (2.47), for each $K \in \mathcal{T}_h$ and $\Gamma \in \mathcal{F}_h^I$ we have

$$
\begin{aligned}
u|_{\partial\Omega} &\in H^{1+\gamma-1/2}(\partial\Omega), \\
u|_{\partial K} &\in H^{1+\gamma-1/2}(\partial K), \quad [u]_\Gamma = 0, \\
\nabla u &\in H^\gamma(\Omega), \\
\nabla u|_{\partial K} &\in H^{\gamma-1/2}(\partial K) \subset L^2(\partial K), \quad [\nabla u]_\Gamma = 0, \quad \langle \nabla u \rangle_\Gamma = \nabla u|_\Gamma.
\end{aligned} \tag{3.8}
$$

Moreover, we have $u \in V^{2,2}(K, 1 - H_0 + \varepsilon)$ and by Theorem 3, the following Green's formula holds for every $K \in \mathcal{T}_h$:

$$
\begin{aligned}
\int_K Lu \, v \, \mathrm{d}x &= \int_K \sum_{i,j=1}^2 a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \, \mathrm{d}x + \int_K cuv \, \mathrm{d}x \\
&\quad + \int_{\partial K} \sum_{i,j=1}^2 a_{ij} \frac{\partial u}{\partial x_j} n_i v \, \mathrm{d}x \quad \forall v \in V^{1,2}(K, H_0 - \varepsilon).
\end{aligned} \tag{3.9}
$$

The discontinuous Galerkin discrete problem is derived in a standard way. We test equation (1.1) by any $v \in H^2(\Omega, \mathcal{T}_h)$, take into account that $H^2(K) \subset V^{1,2}(K, H_0 - \varepsilon)$, use the generalized Green's theorem and the boundary condition (1.3) and to the resulting relation we add terms vanishing or cancelling with each other due to (3.8) and the Dirichlet condition (1.2). After simple manipulation we get the identity

$$
A_h(u, v) = \ell_h(v), \quad v \in H^2(\Omega, \mathcal{T}_h), \tag{3.10}
$$

where

$$
A_h(w, v) = a_h(w, v) + \mu_1 J_h(w, v), \tag{3.11}
$$

and

$$
\begin{aligned}
a_h(w, v) &= \sum_{K \in \mathcal{T}_h} \int_K \left(\sum_{i,j=1}^2 a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} + cwv \right) \mathrm{d}x \\
&\quad - \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \left(\sum_{i,j=1}^2 a_{ij} \left\langle \frac{\partial w}{\partial x_j} \right\rangle n_i^\Gamma [v] + \theta \sum_{i,j=1}^2 a_{ij} \left\langle \frac{\partial v}{\partial x_j} \right\rangle n_i^\Gamma [w] \right) \mathrm{d}S \\
&\quad - \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \left(\sum_{i,j=1}^2 a_{ij} \frac{\partial w}{\partial x_j} n_i^\Gamma v + \theta \sum_{i,j=1}^2 a_{ij} \frac{\partial v}{\partial x_j} n_i^\Gamma w \right) \mathrm{d}S,
\end{aligned} \tag{3.12}
$$

$$J_h(w,v) = \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma[w]\,[v]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma \sigma wv\,\mathrm{d}S, \tag{3.13}$$

$$\ell_h(v) = \int_\Omega fv\,\mathrm{d}x + \int_{\Gamma_N} qv\,\mathrm{d}S \tag{3.14}$$

These forms make sense for $w \in H^{1+\gamma}(\Omega,\mathcal{T}_h)$ and $v \in H^2(\Omega,\mathcal{T}_h)$. The form J_h is called interior and boundary penalty. We define the weight σ by the relations

$$\sigma|_\Gamma = h(\Gamma)^{-1}, \quad h(\Gamma) = d(\Gamma)/C_W, \tag{3.15}$$

where $C_W > 0$ is a suitable constant.

As for the choice of the parameter θ, three possibilities are used: $\theta = -1$, $\theta = 0$ and $\theta = 1$, which lead to the nonsymmetric, incomplete and symmetric approximation of diffusion terms, combined with interior and boundary penalty, called NIPG, IIPG and SIPG method, respectively.

On the basis of (3.10) we define an *approximate solution* as a function $u_h \in S_{hp}$ such that

$$A_h(u_h, v_h) = \ell_h(v_h) \quad \forall\, v_h \in S_{hp}. \tag{3.16}$$

It follows from (3.10) and (3.16) that the error $e_h = u_h - u$ of the DG method (3.16) satisfies the condition

$$A_h(e_h, v_h) = 0 \quad \forall\, v_h \in S_{hp}. \tag{3.17}$$

In what follows we shall be concerned with the estimation of the error e_h.

4 Some auxiliary results

Let us consider a system of triangulations $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ $(h_0 > 0)$ of the domain Ω with the following property:

(A1) $\{\mathcal{T}_n\}_{h\in(0,h_0)}$ is regular. This means that there exists a constant $C_R > 0$ such that

$$h_K/\rho_K \le C_R \quad \forall\, K \in \mathcal{T}_h,\ \forall\, h \in (0,h_0). \tag{4.1}$$

In what follows we shall always assume that assumption (A1) is satisfied.

4.1 Interpolation

By $(\cdot,\cdot)$ and $(\cdot,\cdot)_K$ we denote the $L^2(\Omega)$- and $L^2(K)$-scalar product, respectively.

Because of the error analysis we shall introduce an S_{hp}-interpolation operators π_h as the $L^2(\Omega)$-projection on the space S_{hp}: if $v \in L^2(\Omega)$, then

$$\pi_h v \in S_{hp}, \quad (\pi_h v - v,\, \varphi) = 0 \quad \forall\, \varphi \in S_{hp}. \tag{4.2}$$

In other words,

$$\begin{aligned} &(\pi_h v)\big|_K \in P^p(K) \quad \forall\, K \in \mathcal{T}_h, \\ &\big((\pi_h v)\big|_K - v\big|_K,\, \varphi\big)_K = 0 \quad \forall\, \varphi \in P^p(K),\ \forall\, K \in \mathcal{T}_h. \end{aligned} \tag{4.3}$$

It is important to analyze *approximation properties of the interpolation* π_h. Using similar techniques as in [14] and [26], we can obtain the following result.

Lemma 2. *Let $\kappa \ge 1$ be integer, $\gamma \in (0,1)$ and $\nu = \min(p,\kappa)$. Then there exist constants C_A, $\tilde{C}_A > 0$ such that*

$$\begin{aligned} &\|v - \pi_h v\|_{L^2(K)} \le C_A\, h_K^{\nu+1} |v|_{H^{\nu+1}(K)}, \\ &|v - \pi_h v|_{H^1(K)} \le C_A\, h_K^{\nu} |v|_{H^{\nu+1}(K)}, \\ &|v - \pi_h v|_{H^2(K)} \le C_A\, h_K^{\nu-1} |v|_{H^{\nu+1}(K)}, \\ &\qquad \forall\, v \in H^{\kappa+1}(K), \quad \forall\, K \in \mathcal{T}_h, \quad \forall\, h \in (0,h_0) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&\|v - \pi_h v\|_{L^2(K)} \le \tilde{C}_A\, h_K^{\gamma+1} |v|_{H^{\gamma+1}(K)}, \\
&|v - \pi_h v|_{H^1(K)} \le \tilde{C}_A\, h_K^{\gamma} |v|_{H^{\gamma+1}(K)}, \\
&|v - \pi_h v|_{H^{\gamma+1}(K)} \le \tilde{C}_A\, |v|_{H^{\gamma+1}(K)} \\
&\qquad \forall\, v \in H^{\gamma+1}(K), \quad \forall\, K \in \mathcal{T}_h, \quad \forall\, h \in (0, h_0).
\end{aligned} \tag{4.5}$$

4.2 Inequalities

Let us introduce some inequalities, which will be used in the following considerations.

Multiplicative trace inequality: There exists a constant $C_M > 0$ such that

$$\begin{aligned}
\|v\|^2_{L^2(\partial K)} \quad &\le \quad C_M \left(\|v\|_{L^2(K)}\, |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right) \\
&\qquad \forall\, v \in H^1(K), \quad \forall\, K \in \mathcal{T}_h, \quad \forall\, h \in (0, h_0).
\end{aligned} \tag{4.6}$$

(For the proof see [23].) The use of Young's inequality in (4.6) yields

$$\|v\|^2_{L^2(\partial K)} \le \tilde{C}_M \left(h_K |v|^2_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right) \tag{4.7}$$

with a constant $\tilde{C}_M$ independent of $v,\ K,\ h$.

Inverse inequality: There exists a constant $C_I > 0$ such that

$$|v|_{H^1(K)} \le C_I\, h_K^{-1} \|v\|_{L^2(K)} \quad \forall\, v \in P^p(K),\, \forall K \in \mathcal{T}_h, \quad \forall\, h \in (0, h_0) \tag{4.8}$$

(cf. e. g. [14]).

Coercivity of the form A_h is represented by the inequality

$$A_h\left(\varphi_h, \varphi_h\right) \ge \alpha \|\varphi_h\|^2_{DG}, \quad \forall\, \varphi_h \in S_{hp},\, \forall\, h \in (0, h_0), \tag{4.9}$$

where $\alpha = \nu_1/2 > 0$ and

$$\|w\|_{DG} = \left(|w|^2_{H^1(\Omega, \mathcal{T}_h)} + J_h(w, w) \right)^{1/2} \tag{4.10}$$

is a norm in $H^1(\Omega, \mathcal{T}_h)$. In [27] it was proved that (4.9) is valid provided

$$\begin{aligned}
C_W \quad &> \quad 0 \quad \text{for NIPG}, \\
C_W \quad &\ge \quad 2C_M(1 + C_I) \quad \text{for IIPG}, \\
C_W \quad &\ge \quad 4C_M(1 + C_I) \quad \text{for SIPG},
\end{aligned} \tag{4.11}$$

where $C_W,\ C_M$ and C_I are constants from (3.15), (4.6) and (4.8), respectively.

Broken Poincaré inequality: There exists a constant $C_P > 0$ such that

$$\begin{aligned}
\|v_h\|^2_{L^2(\Omega)} \quad &\le \quad C_P \left(|v_h|^2_{H^1(\Omega, \mathcal{T}_h)} + J_h(v_h, v_h) \right) \\
&\qquad \forall\, v_h \in S_{hp},\, \forall\, h \in (0, h_0).
\end{aligned} \tag{4.12}$$

The proof follows from [10] and assumption (A1).

In what follows, for the sake of simplicity, by C we shall denote a positive generic constant, independent of $\varphi,\ v,\ K,\ h$, attaining different values in different places.

4.3 Generalization of inequality (4.7)

In our further considerations, we shall need a similar estimate to (4.7) valid for functions $v \in H^\gamma(K)$. In virtue of (2.47), $\gamma - 1/2 > 0$. The space $H^{\gamma-1/2}(\partial K)$ is continuously imbedded in $L^2(\partial K)$.

By $\hat{K}$ we denote the reference triangle with vertices $(0,0),\ (0,1),\ (1,0)$. For each $K \in \mathcal{T}_h$ there exists a one-to-one affine mapping $F_K : \hat{K} \stackrel{\text{onto}}{\longrightarrow} K,\ \hat{x} \in \hat{K} \to x = F_K(\hat{x}) = \mathbb{B}_K \hat{x} + b_K$, where $\mathbb{B}_K$ is a nonsingular 2×2 matrix and $b_K \in \mathbb{R}^2$. In virtue of assumption (4.1), there exist constants $c_1,\ c_2,\ c_3 > 0$ such that (see, e. g. [14])

$$\begin{array}{llll} \text{a)} & \|\mathbb{B}_K\| \le c_1\, h_K, & \text{b)} & \|\mathbb{B}_K^{-1}\| \le c_2/\rho_K, \\ \text{c)} & |\det \mathbb{B}_K| = 2|K|, & \text{d)} & c_3\,\rho_K^2 \le |K| \le c_3\, h_K^2 \\ & \forall\, K \in \mathcal{T}_h,\ \forall\, h \in (0,h_0). & & \end{array} \tag{4.13}$$

(It can be shown that the constant c_3 is the area of the unit circle in $\mathbb{R}^2$.)

Lemma 3. *Let $\gamma \in (\frac{1}{2}, 1]$. Then there exists a constant $C > 0$ independent of $\varphi,\ K,\ h$ such that*

$$\begin{aligned} \|\varphi\|_{L^2(\partial K)} \quad \le \quad & C\left(h_K^{-1/2}\|\varphi\|_{L^2(K)} + h_K^{\gamma-1/2}|\varphi|_{H^\gamma(K)}\right) \\ & \qquad \forall\, \varphi \in H^\gamma(K),\ \forall\, K \in \mathcal{T}_h,\ \forall\, h \in (0,h_0). \end{aligned} \tag{4.14}$$

Proof. We shall use some results from [26]. By Theorem 2.12 from [26], $\varphi \in H^\gamma(K)$ if and only if $\hat{\varphi} = \varphi \circ F_K \in H^\gamma(\hat{K})$. By the continuous imbedding $H^{\gamma-1/2}(\partial \hat{K}) \hookrightarrow L^2(\partial \hat{K})$ and (2.7),

$$\|\hat{\varphi}\|_{L^2(\partial \hat{K})} \le C\|\hat{\varphi}\|_{H^{\gamma-1/2}(\partial \hat{K})} \le C\|\hat{\varphi}\|_{H^\gamma(\hat{K})}, \tag{4.15}$$

where the constant $C = C(\hat{K}) > 0$ is independent of $\varphi,\ \hat{\varphi},\ K,\ h$. Moreover, using the substitution theorem, we can show that

$$\|\varphi\|^2_{L^2(\partial K)} = \tilde{C}\, h_K \|\hat{\varphi}\|^2_{L^2(\partial \hat{K})}, \tag{4.16}$$

with a constant $\tilde{C} = \tilde{C}(\hat{K})$ independent of $\varphi,\ \hat{\varphi},\ K,\ h$.

By (4.15), (4.16), (2.2) and (2.3),

$$\|\varphi\|_{L^2(\partial K)} \le C\, h_K^{1/2}\left(\|\hat{\varphi}\|_{L^2(\hat{K})} + |\hat{\varphi}|_{H^\gamma(\hat{K})}\right). \tag{4.17}$$

Now, in virtue of relations (2.13) and (2.14) from [26], where we set $N = p = 2$, $k = 0$, we find that

$$|\hat{\varphi}|_{H^\gamma(\hat{K})} \le C\|\mathbb{B}_K\|^{1+\gamma}\, |\det \mathbb{B}_K|^{-1}\, |\varphi|_{H^\gamma(K)}.$$

This and (4.13), a) c) and d) imply that

$$|\hat{\varphi}|_{H^\gamma(\hat{K})} \le C\, h_K^{-1+\gamma}|\varphi|_{H^\gamma(K)}. \tag{4.18}$$

Further, using the inequalities

$$\|\hat{\varphi}\|_{L^2(\hat{K})} \le C|\det \mathbb{B}_K|^{-1/2}\, \|\varphi\|_{L^2(K)} \le C\, h_K^{-1}\|\varphi\|_{L^2(K)}, \tag{4.19}$$

(4.13), c), d) and (4.1), we get (4.14). □

Remark. We see that for $\gamma = 1$, inequality (4.14) is equivalent to (4.7).

5 Abstract error estimate

In order to derive the estimate of the method error $e_h = u_h - u$, we write it in the form

$$e_h = \xi + \eta, \tag{5.1}$$

where

$$\xi = u_h - \pi_h u \in S_{hp}, \quad \eta = \pi_h u - u. \tag{5.2}$$

Then identity (3.17) with $v_h := \xi$ can be written in the form

$$A_h(\xi, \xi) = -A_h(\eta, \xi). \tag{5.3}$$

In virtue of the coercivity inequality (4.9) valid under assumption (4.11), we have

$$\alpha \|\xi\|_{DG}^2 \le |A_h(\eta, \xi)|. \tag{5.4}$$

Our first goal will be the estimation of $\|\xi\|_{DG}$. Taking into account (3.11), we see that we need to estimate the expressions $a_h(\eta, \xi)$ and $J_h(\eta, \xi)$. By (3.12),

$$a_h(\eta, \xi) = \vartheta_1 + \vartheta_2, \tag{5.5}$$

where

$$\begin{aligned} \vartheta_1 &:= \sum_{K \in \mathcal{T}_h} \int_K \left(\sum_{i,j=1}^2 a_{ij} \frac{\partial \eta}{\partial x_j} \frac{\partial \xi}{\partial x_i} + c\,\eta\,\xi \right) \mathrm{d}x, \qquad (5.6) \\ \vartheta_2 &:= -\sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \left(\sum_{i,j=1}^2 a_{ij} \left\langle \frac{\partial \eta}{\partial x_j} \right\rangle n_i^\Gamma [\xi] + \theta \sum_{i,j=1}^2 a_{ij} \left\langle \frac{\partial \xi}{\partial x_j} \right\rangle n_i^\Gamma [\eta] \right) \mathrm{d}S \\ &\quad - \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \left(\sum_{i,j=1}^2 a_{ij} \frac{\partial \eta}{\partial x_j} n_i^\Gamma \xi + \theta \sum_{i,j=1}^2 a_{ij} \frac{\partial \xi}{\partial x_j} n_i^\Gamma \eta \right) \mathrm{d}S. \end{aligned}$$

The use of the Cauchy inequality, (1.7), the assumption that $c \in L^\infty(\Omega)$ and the broken Poincaré inequality (4.12) immediately yield the estimate

$$|\vartheta_1| \le \mu_2 |\eta|_{H^1(\Omega, \mathcal{T}_h)} |\xi|_{H^1(\Omega, \mathcal{T}_h)} + C_P \|c\|_{L^\infty(\Omega)} \|\eta\|_{L^2(\Omega)} \|\xi\|_{DG}. \tag{5.7}$$

The estimation of ϑ_2 is more complex. By (1.7) and the Cauchy inequality, for $\Gamma \in \mathcal{F}_h^I$ we have

$$\begin{aligned} \left| \int_\Gamma \sum_{i,j=1}^2 a_{ij} \left\langle \frac{\partial \eta}{\partial x_j} \right\rangle n_i^\Gamma [\xi] \, \mathrm{d}S \right| &\le \mu_2 \int_\Gamma |\nabla \eta| \; |[\xi]| \, \mathrm{d}S \\ \le \;\; \mu_2 \left(h(\Gamma) \int_\Gamma |\langle \nabla \eta \rangle|^2 \, \mathrm{d}S \right)^{1/2} & \left(h(\Gamma)^{-1} \int_\Gamma [\xi]^2 \, \mathrm{d}S \right)^{1/2}. \end{aligned}$$

Using similar estimates for $\Gamma \in \mathcal{F}_h^D$ and other expressions appearing in ϑ_2 and the definition of the form J_h, we find that

$$\begin{aligned} |\vartheta_2| \;\; \le \;\; & \mu_2\, J_h(\xi, \xi)^{1/2} \left(\sum_{\Gamma \in \mathcal{F}_h^I} h(\Gamma) \int_\Gamma |\langle \nabla \eta \rangle|^2 \, \mathrm{d}s + \sum_{\Gamma \in \mathcal{F}_h^D} h(\Gamma) \int_\Gamma |\nabla \eta|^2 \, \mathrm{d}S \right)^2 \qquad (5.8) \\ & + \mu_2\, J_h(\eta, \eta)^{1/2} \left(\sum_{\Gamma \in \mathcal{F}_h^I} h(\Gamma) \int_\Gamma |\langle \nabla \xi \rangle|^2 \, \mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h^D} h(\Gamma) \int_\Gamma |\nabla \xi|^2 \, \mathrm{d}S \right)^{1/2}. \end{aligned}$$

Then, (3.15) and (5.8) imply that

$$|\vartheta_2| \leq \sqrt{2}\mu_2 \, J_h(\xi,\xi)^{1/2} \, C_W^{-1/2} \left(\sum_{K\in\mathcal{T}_h} h_K \int_{\partial K} |\nabla\eta|^2 \, \mathrm{d}S \right)^{1/2} + \sqrt{2}\mu_2 \, J_h(\eta,\eta)^{1/2} \, C_W^{-1/2} \left(\sum_{K\in\mathcal{T}_h} h_K \int_{\partial K} |\nabla\xi|^2 \, \mathrm{d}S \right)^{1/2}. \quad (5.9)$$

Using (4.6) and (4.8) (similarly as in [22]), we get

$$\sum_{K\in\mathcal{T}_h} h_K \int_{\partial K} |\nabla\xi|^2 \, \mathrm{d}S \leq C_M(1+C_I)\, |\xi|^2_{H^1(\Omega,\mathcal{T}_h)}. \quad (5.10)$$

Now let us estimate the expressions $\sum_{K\in\mathcal{T}_h} h_K \int_{\partial K} |\nabla\eta|^2 \, \mathrm{d}S$. We have $\nabla\eta = \nabla\Pi_h u - \nabla u \in H^\gamma(\Omega)$ and, hence, $\nabla\eta|_{\partial K} \in H^{\gamma-1/2}(\partial K)$ for $K \in \mathcal{T}_h$. By (4.14)

$$\sum_{K\in\mathcal{T}_h} h_K \int_{\partial K} |\nabla\eta|^2 \, \mathrm{d}S = \sum_{K\in\mathcal{T}_h} h_K \|\nabla\eta\|^2_{L^2(\partial K)} \leq C \sum_{K\in\mathcal{T}_h} \left(|\eta|^2_{H^1(K)} + h_K^{2\gamma} |\eta|^2_{H^{1+\gamma}(K)} \right). \quad (5.11)$$

Further, we shall estimate $J_h(\eta,\eta)$. By the definition (3.13) of the form J_h, the definition (3.15) of $h(\Gamma)$ and the inequality

$$(\varphi+\psi)^2 \leq 2(\varphi^2+\psi^2), \quad \varphi, \, \psi \in \mathbb{R}, \quad (5.12)$$

we get

$$J_h(\eta,\eta) \leq 4C_W \sum_{K\in\mathcal{T}_h} h_K^{-1} \int_{\partial K} |\eta|^2 \, \mathrm{d}S. \quad (5.13)$$

In virtue of $u \in H^{1+\gamma}(\Omega)$, we have $u \in H^1(\Omega)$ and by (4.7),

$$J_h(\eta,\eta) \leq C \sum_{K\in\mathcal{T}_h} \left(|\eta|^2_{H^1(K)} + h_K^{-2} \|\eta\|^2_{L^2(K)} \right). \quad (5.14)$$

Summarizing (5.5), (5.7), (5.9), (5.10), (5.11) and (5.14), we find that

$$|A_h(\eta,\xi)| \leq C\|\xi\|_{DG} \left(\sum_{K\in\mathcal{T}_h} \left(|\eta|^2_{H^1(K)} + h_K^{2\gamma} |\eta|^2_{H^{1+\gamma}(K)} + h_K^{-2} \|\eta\|^2_{L^2(K)} \right) \right)^{1/2}. \quad (5.15)$$

This and (5.4) yield the estimate

$$\|\xi\|_{DG} \leq C \left(\sum_{K\in\mathcal{T}_h} \left(|\eta|^2_{H^1(K)} + h_K^{2\gamma} |\eta|^2_{H^{1+\gamma}(K)} + h_K^{-2} \|\eta\|^2_{L^2(K)} \right) \right)^{1/2}. \quad (5.16)$$

Finally, using the triangle inequality

$$\|e_h\|_{DG} \leq \|\xi\|_{DG} + \|\eta\|_{DG} = \|\xi\|_{DG} + \left(\sum_{K\in\mathcal{T}_h} |\eta|^2_{H^1(K)} + J_h(\eta,\eta) \right)^{1/2}$$

and estimate (5.14), we get the *abstract error estimate*:

Theorem 4. *Let assumption (A1) be satisfied and let the exact solution $u \in H^{1+\gamma}(\Omega)$. Then the error of the DG method satisfies the estimate*

$$\|e_h\|_{DG} \leq C \sum_{K\in\mathcal{T}_h} \left(|\eta|^2_{H^1(K)} + h_K^{2\gamma} |\eta|^2_{H^{1+\gamma}(K)} + h_K^{-2} \|\eta\|^2_{L^2(K)} \right)^{1/2}, \quad h \in (0,h_0). \quad (5.17)$$

Remark. The parameter $\gamma \in \left(\frac{1}{2}, 1\right]$ corresponds to the regularity of the exact solution u. As follows from Section 2.2, u is regular inside the domain Ω, namely $u \in H^{\kappa+1}(K)$ for such $K \in \mathcal{T}_h$ for which $K \cap M = \emptyset$ and $u \in H^{1+\gamma}(K)$ with $\gamma \in (1/2, 1)$, if $K \cap M \neq \emptyset$. Moreover, from the above estimations one can see that for each $K \in \mathcal{T}_h$ we can write $u \in H^{s_K+1}(K)$, $K \in \mathcal{T}_h$, where $s_K = \kappa$, if $K \cap M = \emptyset$. Further, $s_K = \gamma \in \left(\frac{1}{2}, 1\right)$, if $K \cap M \neq \emptyset$. This leads us to the notation

$$\begin{aligned} \mathcal{T}_h^M &= \{K \in \mathcal{T}_h; K \cap M \neq \emptyset\}, \\ \mathcal{T}_h^0 &= \{K \in \mathcal{T}_h; K \cap M = \emptyset\}. \end{aligned} \tag{5.18}$$

Now we can write the abstract error estimate in the form

$$\begin{aligned} \|e_h\|_{DG} \le\; & C\Big(\sum_{K \in \mathcal{T}_h^0} \Big(|\eta|^2_{H^1(K)} + h_K^2 |\eta|^2_{H^2(K)} + h_K^{-2} |\eta|^2_{L^2(K)} \Big) \\ & + \sum_{K \in \mathcal{T}_h^M} \Big(|\eta|^2_{H^1(K)} + h_K^{2\gamma} |\eta|^2_{H^{1+\gamma}(K)} + h_K^{-2} \|\eta\|^2_{L^2(K)} \Big) \Big)^{1/2}, \quad h \in (0, h_0). \end{aligned} \tag{5.19}$$

6 Error estimates in terms of h_K

In order to obtain error estimates in terms of h_K for $K \in \mathcal{T}_h$, we have to estimate the norms of η appearing in (5.19) and take into account the regularity properties of the exact solution on elements $K \in \mathcal{T}_h^0$ and $K \in \mathcal{T}_h^M$, namely $u \in H^{s_K+1}(K)$. By (4.4), under the notation $\nu = \min(p, \kappa)$, for $K \in \mathcal{T}_h^0$ we have

$$\begin{aligned} \|\eta\|_{L^2(K)} &\le C_A\, h_K^{\nu+1} |u|_{H^{\nu+1}(K)}, \\ |\eta|_{H^1(K)} &\le C_A\, h_K^{\nu} |u|_{H^{\nu+1}(K)}, \\ |\eta|_{H^2(K)} &\le C_A\, h_K^{\nu-1} |u|_{H^{\nu+1}(K)}, \end{aligned} \tag{6.1}$$

with $\nu = \min(p, \kappa)$. If $K \in \mathcal{T}_h^M$, then by (4.5)

$$\begin{aligned} \|\eta\|_{L^2(K)} &\le C_A\, h_K^{1+\gamma} |u|_{H^{1+\gamma}(K)}, \\ |\eta|_{H^1(K)} &\le \tilde{C}_A\, h_K^{\gamma} |u|_{H^{1+\gamma}(K)}, \\ |\eta|_{H^{1+\gamma}(K)} &\le \tilde{C}_A\, |u|_{H^{1+\gamma}(K)}. \end{aligned} \tag{6.2}$$

Now (6.1), (6.2) and (5.19) immediately yield the estimate

$$\|e_h\|_{DG} \le C \left(\sum_{K \in \mathcal{T}_h^0} h_K^{2\nu} |u|^2_{H^{\nu+1}(K)} + \sum_{K \in \mathcal{T}_h^M} h_K^{2\gamma} |u|^2_{H^{1+\gamma}(K)} \right)^{1/2}, \quad h \in (0, h_0). \tag{6.3}$$

Using the inequality $h_K \le h$, we get

$$\|e_h\|_{DG} \le C \left(h^{2\nu} |u|^2_{H^{\nu+1}(\tilde{\Omega})} + h^{2\gamma} |u|^2_{H^{\gamma+1}(\Omega)} \right)^{1/2}, \tag{6.4}$$

where $\tilde{\Omega}$ is a a subdomain of Ω satisfying the condition $\operatorname{dist}(\tilde{\Omega}, M) > 0$. Hence, since $\gamma < \nu$, $\|e_h\|_{DG} = O(h^{\gamma})$. On the other hand, in the case that the exact solution u would satisfy the regularity condition $u \in H^{\kappa+1}(\Omega)$, $\kappa \ge 1$, we would get the standard error estimate $\|e_h\|_{DG} = O(h^{\nu})$. Our further goal is to obtain this estimate also in the case of singular behaviour of the exact solution in the vicinity of the set M. To this end we use suitable *graded meshes*.

7 Discontinuous Galerkin method on graded partitions

We again consider a family of partitions $\mathcal{T}_h$ of $\overline{\Omega}$ with the properties introduced in Sections 3 and 4. We shall use the notation $k = \nu - 1$ and $\beta = 1 - H_0 + \varepsilon$ (with $\varepsilon > 0$ arbitrarily small). It is possible to write $\gamma = 1 - \beta$.

In order to treat the singularities of the solution near the irregular part M of the boundary, we assume that the partition $\mathcal{T}_h$ is graded in the following way:

$$\begin{array}{lll} \text{if } K \in \mathcal{T}_h^M, & \text{then} & \underline{C}_1 h^{1/\mu} \leq h_K \leq \overline{C}_1 h^{1/\mu}, \\ \text{if } K \in \mathcal{T}_h^0, & \text{then} & \underline{C}_2 h r_K^{1-\mu} \leq h_K \leq \overline{C}_2 h r_K^{1-\mu}, \end{array} \tag{7.1}$$

where we use the notation from (5.18), $r_K = \operatorname{dist}(K, M)$ and $\mu \in (0, 1]$ is a parameter to control the grading. Note that for $\mu = 1$ an unrefined partition is produced.
Examples for the construction of graded partitions can be found in [4],[2],[3].

In order to obtain the optimal error estimate of order $O(h^\nu)$, we shall analyze the individual terms in the abstract error estimate (5.19). By ∇_ℓ we shall denote the vector of all partial derivatives of order ℓ. We shall prove the following result.

Theorem 5. *Let u be the exact solution of problem (1.1) - (1.3) and let $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ be a regular system of triangulations of the domain Ω satisfying assumption (A1) and graded towards the set M according to (7.1). If u_h is the approximate solution of problem (1.1) - (1.3) obtained by the discontinuous Galerkin method (3.16) under condition (2.48), then there exists a constant $C > 0$ such that the error $e_h = u_h - u$ satisfies the estimate*

$$\|e_h\|_{DG} \leq C h^{\tilde{\alpha}} \left(\|u\|^2_{V^{2+k,2}(\Omega,\beta+k)} + |u|^2_{H^{1+\gamma}(\Omega)} \right)^{1/2}, \quad h \in (h, \mathit{min}(h_0, 1)), \tag{7.2}$$

where

$$\tilde{\alpha} = \begin{cases} k+1 & \textit{for } \mu < \frac{H_0}{k+1}, \\ \frac{H_0 - \varepsilon}{\mu} & \textit{for } \mu \geq \frac{H_0}{k+1}, \end{cases} \tag{7.3}$$

with $\varepsilon > 0$ arbitrarily small.

Proof. The derivation of the error estimate (7.2) is based on estimate (5.19). It is necessary to distinguish two cases.

I) Let us assume that an element K is a subset of the regularity region, i.e. $K \in \mathcal{T}_h^0$ and the exact solution $u \in H^{\nu+1}(K)$, $\nu \geq 1$. We set $k = \nu - 1$. In view of (5.19), it is necessary to estimate the interpolation error $\eta = \pi_h u - u$ in the seminorms $|\cdot|_{H^1(K)}$ and $|\cdot|_{H^2(K)}$ and in the norm $\|\cdot\|_{L^2(K)}$.

Using (4.5) and setting $\beta = 1 - H_0 + \varepsilon$ (where $\varepsilon > 0$ is arbitrarily small), we get

$$\begin{aligned} \|\eta\|^2_{H^1(K)} &\leq C h_K^{2(k+1)} \|\nabla_{k+2} u\|^2_{L_2(K)} \\ &\leq C h_K^{2(k+1)} r_K^{-2(\beta+k)} \int_K r_K^{2(\beta+k)} |\nabla_{2+k} u|^2 \mathrm{d}x \\ &\leq C h_K^{2(k+1)} r_K^{-2(\beta+k)} \int_K r^{2(\beta+k)} |\nabla_{2+k} u|^2 \mathrm{d}x \\ &\leq C h_K^{2(k+1)} r_K^{-2(\beta+k)} \|u\|^2_{V^{2+k,2}(K,\beta+k)}. \end{aligned} \tag{7.4}$$

Similarly we get

$$\begin{aligned} h_K^2 |\eta|^2_{H^2(K)} &\leq C h_K^{2(k+1)} \|\nabla_{k+2} u\|^2_{L^2(K)} \\ &\leq C h_K^{2(k+1)} r_K^{-2(\beta+k)} \|u\|^2_{V^{2+k,2}(K,\beta+k)} \end{aligned}$$

and

$$h_K^{-2} \|\eta\|_{L^2(K)} \leq C h_K^{2(k+1)} r_K^{-2(\beta+k)} \|u\|^2_{V^{2+k,2}(K,\beta+k)}.$$

Hence, we have

$$\Phi_K(\eta) := |\eta|^2_{H^1(K)} + h_K^2 |\eta|^2_{H^2(K)} + h_K^{-2} \|\eta\|^2_{L^2(K)} \leq C h_K^{2(k+1)} r_K^{-2(\beta+k)} \|u\|^2_{V^{2+k,2}(K,\beta+k)}. \tag{7.5}$$

Due to Corollary 1, we have that $u \in V^{2+k,2}(\Omega, \beta+k)$, which implies that that $\|u\|^2_{V^{2+k,2}(K,\beta+k)} < \infty$.

Now, we exploit the grading of the mesh, namely, the inequality $h_K \le \overline{C}_2 h r_K^{1-\mu}$. This implies that

$$h_K^{2(k+1)} r_K^{-2(\beta+k)} \le Ch^{2(k+1)} r_K^{(1-\mu)2(k+1)-2(\beta+k)} \le Ch^{2(k+1)}$$

for

$$\mu \le \frac{1-\beta}{k+1} = \frac{H_0 - \varepsilon}{k+1}. \tag{7.6}$$

Because $\varepsilon > 0$ is arbitrarily small this condition reduces to $\mu < \frac{H_0}{k+1}$. Thus, we get in the case $K \cap M = \emptyset$ the estimate

$$\Phi_K(\eta) \le Ch^{2(k+1)} \|u\|^2_{V^{2+k,2}(K,\beta+k)} \quad \text{for } \mu < \frac{H_0}{k+1}. \tag{7.7}$$

For $\mu \ge \frac{H_0}{k+1}$ we use the inequality $h_K \le C r_K$ and for any $\alpha \in (0,1)$ we get

$$h_K^{2(k+1)} r_K^{-2(\beta+k)} = h_K^{2(k+\alpha)} h_K^{2(1-\alpha)} r_K^{-2(\beta+k)} \le Ch^{2(k+\alpha)} r_K^{2(\alpha+k)(1-\mu)} r_K^{2(1-\alpha)} r_K^{-2(\beta+k)}.$$

Now we choose $\alpha \in (0,1)$ in such a way, that the exponent of r_K vanishes, which means that

$$\alpha + k = \frac{H_0 - \varepsilon}{\mu}. \tag{7.8}$$

Since $\mu \ge \frac{H_0}{k+1}$, such α exists. Thus, we get the estimate

$$\Phi_K(\eta) \le Ch^{2(k+\alpha)} \|u\|^2_{V^{2+k,2}(K,\beta+k)} \quad \text{for } \mu \ge \frac{H_0}{k+1}. \tag{7.9}$$

From (7.5), (7.8) and (7.9) we see that

$$\Phi_K(\eta) \le Ch^{2\tilde{\alpha}} \|u\|^2_{V^{2+k,2}(K,\beta+k)}, \tag{7.10}$$

where $\tilde{\alpha}$ is defined by (7.3).

II) If $K \cap M \ne \emptyset$, i.e. $K \in \mathcal{T}_h^M$, then $u \notin H^{k+2}(K)$ in general. We proceed analogously to [4], proof of Theorem 3.3. In view of (5.19), it is necessary to treat the expressions $|\eta|^2_{H^1(K)}, h_K^{-2}\|\eta\|^2_{L^2(K)}$ and $h_K^{2\gamma}|\eta|_{H^{1+\gamma}(K)}$, where $\gamma \in (\frac{1}{2}, 1)$. We know that $u \in H^{1+\gamma}(K)$ as well as $u \in V^{2,2}(K,\beta)$. Moreover, $\beta = 1 - H_0 + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. We shall take into account the relations $\gamma = H_0 - \varepsilon = 1 - \beta > 0$.

Let us set

$$\Psi_K(\eta) = |\eta|^2_{H^1(K)} + h_K^{2\gamma}|\eta|_{H^{1+\gamma}(K)} + h_K^{-2}\|\eta\|^2_{L^2(K)}. \tag{7.11}$$

We have the inequality $\|\eta\|^2_{L^2(K)} \le 2\|u\|^2_{L^2(K)} + 2\|\pi_h u\|^2_{L^2(K)}$ and similar inequalities for $|\eta|^2_{H^1(K)}$ and $|\eta|^2_{H^2(K)}$. Since $1-\beta > 0$ and $r < h_K$, it holds

$$\begin{aligned} |u|^2_{H^1(K)} &= \int_K r^{2(1-\beta)} r^{2(\beta-1)} |\nabla u|^2 \,\mathrm{d}x \\ &\le h_K^{2(1-\beta)} \int_K r^{2(\beta-1)} |\nabla u|^2 \,\mathrm{d}x \\ &\le h_K^{2(1-\beta)} \|u\|^2_{V^{2,2}(K,\beta)}. \end{aligned} \tag{7.12}$$

Further,

$$\begin{aligned} h_K^{-2}\|u\|^2_{L^2(K)} &= h_K^{-2} \int_K r^{2(2-\beta)} r^{2(\beta-2)} |u|^2 \,\mathrm{d}x \\ &\le h_K^{-2} h_K^{2(2-\beta)} \int_K r^{2(\beta-2)} |u|^2 \,\mathrm{d}x \\ &\le h_K^{2(1-\beta)} \|u\|^2_{V^{2,2}(K,\beta)}. \end{aligned} \tag{7.13}$$

In order to estimate $\|\pi_h u\|_{L^2(K)}$, we use the relation

$$\|\pi_h u\|_{L^2(K)} \leq \|u\|_{L^2(K)}, \tag{7.14}$$

which is a consequence of the fact that $\pi_h u|_K$ is the $L^2(K)$-projection of $u|_K$ onto the space S_{hp}. Now we apply (7.13) and get

$$h_K^{-2}\|\pi_h u\|^2_{L^2(K)} \leq h_K^{2(1-\beta)}\|u\|^2_{V^{2,2}(K,\beta)}. \tag{7.15}$$

For the estimation of $|\pi_h u|_{H^1(K)}$ we use the inverse inequality (4.8), inequality (7.14) and estimate (7.13):

$$\begin{aligned} |\pi_h u|_{H^1(K)} &\leq Ch_K^{-1}\|\pi_h u\|_{L^2(K)} \leq h_K^{-1}\|u\|_{L^2(K)} \\ &\leq h_K^{1-\beta}\|u\|_{V^{2,2}(K,\beta)}. \end{aligned} \tag{7.16}$$

Now, it remains to pay attention to the estimation of the term $h_K^\gamma|\eta|_{H^{1+\gamma}(K)} \leq \tilde{C}_A h_K^\gamma |u|_{H^{1+\gamma}(K)}$, as follows from (6.2). Taking into account that $\gamma = H_0 - \varepsilon = 1 - \beta$, we get

$$h_K^\gamma|\eta|_{H^{1+\gamma}(K)} \leq Ch_K^{1-\beta}|u|_{H^{1+\gamma}(K)}. \tag{7.17}$$

Summarizing (7.11), (7.12), (7.13), (7.15) and (7.17), we obtain

$$\Psi_K(\eta) \leq Ch_K^{2(1-\beta)}\left(\|u\|^2_{V^{2,2}(K,\beta)} + |u|^2_{H^{1+\gamma}(K)}\right). \tag{7.18}$$

Now, this estimate, the relation $1-\beta = H_0 - \varepsilon$ and (7.1) (provided $0 < h \leq 1$), imply that

$$\Psi_K(\eta) \leq Ch^{2(k+1)}\left(\|u\|^2_{V^{2,2}(K,\beta)} + |u|^2_{H^{1+\gamma}(K)}\right), \text{ if } \mu < \frac{H_0}{k+1}, \tag{7.19}$$

and

$$\Psi_K(\eta) \leq Ch^{2\frac{H_0-\varepsilon}{\mu}}\left(\|u\|^2_{V^{2,2}(K,\beta)} + |u|^2_{H^{1+\gamma}(K)}\right), \text{ if } \mu \geq \frac{H_0}{k+1}, \tag{7.20}$$

In virtue of the imbedding (2.10), for $K \in \mathcal{T}_h^M$ we get

$$\Psi_K(\eta) \leq Ch^{2\tilde{\alpha}}\left(\|u\|^2_{V^{2+k,2}(K,\beta+k)} + |u|^2_{H^{1+\gamma}(K)}\right) \tag{7.21}$$

Finally, since

$$\|e_h\|^2_{DG} \leq C\left(\sum_{K\in\mathcal{T}_h^0}\Phi_K(\eta) + \sum_{K\in\mathcal{T}_h^M}\Psi_K(\eta)\right),$$

the use of (7.10) and (7.21) yield (7.2), what we wanted to prove. □

□

8 Conclusion

In this paper we have presented theoretical analysis of error estimates for the discontinuous Galerkin finite element method applied on graded meshes to the numerical solution of a 2D elliptic equation in a polygonal domain, equipped with Dirichlet or mixed Dirichlet-Neumann boundary conditions. We considered here the realistic regularity of the exact solution by detailed description of the leading singular terms of the solution. In this way we characterize the behaviour of the solutions near singular boundary points, i.e. corners and the points, where different types of boundary conditions meet. The paper contains an analysis of the regularity of the exact solution, a generalization of some approximation results to functions from Sobolev-Slobodetskii spaces, which allowed to prove an abstract error estimate. With the use of a graded mesh refinement and weighted Sobolev spaces the main result was obtained. It consists in the proof of optimal error estimates of the same order in the maximal mesh size as in the case with a sufficient global regularity of the exact solution.

There are the following subjects for further work:

- analysis of optimal error estimates of the DGFEM in problems with singular boundary points in the L^2-norm,
- treatment of the case when the leading singularity has an exponent $H_0 \leq \frac{1}{2}$.
- analysis of the graded mesh method used in the framework of other versions of the DGFEM, as e.g. local discontinuous Galerkin method (cf. [13]),
- analysis of the effect of numerical integration in the graded mesh discontinuous Galerkin method,
- demonstration of theoretical results by numerical experiments,
- application of graded mesh refinement to the discontinuous Galerkin solution of nonlinear and non-stationary problems,
- extension of the method to three-dimensional problems,
- application of graded mesh discontinuous Galerkin technique to some technically relevant problems.

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